Mathematics 307—November 16, 1995

Matrices and quadratic functions II

The main points of the earlier discuss about quadratic curves

$$ax^2 + bxy + cy^2 = 1$$

were (1) the best way to draw it is to perform an orthogonal change of coordinates so that in the new coordinates (x_*, y_*) it becomes

$$\lambda x_*^2 + \mu y_*^2 = 1$$

which is simple to draw; (2) in order to find the proper change of coordinates and the constants λ, μ we must find the eigenvectors and eigenvalues of a certain symmetric matrix associated to the quadratic function. The particular matrix that came up in that discussion had something to do with gradients. In this section we shall see these points covered more formally, using a slightly different idea in order to relate quadratic functions to symmetric matrices. Furthermore, we shall see that something similar can be done in any number of dimensions.

The quadratic function $ax^2 + bxy + cy^2$ is called **homogeneous** because all terms are of degree two. Other quadratic functions might include some linear terms or constant terms, for example $x^2 + y$, but I won't consider those here. If we are given the (homogeneous) quadratic function $Q(x, y) = ax^2 + bxy + cy^2$ then we associate to it the matrix

$$A_Q = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} .$$

In other words, we lay the coefficients of x^2 and y^2 along the diagonal and spread the coefficient of xy into two halves, each in one of the off-diagonal locations. This choice is motivated by these two considerations: (1) the matrix we get is symmetric; (2) the relation betwen the quadratic function and the matrix is that

$$Q(x, y) = ax^2 + bxy + cy^2 = [\,x \quad y\,] \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

as you can check for yourself by multiplying it out. In vector notation

$$Q(v) = {}^t v \, A_Q \, v$$

where v is a column vector.

It is extremely important to realize that *this process gives essentially a new interpretation of matrices*, at least the symmetric ones. Another important thing to realize is that this way of constructing a matrix depends on the coordinate system being used. The function $Q(x, y)$ is something with an existence independent of a choice of coordinates; it just asigns a number to each point of the plane, and if the point has (x, y) as its coordiantes in the conventional coordinate system the value of this number is $ax^2 + bxy + cy^2$. If we change coordinates we shall get a new expression. If, for example we make a change of coordinates to (x_*, y_*) here $x_* = 2x$, $y_* = y/2$ then the expression for Q in the new coordinates is

$$\left(\frac{x_*}{2}\right)^2 + \left(\frac{x_*}{2}\right)(2y_*) + (2y_*)^2 = \frac{x_*^2}{4} + x_* y_* + 4y_*^2 .$$

There is a fairly simple formula for changing coordinates in terms of matrix multiplication.

Proposition. *If A_E is the symmetric matrix associated to Q in the E-coordinate system and F is another basis, then*

$$A_F = {}^t F \, A_E \, F .$$

Compare this with the formula for changing coordinates for linear transformations:

$$A_F = F^{-1}\, A_E\, F$$

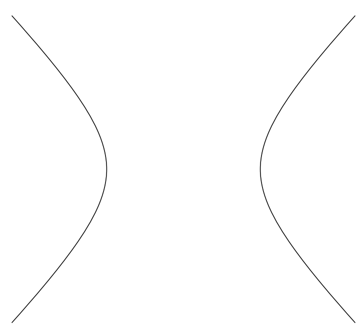
The two formulas give the same result when it happens that ${}^tF = F^{-1}$*, or in other words precisely when* F *is an orthogonal matrix, or when we are making an orthogonal chnage of coordinates.*

Since A is a symmetric matrix, we know that if we let F be a matrix whose columns are normalized eigenvectors, then F will be an orthogonal matrix and $F^{-1}A_E F$ will be diagonal, with entries equal to the eigenvalues λ, μ. If we make this coordinate change, therefore, the new matrix will be

$$\lambda x_*^2 + \mu y_*^2$$

In making this coordinate change all formulas for distances remain the same, so we can read off the exact shape of the curve $Q = 1$ from the information we now have.

In particular: (1) if λ and μ are both positive, then the curve is an ellipse. Its axes lie along the lines of eigenvectors, and the lengths of its semi-axes are $1/\sqrt{\lambda}$, $1/\sqrt{\mu}$. (2) If one is negative and the other positive, then we have an hyperbola. How it lies depends on which is positive. For example, the curve $x^2 - y^2 = 1$ looks like this:



The same ideas work in any number of dimensions.A homogeneous quadratic function looks like this:

$$Q(x_1, x_2, \ldots, x_n) = q_{1,1}x_1^2 + q_{1,2}x_1x_2 + \cdots + q_{n,n}x_n^2$$

and corresponds to the matrix

$$A_Q = \begin{bmatrix} q_{1,1} & q_{1,2}/2 & \ldots \\ q_{1,2}/2 & q_{2,2} & \ldots \\ \ldots & & \end{bmatrix}$$

The way this works depends on which coordinate system we are using. The exact relationship between coordinates, quadratic functions, and symmetric matrices is thus

$$Q(x) = {}^tx_E\, A_{Q,E}\, x_E$$

If we change coordinates we have

$$\begin{aligned} x_E &= F x_F \\ {}^tx_E\, A_{Q,E}\, x_E &= {}^tx_F\, {}^tF\, A_{Q,E}\, F x_F \\ A_{Q,F} &= {}^tF\, A_{Q,E}\, F \end{aligned}$$

To write Q in diagonal form we let F have a normalixed orthogonal set of eigenvectors as its columns. But is important to realize that we can also use non-orthogonal changes of coordinates, and still get some interesting information.

In $3D$ the equation $Q(x) = 1$ generally describes a surface. We have the following classification:

Sign distribution	example	surface type
$+++$	$x^2 + y^2 + z^2 = 1$	ellipsoid
$++-$	$x^2 + y^2 - z^2 = 1$	one-sheeted hyperboloid
$+++$	$x^2 - y^2 - z^2 = 1$	two-sheeted hyperboloid

I'm afraid they are a bit too complicated for me to illustrate here. Next year, perhaps.

Gauss elimination and symmetric matrices

The point is that if we have a positive definite matrix A we can apply Gaussian elimination to it to get

$$A = L\,D\,{}^tL$$

where D has all positive entries, so we can take a square root and write

$$A = L_*{}^tL_*$$

where now L does not necessarily have 1's down the diagonal.

This can be used to solve the **generalized eigenvalue** problem

$$Kv = \lambda M v$$

where M is positive definite, by factoring M in this way and bringing the factors to the left.