Two Proofs of Ceva's Theorem

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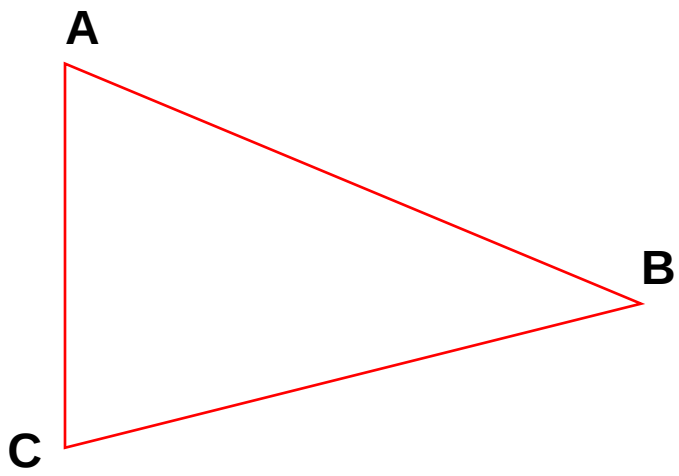
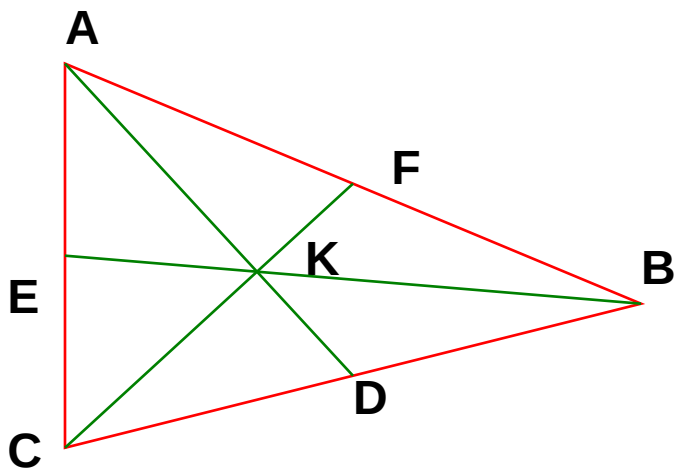
Ceva’s Theorem

Given the triangle ABC,



Ceva's Theorem

Given the triangle ABC,

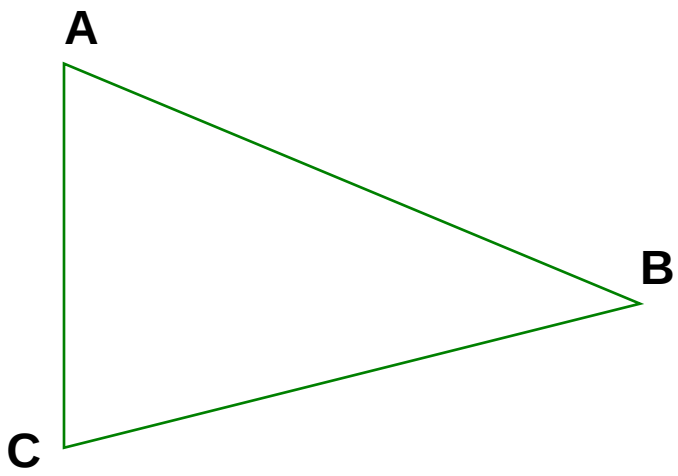


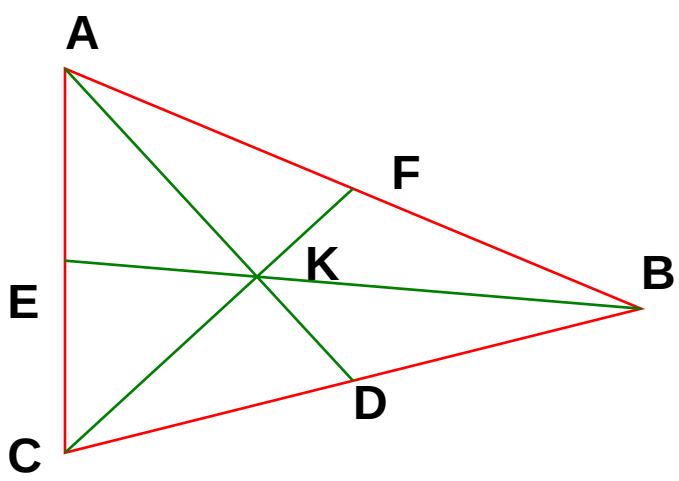
The three lines AD, BE and CF intersect

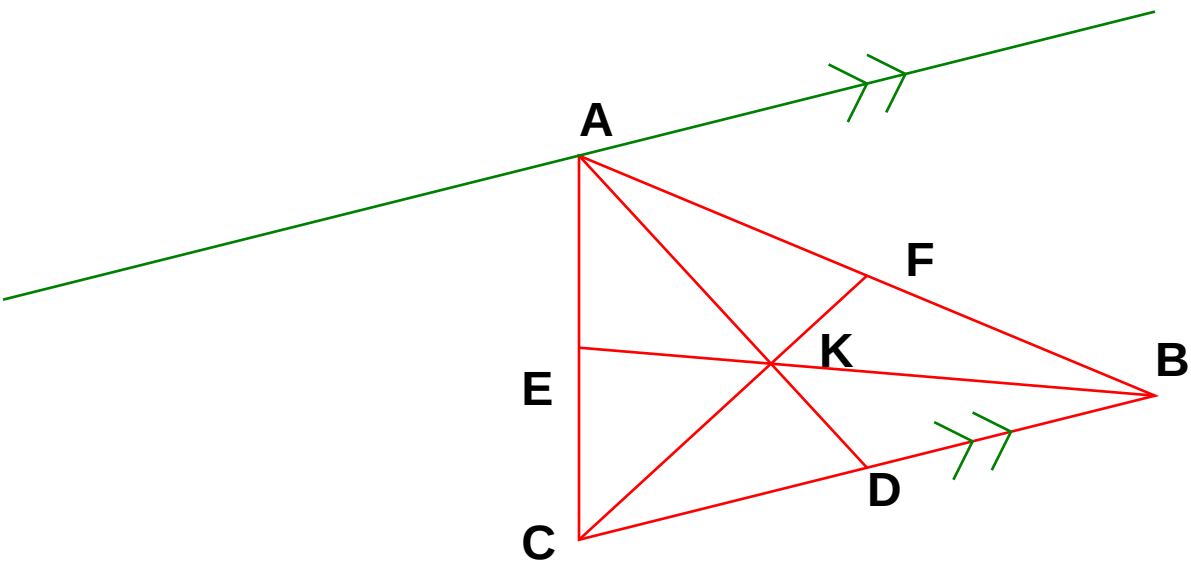
at the point K, if and only if:

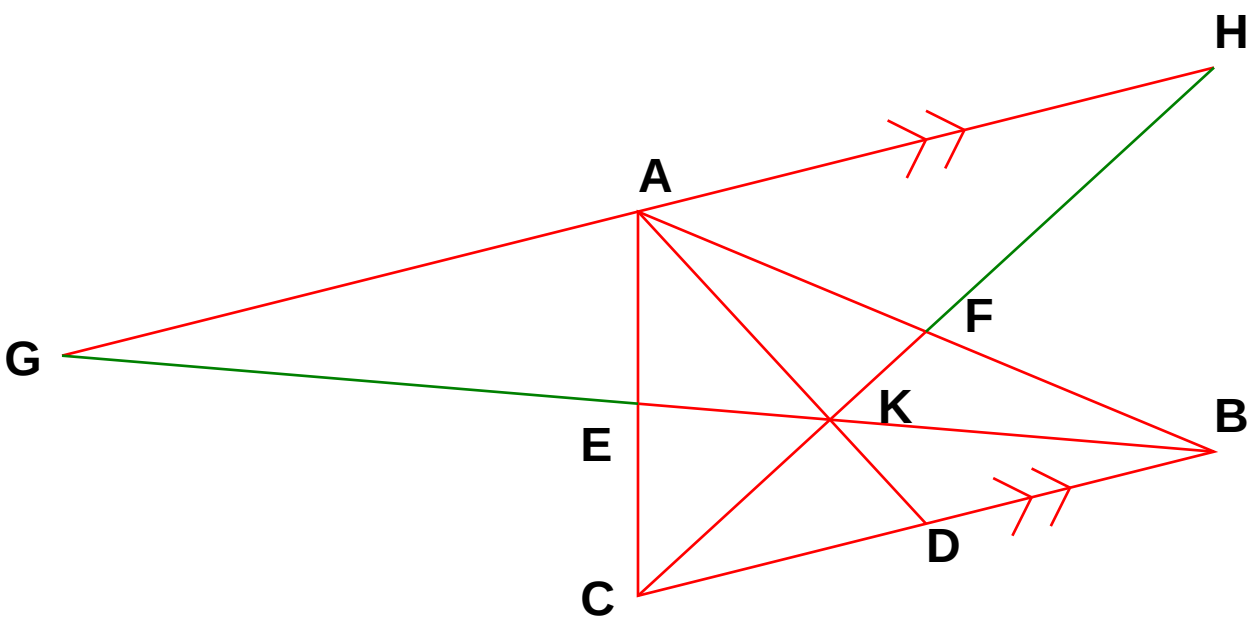
$$AF/FB * BD/DC * CE/EA = 1$$

Proof 1

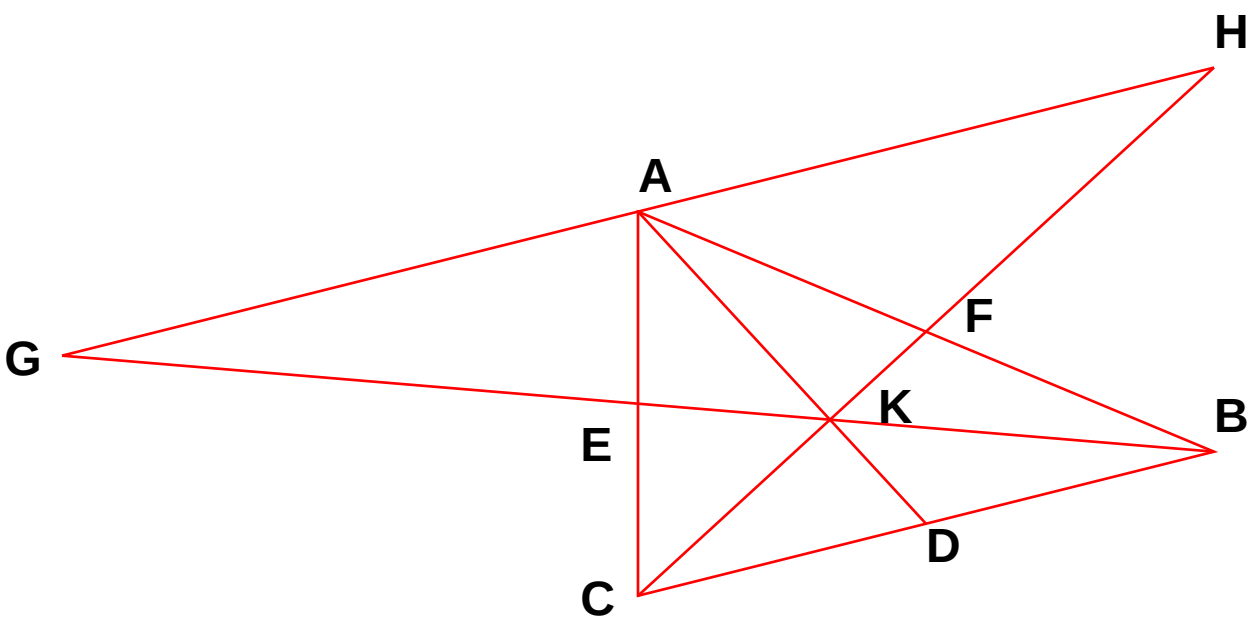



A
F
K
B
E
D
C


A
F
K
B
E
D
C


H
A
F
G
K
B
E
D
C

We now have several pairs of similar triangles,

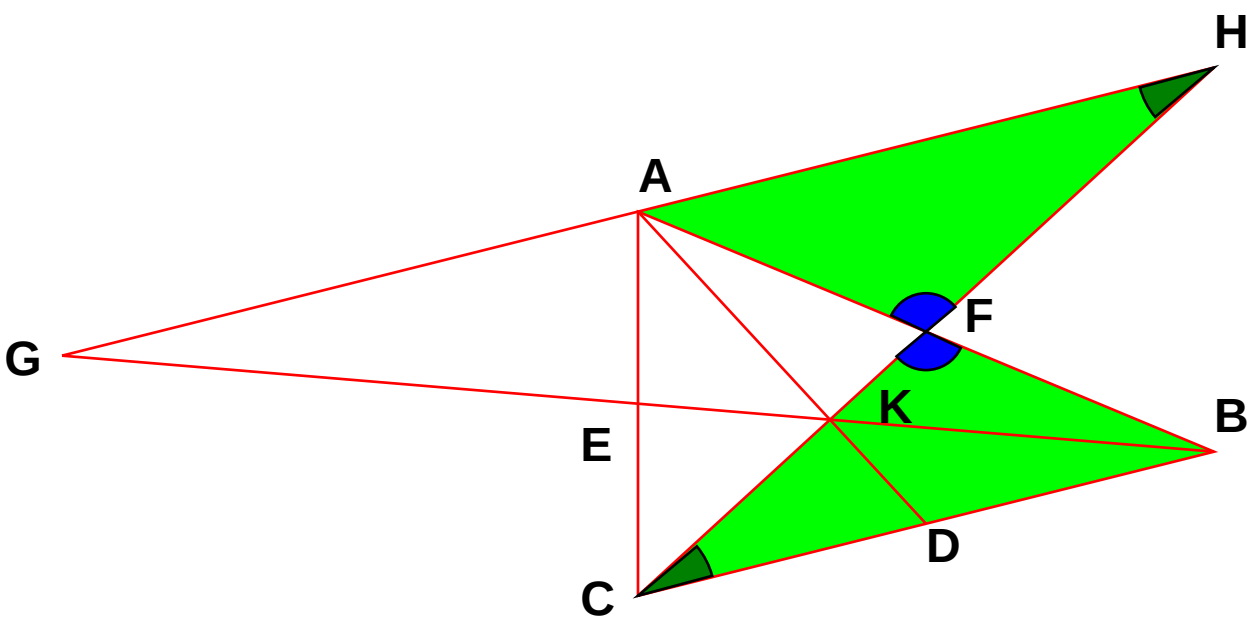


AHF is similar to BCF

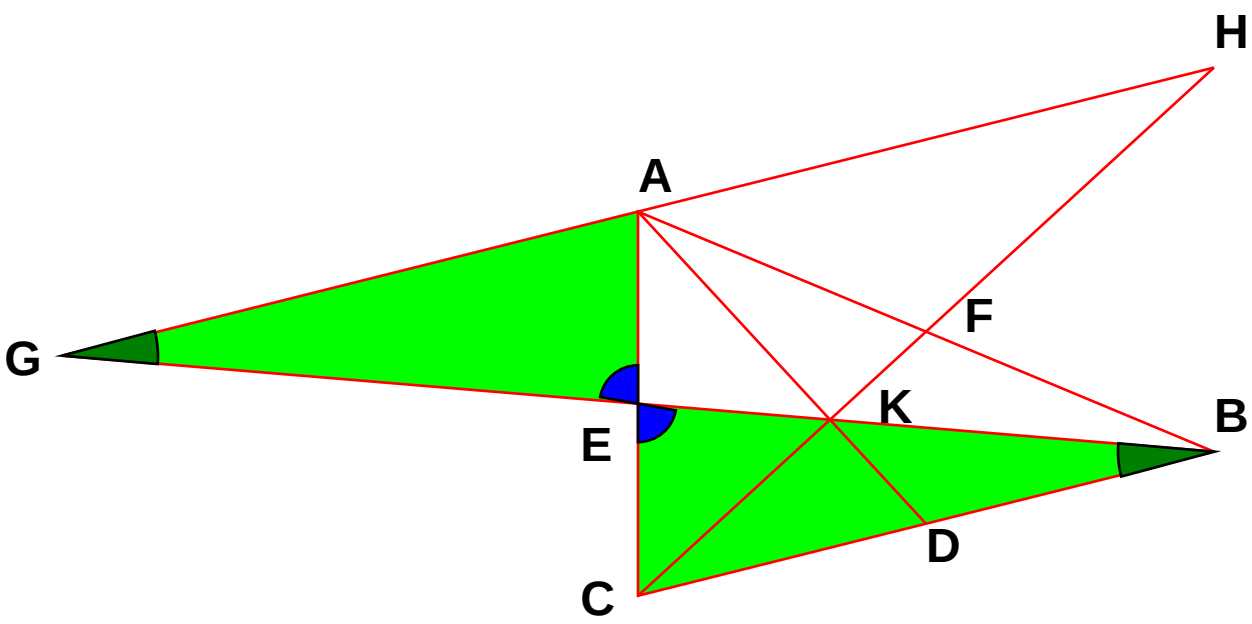
AEG is similar to BCE

AGK is similar to BDK

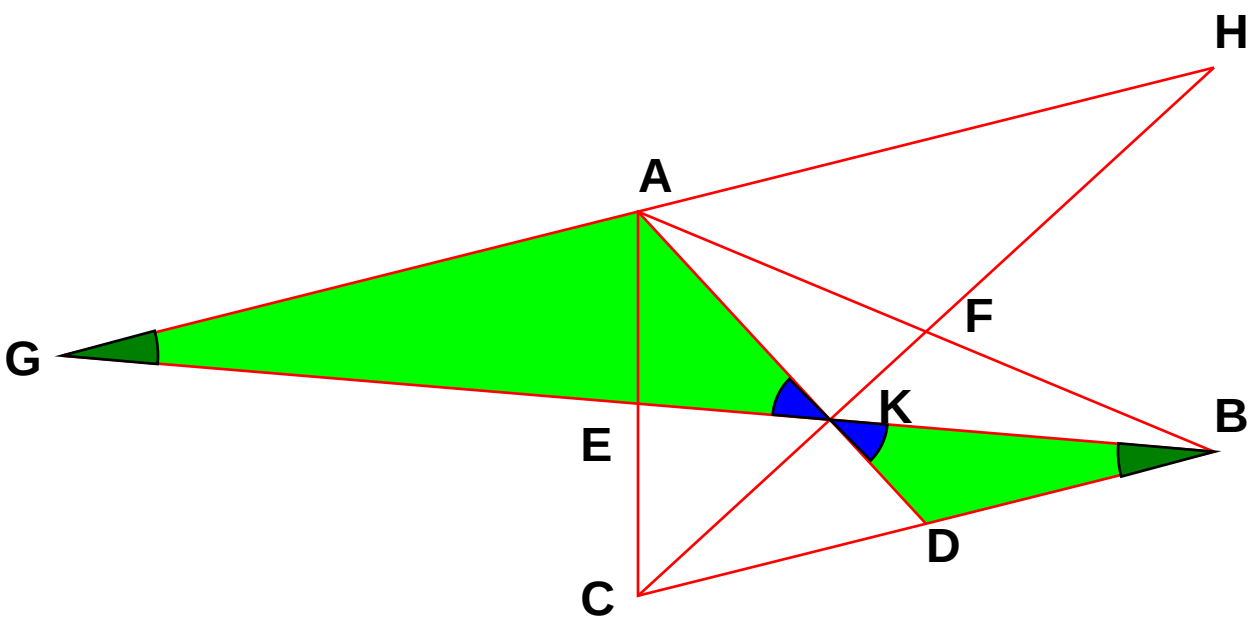
CDK is similar to AHK



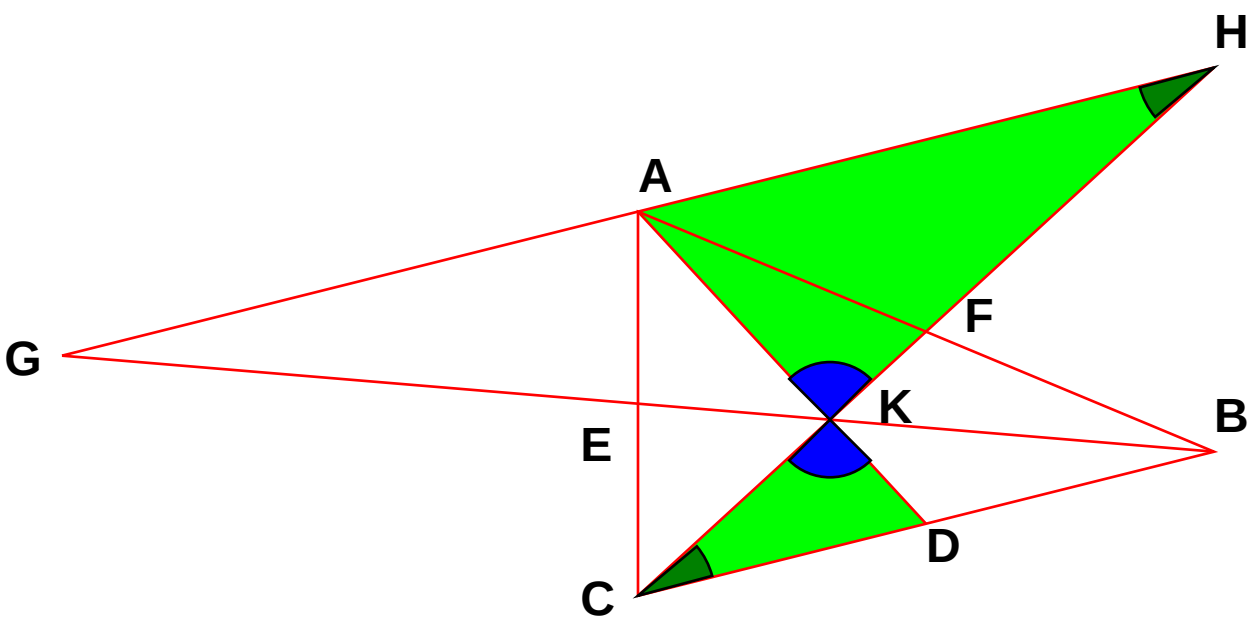
AHF is similar to BCF



AEG is similar to BCE

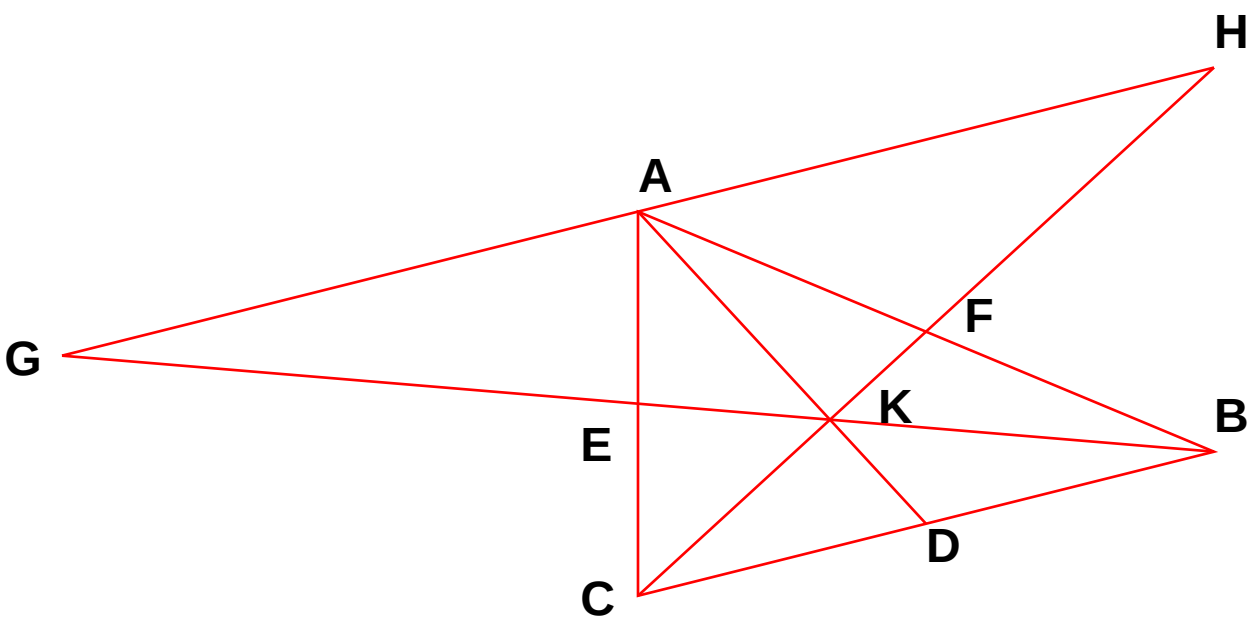


AGK is similar to BDK



CDK is similar to AHK

We now have the following proportions,

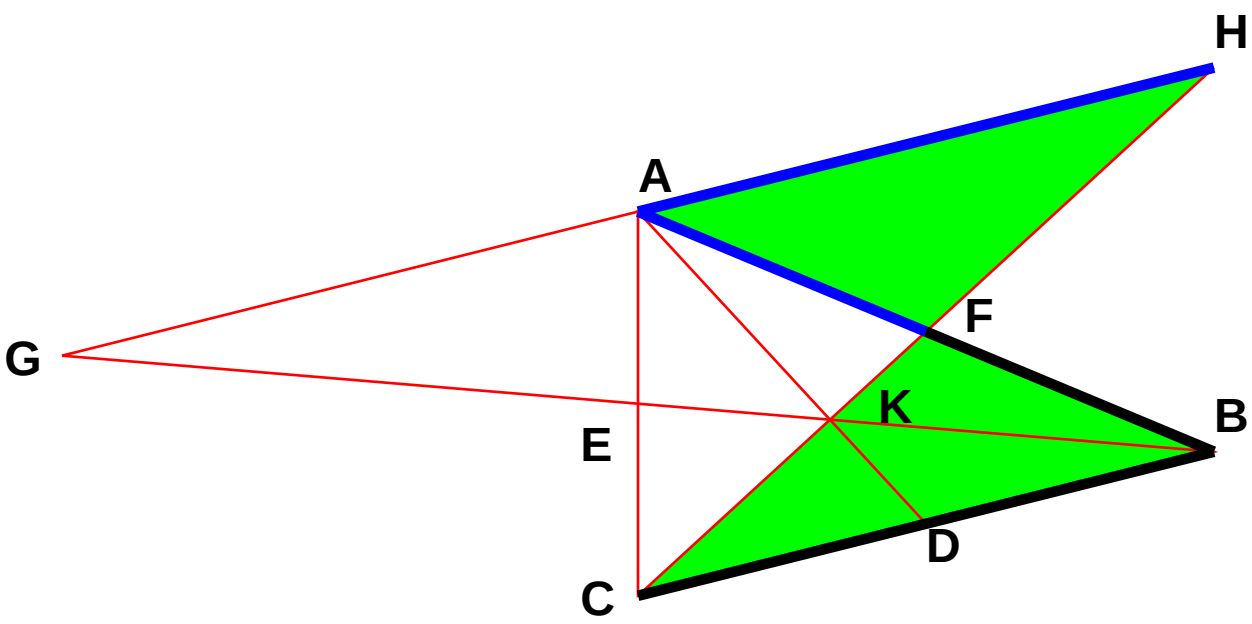


AF/FB = AH/BC

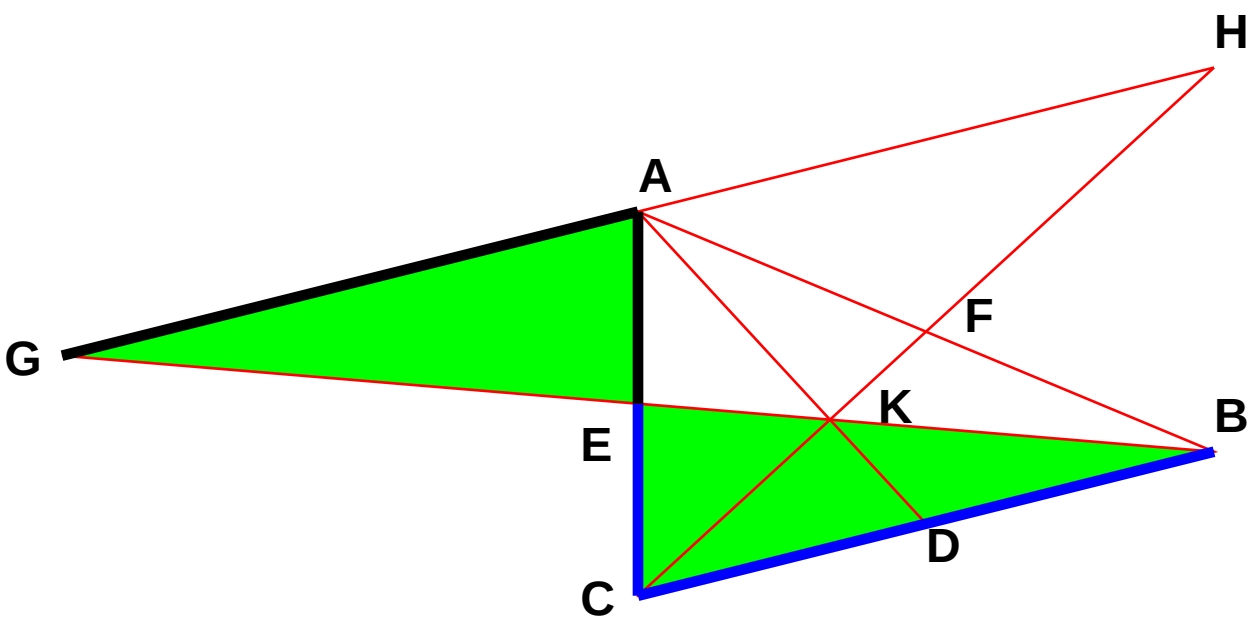
CE/EA = BC/AG

AG/BD = AK/DK

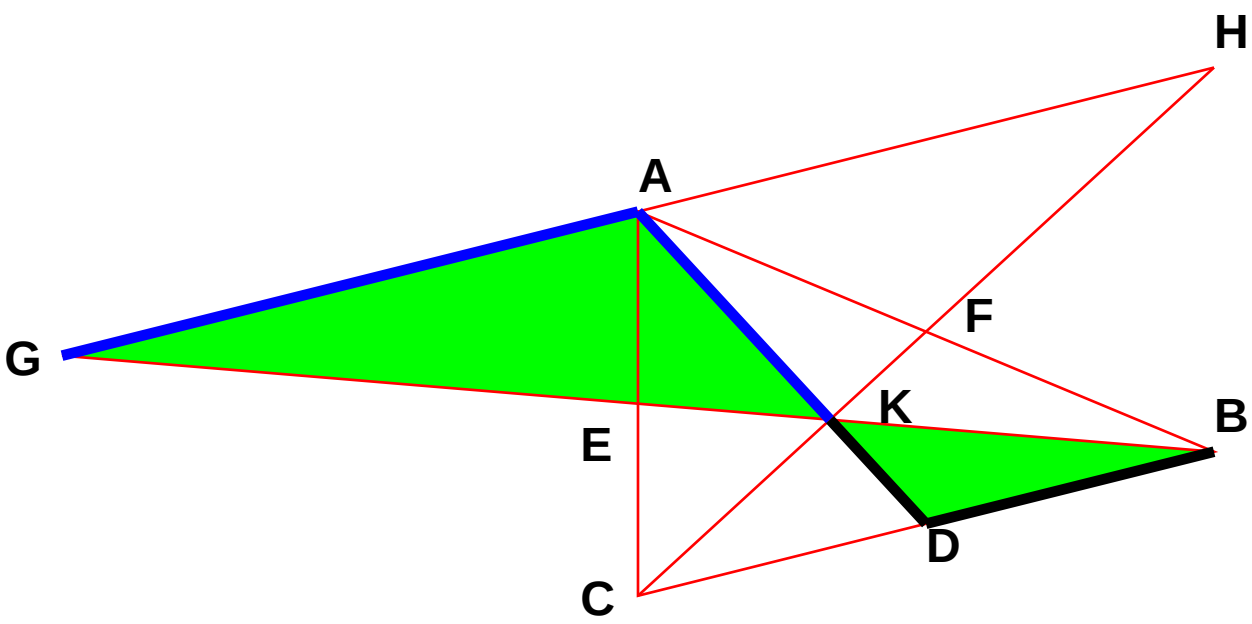
AH/DC = AK/DK



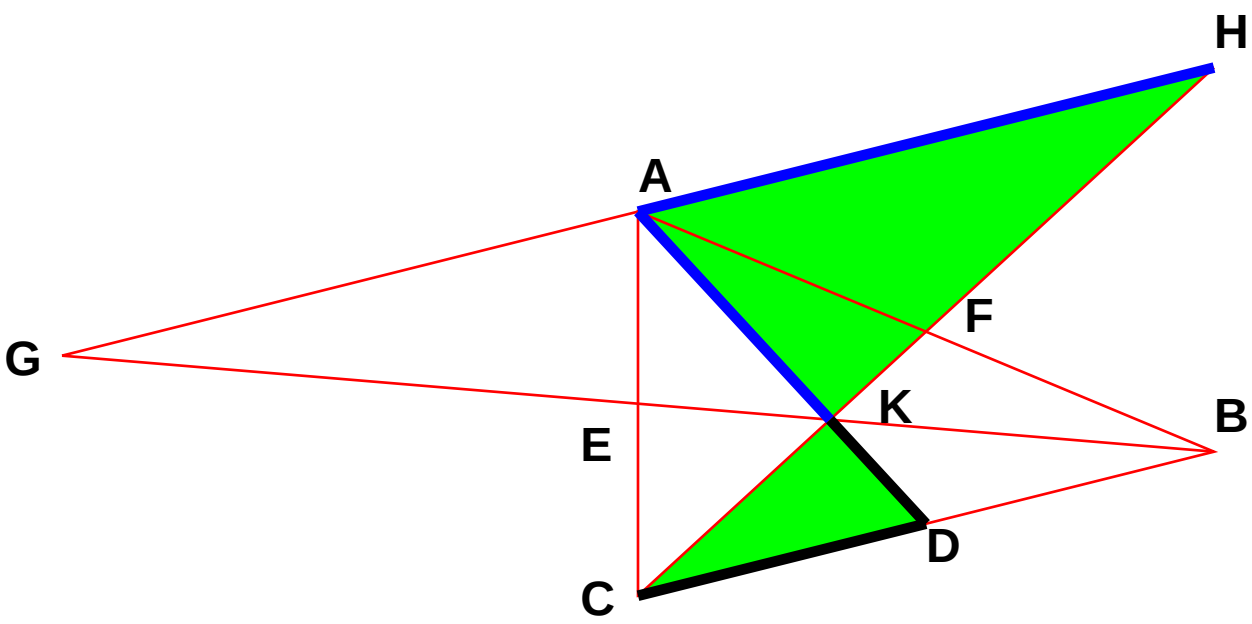
AF/FB = AH/BC



CE/EA = BC/AG

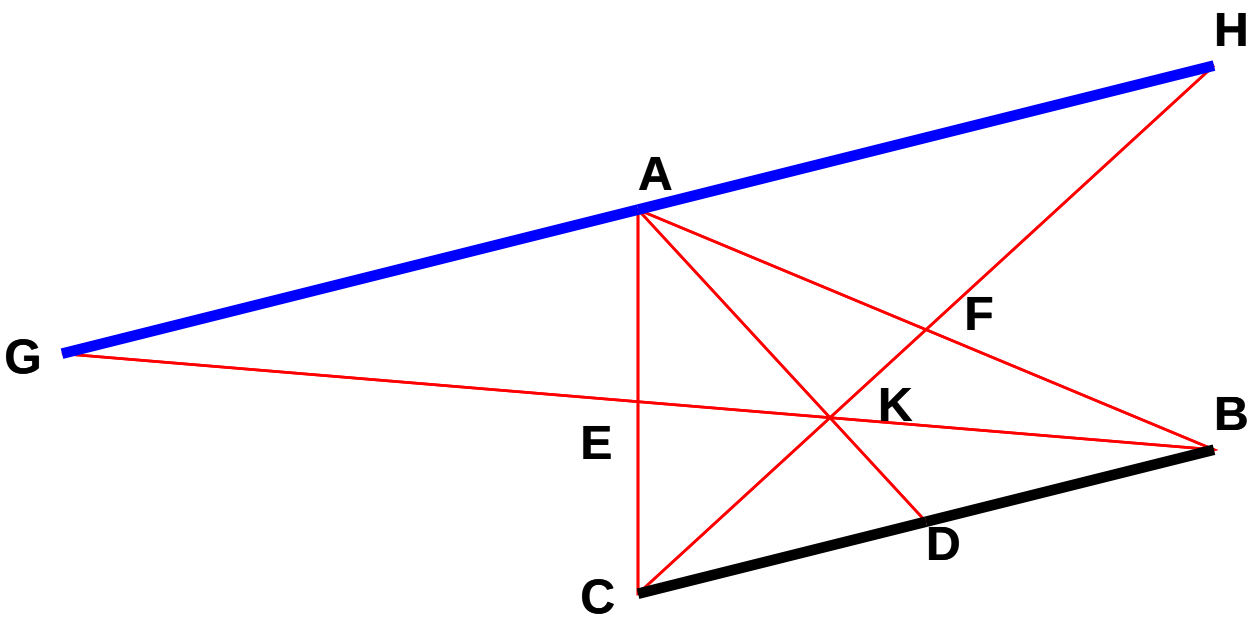


AG/BD = AK/DK



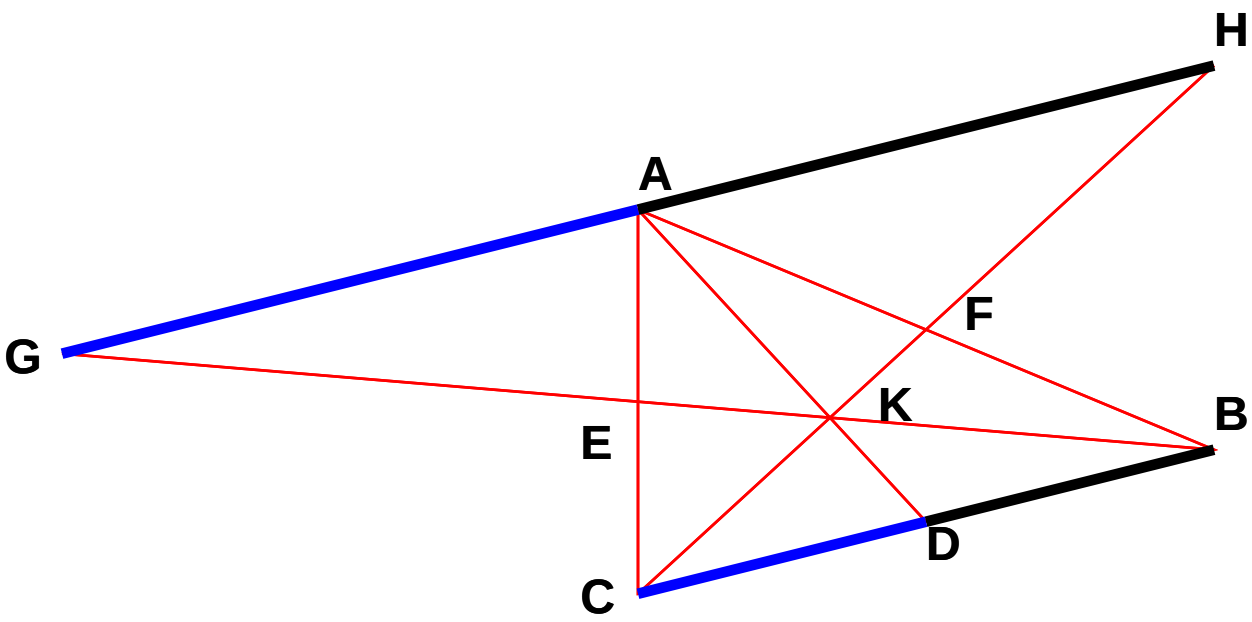
AH/DC = AK/DK

From the last two proportions, we have



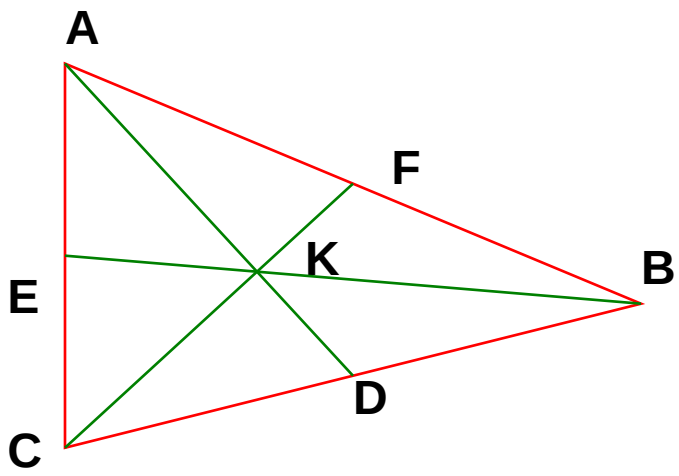
AG/BD = AH/DC

And therefore, we have



BD/DC = AG/AH

Consider three of the identities that we derived,

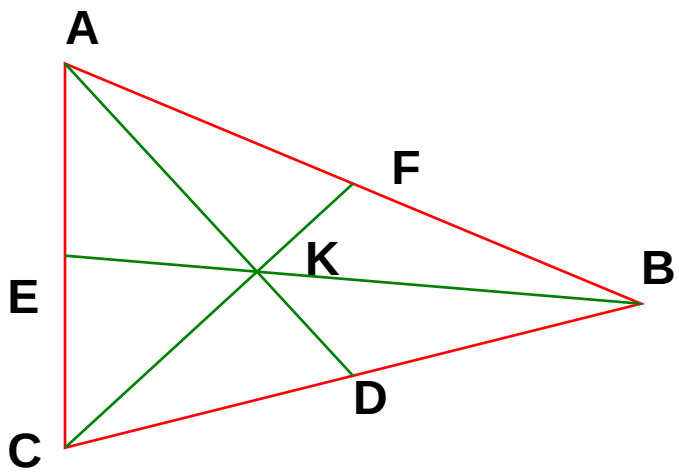


AF/FB = AH/BC

CE/EA = BC/AG

BD/DC = AK/DK

Multiplying them together, we get



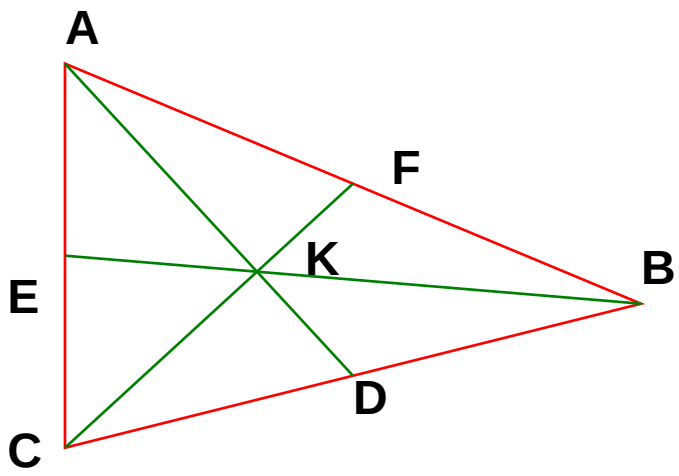
AF/FB * BD/DC * CE/EA

= AH/BC * BC/AG * AG/AH

= (AH * BC * AG)/(BC * AG * AH)

= 1

From the diagram,



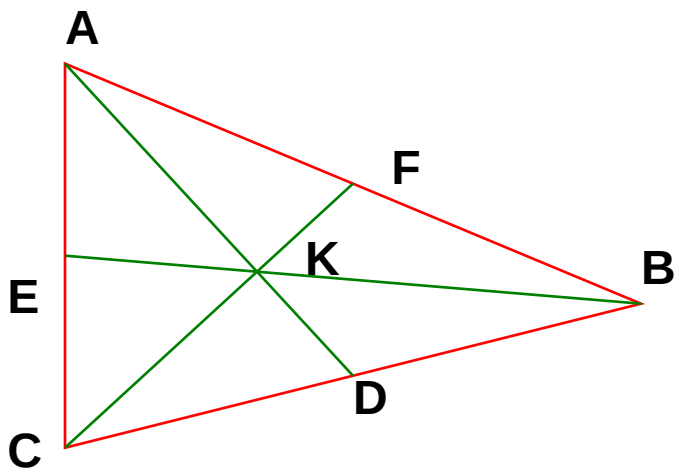
AD BE and CF all intersect at the point K.

Since we have shown that

$$AF/FB * BD/DC * CE/EA = 1$$

the => direction of Ceva's theorem has been proven.

We must now prove the reverse.

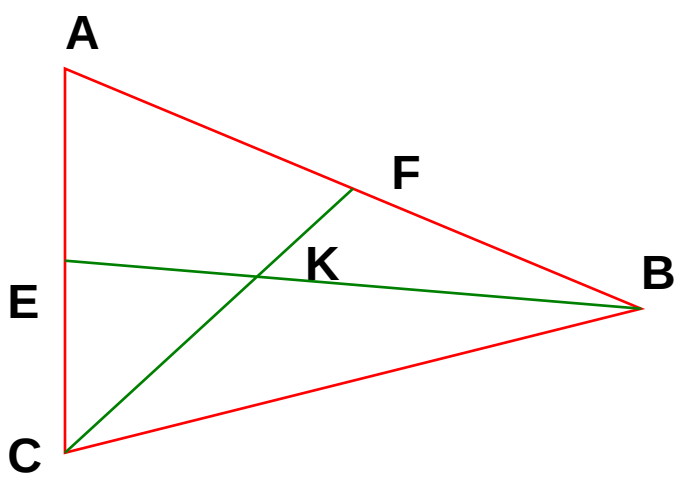


That is, if

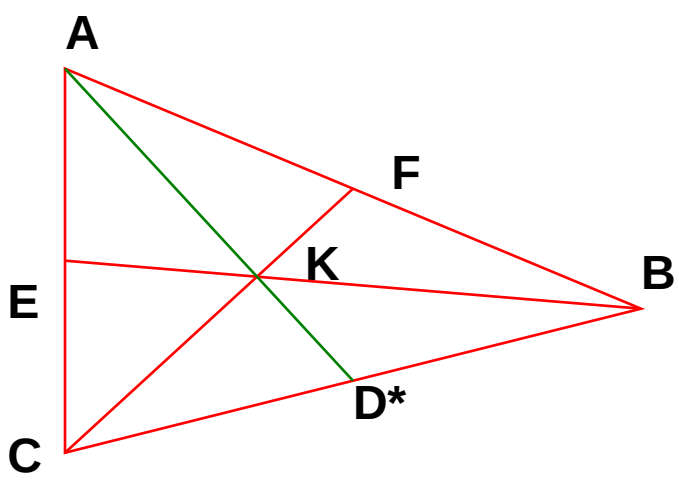
$$AF/FB * BD/DC * CE/EA = 1$$

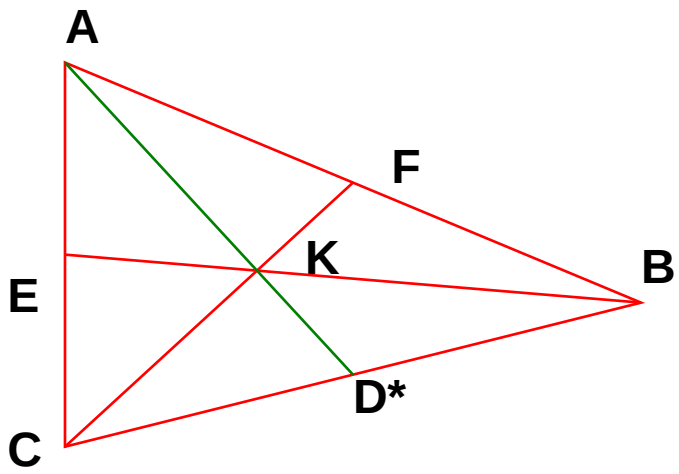
then AD BE and CF all intersect at one point.

Consider the following situation,



Draw AK and continue until it meets BC at D*





We are given

$$AF/FB * BD/DC * CE/EA = 1$$

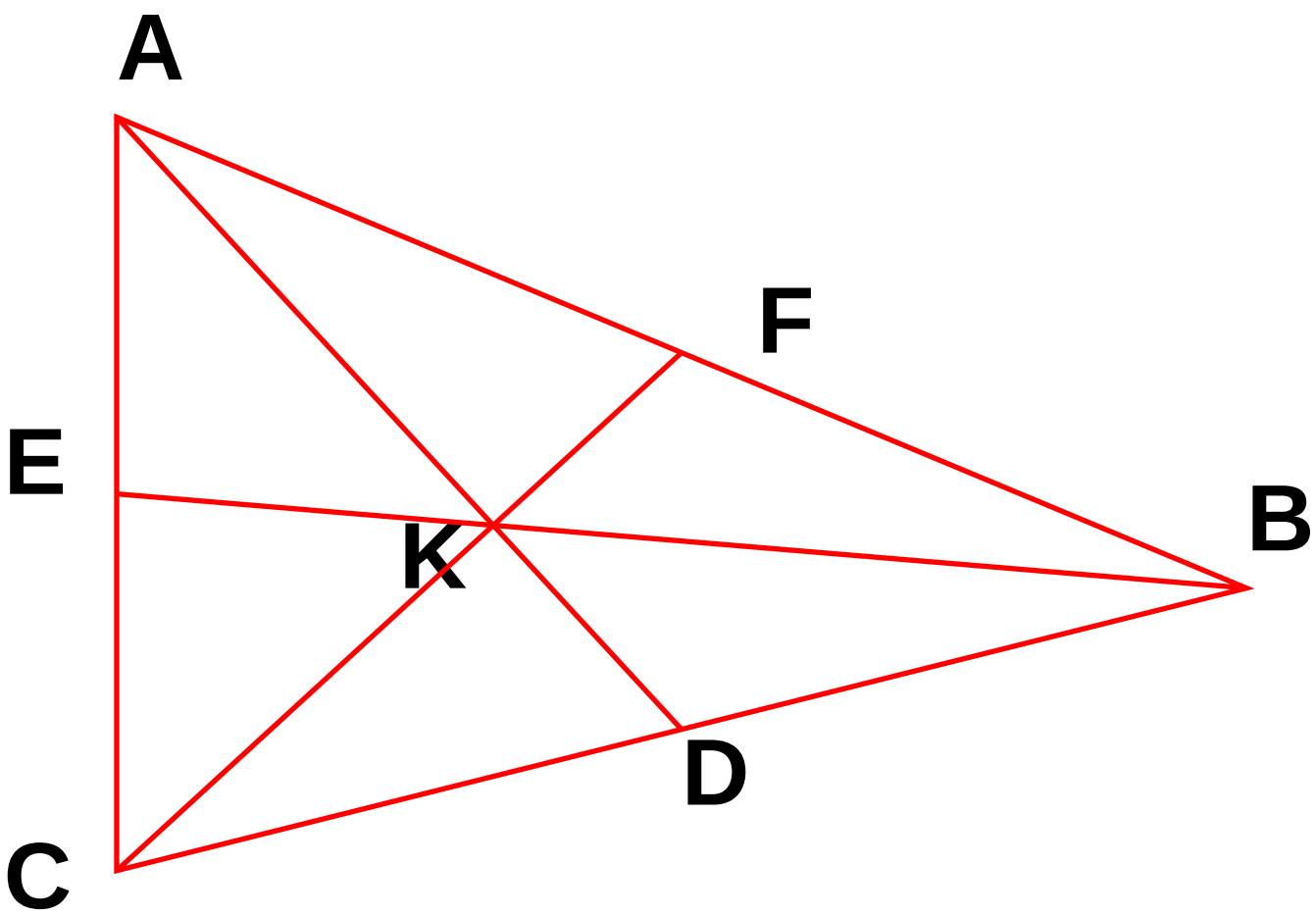
And from the => direction of Ceva's theorem

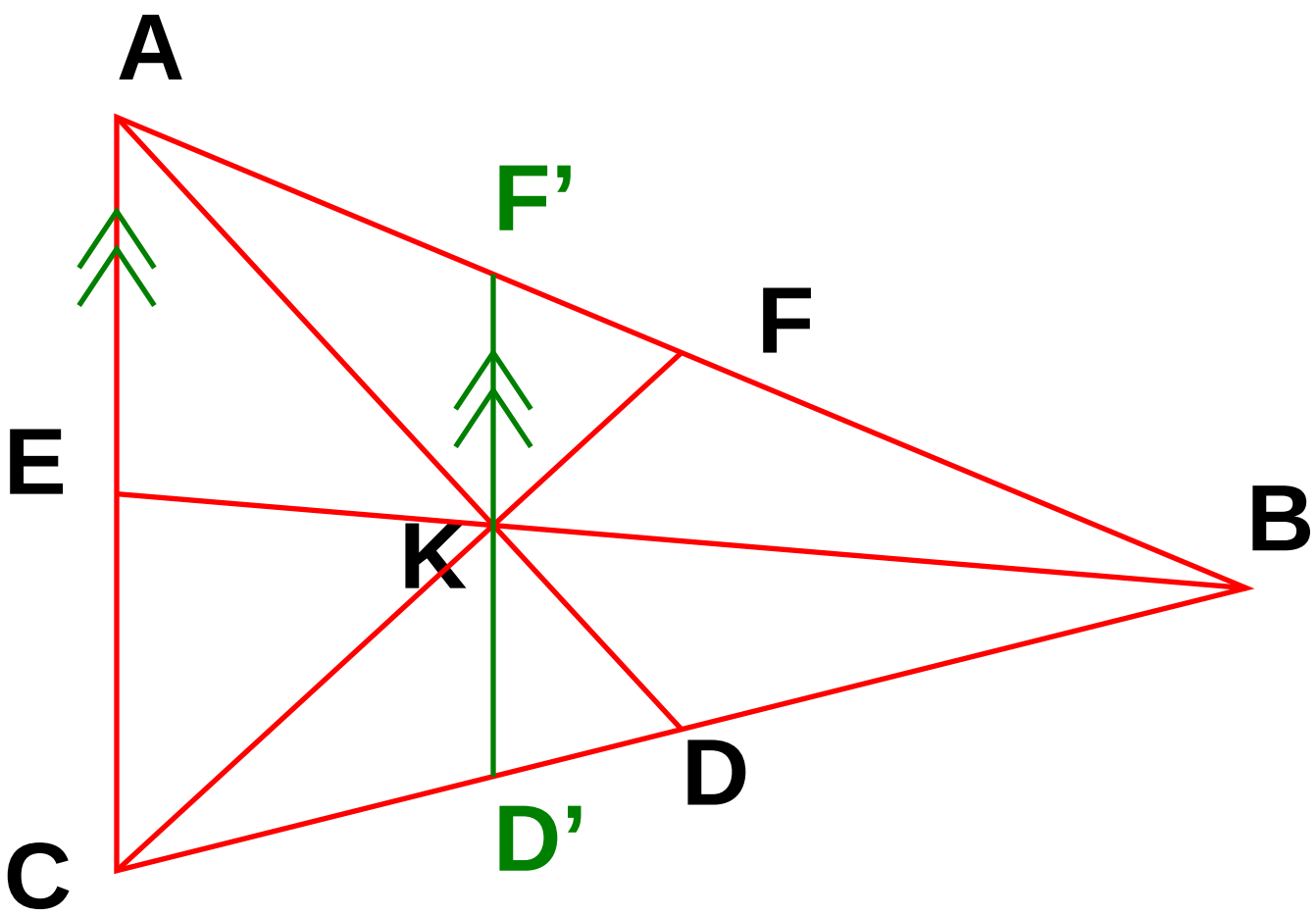
$$AF/FB * BD^*/D^*C * CE/EA = 1$$

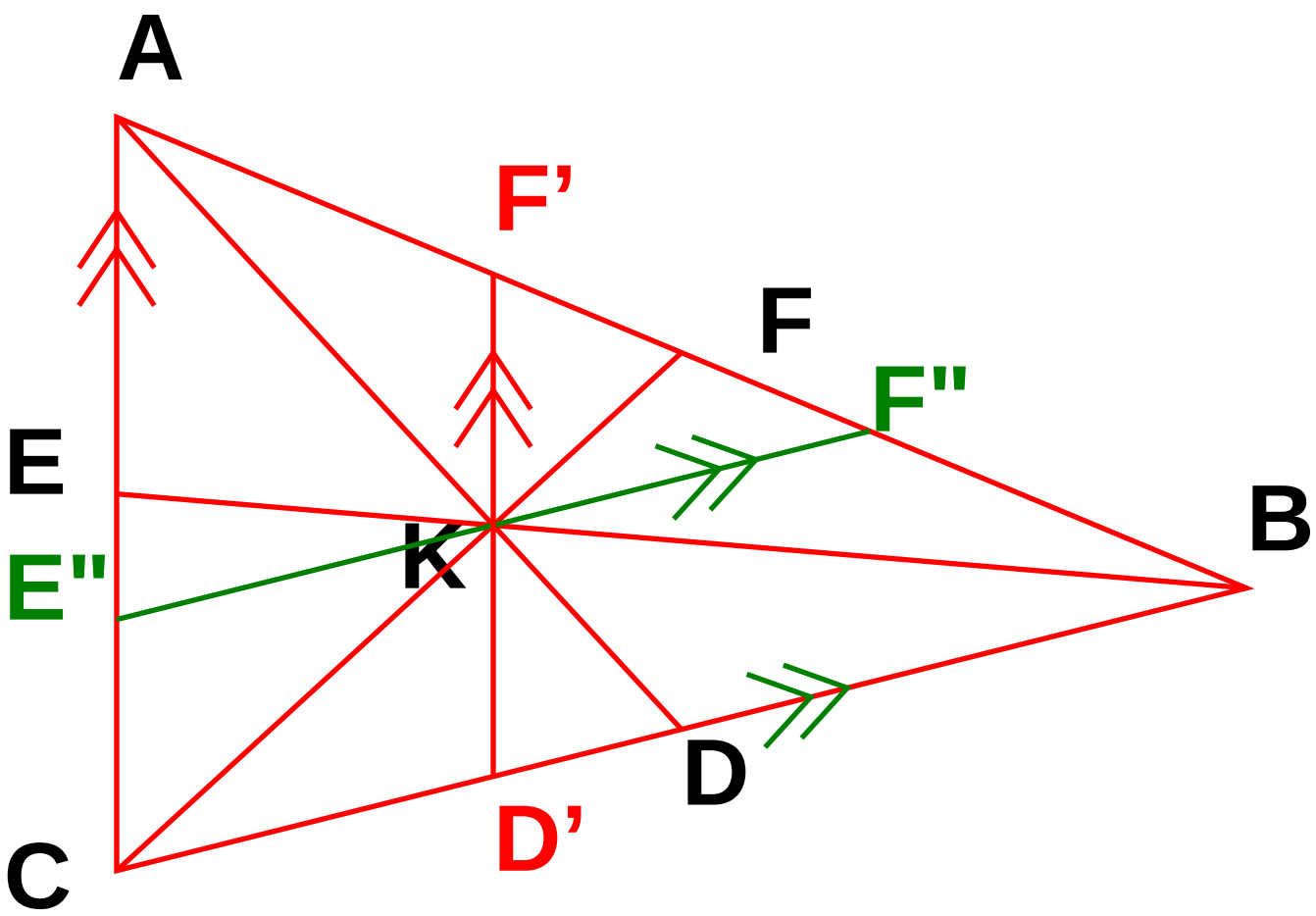
It follows that D*=D and thus AD BE and CF intersect at K.

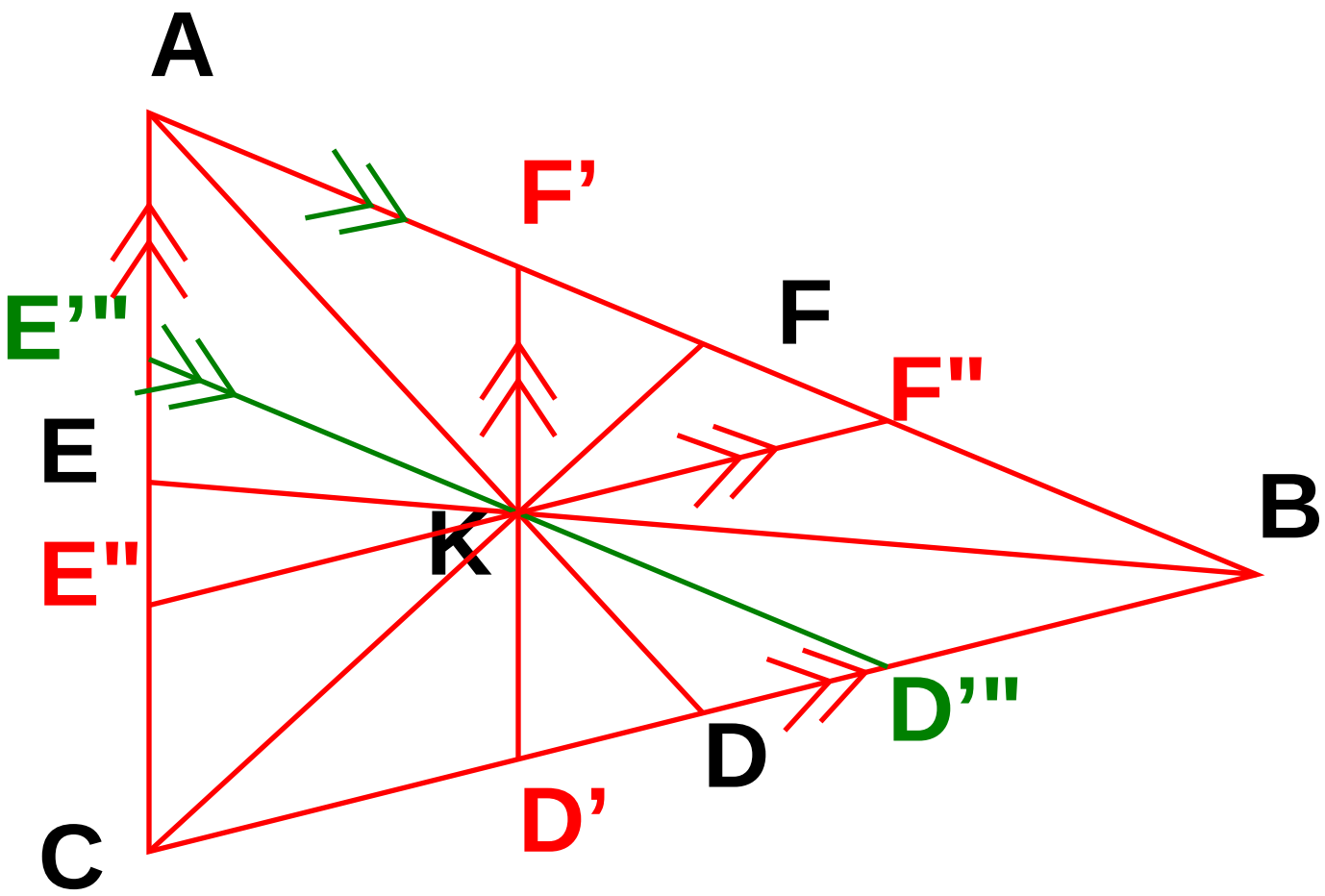
The <= direction of Ceva's theorem has been proven.

Proof 2

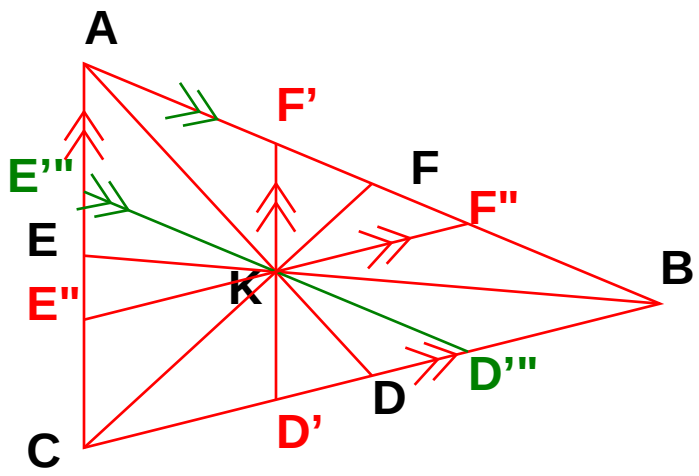



A
F’
F
E
B
K
D
C
D’


A
F’
F
F"
E
E"
K
B
D
D’
C


A
F’
E’"
F
F"
E
B
K
E"
D’"
D
C
D’

Again, we have pairs of similar triangles,

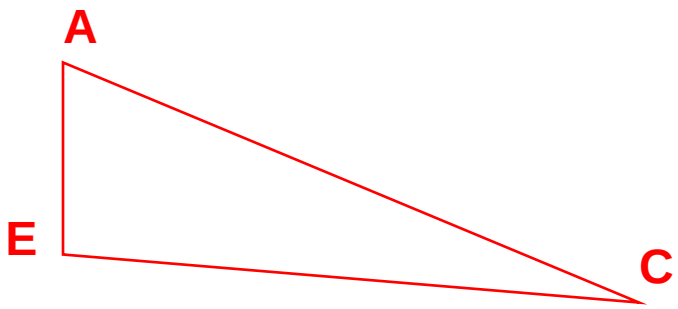


AEC is similar to F’KC

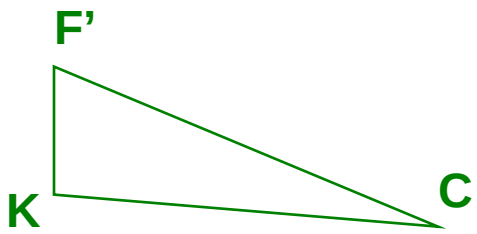
EBC is similar to KD’C

ACF is similar to F’KF

ACD is similar to KD’D

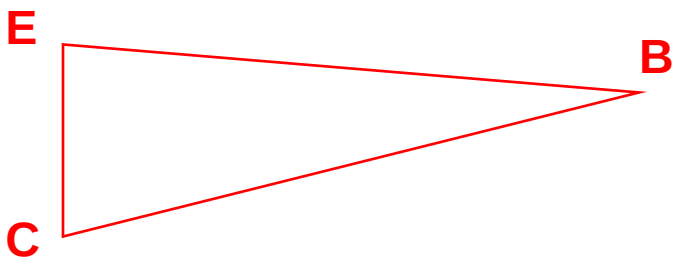


is similar to

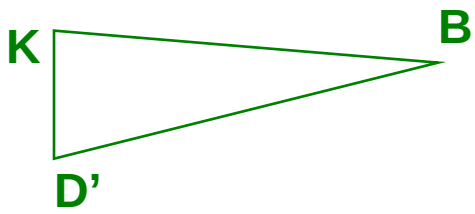


Therefore,

AE/F’K = CE/CK

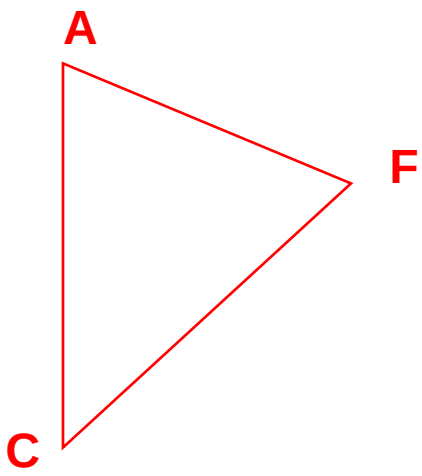


is similar to

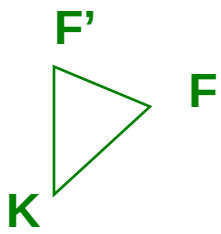


Therefore,

EB/KD’ = CE/CK

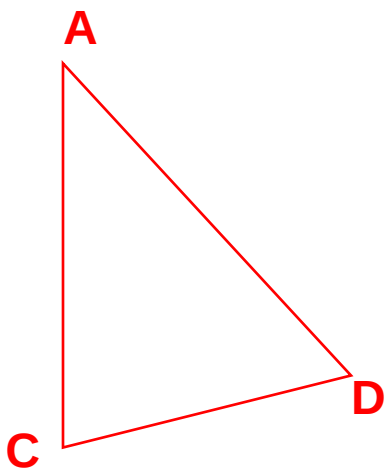


is similar to

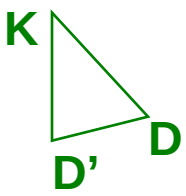


Therefore,

$$F'K/AC = KF/CF$$



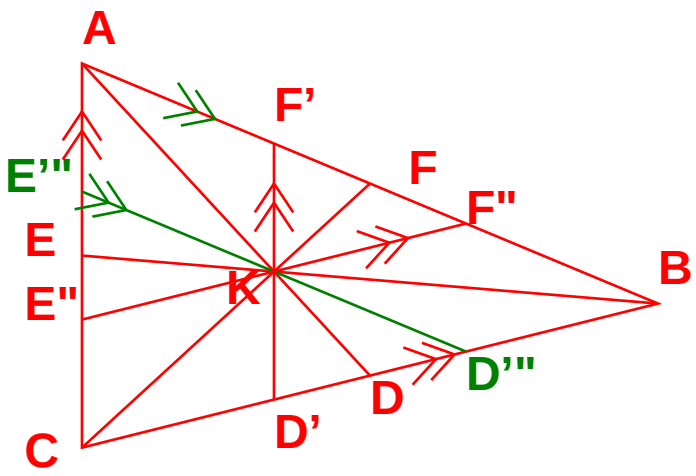
is similar to



Therefore,

KD’/AC = KD/AD

Our identities are,



AE/F’K = CE/CK

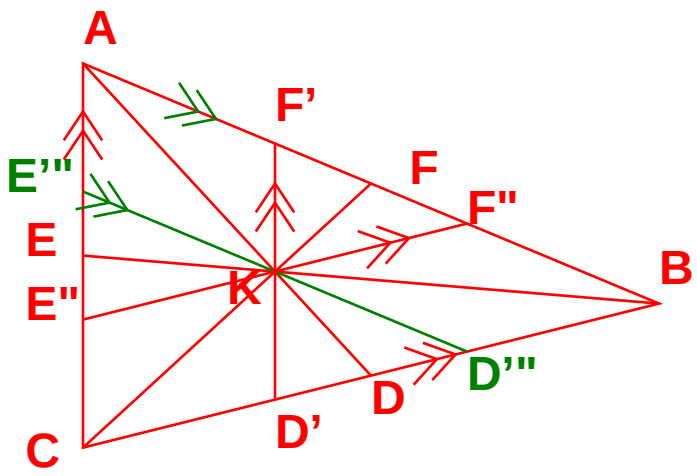
EB/KD’ = CE/CK

F’K/AC = KF/CF

KD’/AC = KD/AD

AE/EB = F’K/KD’ [from the first two]

Maniputlating the identities, we have



AE/EB = [KF/KD]/[BF/AD]

BD/DC = [KE/KF]/[CE/BF]

CF/FA = [KD/KE]/[AD/CE]

Multiplication of the right sides gives 1 as all the terms cancel each other out. We are left with Ceva’s identity:

AF/FB * BD/DC * CE/EA = 1

Thus Ceva’s theorem is proven.

Two Proofs of Ceva's Theorem

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References:

Ceva's Theorem by Alexander Bogomolny of cut-the-knot.org

http//www.cut-the-knot.org/Generalization/ceva.shtml

Note: cut-the-knot.org recognizes Darij Grinberg as the source for the second proof