THE NINE-POINT CIRCLE

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Math 308 - Project
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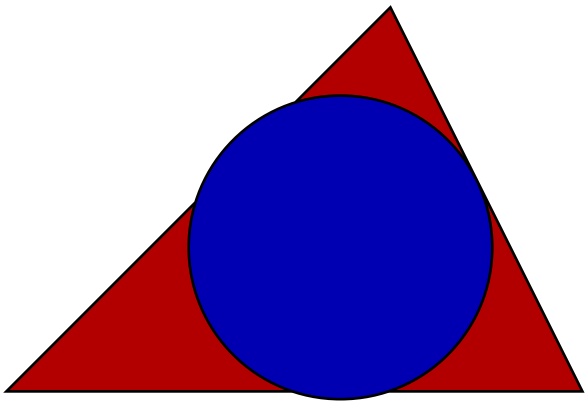


PREFACE

Following the completion of my Euclidean Geometry course, it has become apparent to me the importance of visual presentation in math. An artistic perspective in math is far more elegant than the detailed algebra that mathematicians address daily. Sometimes, like in Euclidean Geometry, an artistic method of presentation is essential to convey the complexity of 3-Dimensional images.

One of the greatest challenges an educator must face in an academic setting is to confidently convince his audience that proofs hold true. Precision and accuracy are critical. Fortunately the programming language PostScript has helped surmount this challenge. Dr. Casselman has demonstrated complex proofs through PostScript which would otherwise be painstakingly malformed on the blackboard. Perhaps even more educationally beneficial, is that students are able to manipulate their own creations. In this project I will present the nine-point circle in PostScript code. It will also encompass many elements of Euclidean Geometry.

1

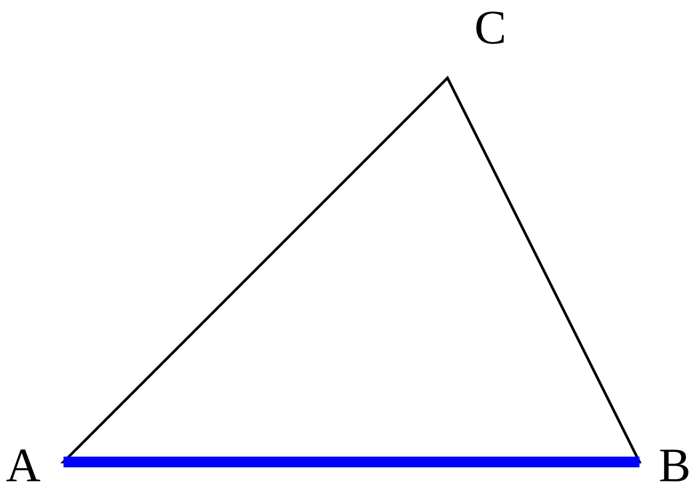
TRIANGLES:
FUNDAMENTAL THEOREMS IN REVIEW

Triangles have always been somewhat fascinating to me. There are so many naturally occuring properties, several of which include a set of lines that converge to a unique point. This point defines the various centres of a triangle such as the theorems we are concerned with; The Circumcentre and the Orthocentre.

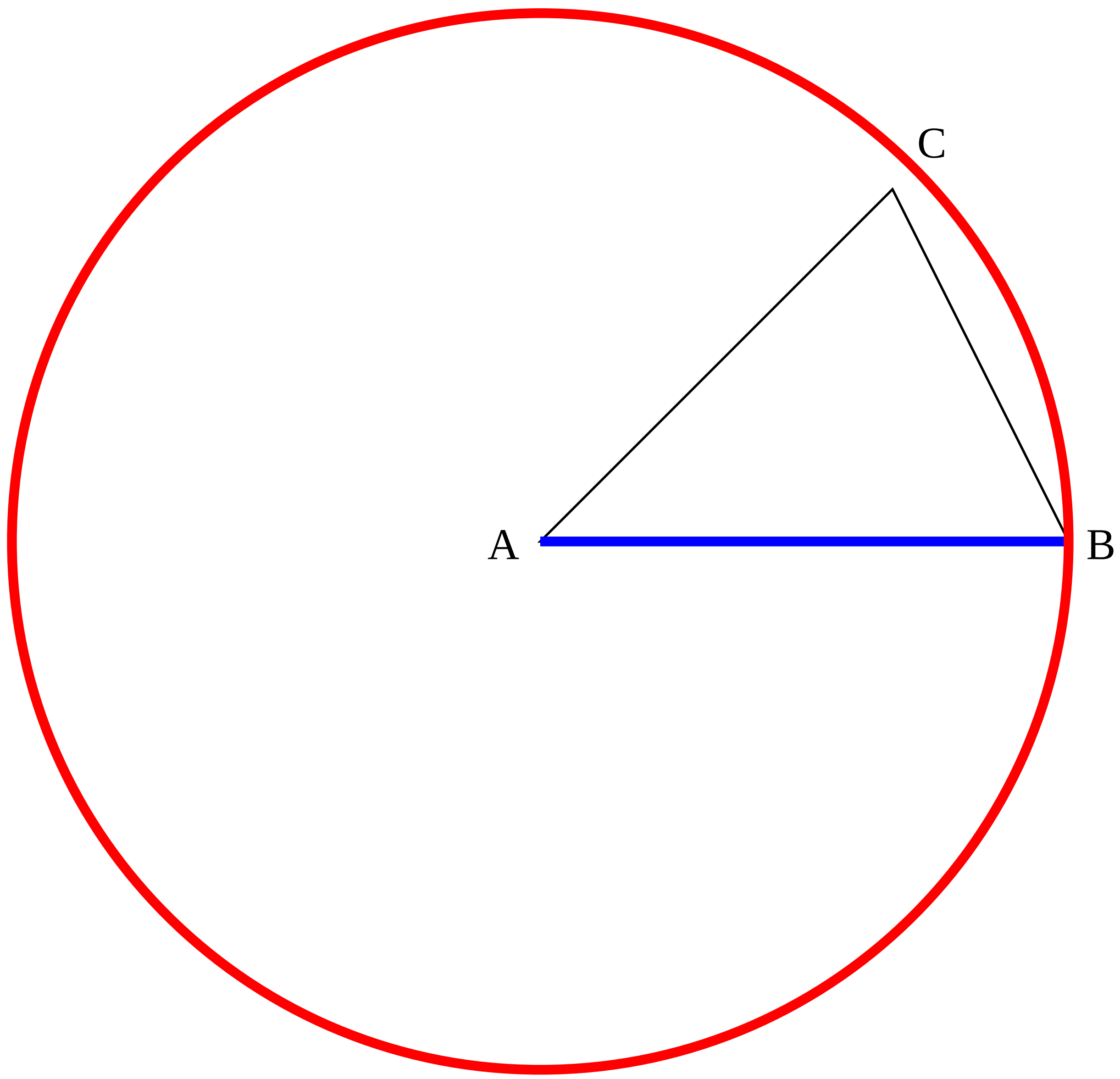
1.1 - THE CIRCUMCENTRE "O"

THEOREM: It is known that 3 non-collinear points A, B and C lie on a unique circle. This circle's centre is called the circumcentre "O" of the triangle ABC. Amazingly, the 3 perpendicular bisectors to this triangle's sides AB, BC and AC meet at precisely this point O.

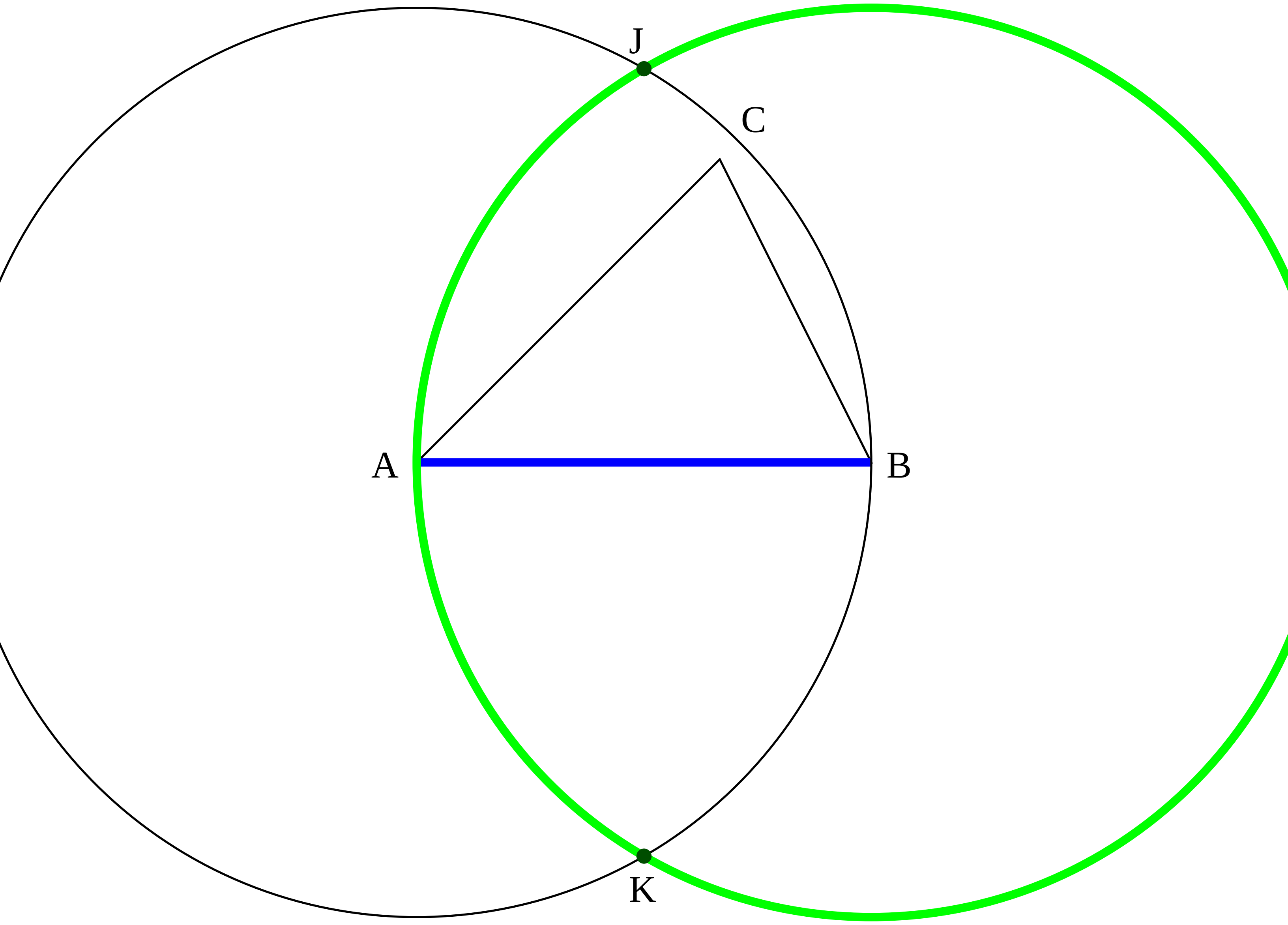
1.1.1 - CONSTRUCTION OF A PERPENDICULAR BISECTOR



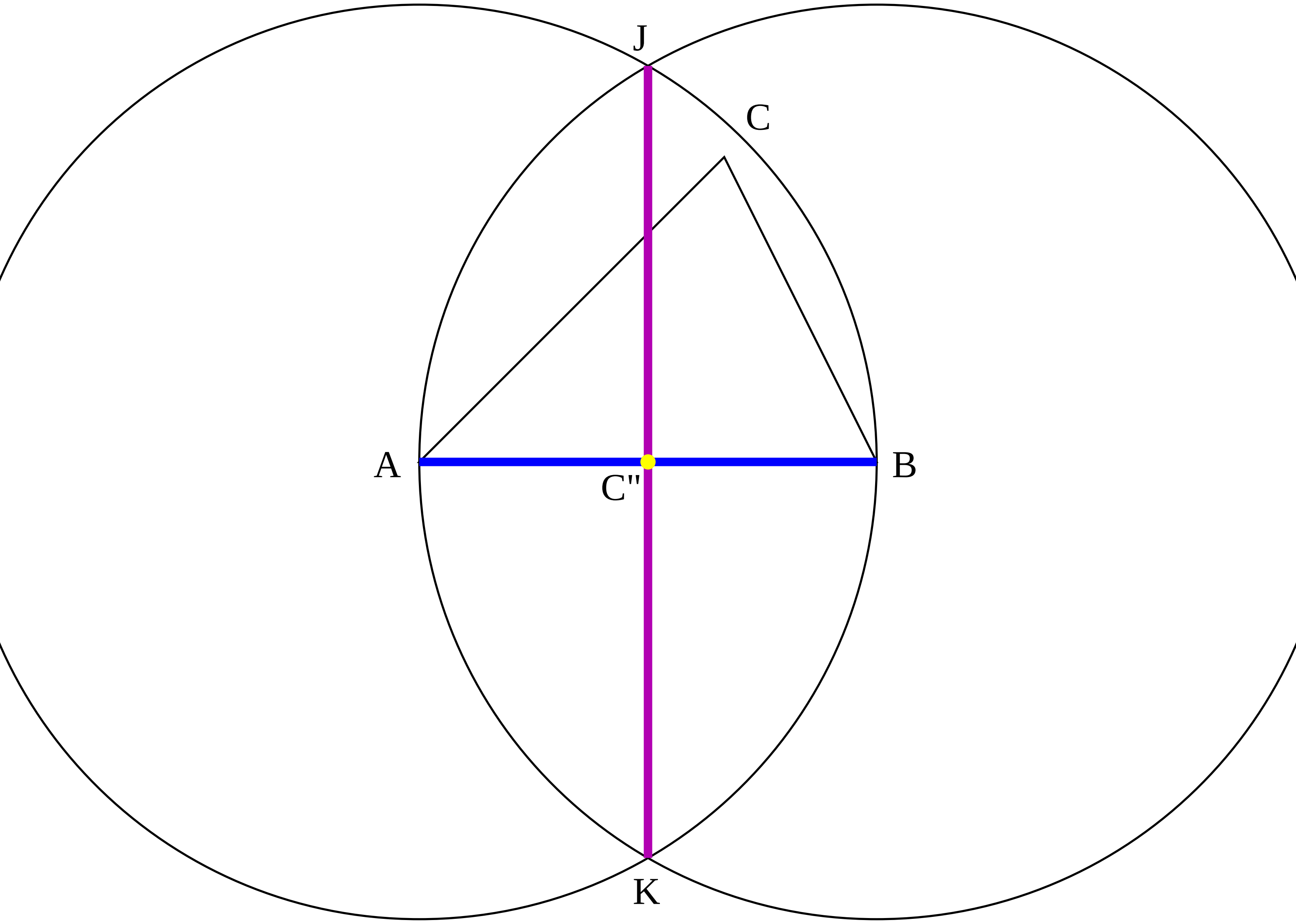
i. Start with any line, for example AB, of any triangle ABC.



ii. Construct a circle of radius AB centred at A.



iii. Construct a circle of radius AB centred at B and let the two points of intersection with the previous circle be J and K.

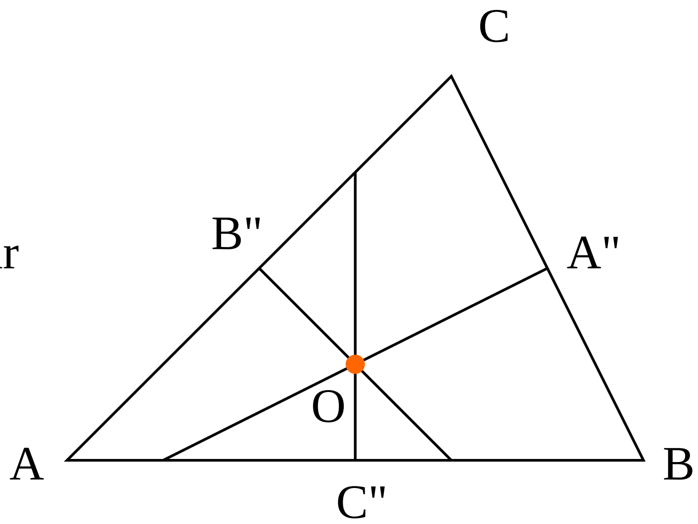


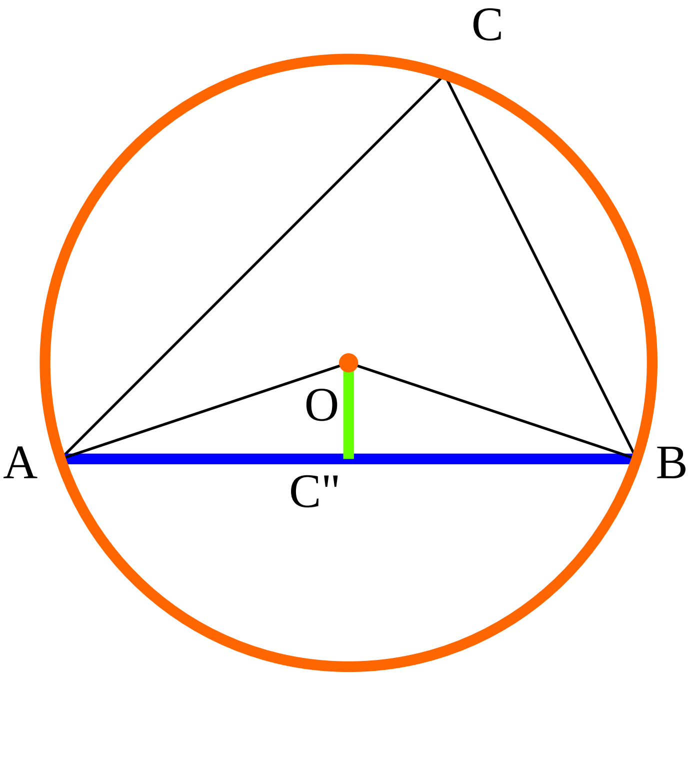
iv. Now produce the line JK through the line AB and let the point of intersection be C".

NOTE: The Proof that the line JK runs perpendicular to the line AB is non-trivial using similar triangles. ie. Triangle ABJ = ABK Also notice that they are equilateral since all the sides are equal to the radius of the circle. Therefore AB = AJ = BJ = AK = BK

1.1.2 - APPLICATION OF THE CIRCUMCENTRE THEOREM

v. Similarly one can constuct the perpendicular bisector to AC and BC. After doing so we can see that the 3 perpendicular bisectors do in fact meet at a common point O, otherwise referred to as the circumcentre.





PROOF: Let AB be the chord to a circle with circumcentre O. Then by the Isosceles Triangle AOB, the line AB has a perpendicular bisector passing through O.

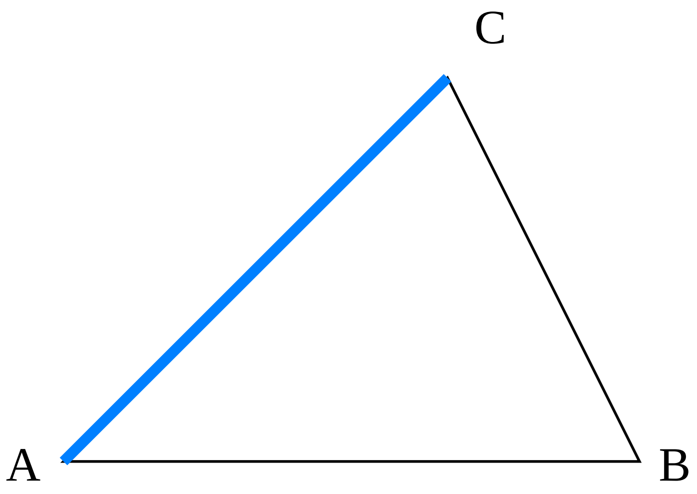
Therefore the Circumcentre does lie on all 3 perpendicular bisectors.

1.2 - THE ORTHOCENTRE "H"

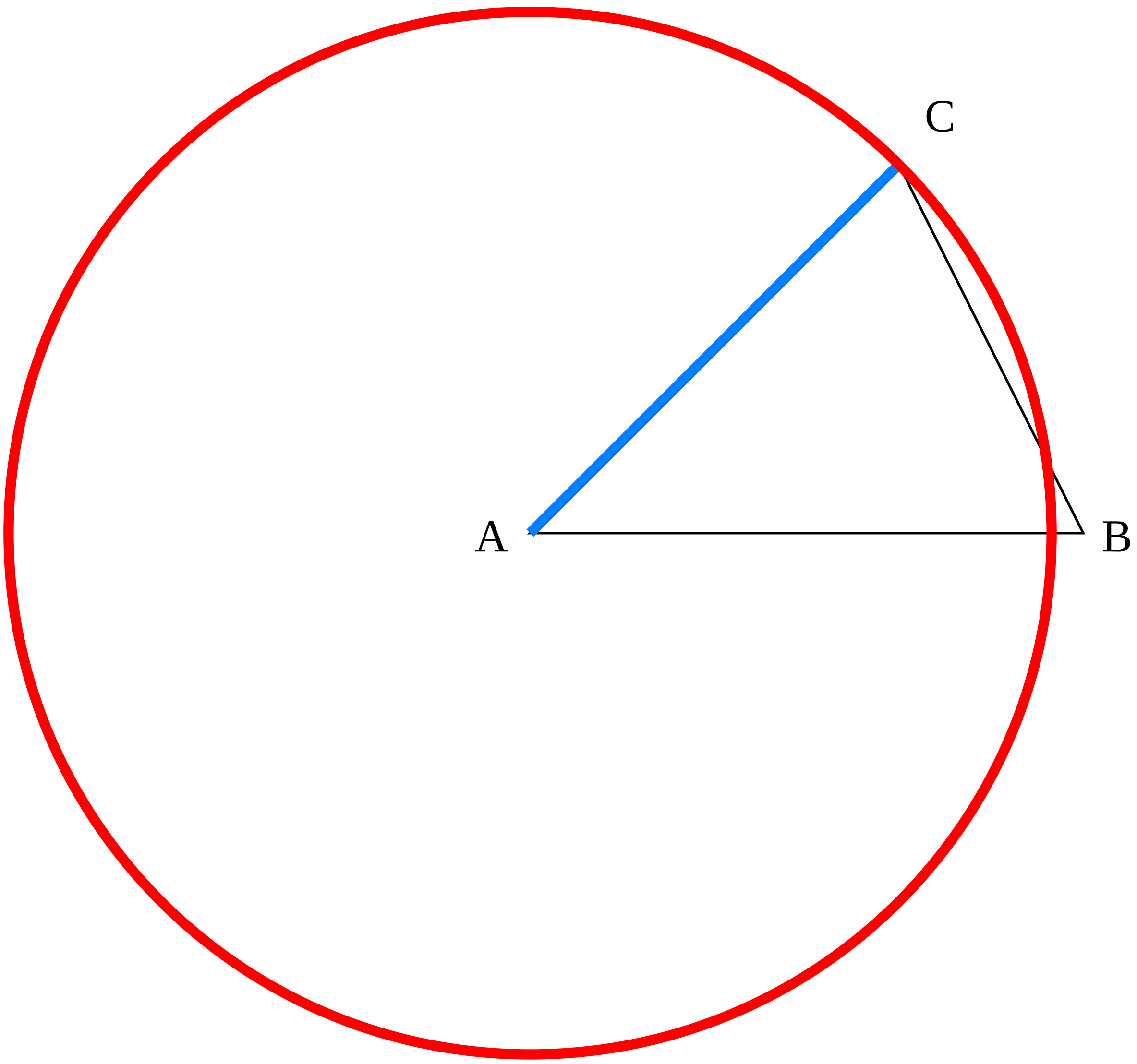
THEOREM: All 3 altitudes of any triangle ABC all meet at exactly one point known as the Othrocentre "H".

DEFINITION: Altitide - An alititude is a line that runs from any vertex on a triangle ABC to its opposite edge and that is perpendicular to that edge.

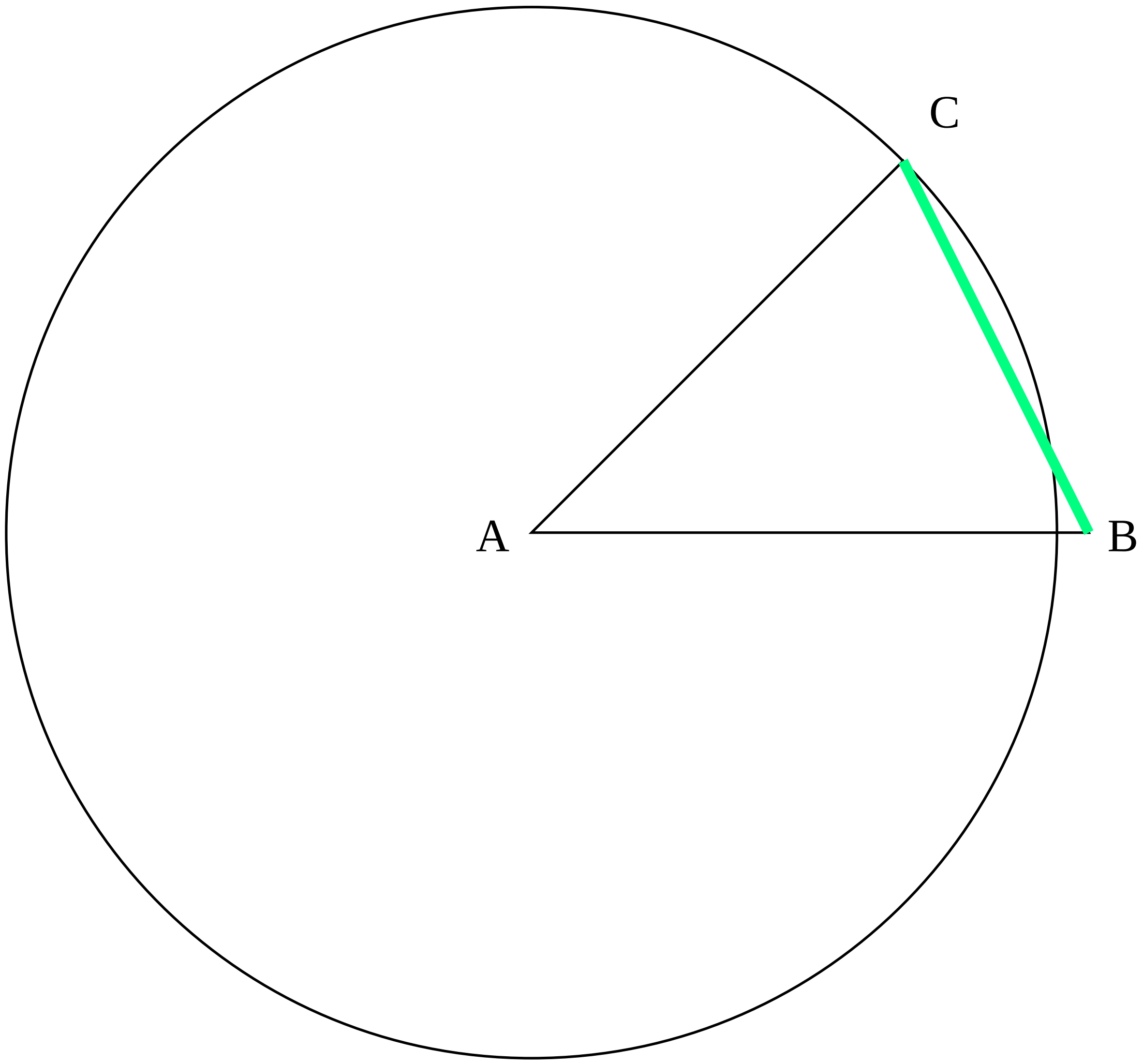
1.2.1 - CONSTRUCTION OF AN ALTITUDE



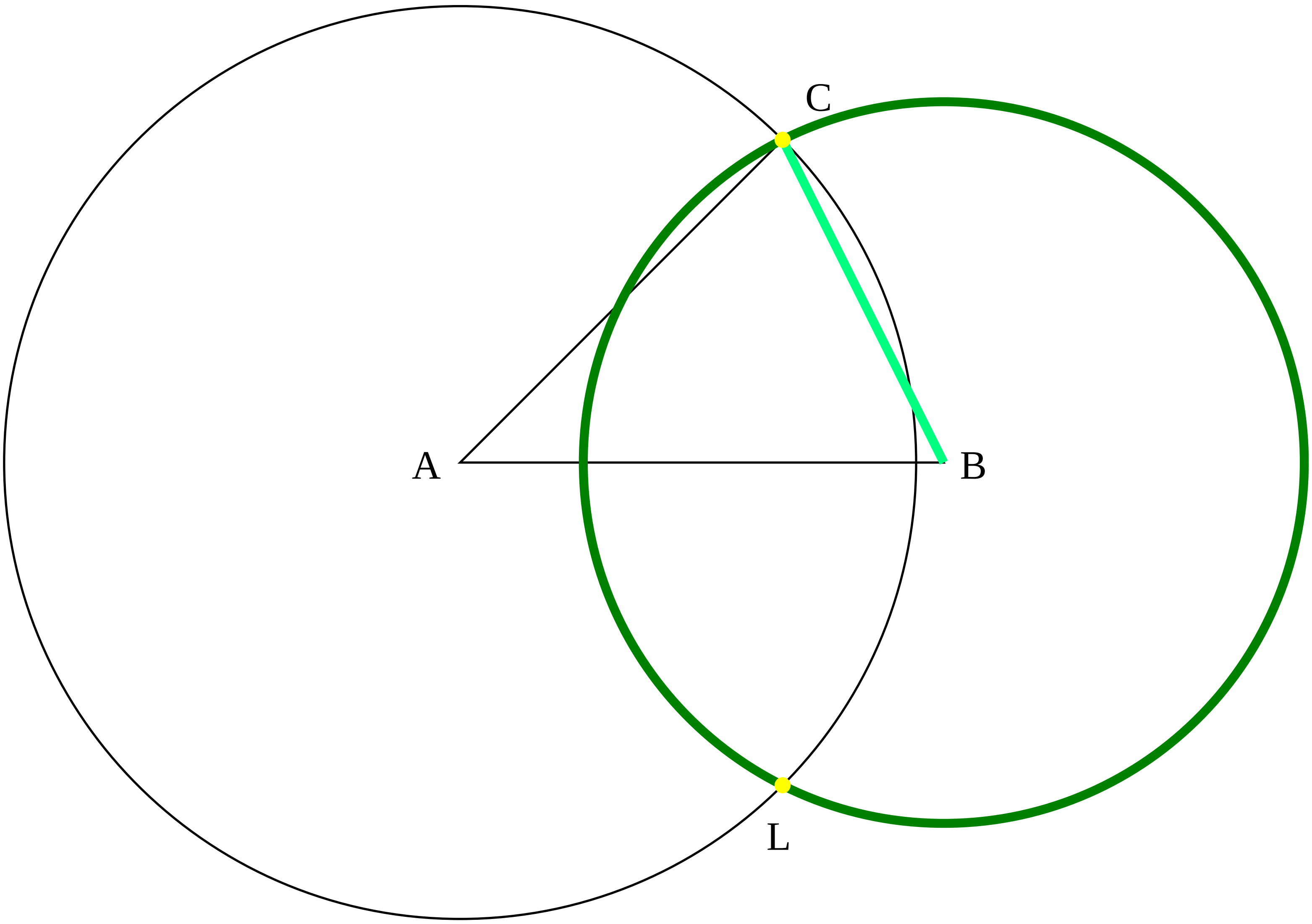
i. Start with any line, for example AC, of any triangle ABC.



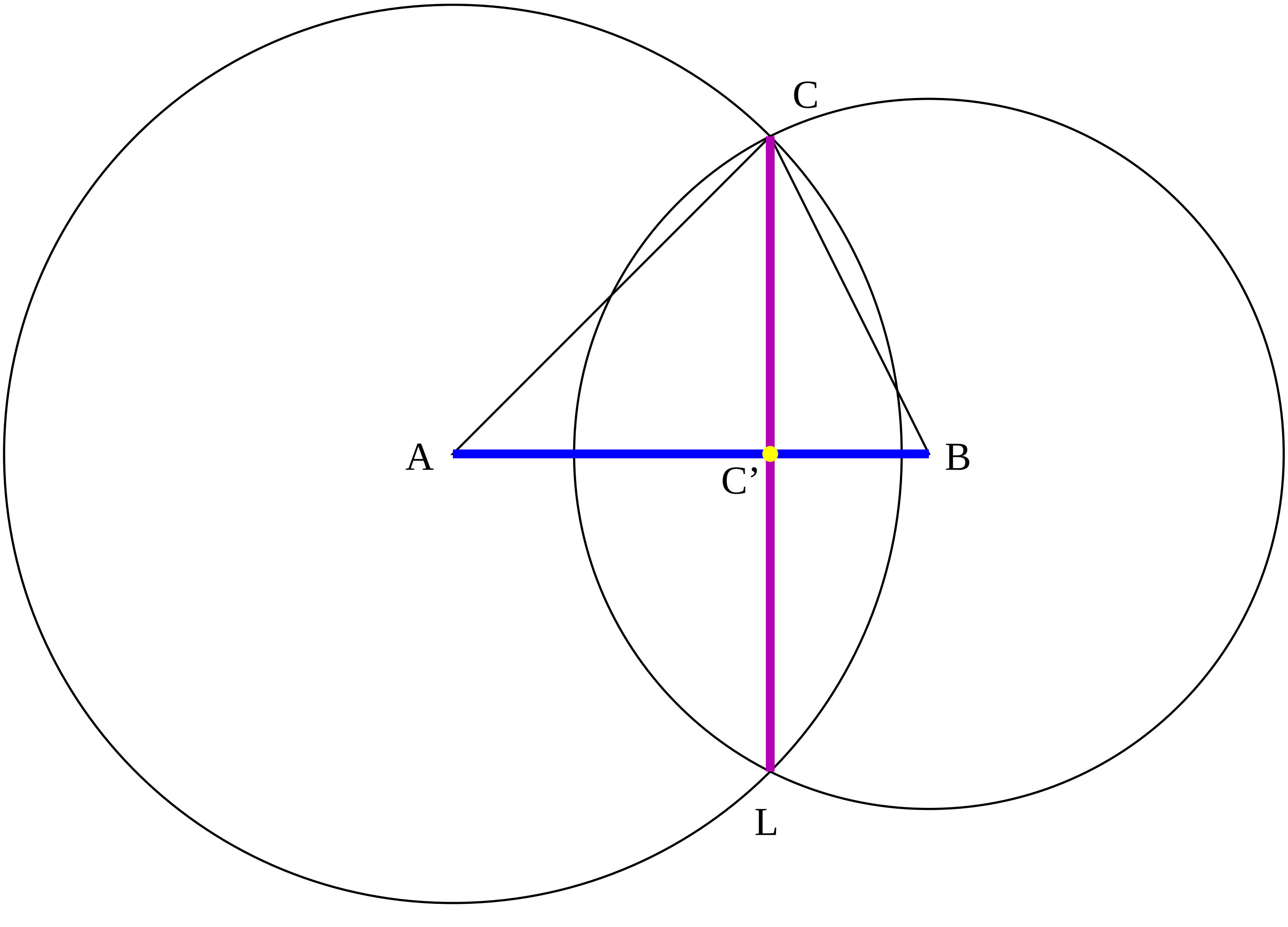
ii. Construct a circle of radius AC centred at A.



iii. Now take a look at the line BC .



iv. Construct a circle of radius BC centred at B and let the two points of intersection with the previous circle be C and L.

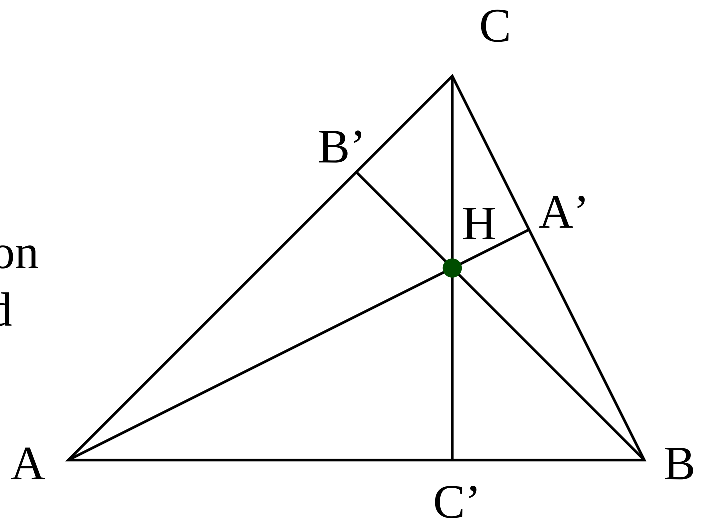


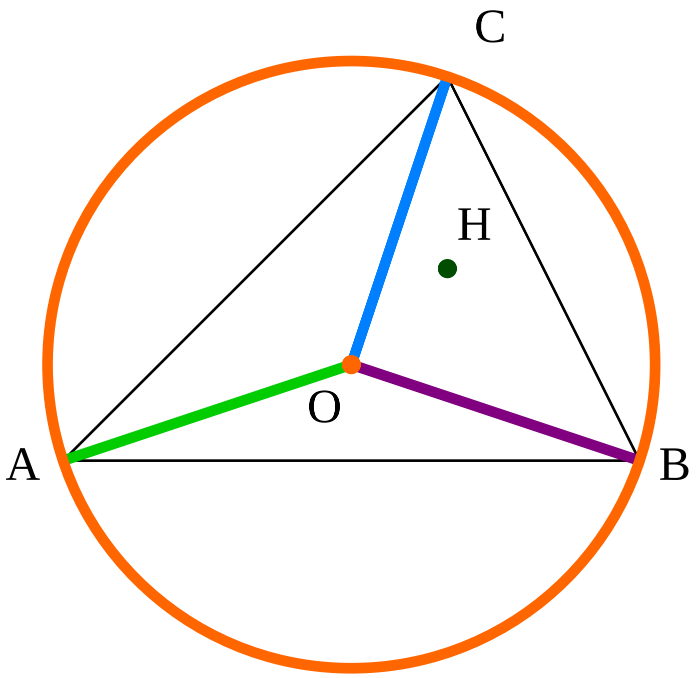
v. Now produce the line CL through the line AB and let the point of intersection be C’.

NOTE: The Proof that the line CL runs perpendicular to the line AB is non-trivial using similar triangles. ie. Triangle BC’C = BC’L

1.2.2 - APPLICATION OF THE ALTITUDE THEOREM

vi. Similarly one can constuct the altitude to A and B. After doing so we can see that the 3 altitudes do in fact meet at a common point H, otherwise referred to as the orthocentre.





PROOF: Using vectors, take the circumcentre "O" to be the origin. Therefore we have $|A| = |B| = |C|$ Therefore let H = A + B + C. It can be shown that H is the orthocentre by veryfying that the line AH is perpendicular to BC, BH is perpendicular to AC, and CH is perpendicual to AB.

Hence H lies on the altitude through A and the other two conditions are similar. From H = A + B + C we can derive the following:

$$
\begin{aligned}
H - A &= B + C \\
[\, H - A \,] \bullet [\, B - C \,] &= [\, B + C \,] \bullet [\, B - C \,] \\
&= B \bullet B - C \bullet C \\
&= |B|^2 - |C|^2 \\
&= 0
\end{aligned}
$$

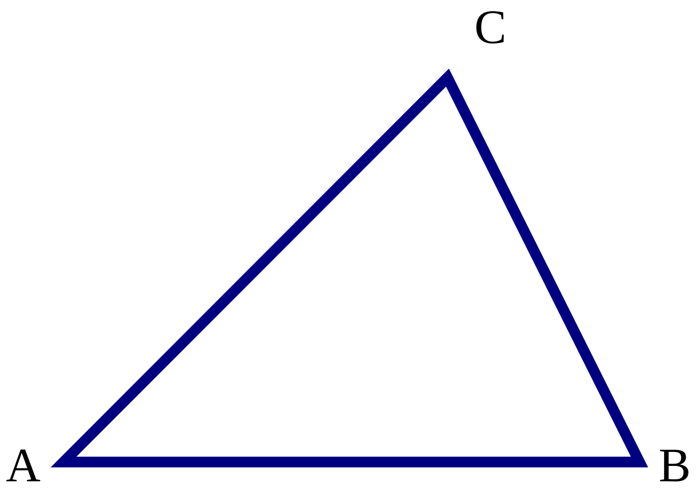
and AH is perpendicular to BC.

2

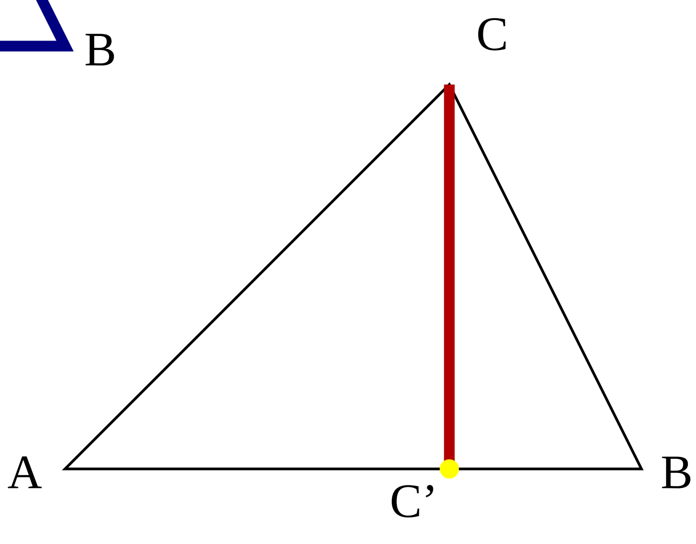
THE EXISTENCE OF THE NINE-POINT CIRCLE

THEOREM: The midpoints to the three sides of any triangle, the midpoints to the lines joining the orthocentre to the three vertices, and the feet of the three altitudes all lie on a circle referred to as the nine-point circle. This circle's centre is found halfway between the triangles circumcentre and the triangles orthocentre.

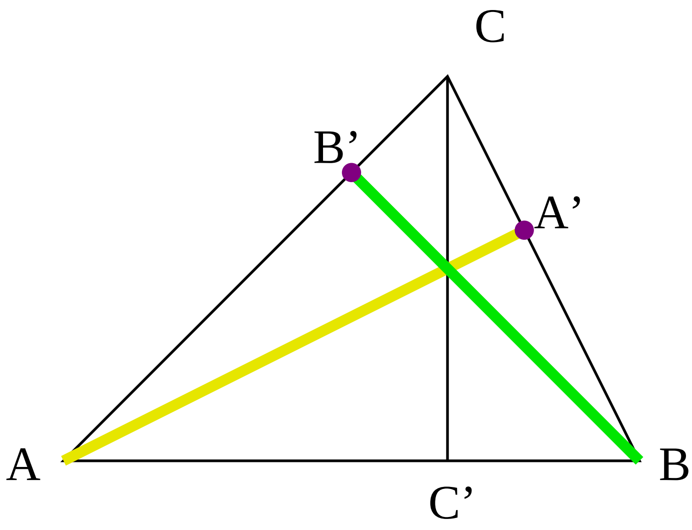
2.1 - CONSTRUCTION OF THE NINE-POINT CIRCLE



i. Start by constructing a triangle ABC of any shape or size

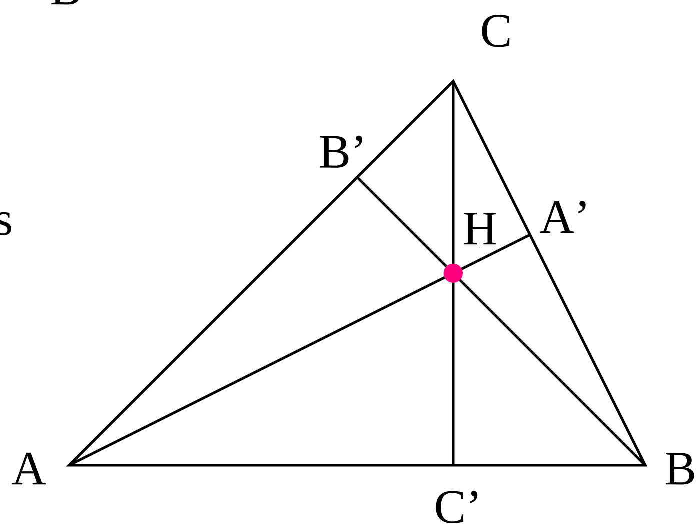


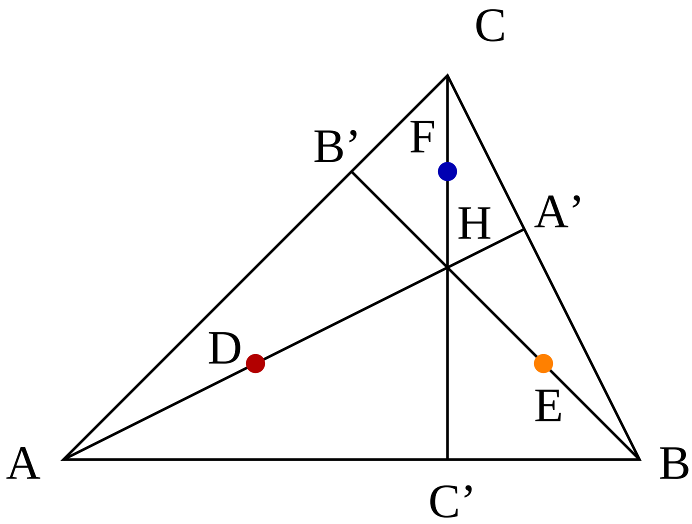
ii. Produce the altitude C and let its foot be C'.



iii. Similarly produce the altitudes A and B and let their feet be A’ and B’ respectively.

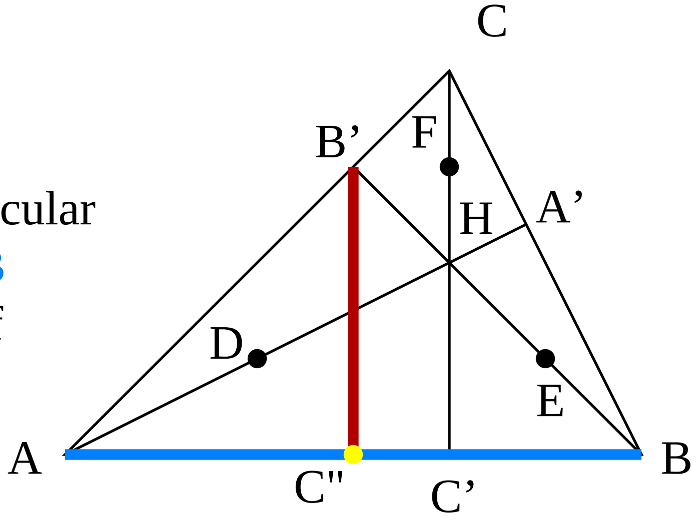
iv. Label the common point between these 3 altitudes H. Let H be the orthocentre to the triangle ABC.

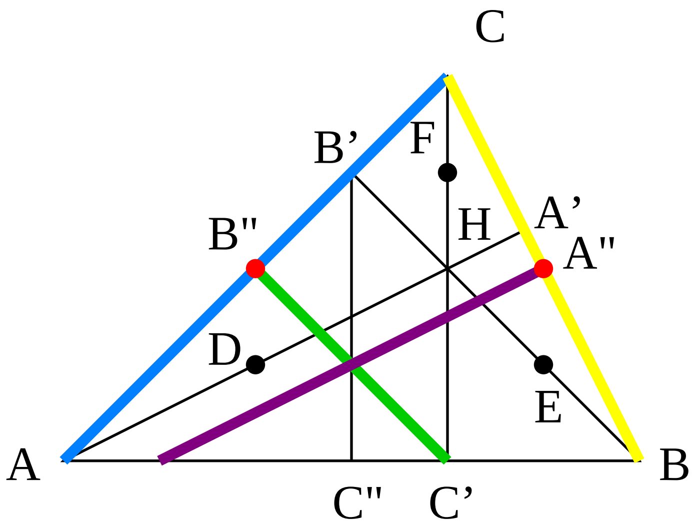




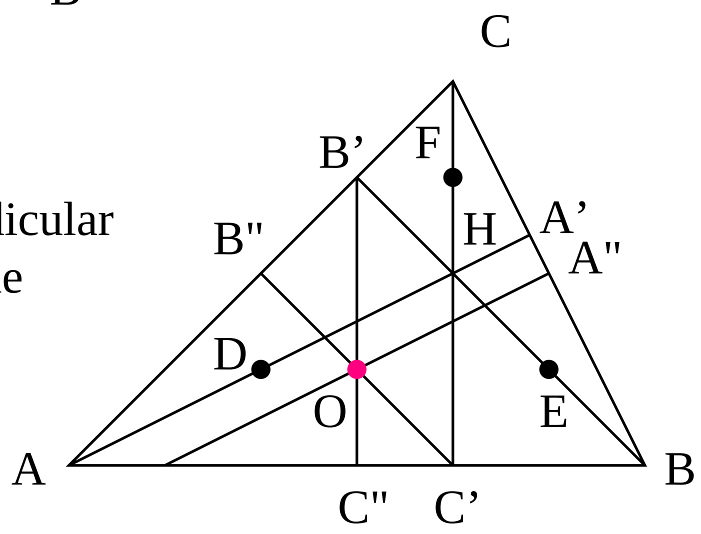
v. Now also label the midpoints of AH, BH and CH. Let them be the points D, E, and F respectively.

vi. Construct the perpendicular bisector to the side AB and let the midpoint of AB be C".

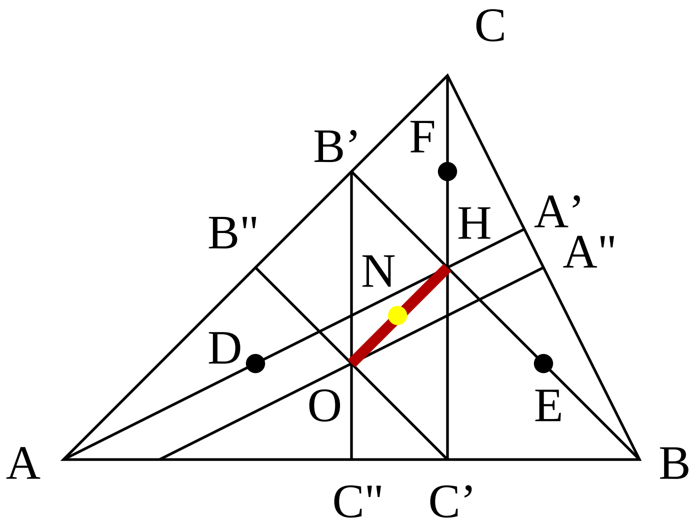




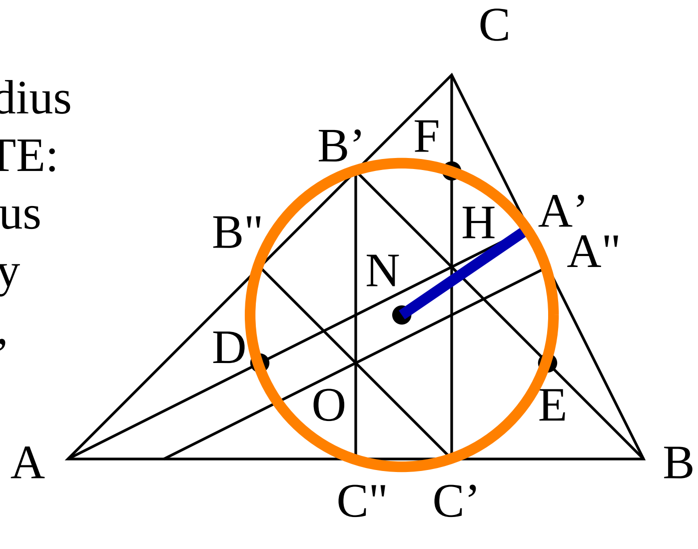
vii. Similarly produce the perpendicular bisectors to the sides AC and BC. Let the midpoints to those sides be B" and A" respectively.



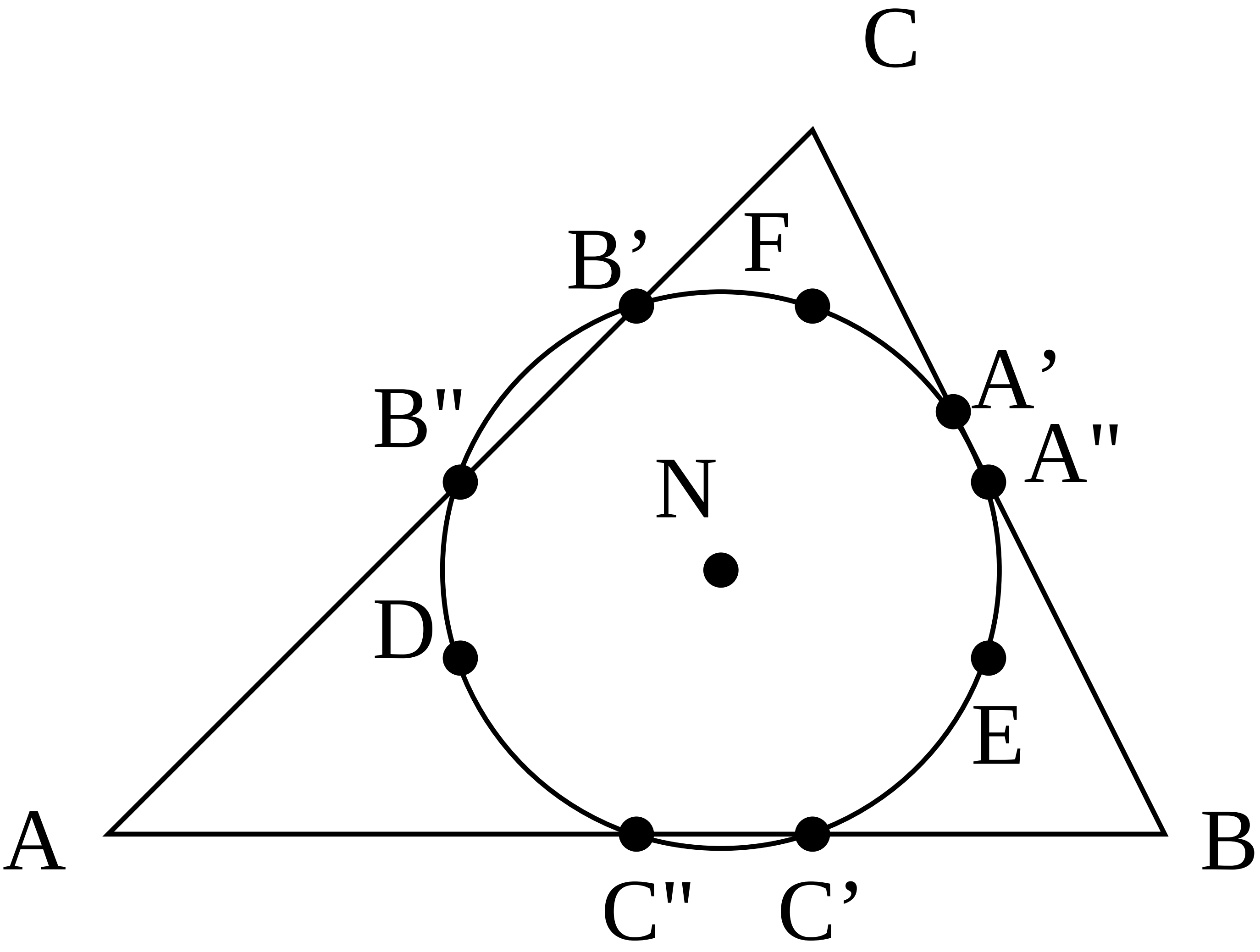
viii. Label the common point between these 3 perpendicular bisectors O. Let O be the circumcentre to the triangle ABC.



ix. Now join the points O and H. Let N be the midpoint to this new line OH and the centre to our nine-point circle.



x. Construct a circle of radius NA’ centred at N. NOTE: The circle can be a radius from the centre N to any of the points A’, B’, C’, A", B", C", D, E or F.



xi. Now we have a circle whose nine points include A’, B’, C’, A", B", C", D, E and F centred at N. This is the nine-point circle to the triangle ABC.

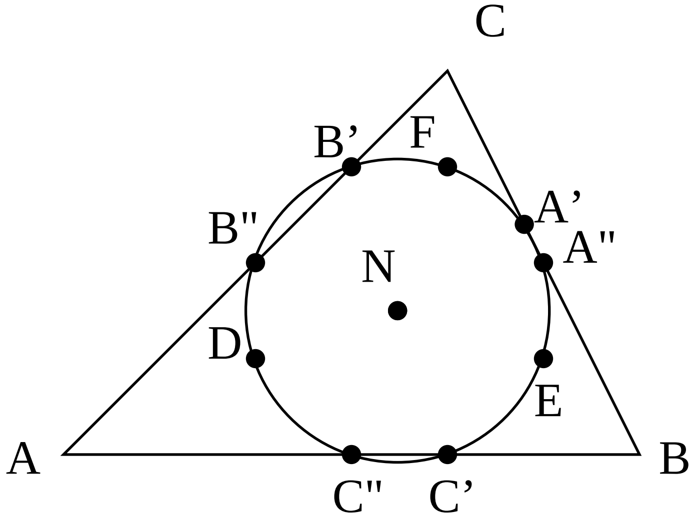
3

A GEOMETRICAL PROOF OF THE NINE-POINT CIRCLE

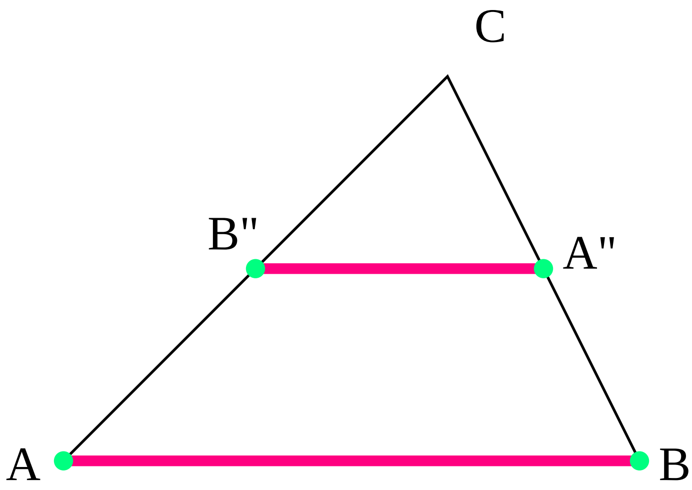
EUCLID’S ELEMENTS - Book VI, Proposition 2: If a straight line is drawn parallel to one of the sides of a triangle, then it cut’s the two adjacent sides of the triangle proportionally. The reverse is also true such that if the sides of a triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle.

3.1 - THE PROOF

REMARK: Let it be known that all the points within this chapter have been predetermined in Chapter 2.

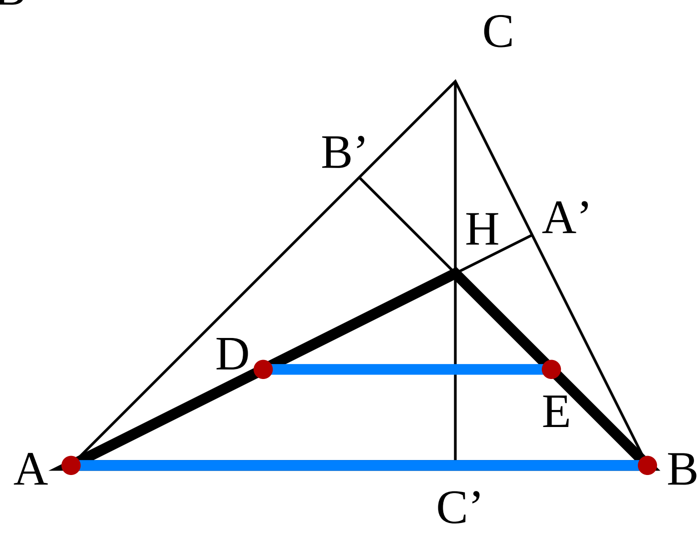


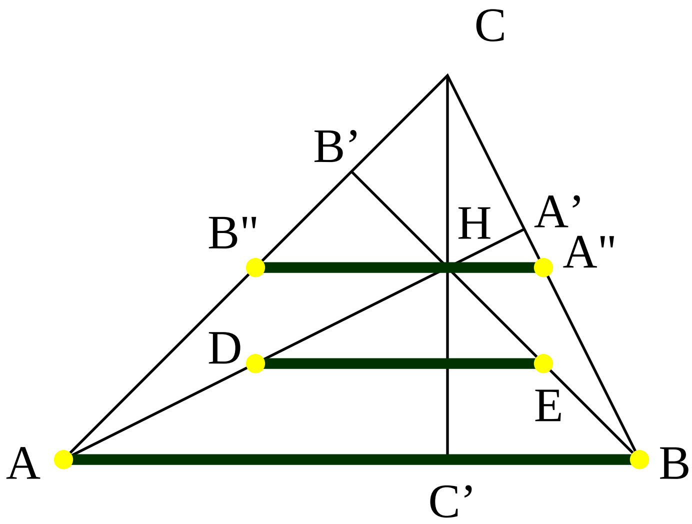
i. To begin start with the nine-point circle of any triangle ABC.



ii. The points A" and B" are the midpoints to the sides AC and BC of the triangle ABC. Therefore according to Euclid VI - Proposition 2, the line A"B" is then parallel to the line AB.

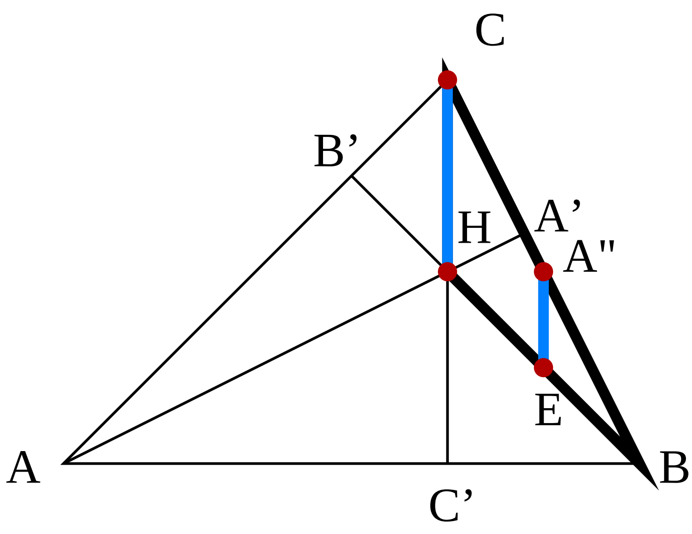
iii. Similarly the points D and E are the midpoints to the lines AH and BH. So by using the triangle ABH, it can be proven that the line DE is parallel to the line AB.

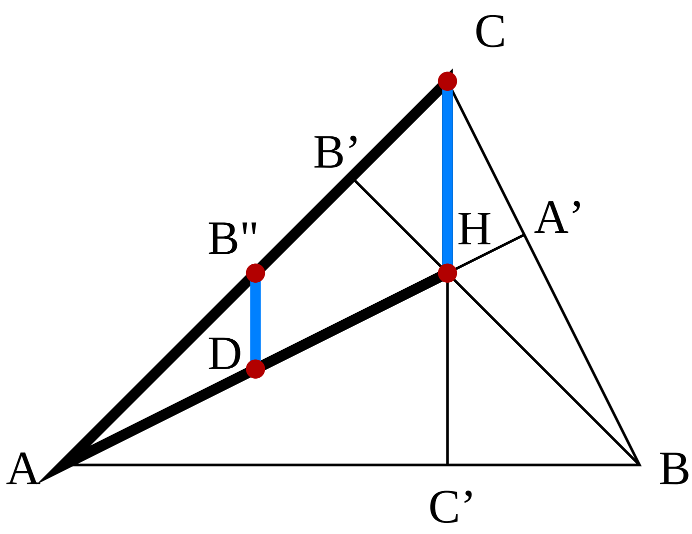




iv. Now by transitivity we can see that AB || DE || A"B".

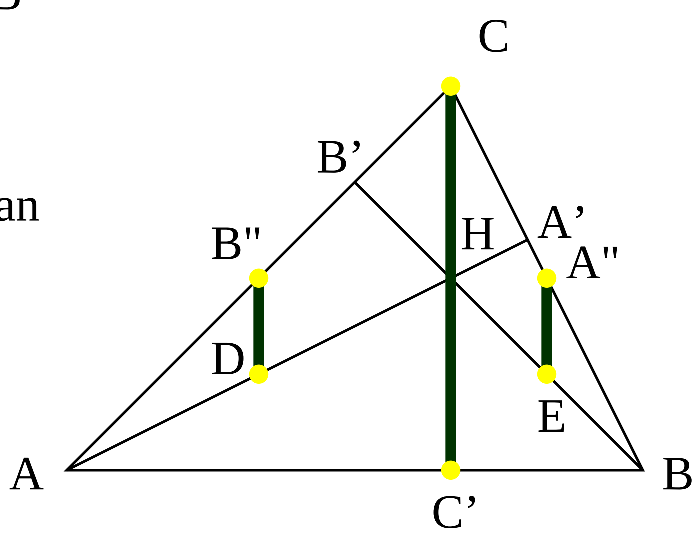
v. Similarly it can be shown that the line A"E is parallel to the line CH since the points A" and E are the midpoints to the sides BC and BH of the triangle BCH.

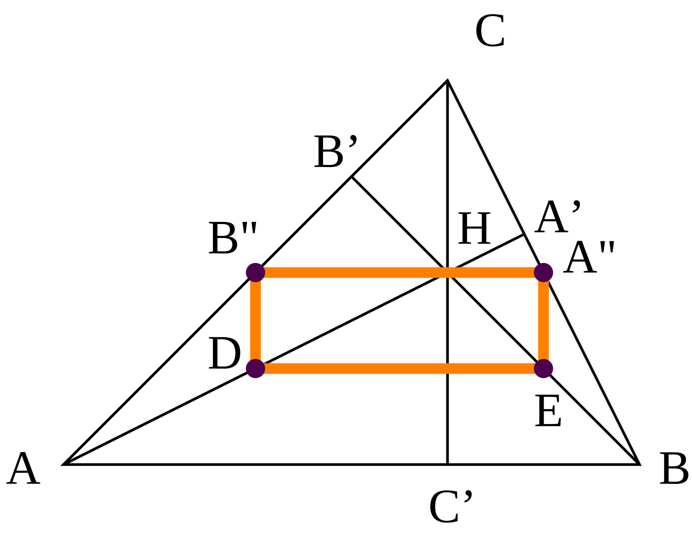




vi. Once again using the triangle ACH, it can be shown that the line B"D is parallel to the line CH. The points B" and D are the midpoints to the lines AC and AH respectively.

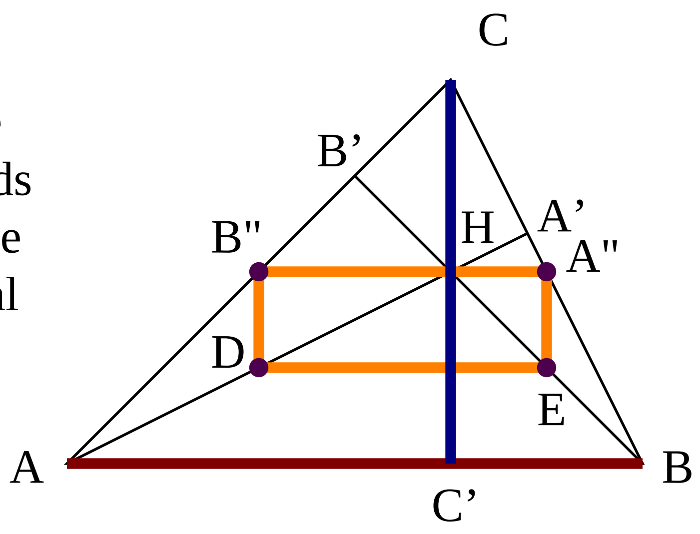
vii. Therefore by transitivity we can see that B"D || CC’ || A"E.

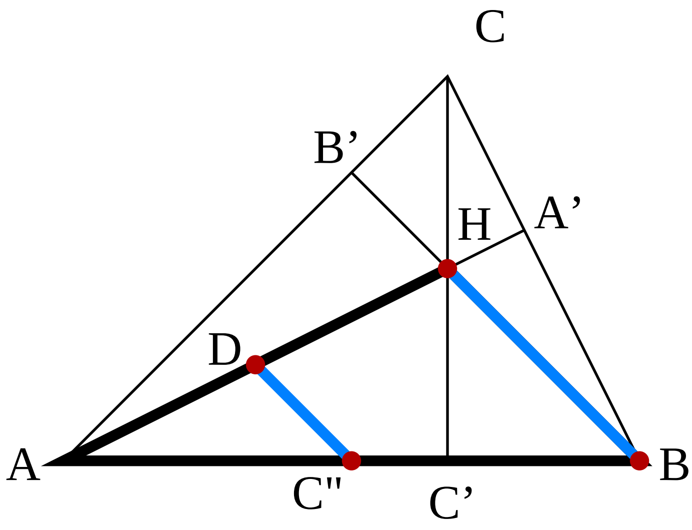




viii. Now notice the quadrilateral A"B"DE.

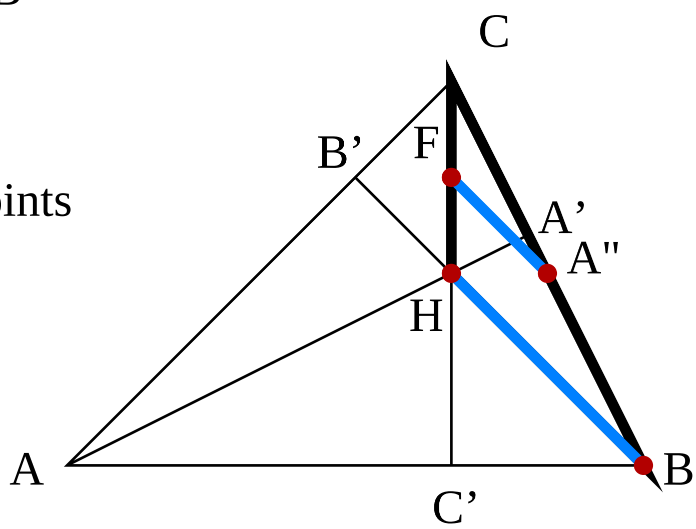
ix. Since the line CC’ is an altitude to the vertex C, or in other words perpendicular to the line AB, we can deduce that the quadrilateral A"B"DE is a rectangle because all the angles are right.

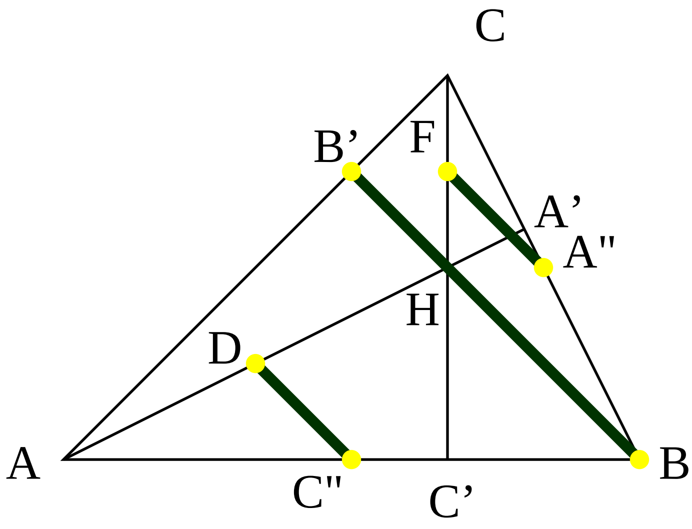




x. The points C" and D are the midpoints to the sides AB and AH of the triangle ABH. Therefore by Euclid VI - Proposition 2, the line C"D is parallel to the line BH.

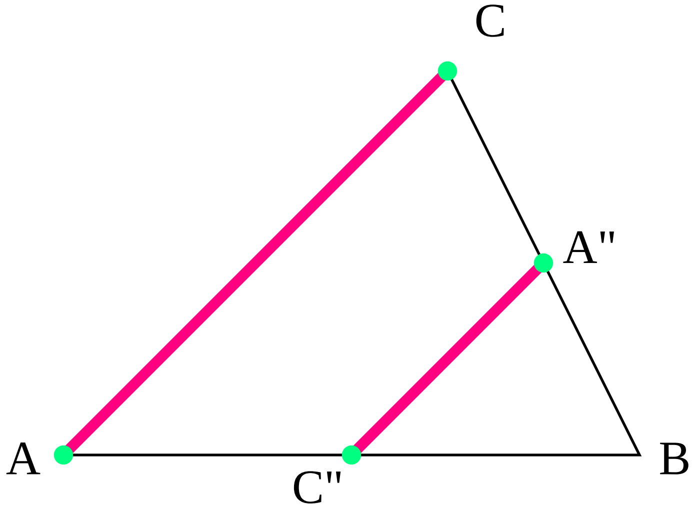
xi. Similarly by joining the midpoints A" and F, we have the line A"F parallel to the line BH.

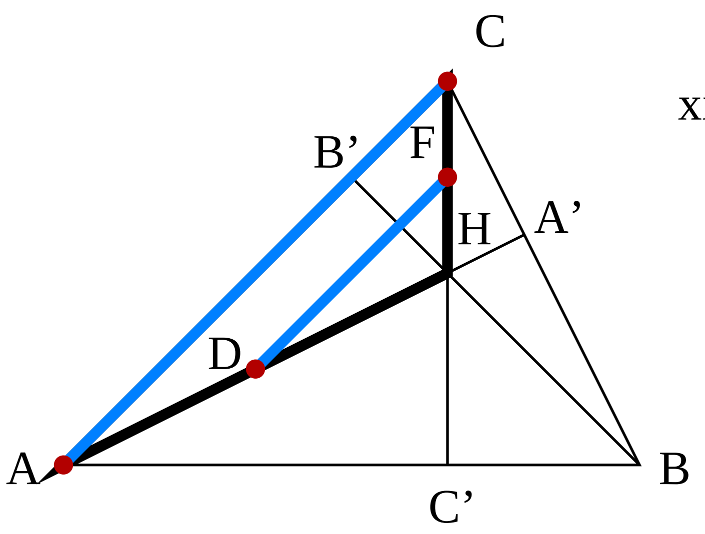




xii. Likewise by transitivity we see that C"D || BB' || A"F.

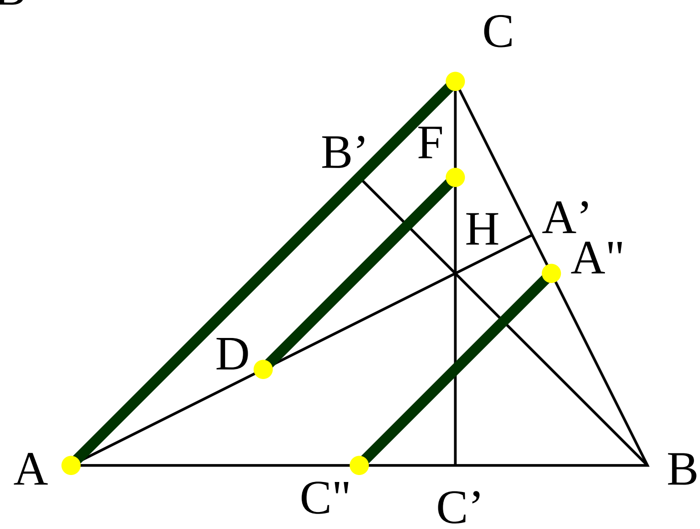
xiii. The points A" and C" are midpoints to the sides AB and BC, so as before the line AC is parallel to the line A"C".

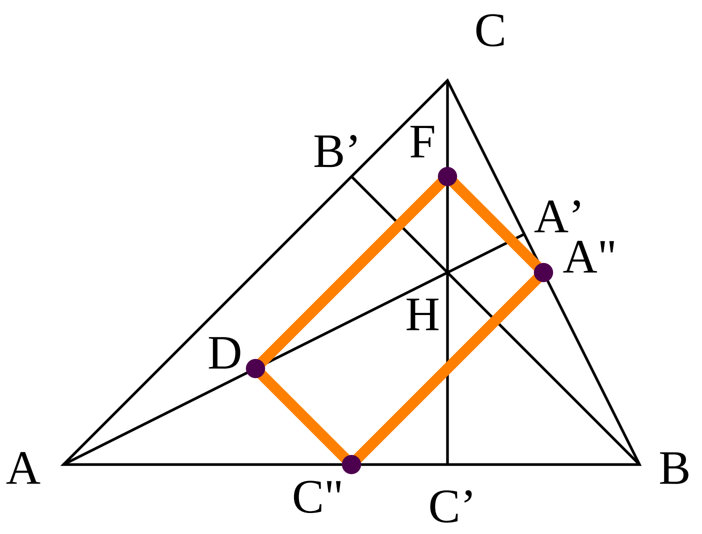




xiv. The points D and F are the midpoints to the sides AH and CH of the triangle ACH. Therefore by Euclid VI - Proposition 2, the line DF is parallel to the line AC.

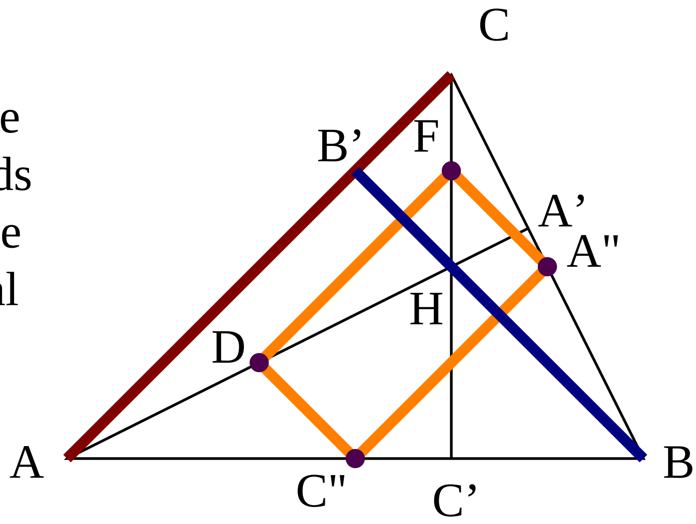
xv. Again by transitivity we can see that AC || DF || C"A".

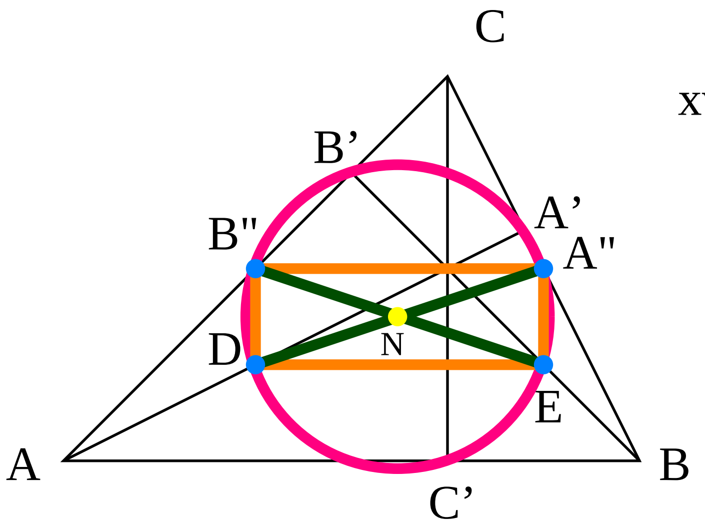




xvi. As before notice the quadrilateral A"C"DF.

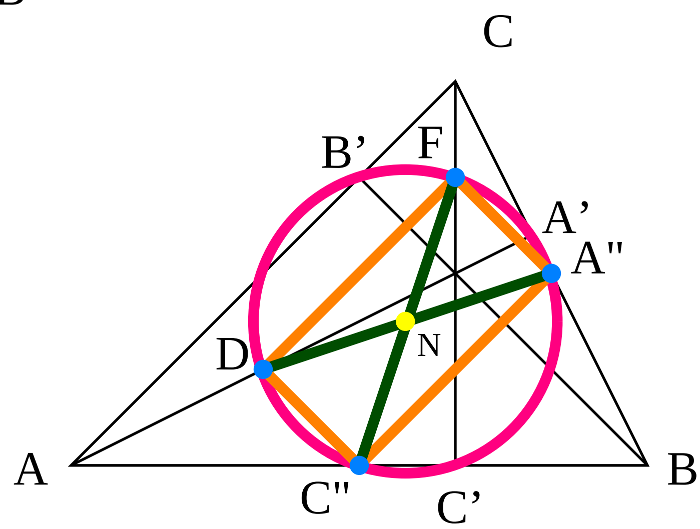
xvii. Since the line BB’ is an altitude to the vertex B, or in other words perpendicular to the line AC, we can deduce that the quadrilateral A"C"DF is a rectangle and all angles are right.

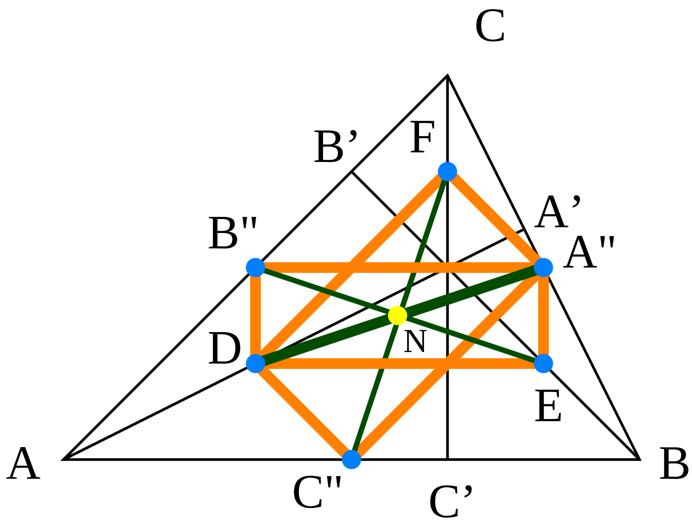




xviii. Now go back to the rectangle A"B"DE. Since it is a rectangle, than we can draw a circle with the lines A"D and B"E as diameters. The centre to this circle is the common point N.

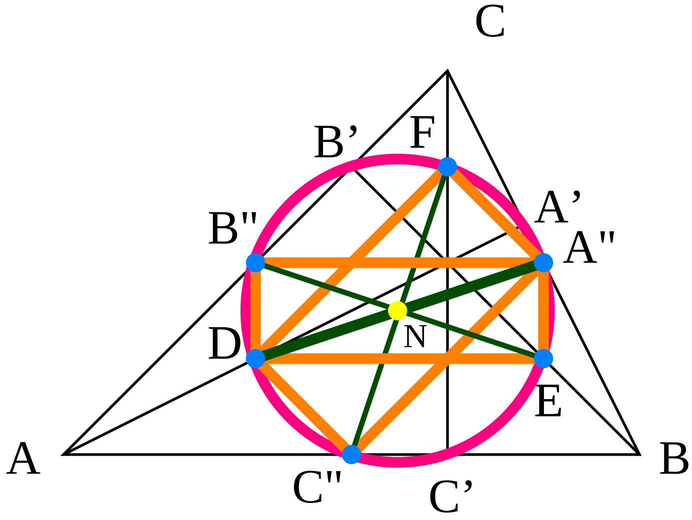
xix. A similar argument can be shown with the rectangle A"C"DF.

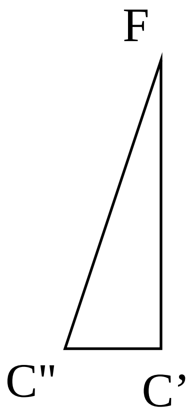




xx. Overlap the two above rectangles A"C"DF and A"B"DE. Notice that they share the common diameter A"D.

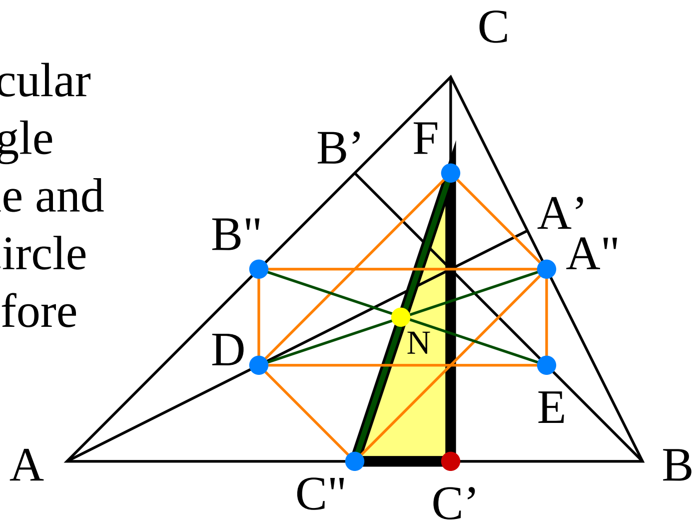
xxi. Therefore these six points A", B", C", D, E and F all lie on a circle centred at N.

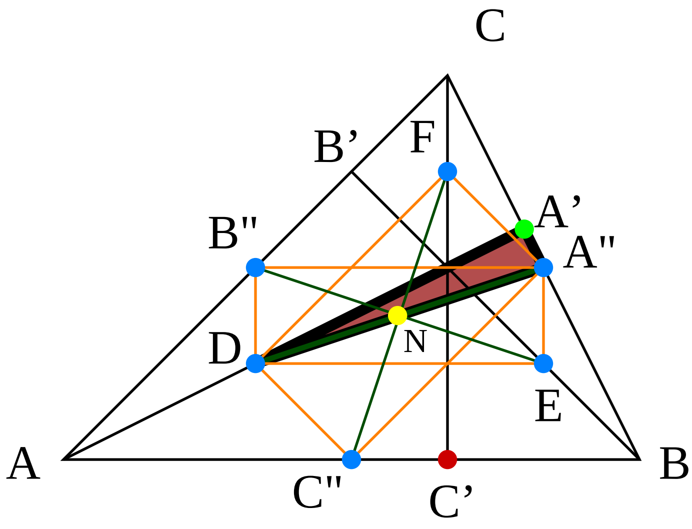




xxii. To determine the last three points A’, B’ and C’ we look at the triangle C’C"F. NOTE: The line C’F is the altitude to the vertex F.

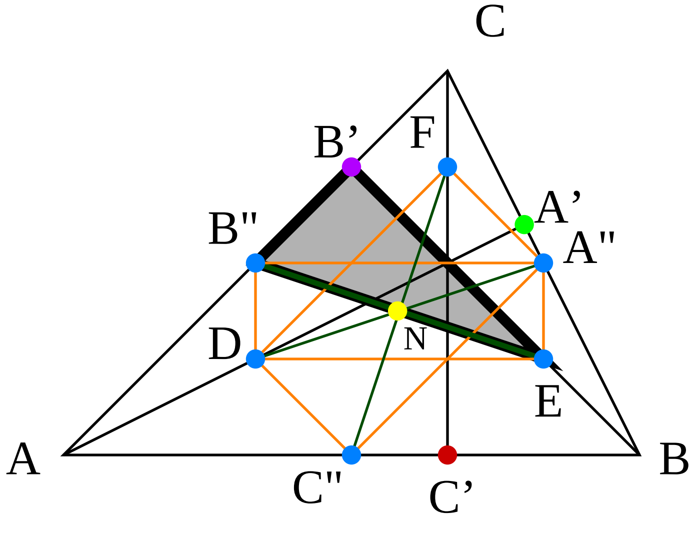
xxiii. Since the line C’F is perpendicular to the line C’C", then the triangle C’C"F is a right-angled triangle and half a rectangle. Therefore a circle with diameter C"F exists as before but now includes a seventh point C’.

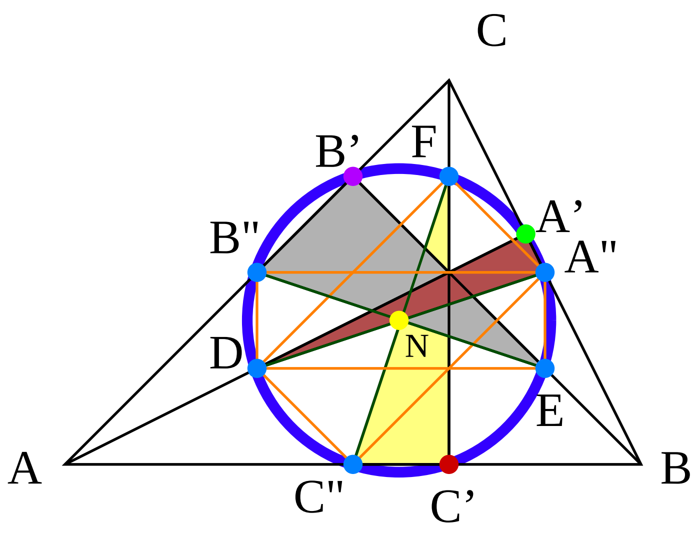




xxiv. Similarly a circle with radius A"D exists as before but now including an eighth point A’.

xxv. Repeat this procedure once more for the triangle B’B"E and deduce that there exists a circle with diameter B"E as before. This finalizes that a ninth point B’ also lies on the circle.





In conclusion Chapter 3 has now proved, in concert with chapters 1 and 2, that the nine distinct points A’, B’, C’, A", B", C", D, E and F of the triangle ABC all lie on a unique circle referred to as the nine-point circle.