Mathematics 307—November 9, 1995

Matrices and quadratic functions

It very often happens in applications that the coefficient matrix A of a system of linear equations is symmetric, and that no row swapping is necessary in order to solve it by Gauss elimination. In this case we shall have the factorization

$$A = L\,U\ .$$

We can write

$$U = D\,L^*$$

where D is a diagonal matrix whose entries are the same as the diagonal of U, and L^* is an upper triangular matrix all of whose diagonal entries are 1. Because A is symmetric, L^* turns out to be the transpose of L, and we can write

$$A = L\,D\,{}^tL\ .$$

In this section I want to explain the significance of this factorization. It involves a new interpretation of matrices, at least of symmetric ones, which is just as important as their interpretation in terms of linear transformations.

Drawing quadratic curves

I will begin with a fairly elementary problem. *What does the curve*

$$F(x,y) = 9x^2 - 4xy + 6y^2 = 1$$

look like?

The equation looks something like the equation for a circle, but it is difficult to see directly how relevant this is. We can answer one question however, without too much trouble: *Are there points on the curve very far from the origin?* Suppose that (x,y) lies on the curve and that $|y|$ is very large. If we divide the equation by y^2 we get

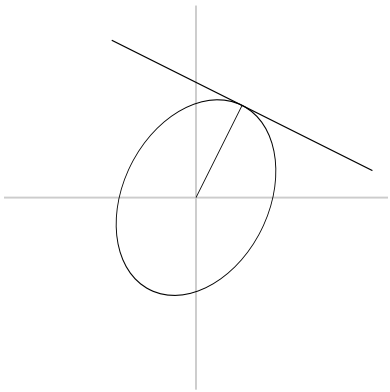
$$9\left(\frac{x}{y}\right)^2 - 4\left(\frac{x}{y}\right) + 6 = \frac{1}{y^2}\ .$$

If y is large enough, the ratio x/y will be arbitrarily close to a root of the equation

$$9r^2 - 4r + 6 = 0\ .$$

However, we can check easily that the roots of this equation are complex. Similarly x cannot be large. Therefore *the curve is bounded.*

Since it is bounded, we can try to locate the points on the curve which are furthest from the origin. Suppose (x,y) is at maximum distance. Then, as the following picture shows, the position vector (x,y) must be parallel to the gradient $(\partial F/\partial x, \partial F/\partial y)$ of the function $F(x,y)$ at that point.



Since

$$\begin{aligned}\frac{\partial F}{\partial x} &= \quad 18x - 4y \\ \frac{\partial F}{\partial y} &= -4x + 12y\end{aligned}$$

we then have

$$\begin{aligned}18x - 4y &= cx \\ -4x + 12y &= cy\end{aligned}$$

for some scalar c. In other words, the vector (x, y) must be an eigenvector for the matrix

$$A = \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix}$$

with eigenvalue c! The eigenvalues of this matrix are

$$c = 10, \quad 20$$

and the eigenvectors are the multiples of

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Notice that they are perpendicular to each other, as they have to be since the matrix is symmetric. To find points on the curve which are eigenvectors we set

$$(x, y) = (a, 2a)$$

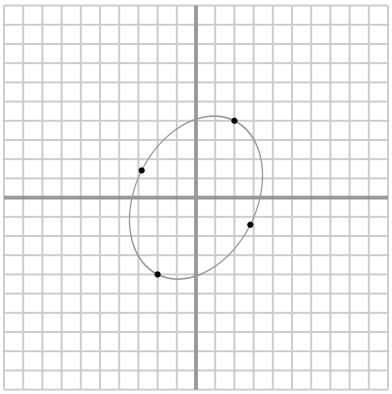
and solve

$$9x^2 - 4xy + 6y^2 = 9a^2 - 8a^2 + 24a^2 = 25a^2 = 1$$

to get $a = \pm 1/5$, $(x, y) = \pm(1/5, 2/5)$, and similarly

$$9x^2 - 4xy + 6y^2 = 36a^2 + 8a^2 + 6a^2 = 50a^2 = 1$$

to get $a = \pm 1/\sqrt{50}$, $(x, y) = \pm(2/\sqrt{50}, -1/\sqrt{50})$. If we plot these points in the plane we get the picture



It certainly looks as if the curve might be an elipse, with major and minor axes through the points in the picture. How can we tell for sure? *We can change coordinates so that the new coordinate axes lie on what we hope to be the axes of the ellipse, and see if the equation of the curve in the new coordinates is recognizably that of an ellipse.*

The new coordinate axes are going to point in the directions of what are $(1, 2)$ and $(-2, 1)$ in the old coordinates. The new coordinates can therefore be chosen as

$$x_* = x + 2y$$
$$y_* = 2x - y$$

since then the equations $x_* = 0$ and $y_* = 0$ are what we want as the new axes. This change of coordinates is not orthogonal, but it is the combination of an orthogonal change of coordinates with a uniform scale change, and therefore preserves shape.

We want to find the new equation for the curve expressed in terms of x_* and y_*, and to do this we must substitute for x and y in terms of x_* and y_*. If we solve the system above we get

$$x = \frac{x_* + 2y_*}{5}$$
$$y = \frac{2x_* - y_*}{5}$$

and if we substitute into the equation of the curve we get

$$\frac{1}{25}[9(x_* + 2y_*)^2 - 4(x_* + 2y_*)(2x_* - y_*) + 6(2x_* - y_*)^2] = 1$$

or

$$x_*^2 + 2y_*^2 = 1$$

This is the equation of an ellipse with its major axis equal to the x_* axis, and minor axis equal to the y_* axis. The sizes are different from that of the original, since the coordinate change we made was not orthogonal. The ratio of the size of the axes is $\sqrt{2}$.