Mathematics 307—September 25, 1995

Symmetric matrices

A **symmetric** matrix is one which is identical to its own transpose.

Proposition. *The eigenvalues of a symmetric matrix are all real. If u and v are eigenvectors for distinct eigenvalues then $u \bullet v = 0$.*

The basic property from which these results follow is this: if M is any matrix and u and v are arbitrary vectors then

$$Mu \bullet v = u \bullet {}^t M v \ .$$

This is because the dot product can be described in terms of a matrix product and transposition: if u and v are column vectors then

$$u \bullet v = {}^t u \, v$$

where the product on the right is the matrix product of the row vector ${}^t U$ and the column vector v. Therefore

$$Mu \bullet v = {}^t(Mu)\, v = {}^t u \, {}^t M v = u \bullet {}^t M v \ .$$

Lemma. *If M is symmetric then*

$$Mu \bullet v = u \bullet Mv \ .$$

Suppose now that M is symmetric, λ an eigenvalue, u a (possibly complex) eigenvector. Then

$$\begin{aligned} Mu &= \lambda u \\ M\overline{u} &= \overline{\lambda u} \\ Mu \bullet \overline{u} &= \lambda \, u \bullet \overline{u} \\ &= u \bullet M\overline{u} \\ &= u \bullet \overline{\lambda u} \\ &= \overline{\lambda} \, u\overline{u} \end{aligned}$$

Here $\overline{x}$ means the complex conjugate of x. Since

$$u \bullet \overline{u} = u_1\overline{u}_1 + \cdots + u_n\overline{u}_n$$

and $z\overline{z} \geq 0$ unless $z = 0$, we see that $\lambda = \overline{\lambda}$, which means that λ is real.

If $Mu = \lambda u$, $Mv = \mu v$ then

$$Mu \bullet v = \lambda\,(u \bullet v) = u \bullet Mv = \mu\,(u \bullet v)$$

so that if $\lambda \neq \mu$, $u \bullet v = 0$. Q.E.D.

Corollary. *If T is a linear transformation with a symmetric matrix then we can find an orthonormal basis of eigenvectors for T.*

Since we can always change the sign of an eigenvector if necessary:

Corollary. *If M is a symmetric matrix then we can find a special orthogonal matrix X such that $X^{-1}MX$ is diagonal.*

In other words, the linear transformations associated to symmetric matrices amount to scale changes in perpendicular directions—perhaps the simplest of all linear transformations to visualize.