Mathematics 307—September 25, 1995

Rotations in $3D$

Specifying a rotation in $3D$ amounts to giving an **axis** and an **angle of rotation**. The axis must be given an **orientation** in order to know the direction of the rotation. Conventionally, this specification follows the **right hand rule**, thumb pointed in the direction of the oriented axis, fingers curled in the direction of positive rotation. The simplest rotations are around one of the axes. For example, rotation around the z-axis fixes all points on the z-axis and rotates points in the (x, y) plane according to a $2D$ rotation. Its matrix is therefore

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first problem I'll attack here is: *given the vector the vector u and the angle θ, what is the matrix of the rotation through angle θ around the axis in the direction of u?*

Suppose we choose a basis consisting of three vectors u_1, u_2, u_3 chosen according to the following recipe: (1) u_3 is the vector u normalized: $u_3 = u/\|u\|$, where $\|u\|$ is the length of the vector u. (2) u_1 is any vector of length 1 in the plane perpendicular to u. (3) $u_2 = u_3 \times u_1$. I will say in a moment how one can find u_1, from which u_2 can be calculated. Then in the U-coordinate system the rotation looks just like rotation around the z-axis, so M_U is the matrix above. If we let U be the matrix whose columns are the coordinate vectors of u_1, u_2, and u_3, then the matrix we are looking for is

$$R(u,\theta) = U M_U U^{-1} = [\, u_1 \quad u_2 \quad u_3 \,] \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} [\, u_1 \quad u_2 \quad u_3 \,]^{-1}$$

We have seen an example on the first homework.

A rotation preserves the lengths of all vectors, only changing their direction. It also preserves the angles between any two vectors. Since the columns of $R(u,\theta)$ are the images of the three vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ the matrix M of $R(u\theta)$ has these properties:

(R1) Each column of M has length 1.
(R2) Any distinct pair of columns are mutually perpendicular.

Equivalently, *a matrix is orthogonal when its inverse is its transpose*, because the calculation of ${}^tX\,X$ involves calculating all the various dot products of its columns.

Furthermore, a rotation preserves orientation in space (as opposed to a reflection in a plane, which reverses it). Whether a linear transformation preserves or changes orientation depends on its determinant. Any linear transformation scales volumes by a factor equal to the absolute value of its determinant. Since a rotation preserves orientation and does not change volumes, the matrix of a rotation also has this property:

(R3) The determinant of M is equal to 1.

A linear transformation is called **orthogonal** if it preserves the lengths of vectors and the angles between them. It is called **special orthogonal** if it preserves orientation as well.

Proposition. *A linear transformation is orthogonal precisely when its matrix with respect to the (x, y, z) coordinate system has properties (R1)–(R2). It is special orthogonal when it satisfies (R3) in addition.*

What may not be so clear is why a linear transformation T is orthogonal if its matrix, say M, has these properties. Suppose it does.

Lemma. *In these circumstances, if u and v are be any 3D vectors, then the dot product $Tu \bullet Tv$ is equal to $u \bullet v$.*

The assumption on M is that this is true if u and v are any of the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Let $u = (x_u, y_u, z_u)$ and $v = (x_v, y_v, z_v)$, so

$$\begin{aligned} u &= x_u\,\mathbf{i} + y_u\,\mathbf{j} + z_u\,\mathbf{k} \\ v &= x_v\,\mathbf{i} + y_v\,\mathbf{j} + z_v\,\mathbf{k} \\ T(u) &= x_u T(\mathbf{i}) + y_u T(\mathbf{j}) + z_u T(\mathbf{k}) \\ T(v) &= x_v T(\mathbf{i}) + y_v T(\mathbf{j}) + z_v T(\mathbf{k}) \end{aligned}$$

Then

$$\begin{aligned} T(u) \bullet T(v) &= (x_u T(\mathbf{i}) + y_u T(\mathbf{j}) + z_u T(\mathbf{k}))(x_v T(\mathbf{i}) + y_v T(\mathbf{j}) + z_v T(\mathbf{k})) \\ &= x_u x_v T(\mathbf{i}) \bullet T(\mathbf{i}) + x_u y_v T(\mathbf{i}) \bullet T(\mathbf{j}) + x_u z_v T(\mathbf{i}) \bullet T(\mathbf{k}) \\ &\quad + y_u x_v T(\mathbf{j}) \bullet T(\mathbf{i}) + y_u y_v T(\mathbf{j}) \bullet T(\mathbf{j}) + y_u z_v T(\mathbf{j}) \bullet T(\mathbf{k}) \\ &\quad + z_u x_v T(\mathbf{k}) \bullet T(\mathbf{i}) + z_u y_v T(\mathbf{k}) \bullet T(\mathbf{j}) + z_u z_v T(\mathbf{k}) \bullet T(\mathbf{k}) \\ &= x_u x_v + y_u y_v + z_u z_v \\ &= u \bullet v\,. \end{aligned}$$

But now both lengths and angles are determined in terms of dot products, so if a linear transformation preserves dot products it must preserve lengths and angles as well. Q.E.D.

Every rotation is an orthogonal linear transformation, and it is relatively simple to visualize how a rotation acts on points in space. If we apply one rotation after another, possibly with different axes, it is not so simple to visualize the combined effect, although the combination certainly also preserves lengths and angles. It may therefore be surprising to know that the combination is also a rotation. In fact:

Proposition. *Every special orthogonal linear transformation is a rotation.*

In other words, if M is a special orthogonal matrix then there exists an axis and an angle θ such that the linear transformation associated to M amounts to rotation of θ around the axis. I shall explain exactly how to find the axis and the angle.

The vectors on the axis are all fixed by a rotation. Therefore in order to find the axis we must see that there is a line of vectors fixed by M, or in other words that 1 is an eigenvalue of M.

We know that every matrix has at least one real eigenvalue, and that it scales the corresponding eigenvectors by that eigenvalue. We also know that there are two possibilities for M—either it has three real eigenvalues or exactly one. If M preserves lengths, the only possible scale factor is ± 1. We argue by cases: (1) suppose M has exactly one real eigenvalue. We know it is then similar to a matrix of the form

$$\begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

where ϵ is the single real eigenvalue. The determinant of this matrix is ϵ, and since we are assuming M special orthogonal, $\epsilon = 1$.

(2) Suppose M has three real eigenvalues equal to ± 1. If all three are -1, then the determinant of M is equal to -1, again a contradiction.

Since M takes vectors on the axis to themselves, and it preserves angles, it takes vectors on the plane perpendicular to the axis into other vectors on that plane. In other words, it acts a two-dimensional linear transformation on that plane. So the original question about three dimensional linear transformations has been replaced by one about two-dimensional ones.

In other words, we now want to show that if T is a special orthogonal linear transformation in two dimensions, it amounts to a rotation. This can be seen geometrically. Let e_1 and e_2 be two unit vectors in the plane, perpendicular to each other. The transformation T must takes e_1 into another unit vector since it preserves lengths. So T rotates e_1 by some angle, say θ. It must take e_2 into a unit vectors perpendicular to $T(e_1)$. Since the determinant of T is 1, it preserves the orientation of the pair, so we must get $T(e_2)$ by rotating $T(e_1)$ by $90°$. In other words, $T(e_2)$ is obtained from e_2 by rotating through θ also.

Exercise. *Find the axis and angle for the special orthogonal matrix*

$$\begin{bmatrix} 0.500000 & -0.734431 & -0.458924 \\ 0.734431 & 0.640407 & -0.224699 \\ 0.458924 & -0.224699 & 0.859593 \end{bmatrix}$$