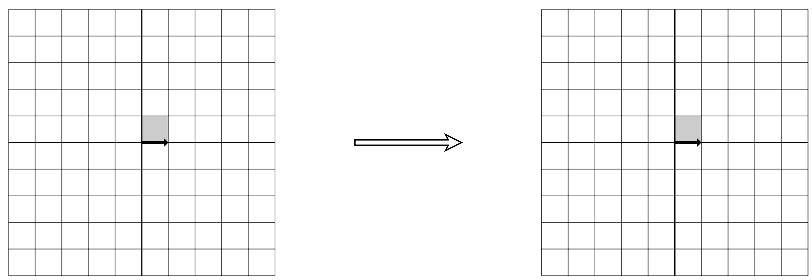
Mathematics 307—October 11, 1995

The geometry of linear transformations in two dimensions

Fix a basis e_1, e_2 for a plane. Having chosen this basis, certain standard linear transformations can be specified.

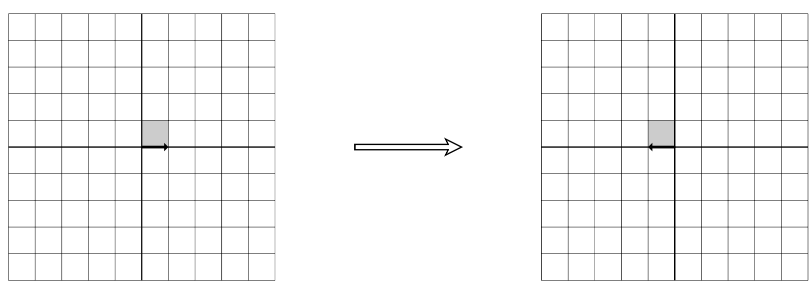
The identity map



It takes any point to itself. It takes e_1 to e_1 and e_2 to e_2. Its matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

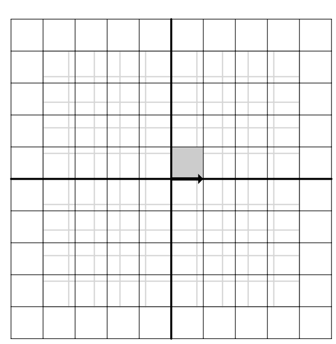
Reflection in the y-axis



It takes e_1 to $-e_1$, e_2 to itself. The matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} .$$

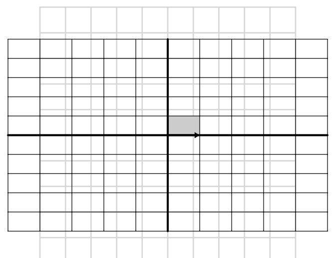
Uniform scaling by c



It takes any x to cx. Its matrix is

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} .$$

Non-uniform scaling



Suppose we scale along the x-axis by a and along the y-axis by b. The matrix is

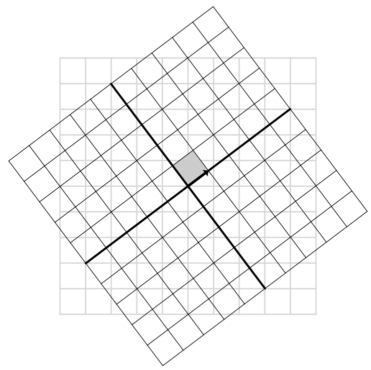
$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} .$$

Projection

A special case of scaling is where we do no scaling in the x-direction, but collapse completely vertically. This amounts to **perpendicular** or **orthogonal projection** onto the x-axis. The matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

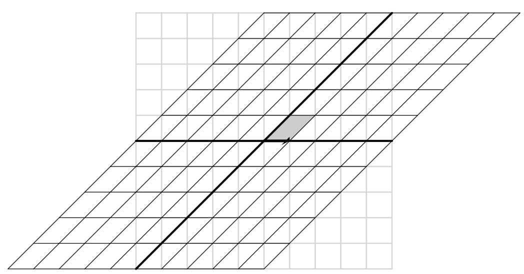
Rotation



Rotation in the positive direction (counter-clockwise) by θ has matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

Shear



Sliding parallel to the x-axis is called a **horizontal shear**. The matrix is of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} .$$