Mathematics 307—December 6, 1995

Linear transformations and coordinate changes in three dimensions

Three dimensional linear transformations are not formally different from those in two dimensions. There are three vectors in a basis, and vectors in a basis must have the property that they do not all lie in one plane.

The earlier results about coordinate change and the way a matrix changes when we change basis can be stated in exactly the same way in three, or even any number, of dimensions.

What is different is that with one more dimension to move around in, there is a slightly wider variety of linear transformations possible. Furthermore, calculations can be quite a bit more difficult.

If

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix}$$

then eigenvectors of M are vectors which are scaled by M:

$$M u = \lambda u$$

for some scaling factor λ. This can be rewritten as

$$(M - \lambda I)\, u = 0\ .$$

For any λ, there is always the solution where u is the 0 vector, but if λ is an eigenvalue there must exist some non-zero solution as well. If the matrix $M - \lambda I$ is non-singular—that is to say if it is invertible—then we can apply its inverse to see that it has a unique solution, which must be just the zero solution. So *if λ is an eigenvalue the matrix $M - \lambda I$ must be singular—non-invertible—and in particular its determinant must vanish.* Conversely, if it is singular then we can find at least a whole one-dimensional family of solutions to the equations above, and λ will be an eigenvalue. This argument is not special to three dimensions.

Proposition. *The eigenvalues of any matrix are the roots of its characteristic polynomial* $\det(M - \lambda I) = 0$, *and the eigenvectors for λ are the solutions of the matrix equation*

$$(M - \lambda I)\, u = 0\ .$$

For calculations, keep in mind that the characteristic polynomial of the 3×3 matrix M is

$$\lambda^3 - (m_{1,1} + m_{2,2} + m_{3,3})\lambda^2 + \left(\begin{vmatrix} m_{2,2} & m_{2,3} \\ m_{3,2} & m_{3,3} \end{vmatrix} + \begin{vmatrix} m_{1,1} & m_{1,3} \\ m_{3,1} & m_{3,3} \end{vmatrix} + \begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \right) \lambda - \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{vmatrix}\ .$$

The coefficient of λ^2 is the negative **trace** of M, and the last term is the negative determinant.

Finding the roots of a polynomial of degree greater than two is not so simple as it is for degree two, and this already complicates the situation.

Another complication in $3D$ is that it is far harder to picture three-dimensional objects than two-dimensional ones. Formally, however, the classification of $3D$ linear transformations is not so different from $2D$ ones.

Suppose M to be a given 3×3 matrix. Consider its characteristic polynomial, of degree three. There are these possibilities: (1) it has three real roots, or (2) it has one real root and two complex conjugate roots. In the first case one of several possibilities may occur: (a) The three real roots are all distinct. In this case we can find three linearly independent eigenvectors. (b) Two of the roots are the same. This breaks up further into cases: (i) we can find two linearly independent eigenvectors for the repeated root; (ii) we cannot. (c) All three of the roots are the same. In this case we have either (i) there exist three linearly independent eigenvectors for this single root; (ii) there exists only two linearly independent eigenvectors for it; (iii) there exists only a line of eigenvectors for it.

We can exhibit examples of each kind of behaviour, which classify all matrices up to similarity. In the first set of examples, a, b, c are meant to be distinct real numbers.

Example. (1)(a)

$$S = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Matrices similar to one of this type are characterized by the property that they have three distinct real eigenvalues. If

$$X = [\, v_1 \quad v_2 \quad v_3 \,]$$

where the v_i are eigenvectors for a etc, then $X^{-1}MX = S$.

Example. (1)(b)(i)

$$S = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

Matrices similar to this matrix are those with two real eigenvalues a and b, where a is a repeated root with multiplicity two, and we can can find two linearly independent eigenvectors for a. In other words, the matrix $M - aI$ has rank one (and $M - bI$ has rank two). Again, if we make up a matrix X whose columns are independent eigenvectors, then $X^{-1}MX = S$.

Example. (1)(b)(ii)

$$S = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

Here a is a repeated root, but $M - aI$ has rank two, so we can only find a one-dimensional line of eigenvectors. Let v_3 be an eigenvector for b, v_1 one for a. We want to find a third vector v_2 such that the matrix of M with respect to the basis v_1, v_2, v_3 is S. This means that

$$Mv_1 = av_1, \quad Mv_2 = av_2 + v_1, \quad Mv_3 = bv_3 \ .$$

In other words

$$(M - aI)v_2 = v_1 \ .$$

So we must solve this equation for v_1. This will be possible because $M - aI$ has rank two.

Proposition. *Suppose that M has eigenvalues $a \neq b$, with a a repeated root. If we can find only one independent eigenvector for a, then M is similar to S.*

Let v_1 be an eigenvector for a, v_3 one for b. We know that the linear transformation associated to M takes the line through v_3 into itself. I claim that we can find a two dimensional plane which intersects this line only at the origin, and which is taken into itself by M. In fact, if we let u_2 be any vector independent of v_1 and v_3 then the matrix of T with respect to the basis v_1, u_2, v_3 looks like this:

$$\begin{bmatrix} a & a_{1,2} & 0 \\ 0 & a_{2,2} & 0 \\ 0 & a_{2,3} & b \end{bmatrix}$$

The characteristic polynomial of M is then $\lambda - b$ times that of the 2×2 matrix A in the upper left, which therefore must be $(\lambda - a)^2$, since a is a repeated root. So $a_{2,2} = a$, and our matrix is

$$\begin{bmatrix} a & a_{1,2} & 0 \\ 0 & a & 0 \\ 0 & a_{2,3} & b \end{bmatrix}$$

We have

$$Tu_2 = au_2 + a_{1,2}v_1 + a_{3,2}v_3$$

I claim now that we can replace u_2 by a new vector of the form $u_2 + xv_3$ so that the new matrix becomes

$$\begin{bmatrix} a & a_{1,2} & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

(with a different $a_{1,2}$). We have

$$\begin{aligned} T(u_2 + xv_3) &= au_2 + a_{1,2}v_1 + a_{3,2}v_3 + bxv_3 \\ &= a(u_2 + xv_3) - axv_3 + a_{1,2}v_1 + a_{3,2}v_3 + bxv_3 \end{aligned}$$

which means that we must solve

$$(b - a)x + a_{3,2} = 0$$

which we can do because $a \neq b$. From here we make a simple scale change of v_1 to get the matrix

$$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

Example. (1)(c)(i)

$$S = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

In order for M to be similar to S, M must in fact be S itself.

Example. (1)(c)(ii)

$$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

This case is distinguished by the property that $M - aI$ has rank one. Suppose we can find independent eigenvectors v_1 and v_2, and let u_3 be any third independent vector. Then the matrix in this basis is

$$\begin{bmatrix} a & 0 & a_{1,3} \\ 0 & a & a_{2,3} \\ 0 & 0 & a \end{bmatrix}$$

where $a_{1,3}$ and $a_{2,3}$ are not both 0. The vector $a_{1,3}v_1 + a_{2,3}v_2$ is therefore non-zero, and we can replace one of v_1 or v_2, say v_1, by it to get a new basis, with $Tv_3 = v_3 + v_1$. For the basis v_1, v_3, v_2 we get the matrix S.

Let me do an example. Suppose

$$M = \begin{bmatrix} -2 & -1 & -2 \\ -3 & 0 & -2 \\ 6 & 2 & 5 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1$$

and it has a single root $\lambda = 1$. The matrix $M - I$ is

$$\begin{bmatrix} -3 & -1 & -2 \\ -3 & -1 & -2 \\ 6 & 2 & 4 \end{bmatrix}$$

It has rank one since all rows are multiples of the first. So the eigenvector equation is

$$3x + y + 2z = 0$$

and the eigenvectors make up a plane. We can find two linearly independent ones. If we set $x = 0$ we get $(0, -2, 1)$ and if we set $y = 0$ we get $(2, 0, -3)$. We can choose $u_3 = (0, 0, 1)$. The matrix in this coordinate system is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

this says that $Tu_3 = u_3 - v_2 + v_1$. We therefore set $v_3 = u_3$, and replace v_2 by $-v_2 + v_1$. Then $Tv_3 = v_3 + v_2$, and still $Tv_2 = v_2$.

Example. (1)(c)(iii)

$$S = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$$

This case is characterized by the condition that the rank of $M - aI$ is two. If we choose v_1 to be an eigenvector and u_2 and u_3 independent, the matrix becomes

$$\begin{bmatrix} a & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & a_{3,2} & a_{3,3} \end{bmatrix}$$

The matrix in the lower right must have an eigenvector, say with coordinates (x, y). then if we replace u_2 by $xu_2 + yu_3$ the matrix will become

$$\begin{bmatrix} a & a_{1,2} & a_{1,3} \\ 0 & a & a_{2,3} \\ 0 & 0 & a \end{bmatrix}$$

Replacing v_1 we can make this

$$\begin{bmatrix} a & 1 & a_{1,3} \\ 0 & a & a_{2,3} \\ 0 & 0 & a \end{bmatrix}$$

and finally replacing v_2 we can make it

$$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$$

Suppose

$$M = \begin{bmatrix} -4 & -3 & -4 \\ -4 & -1 & -3 \\ 9 & 5 & 8 \end{bmatrix}$$

The characteristic polynomial is again

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1$$

with repeated roots. Here

$$M - I = \begin{bmatrix} -5 & -3 & -4 \\ -4 & -2 & -3 \\ 9 & 5 & 7 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 9 & 5 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors are solutions of the pair of equations

$$\begin{aligned} 9x + 5y + 7z &= 0 \\ 2y + z &= 0 \end{aligned}$$

If we set $z = 1$ we get $y = -1/2$, $9x = -5y - 7z$, $x = -1/2$. So $(-1, -1, 2)$ is an eigenvector. Set v_1 equal to it. For v_2 and v_3 we want

$$\begin{aligned} (M - I)v_2 &= v_1 \\ (M - I)v_3 &= v_2 \end{aligned}$$

which we can solve in turn. Each has many solutions, but at least one.

Example. (2)

$$S = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

Here we may as well assume $b > 0$. This case is characterized by the property that M has one real eigenvalue and two conjugate complex ones. We can find X in this case just as we did for the two dimensional case: let U be a matrix whose columns are the eigenvectors $[u\overline{u}v]$, and

$$U_0 = \begin{bmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and set $X = UU_0^{-1}$.

The reasoning above, summarized, shows that in fact

Proposition. *Every $3D$ matrix is similar to exactly one of the types above.*

There are again a small number of types. The basic idea is that we can extend the two-dimensional types to three dimensions by scaling along the third dimension, which gives us most of the typoes above, or we can add one new type which is the combination of a uniform scale change with a new type of shear.

In other words, again the basic classification is into **scale changes**, **rotations**, and **shears**, but with essentially two kinds of shear, illustrated by these examples:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The variety of each type is larger, too. For example, true rotations in $3D$ are specified by a choice of oriented axis and angle of rotation.