Projective Geometry

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Math 308
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Final Project

1. Duality

The principle of Duality is a very important and useful property of projective geometry.In projective geometry, all the propostions exist in dual pairs. If we interchange the words "line" and "point", "collinear" and "concurrent", and "join" and "intersect" in a theorem, the resulting propostion is also true.

Example: Desargues’s theorem is state as: If two triangles have the joins of corresponding vertices concurrent then the intersects of correspoding sides are collinear.

Therefore its dual is: It two triangles have the intersects of corresponding sides collinear then the joins of correspdoing verices are concurrent.

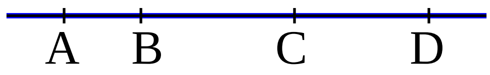
point <=> line
join <=> intersect
collinear <=> concurrent

2. The Cross Ratio

Cross Ratio is an important and fundermental concept in projective geometry. It is defined for 4 collinear points or 4 concurent lines.

The cross-ratio of four collinear points A, B, C, D is defined as:

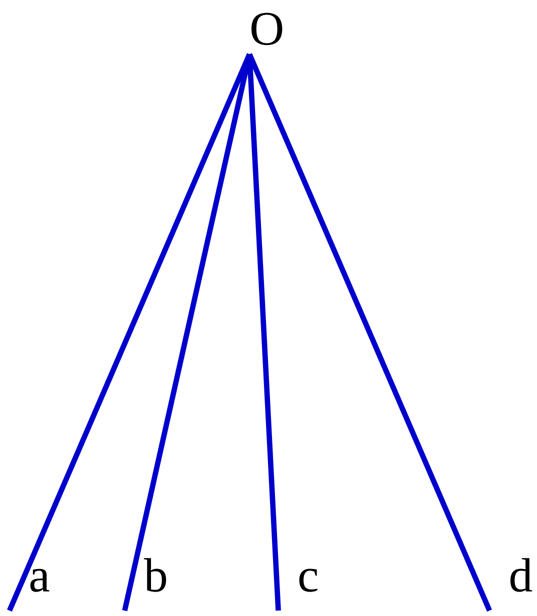
$$R(A,B;C,D) = \frac{CA/CB}{DA/DB}$$



note:CA etc is the signed magnitute of the corresponding segment.

The cross-ratio of four concurrent lines a, b, c, d is defined as:

$$R(a,b;c,d) = \frac{\sin(ca)/\sin(cb)}{\sin(da)/\sin(db)}$$



note: ca is the angle between line c and a. Angles are also signed.

Lemma. 4 concurrent lines a, b, c, d intersect with another straight line at 4 collinear points A, B, C, D. Then R(A,B;C,D) = R(a,b;c,d).

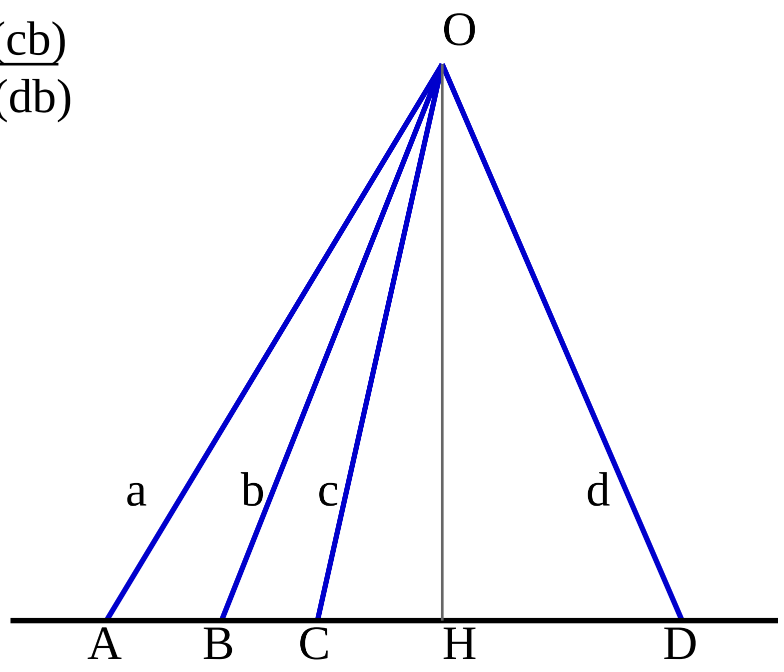
Proof. Draw OH perpendicular to line ABCD

$$R(A,B;C,D) = \frac{CA/CB}{DA/DB} = \frac{1/2*OH*CA \ / \ 1/2*OH*CB}{1/2*OH*DA \ / \ 1/2*OH*DB}$$

$$= \frac{\text{areaCOA / areaCOB}}{\text{areaDOA / areaDOB}}$$

$$= \frac{1/2*OA*OC*\sin(ca) \ / \ 1/2*OB*OC*\sin(cb)}{1/2*OA*OD*\sin(da) \ / \ 1/2*OB*OD*\sin(db)}$$

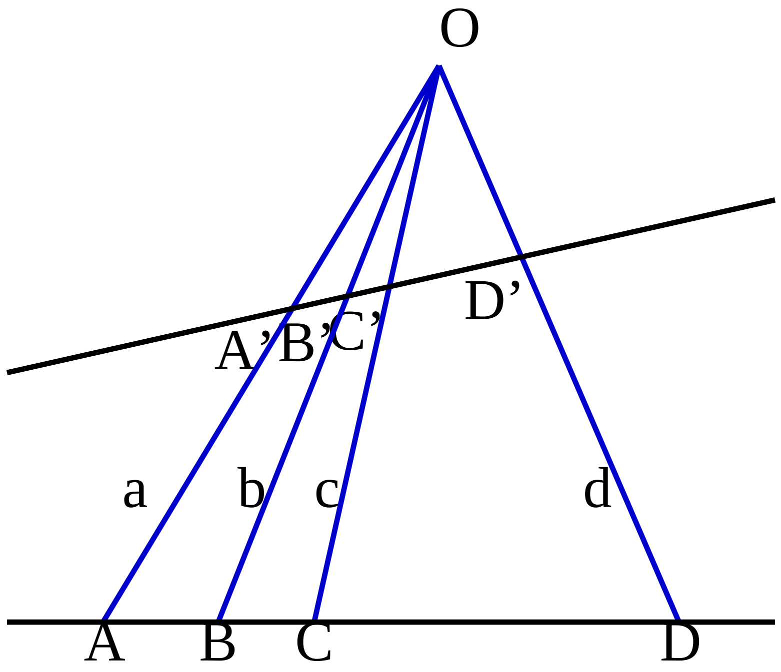
$$= \frac{\sin(ca) \ / \ \sin(cb)}{\sin(da) \ / \ \sin(db)}$$



Theorem. The cross-ratio of collinear points does not change under central projections.

Proof. From the lemma before,

$$R(A, B; C, D) = R(a, b; c, d) = R(A', B'; C', D').$$



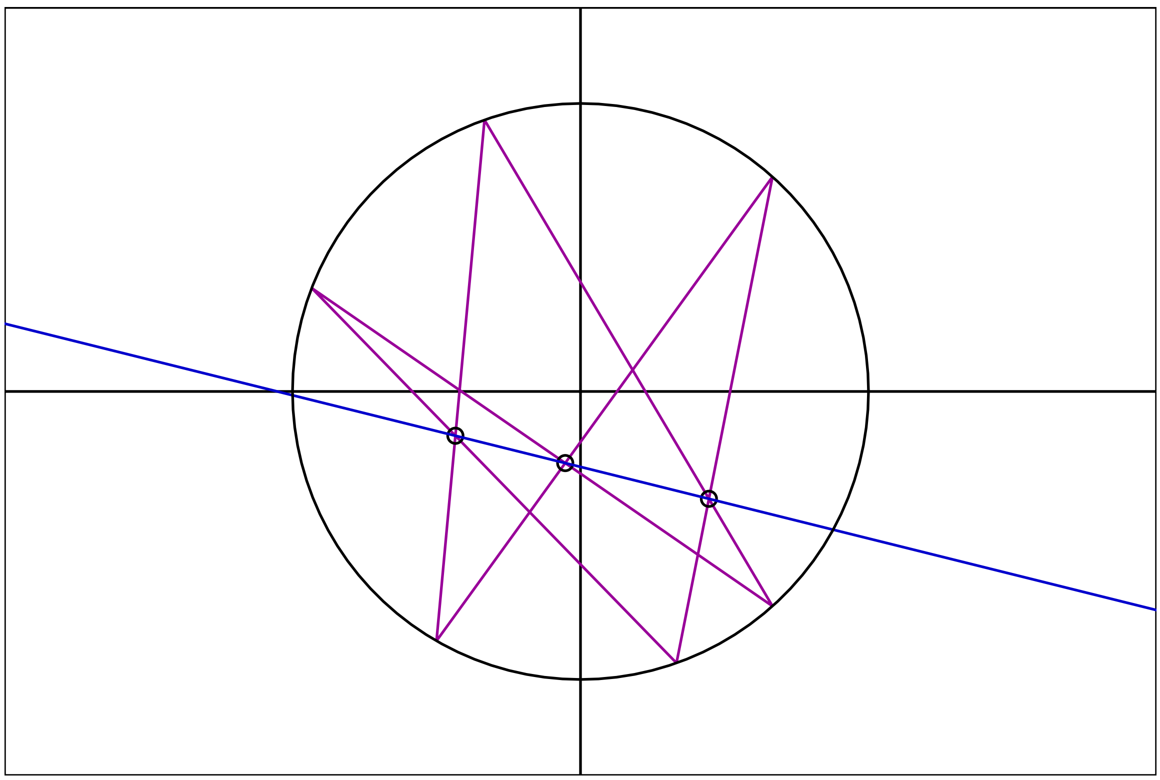
3. Improtant Theorems in projective geometry

3.1 Pascal's theorem and its applications

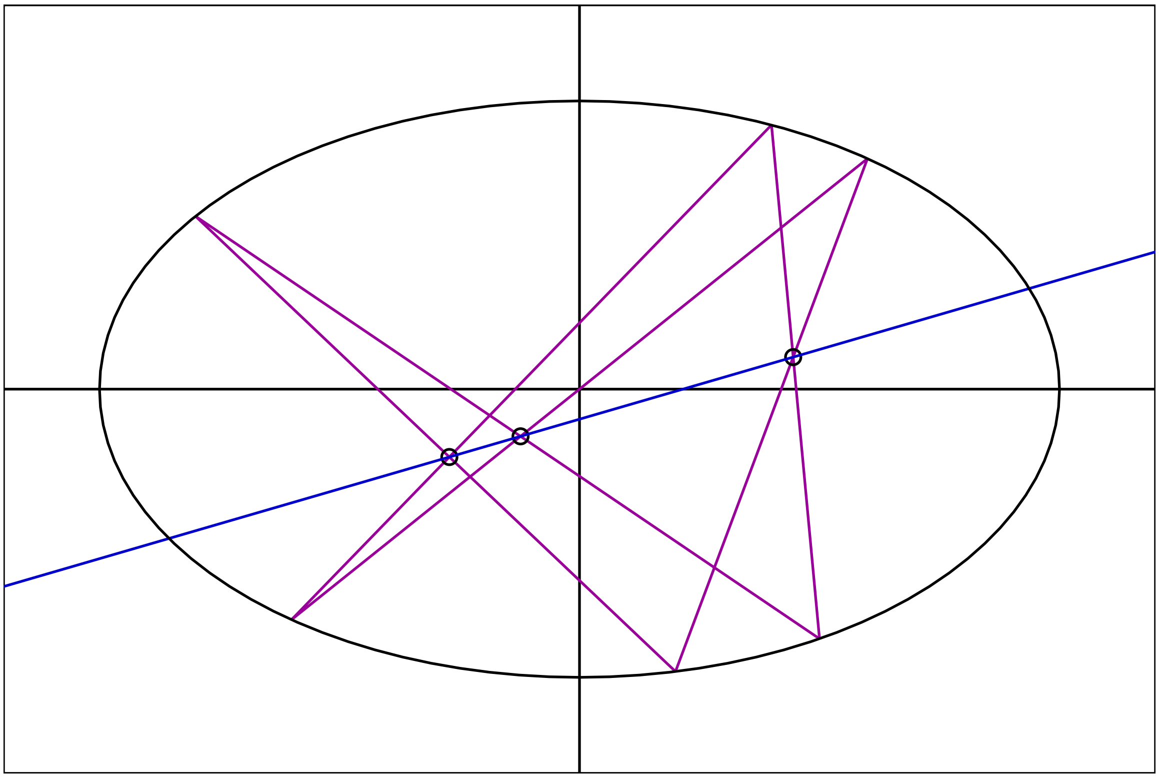
Pascal's theorem states that if a hexagon is inscribed in a conic section, then the continuation of opposite sides intersect in 3 points which are collinear

The following figures show the cases of the Pascal's theorem in Circle, Eclipse, Hyperbola, and Parabola.

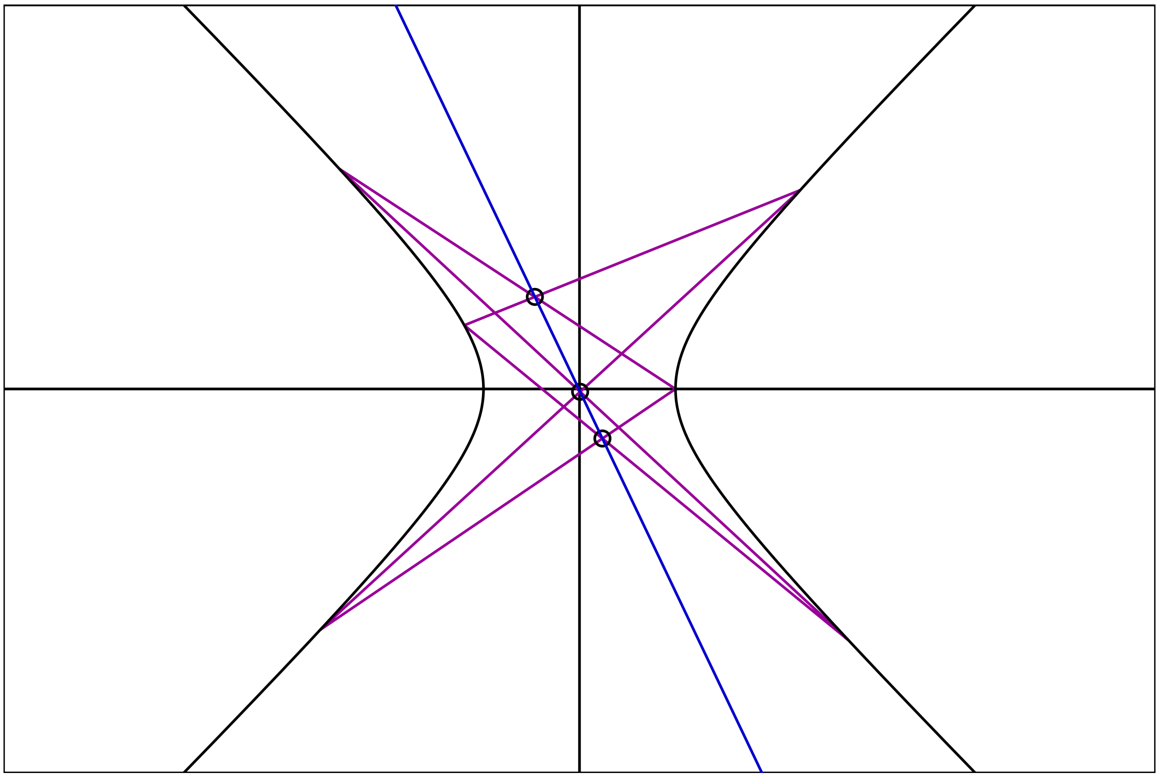
Circle



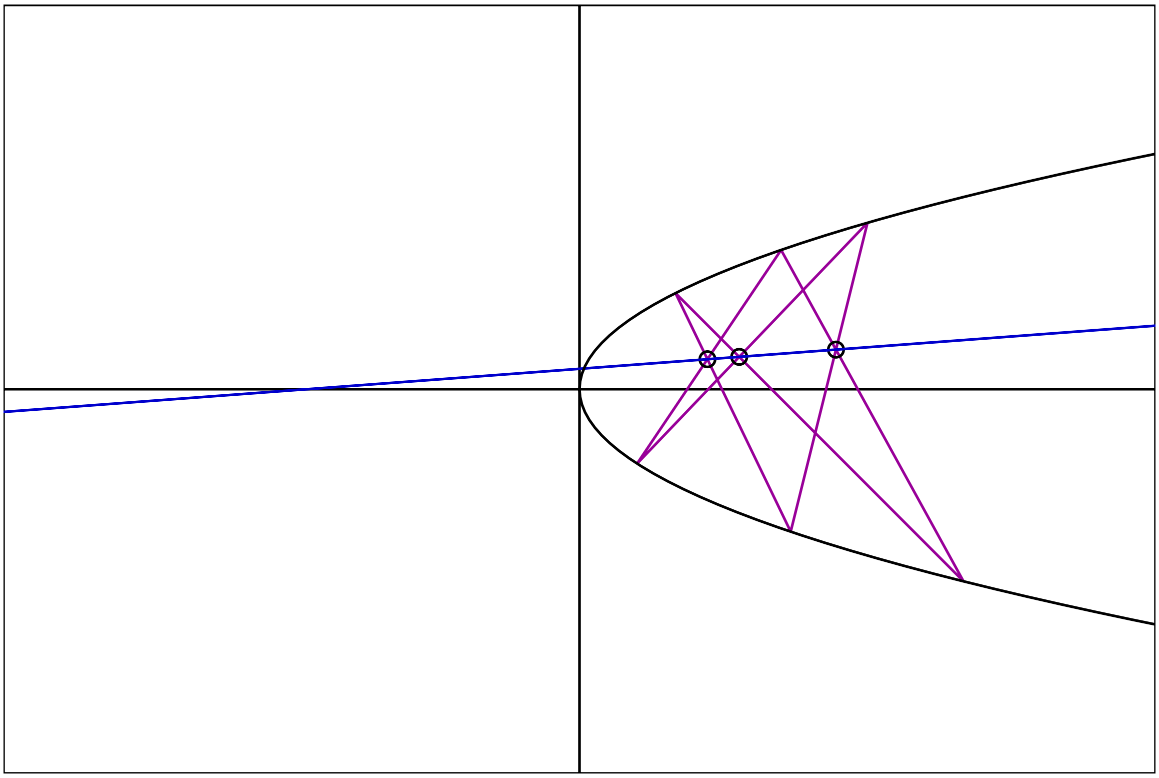
Eclipse



Hyperbola



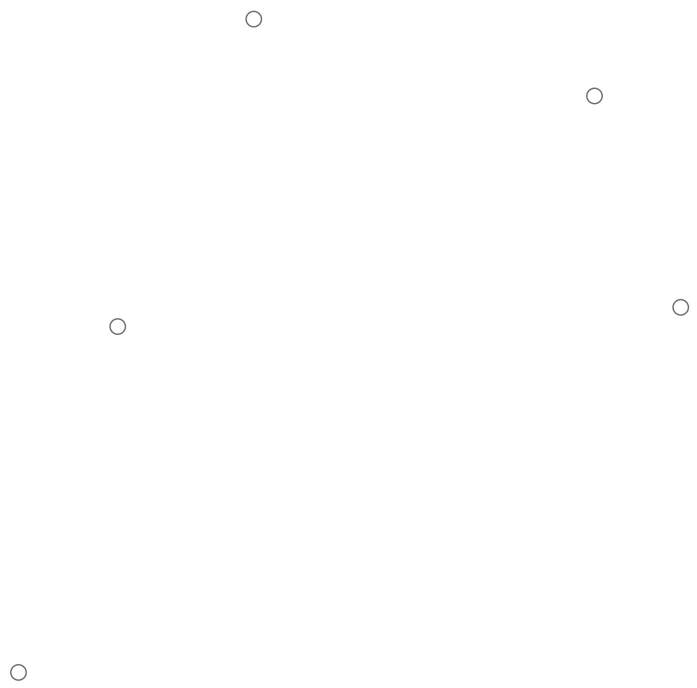
Parabola

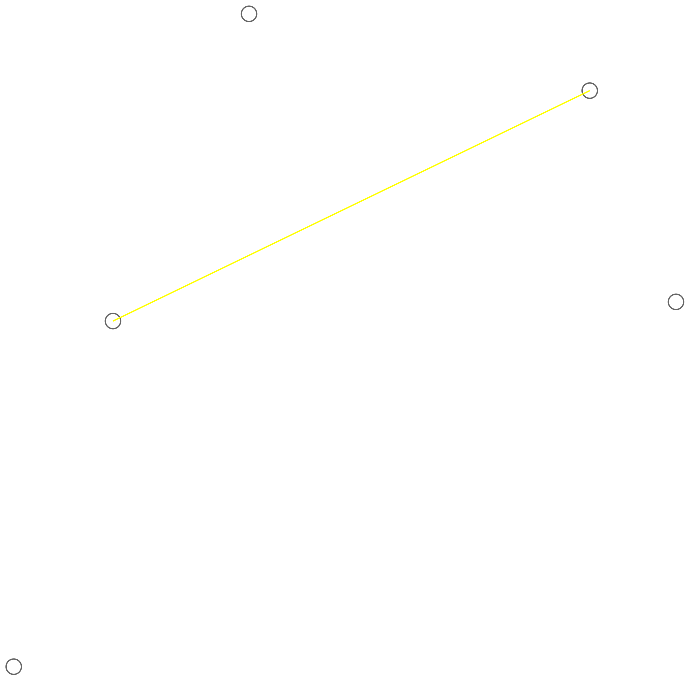


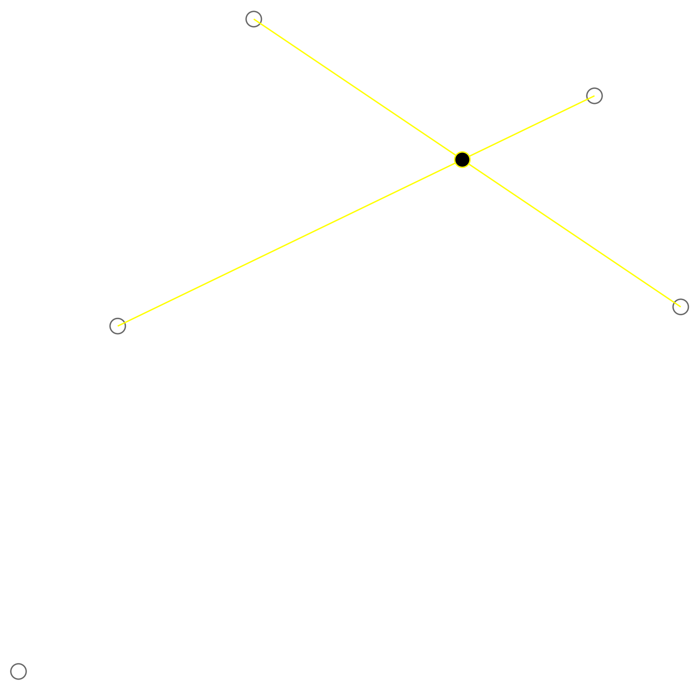
Applications of Pascal’s theorem

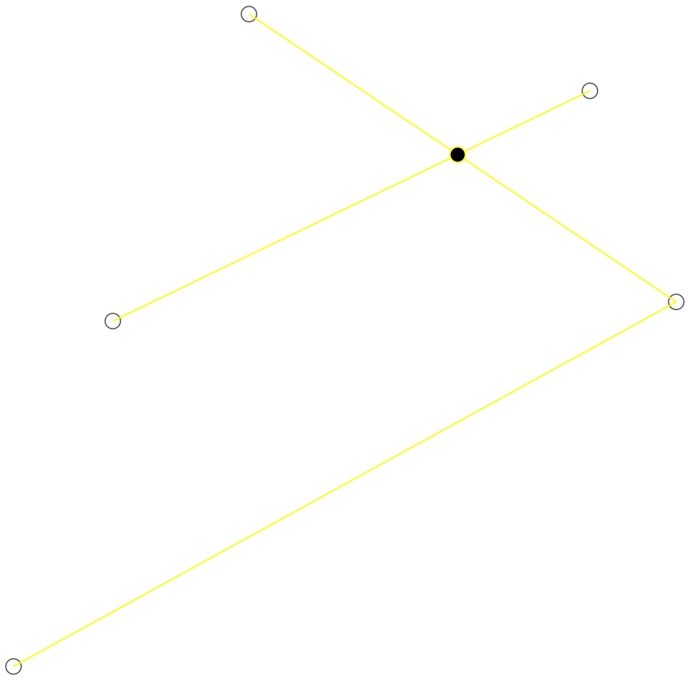
Pascal’s theorem is widely applied. Many of its applications are from geometry construction. I will demonstrate 2 applications of Pascal’s theorem.

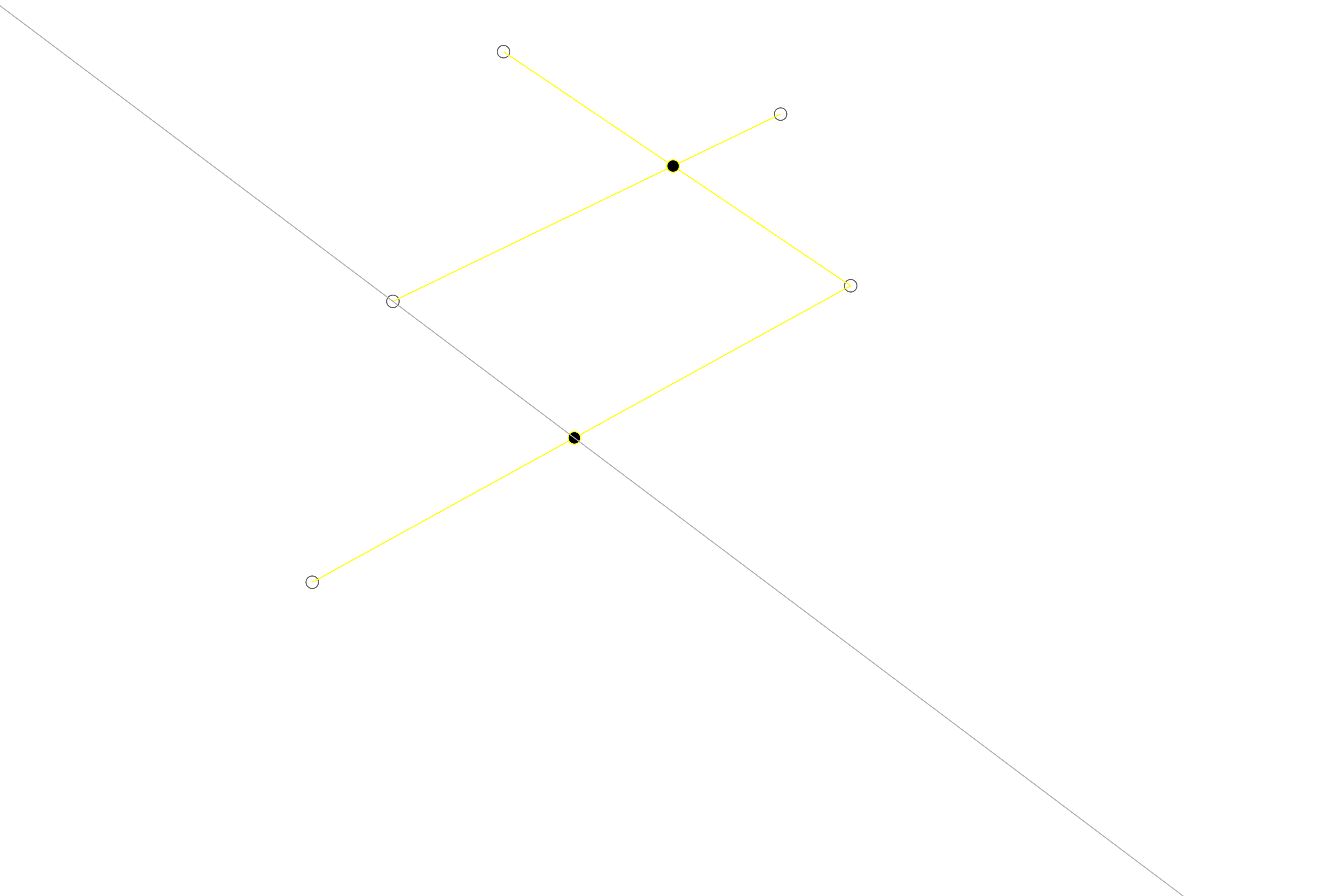
Application 1: Locate the sixth point on a conic when five points on this conic are given.

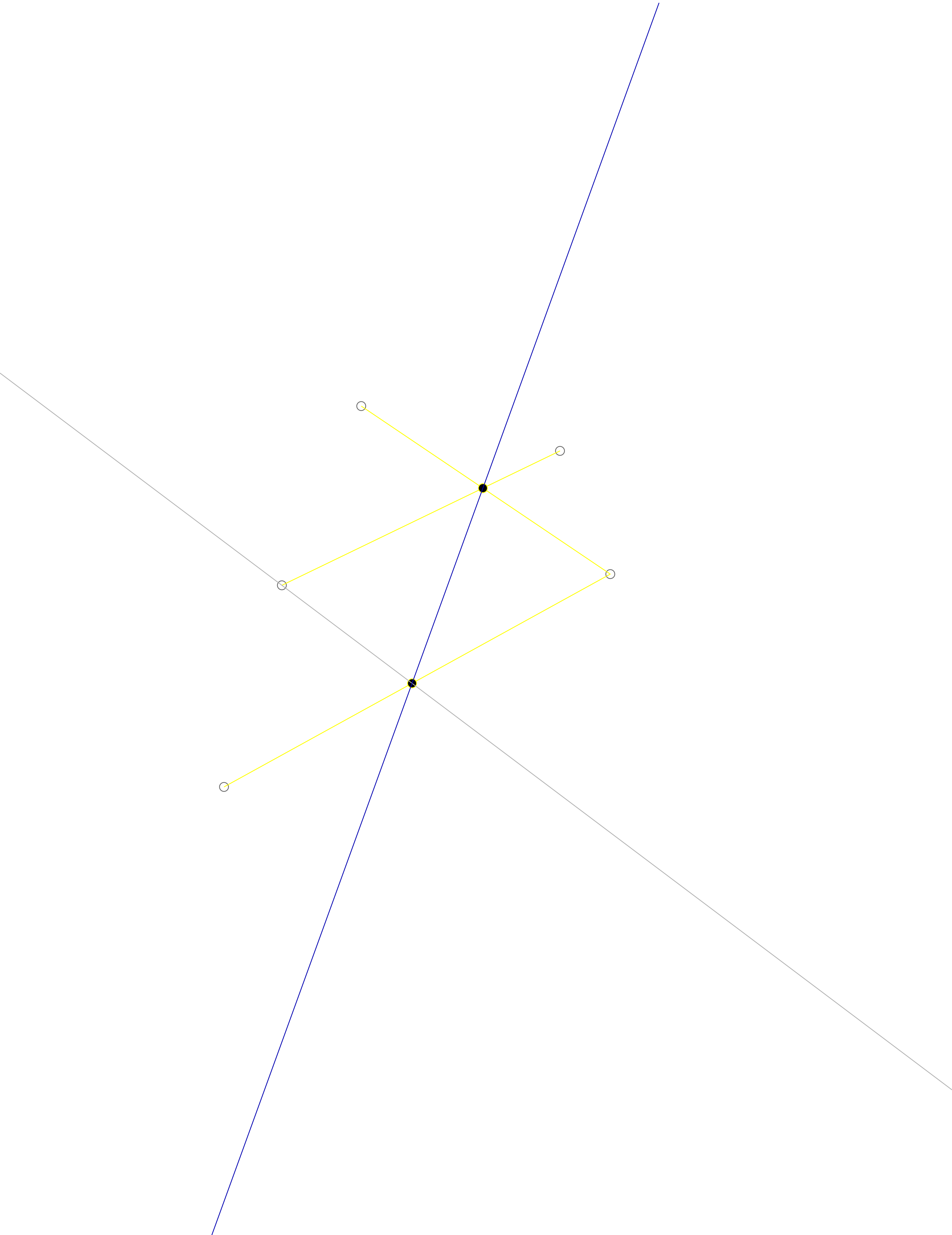


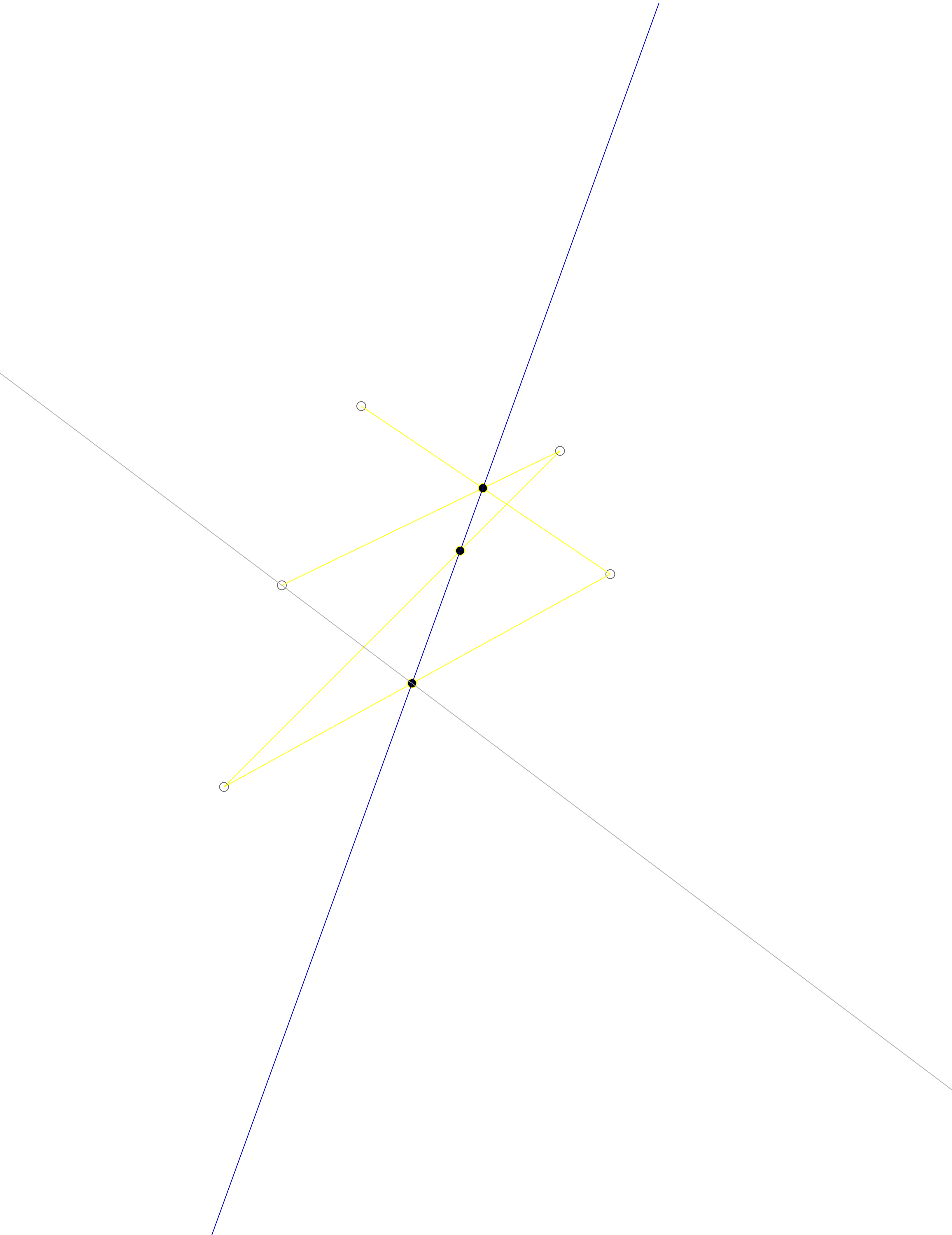


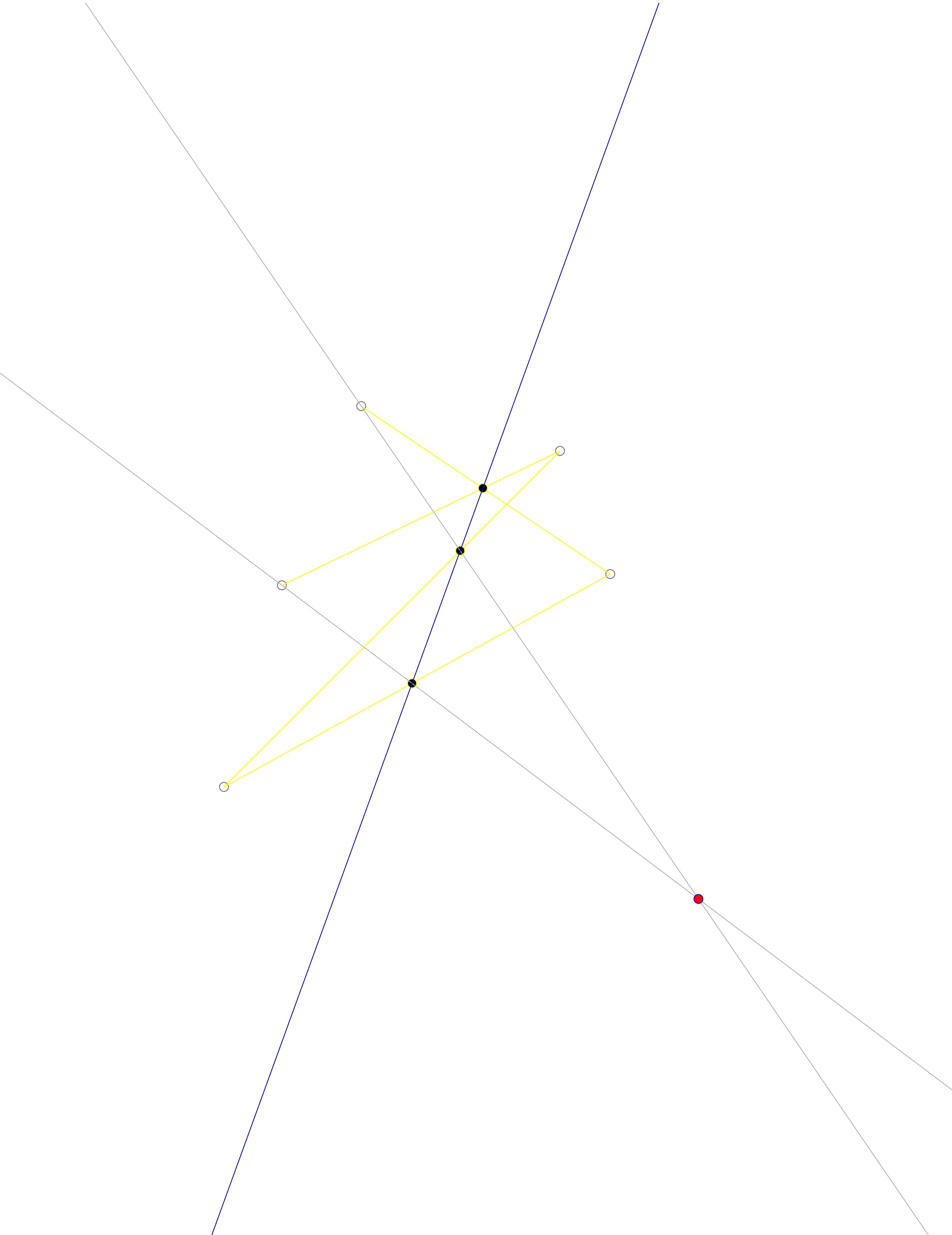


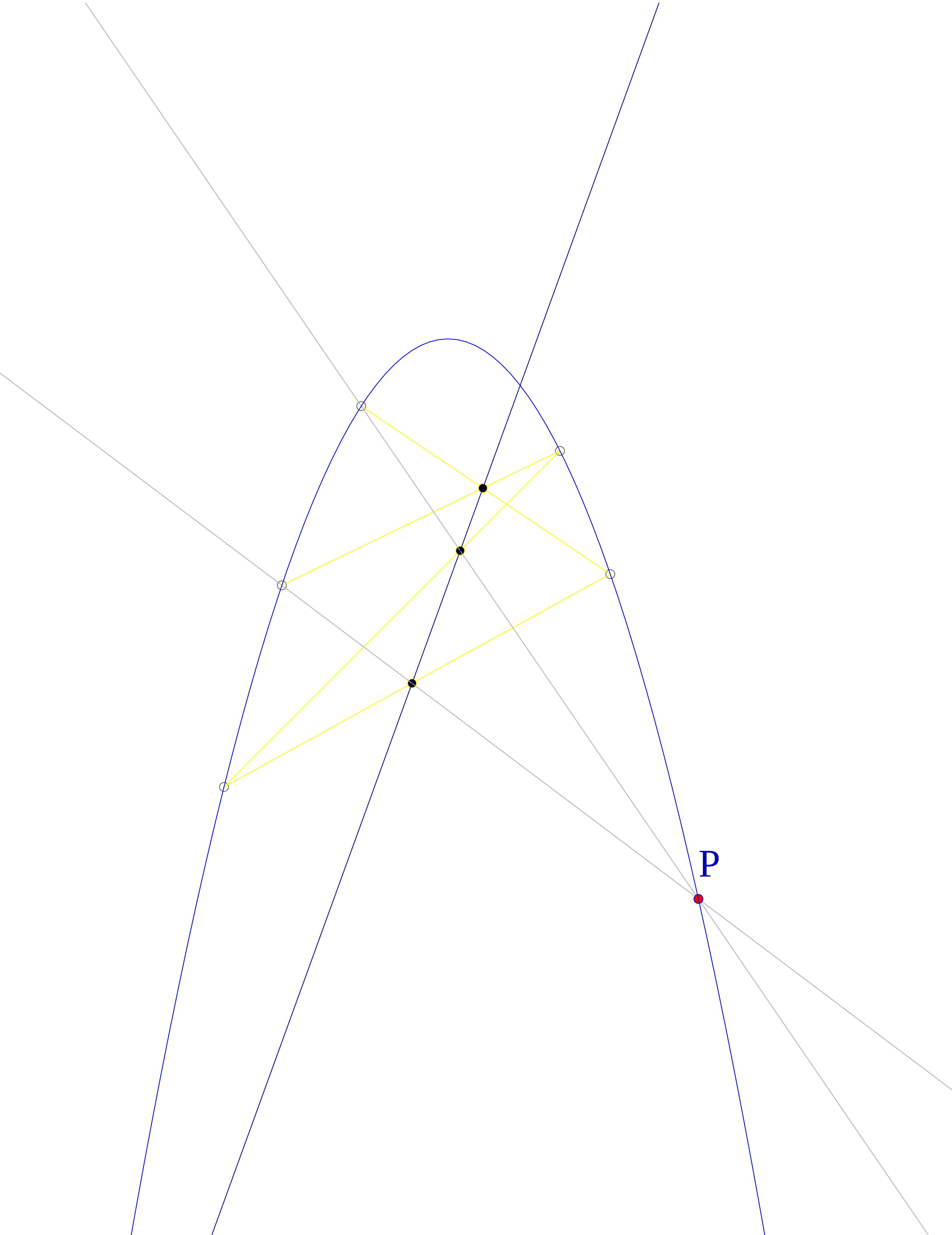








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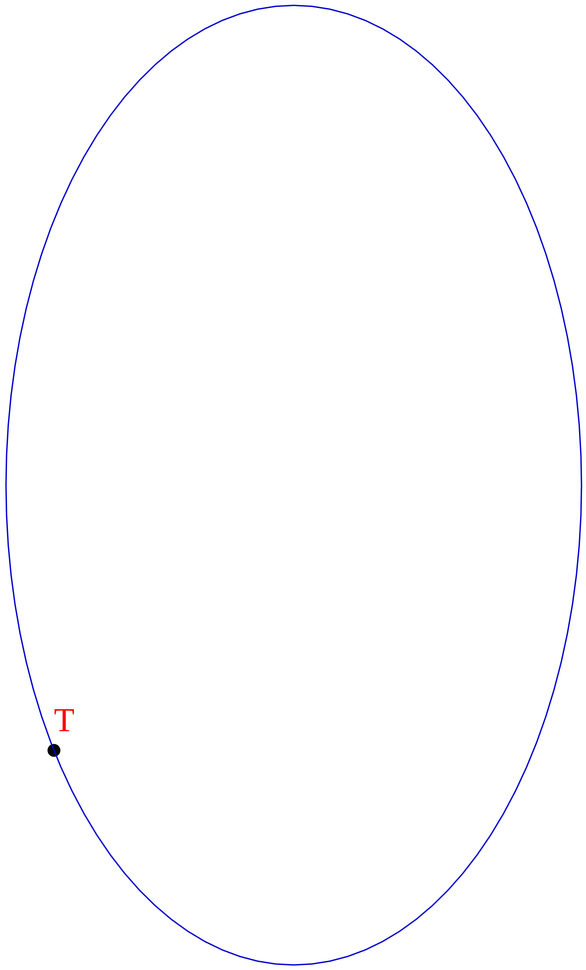
Application 2: Given a conic and five points on it, construct a tangent line at one the points.

Note: This application can be used to draw a tangent line at any point of a given conic.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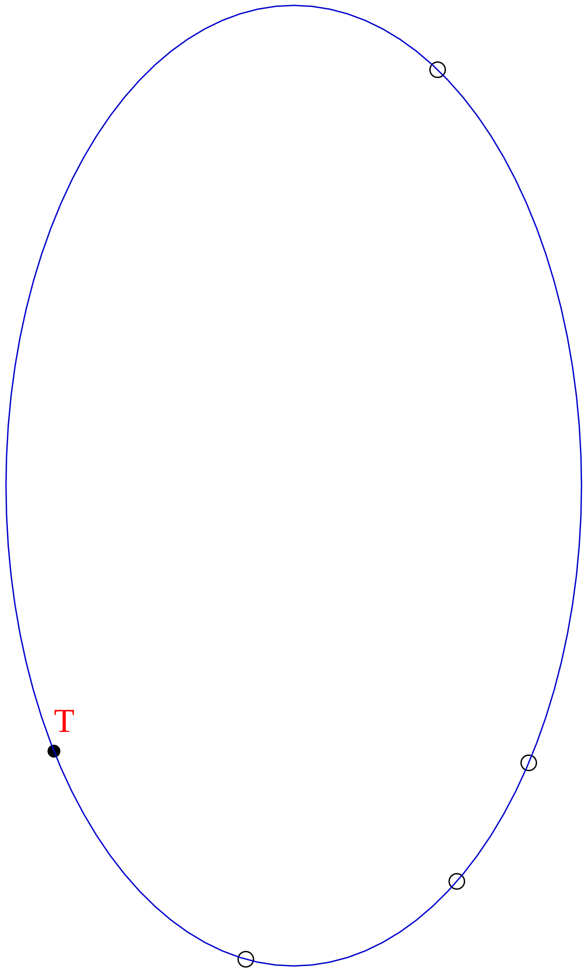
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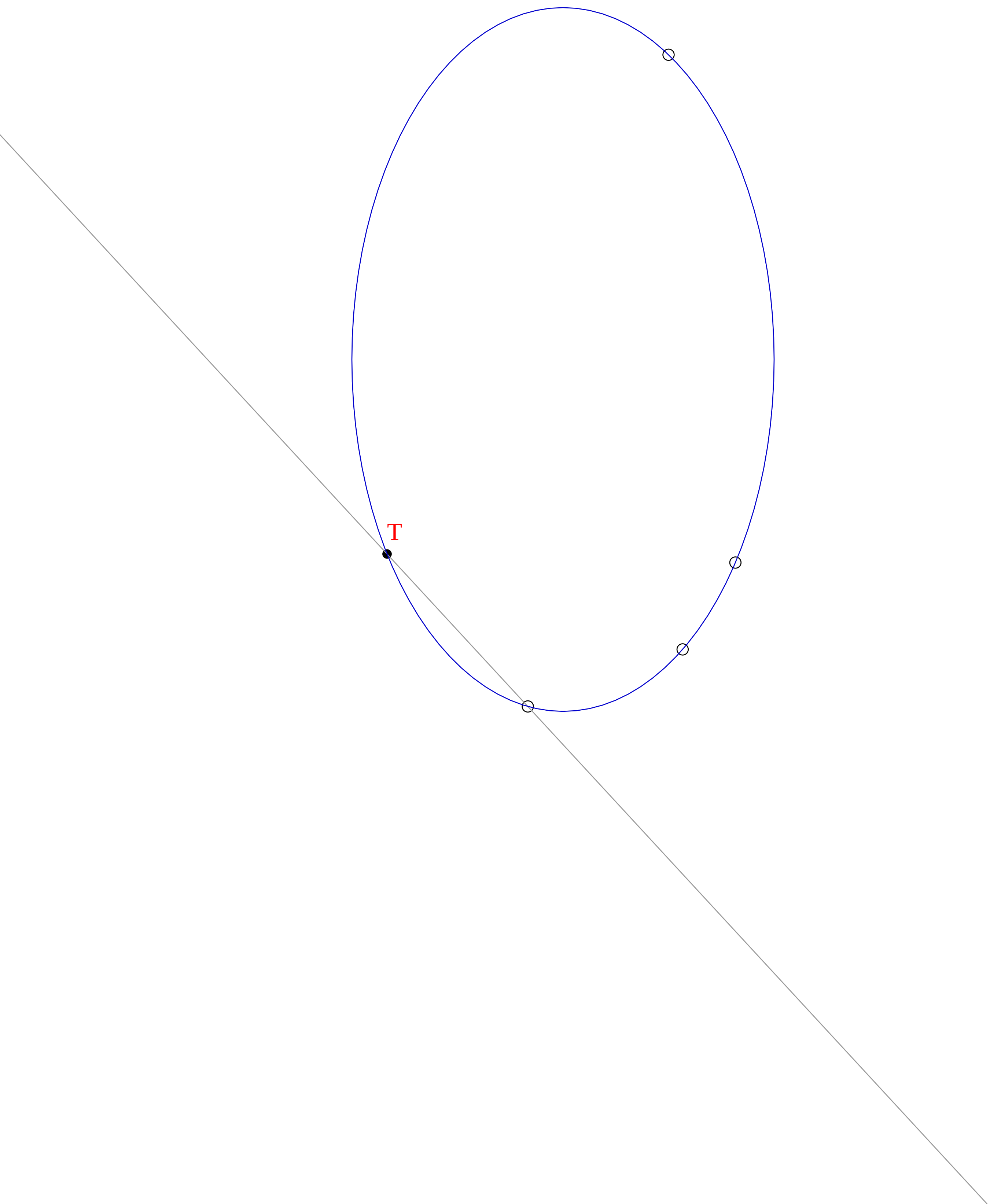
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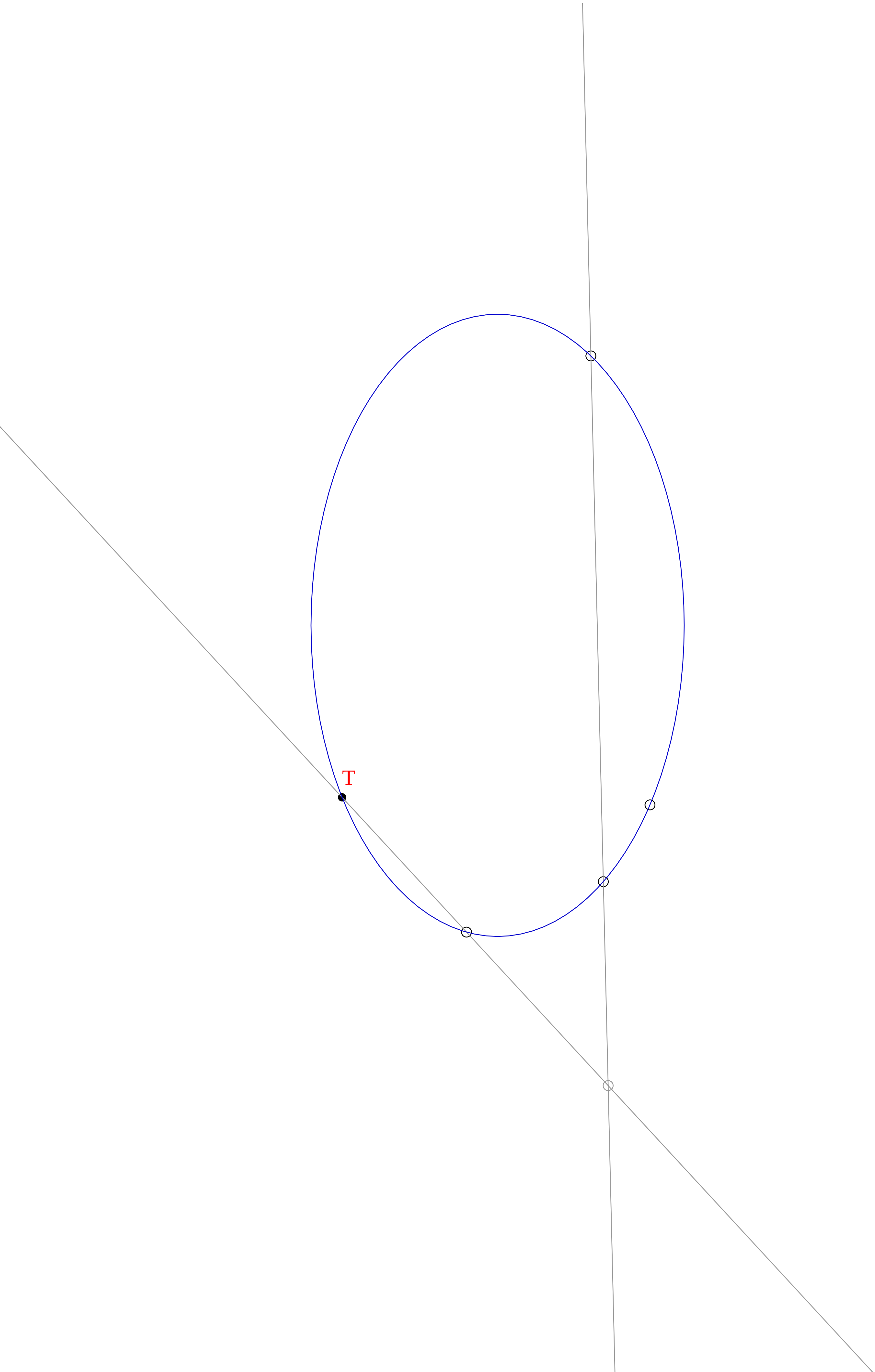
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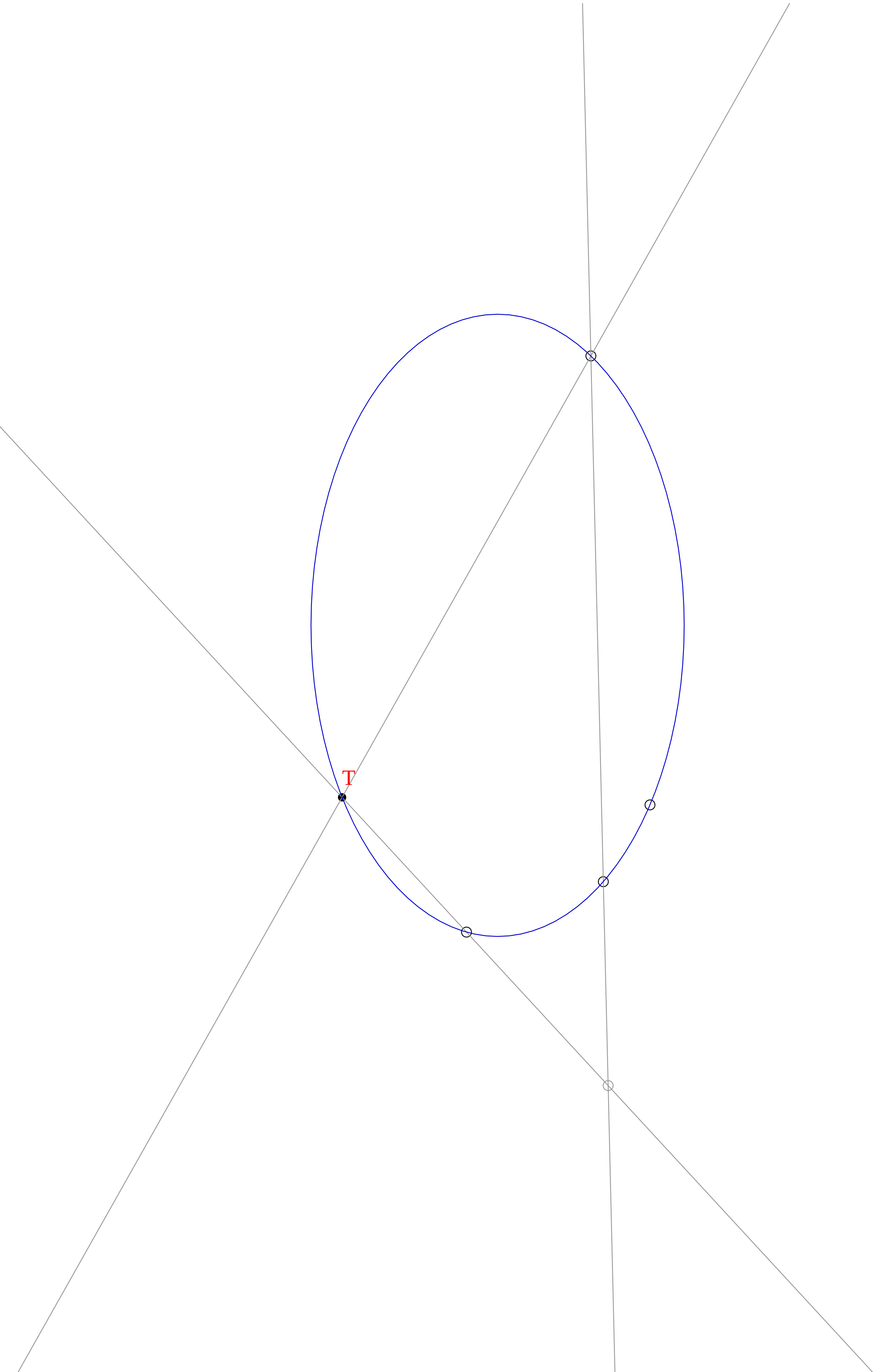
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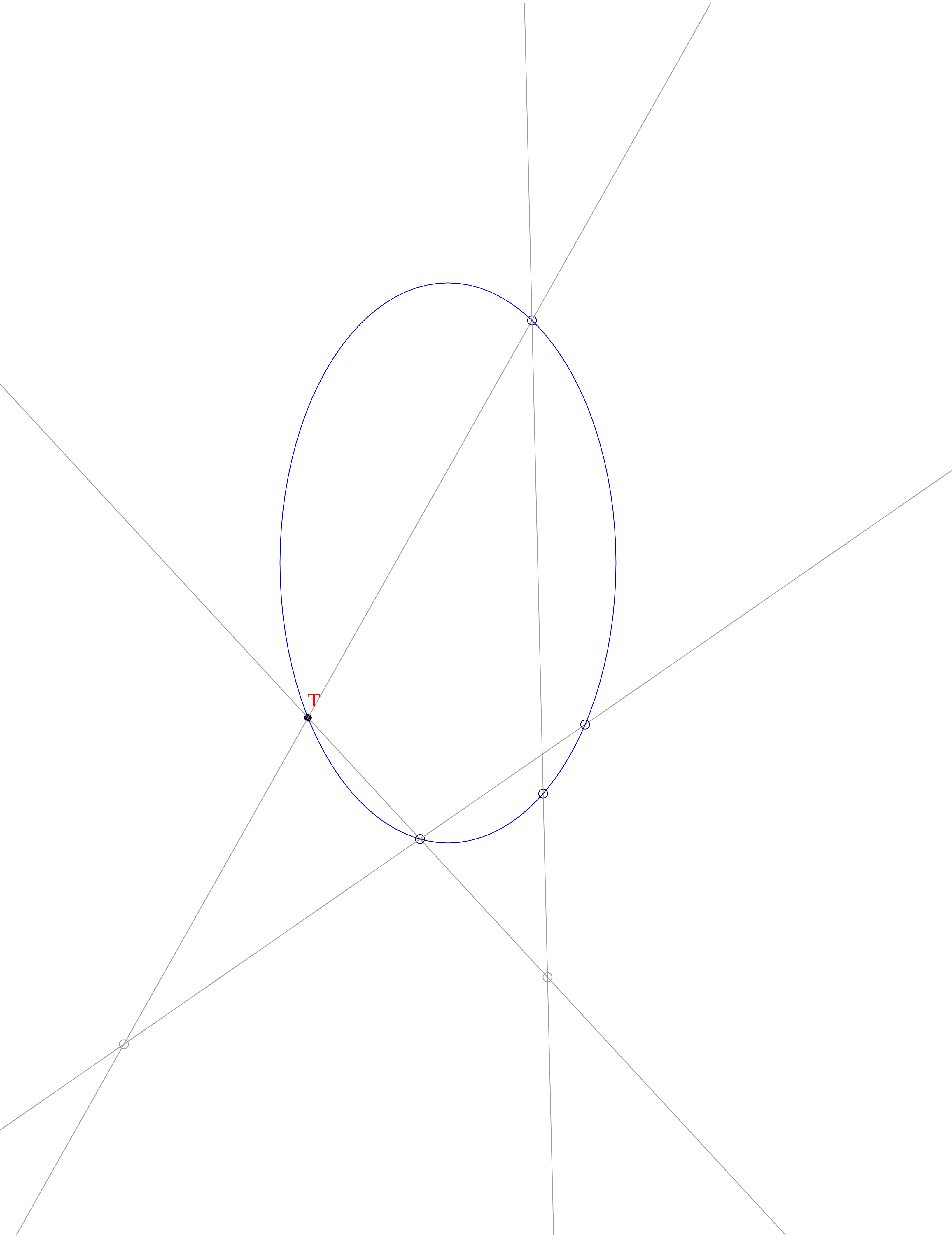
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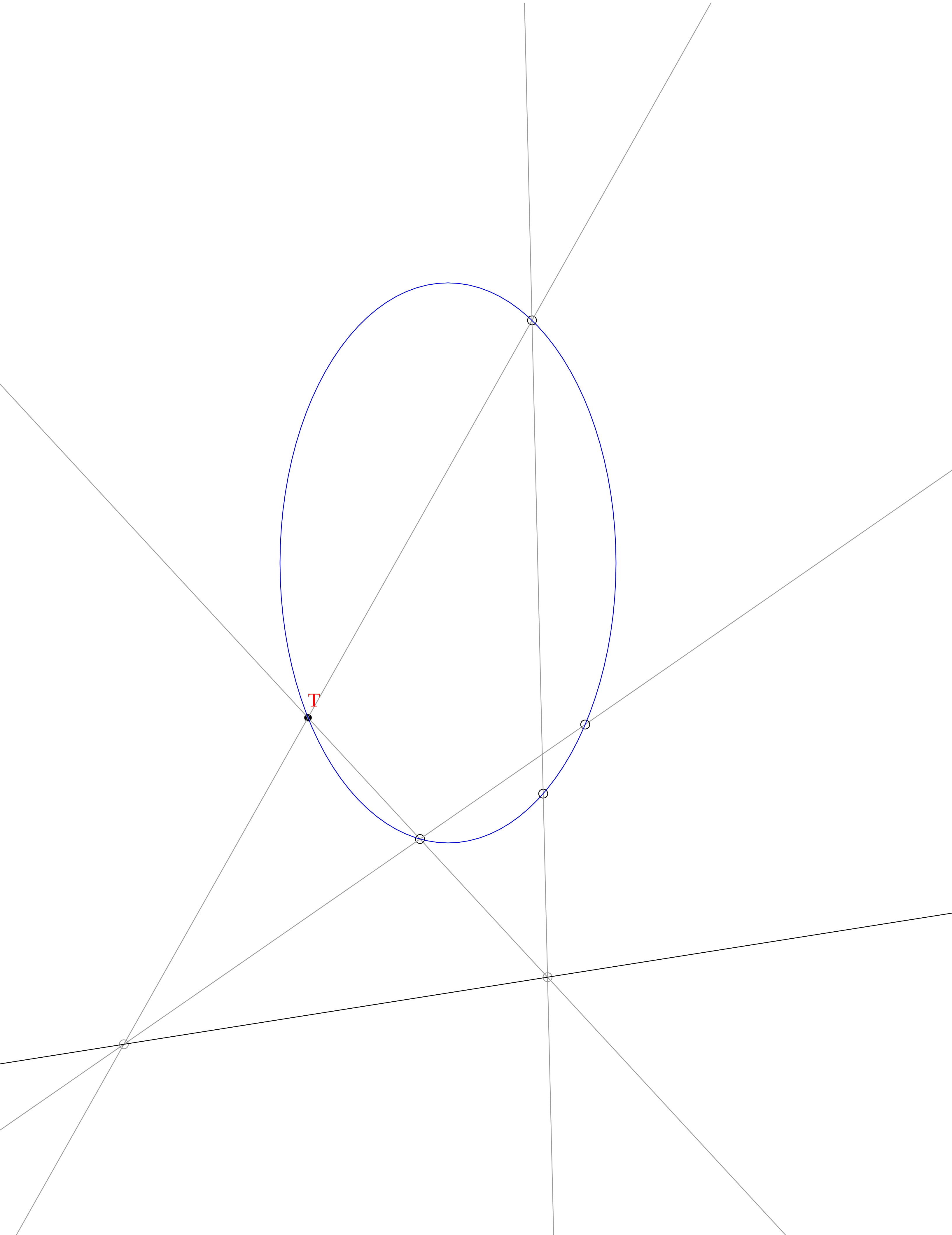
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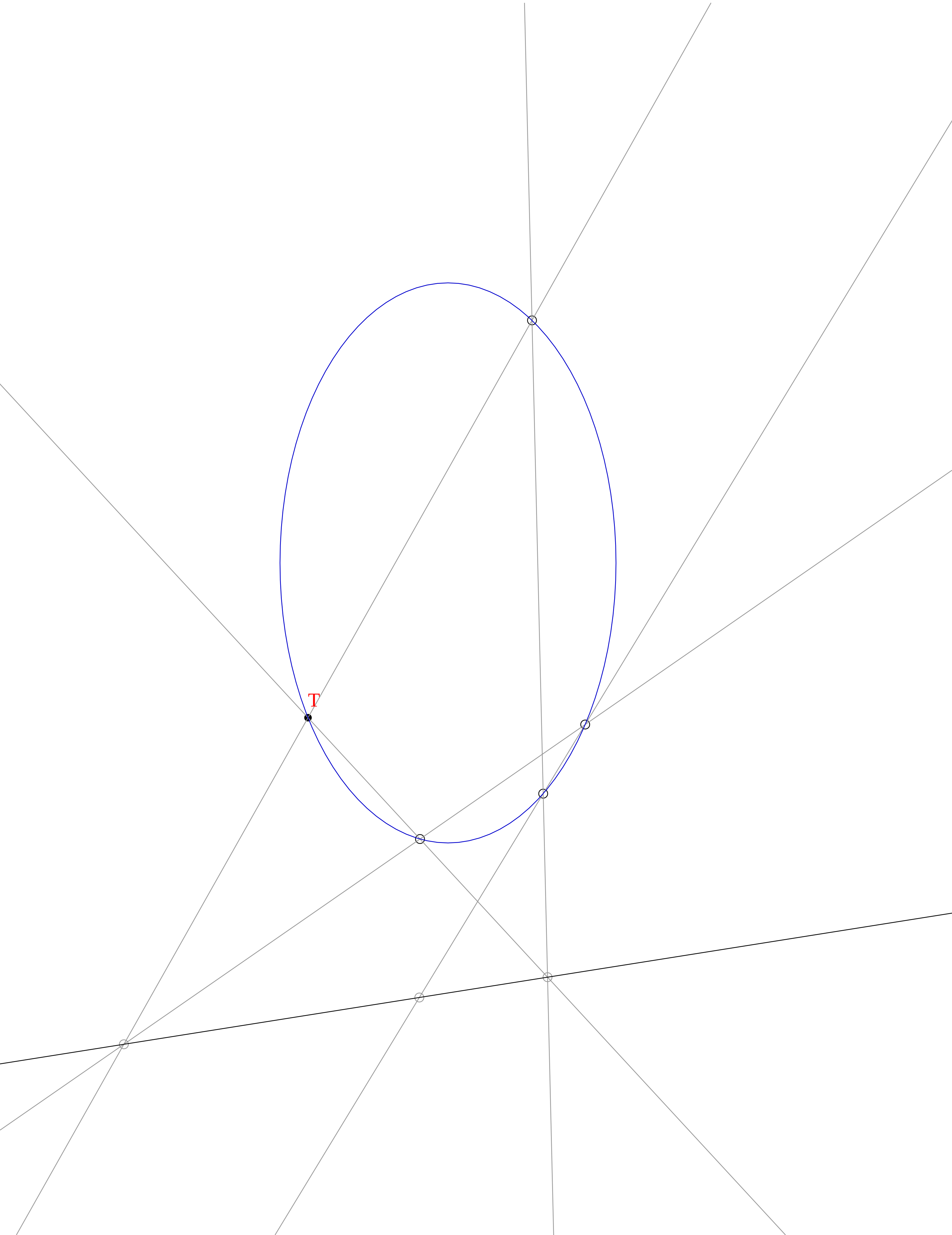
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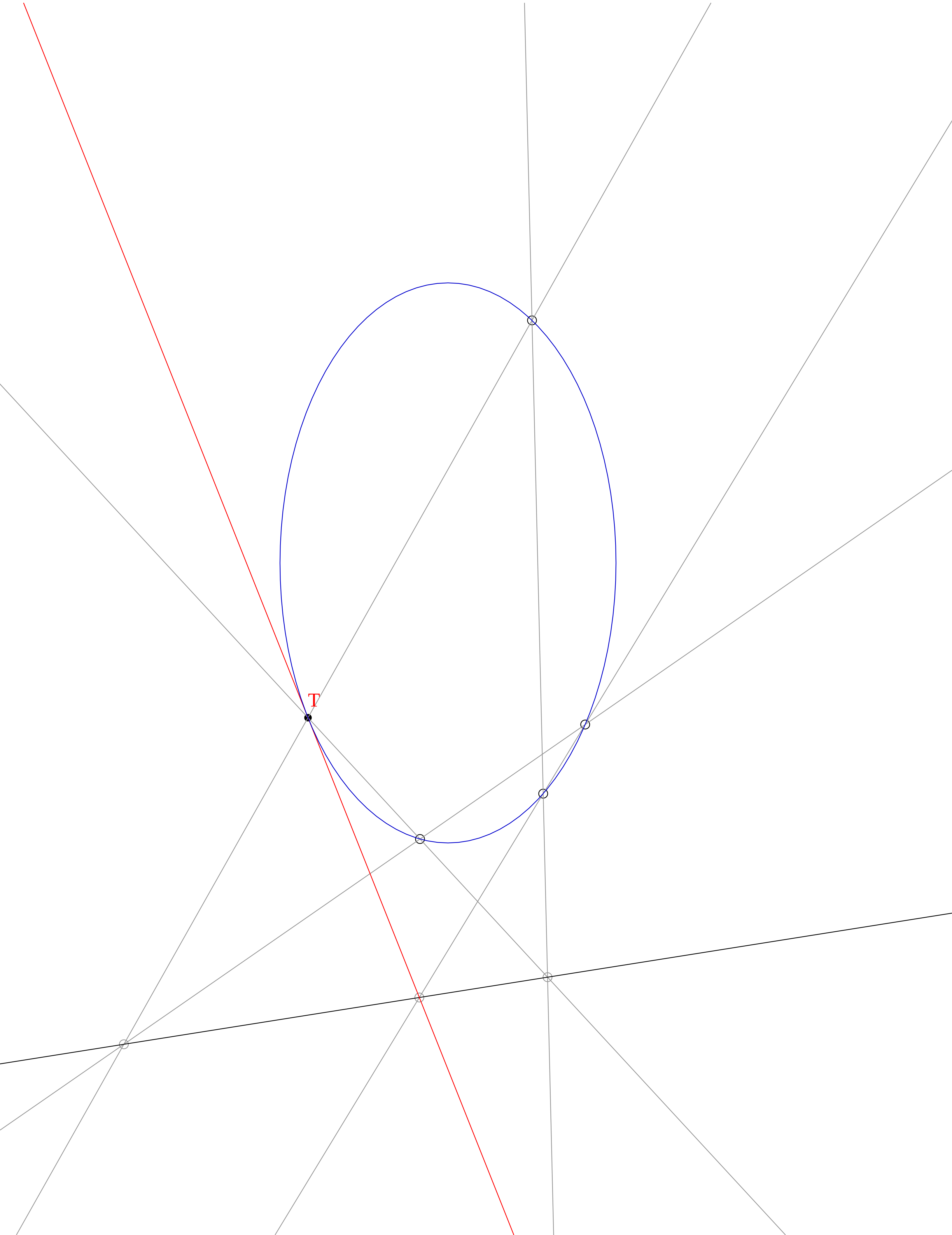
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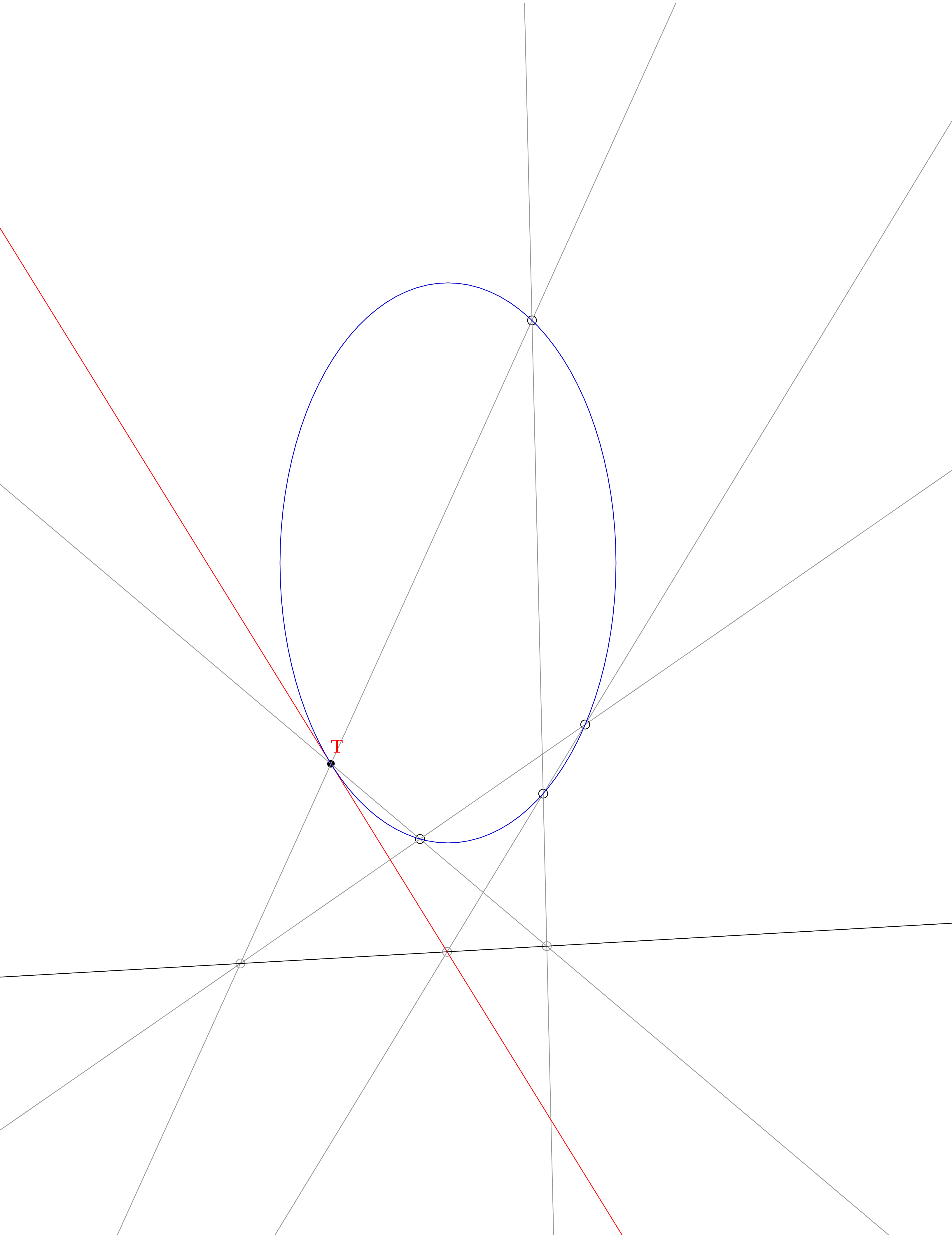
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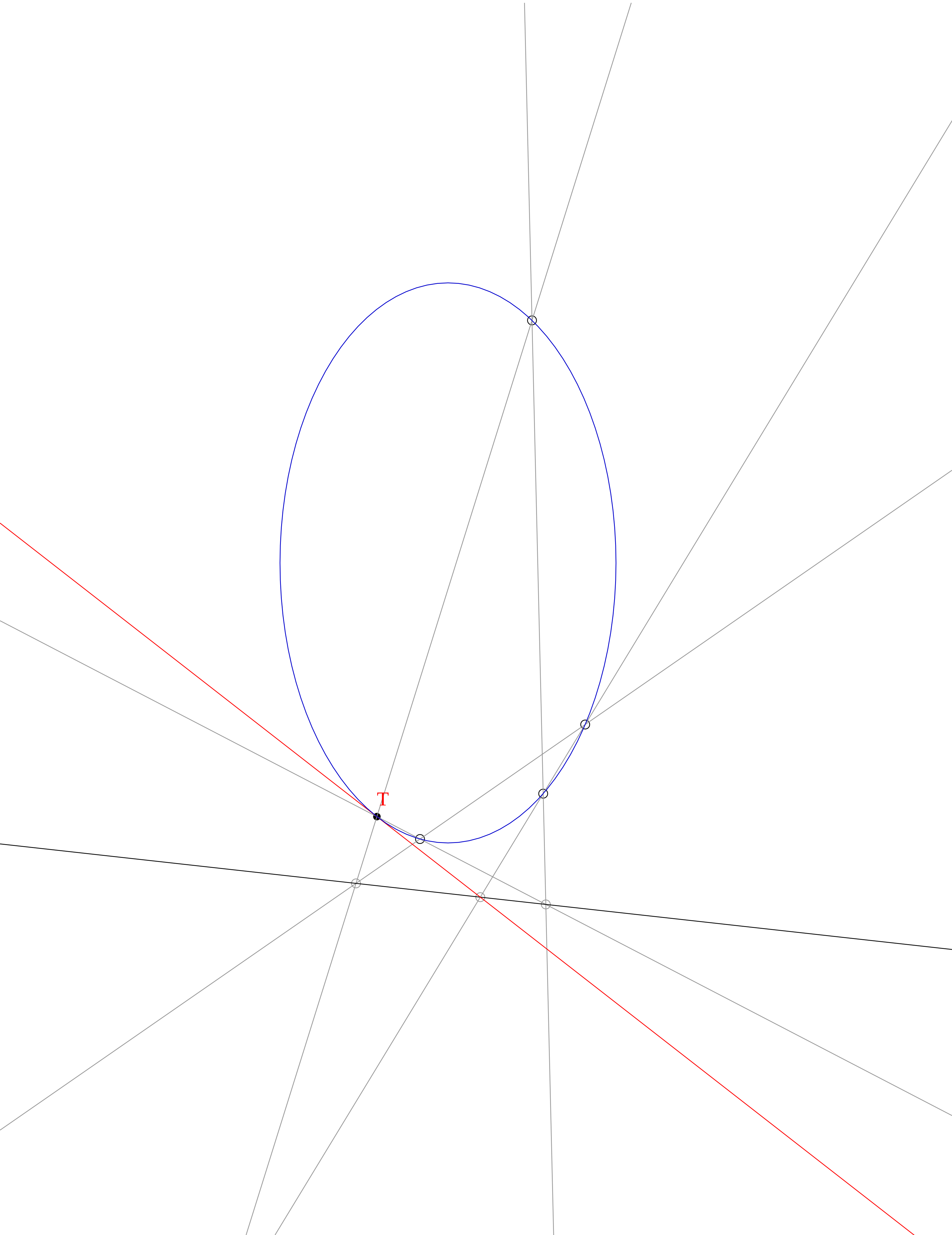
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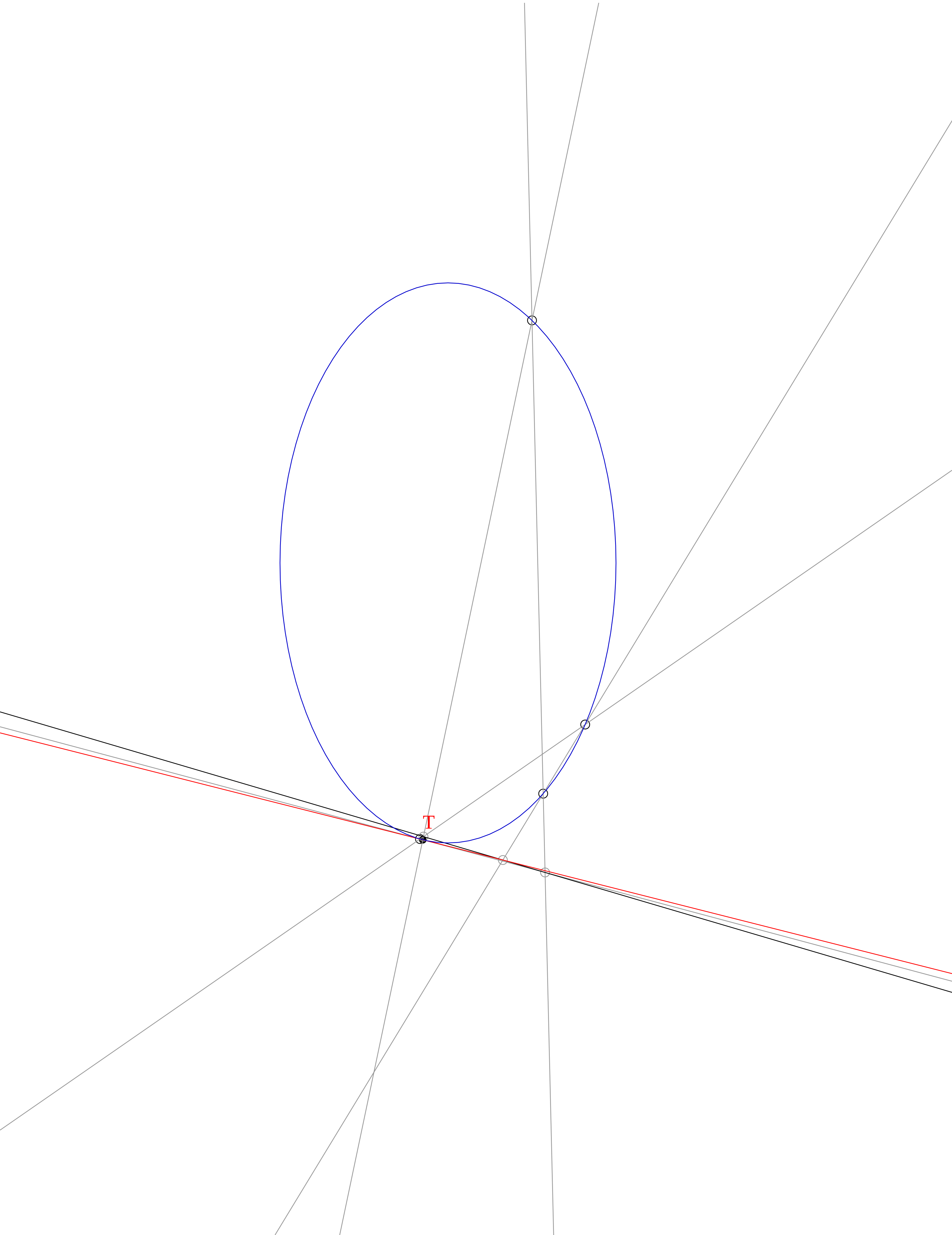
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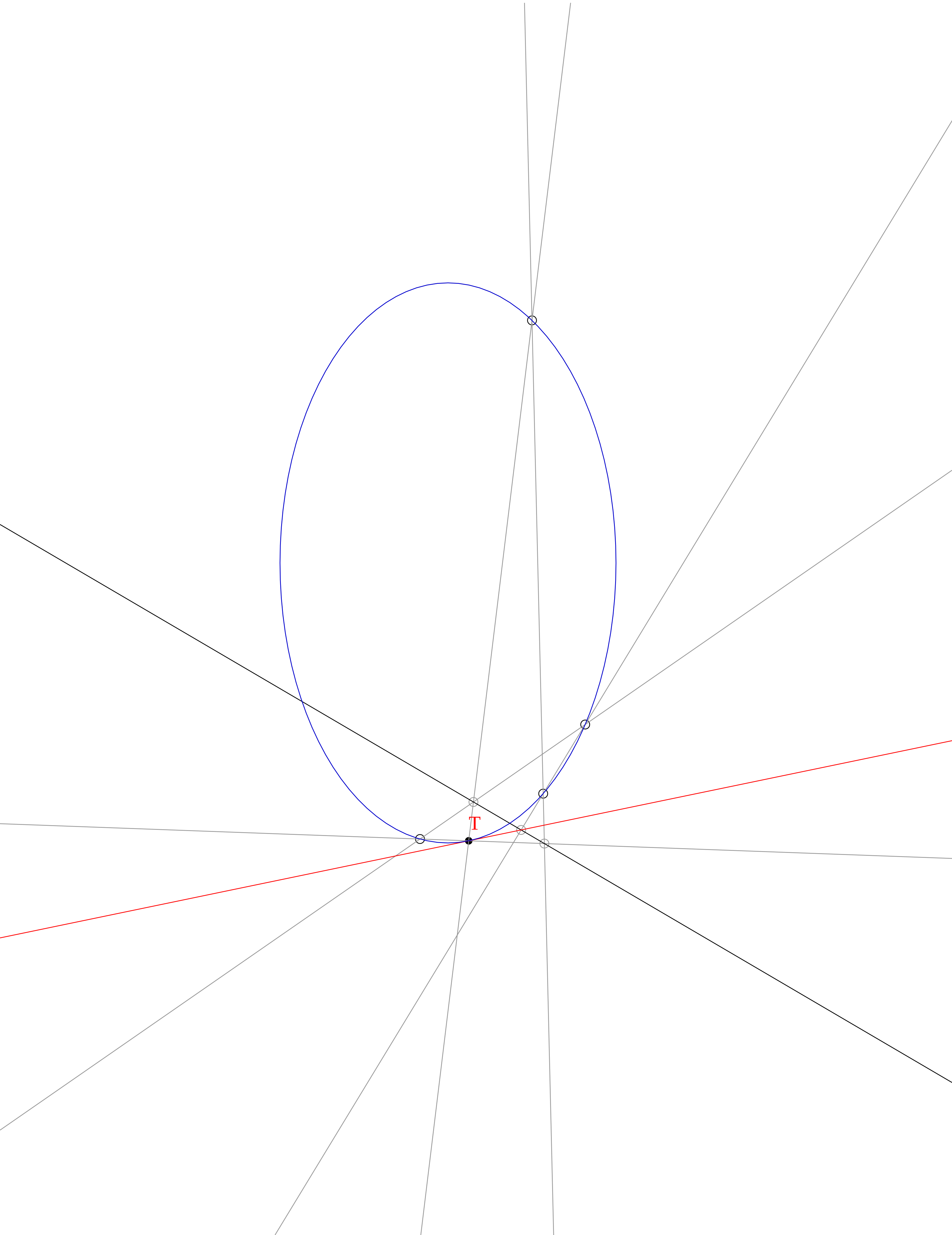
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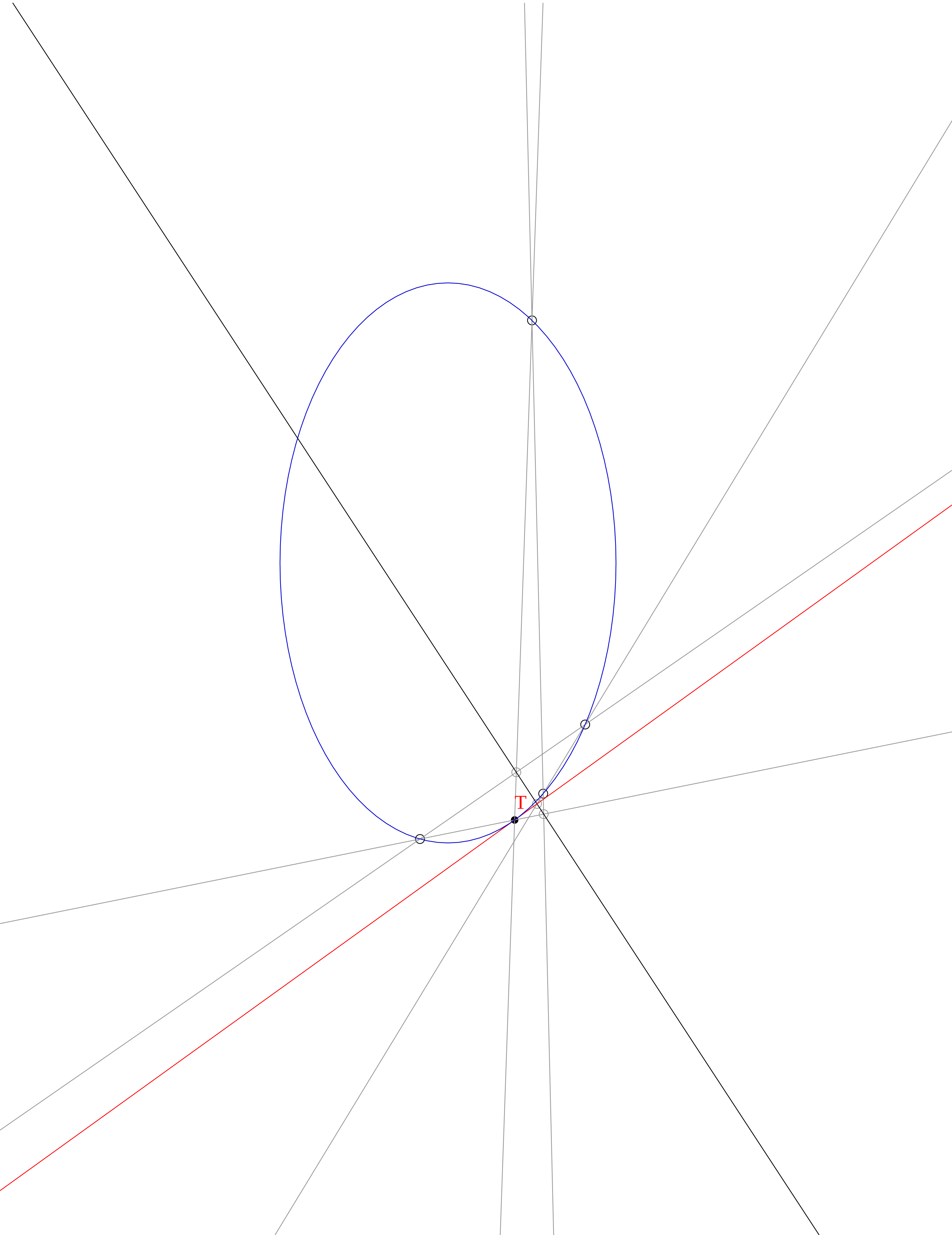
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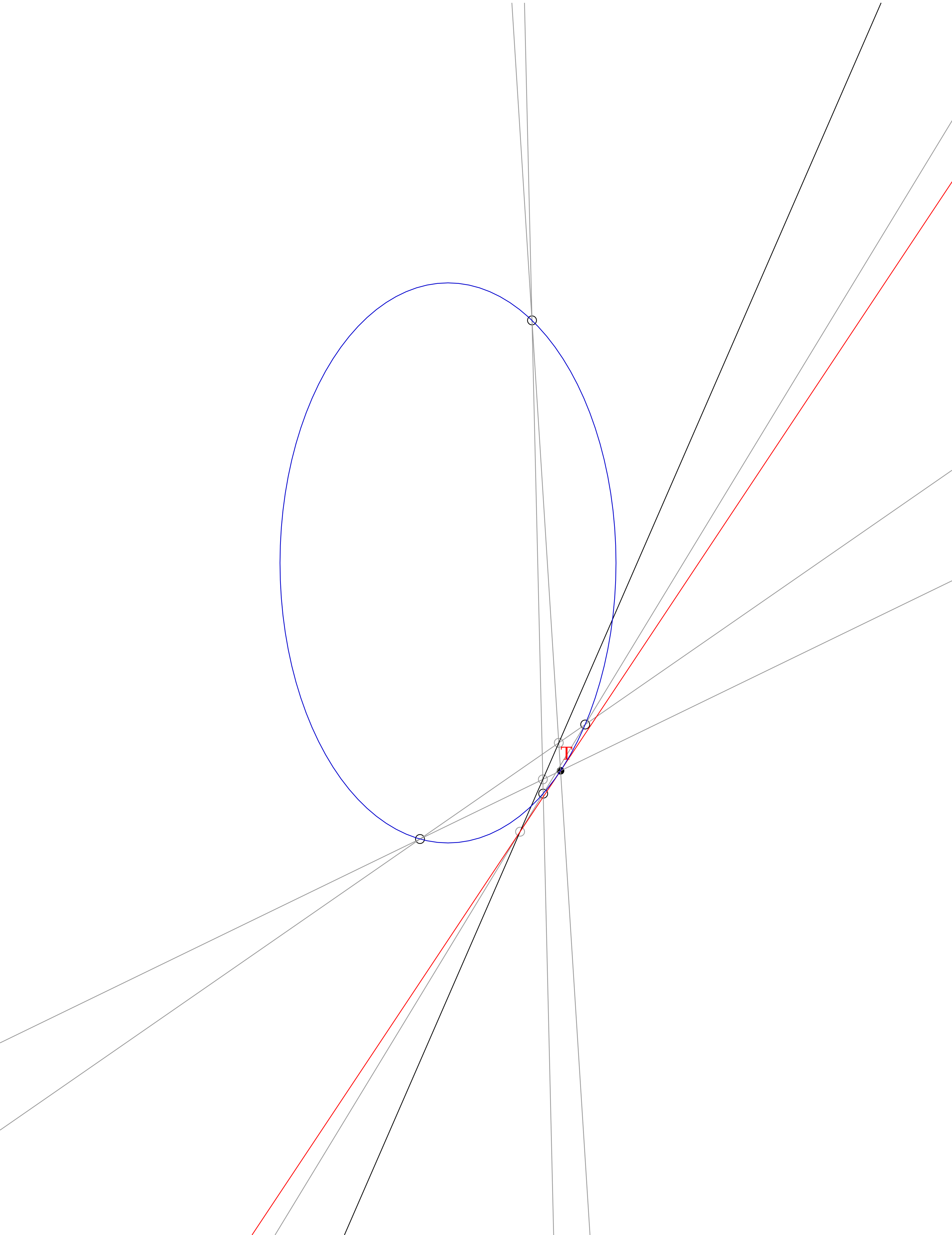
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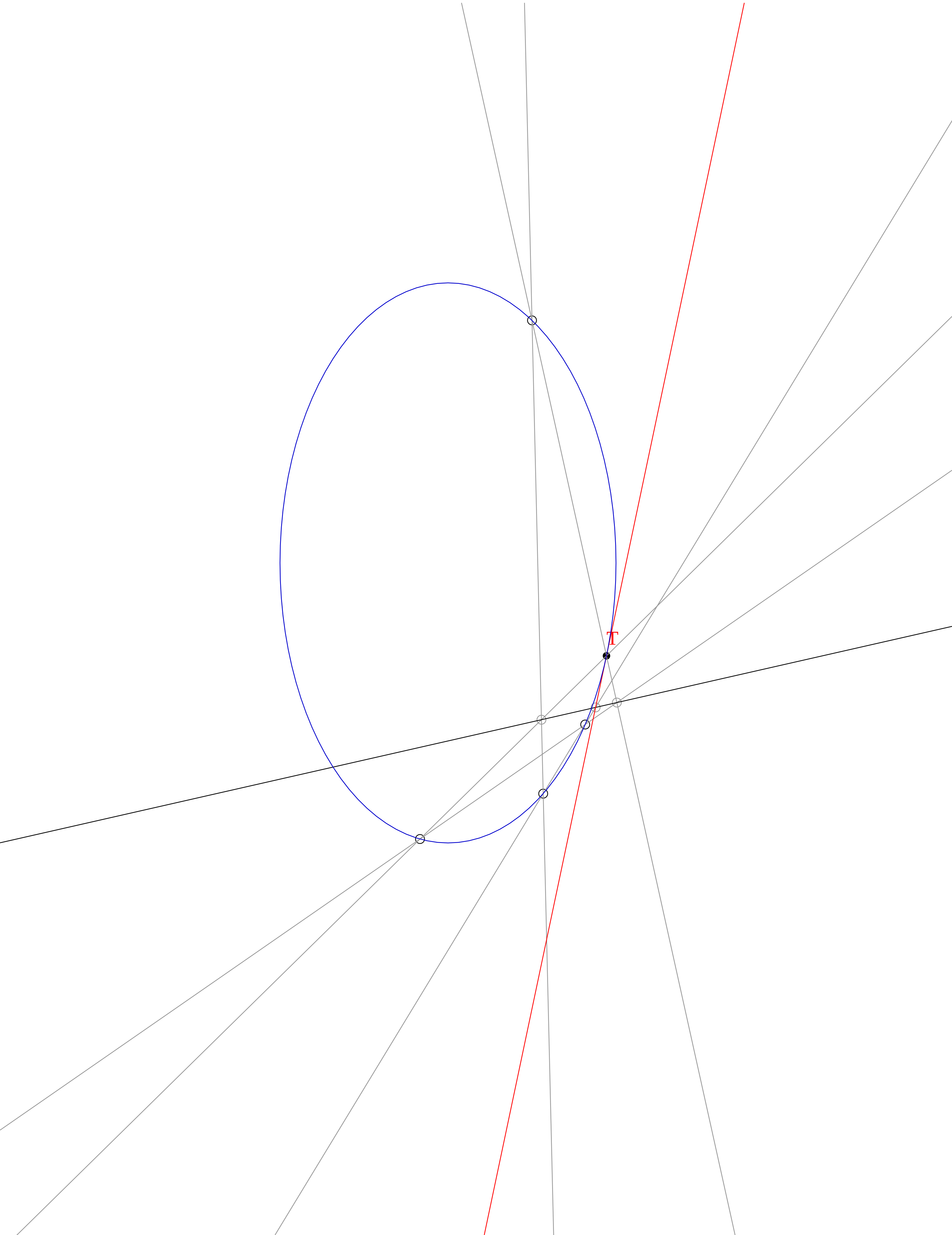
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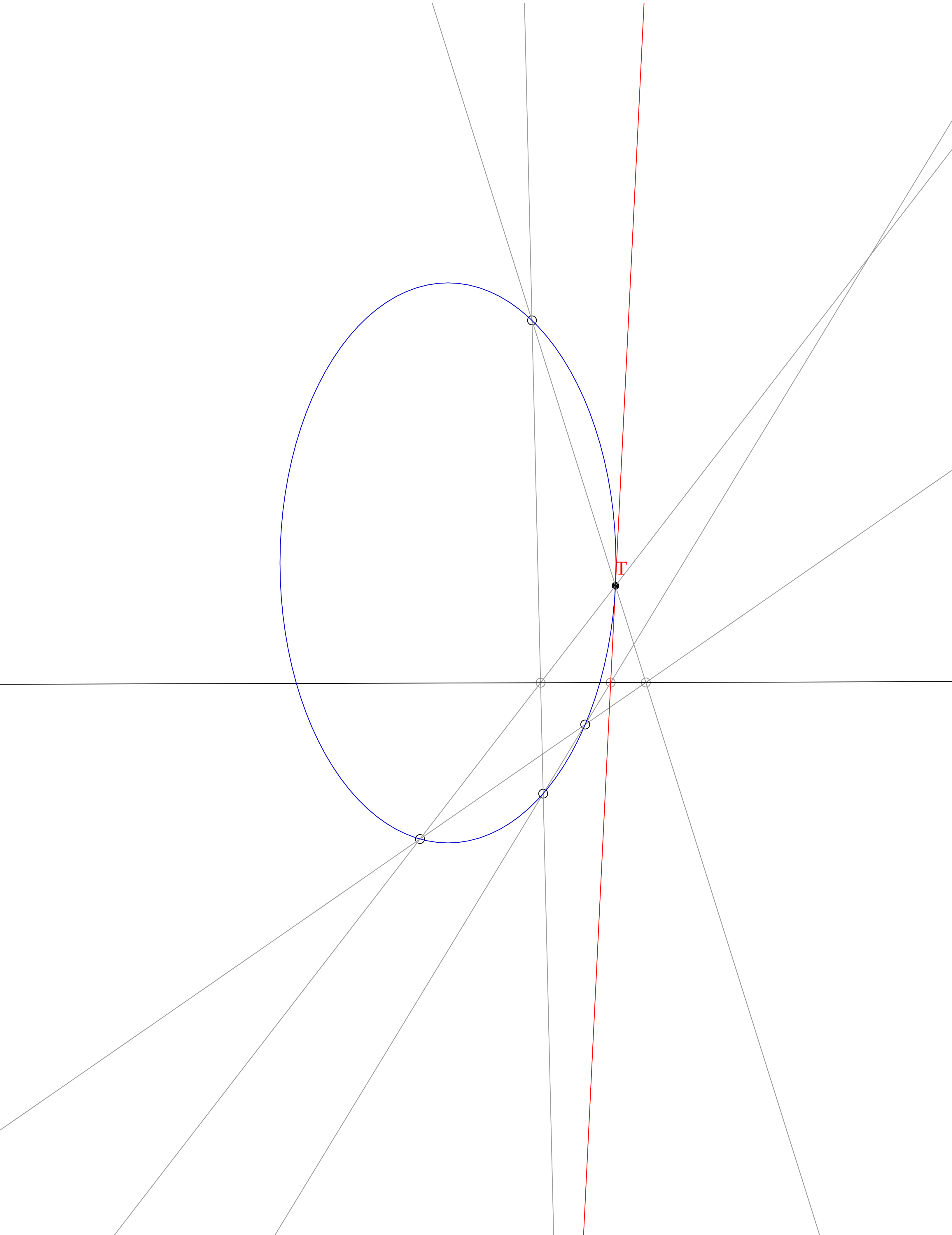
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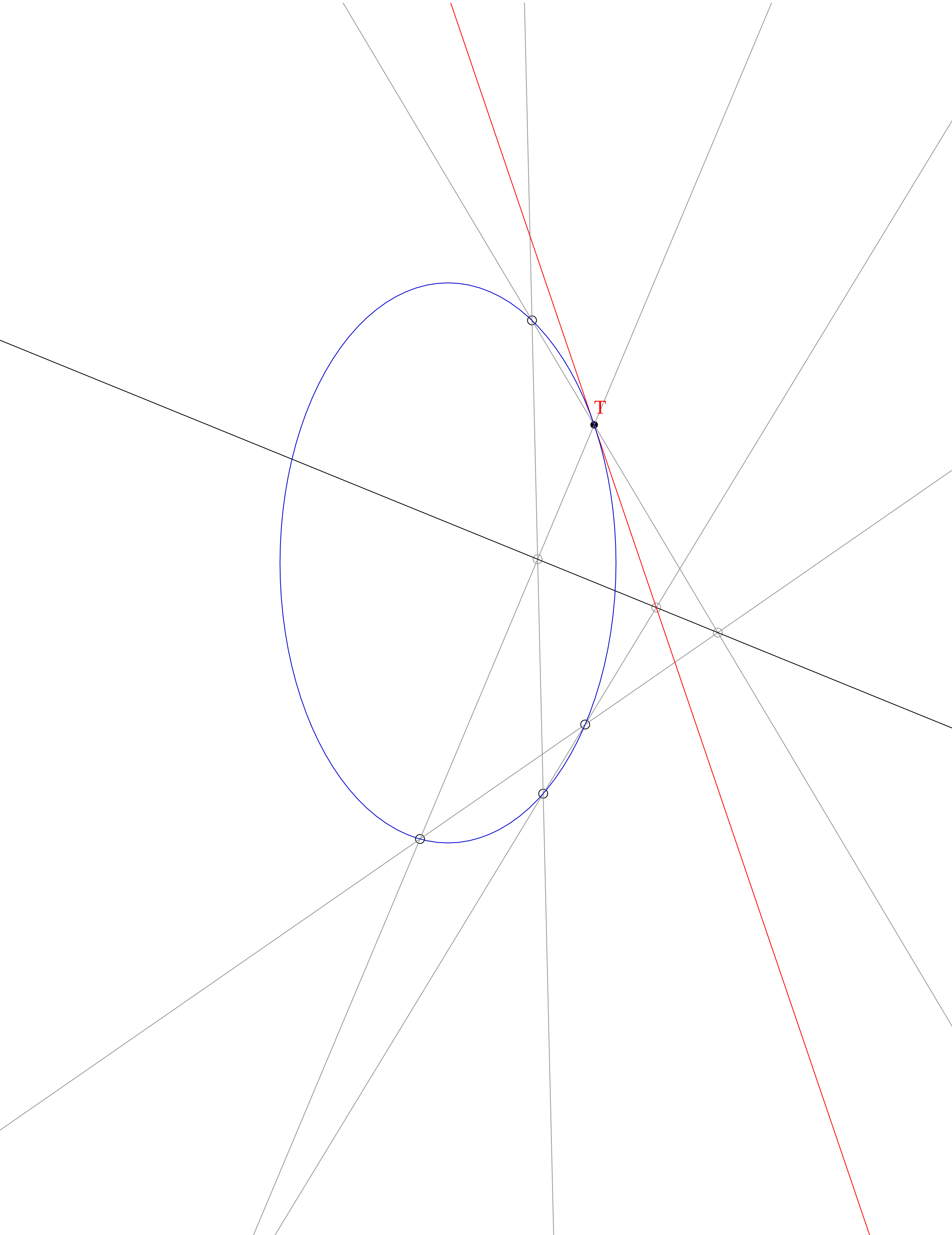
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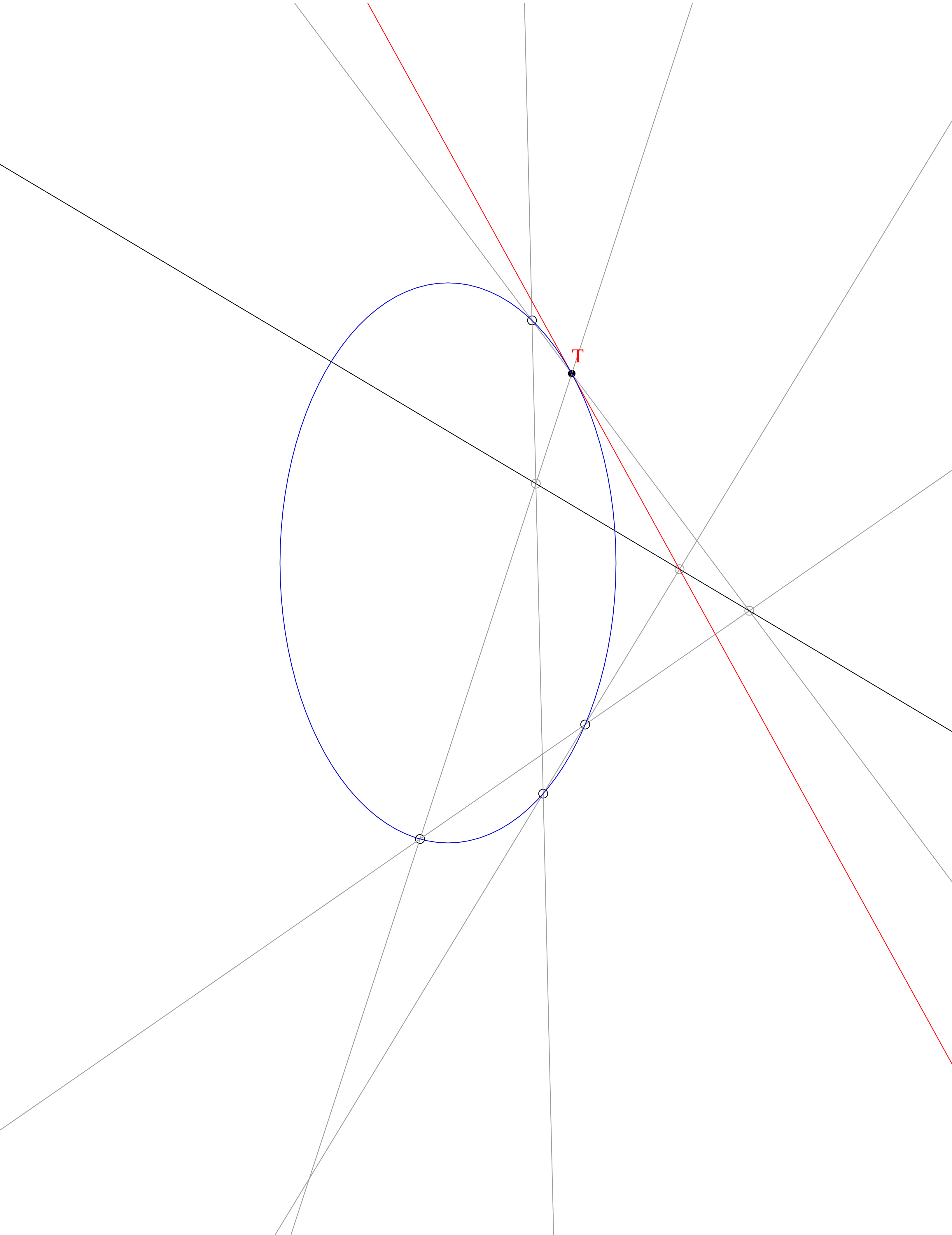
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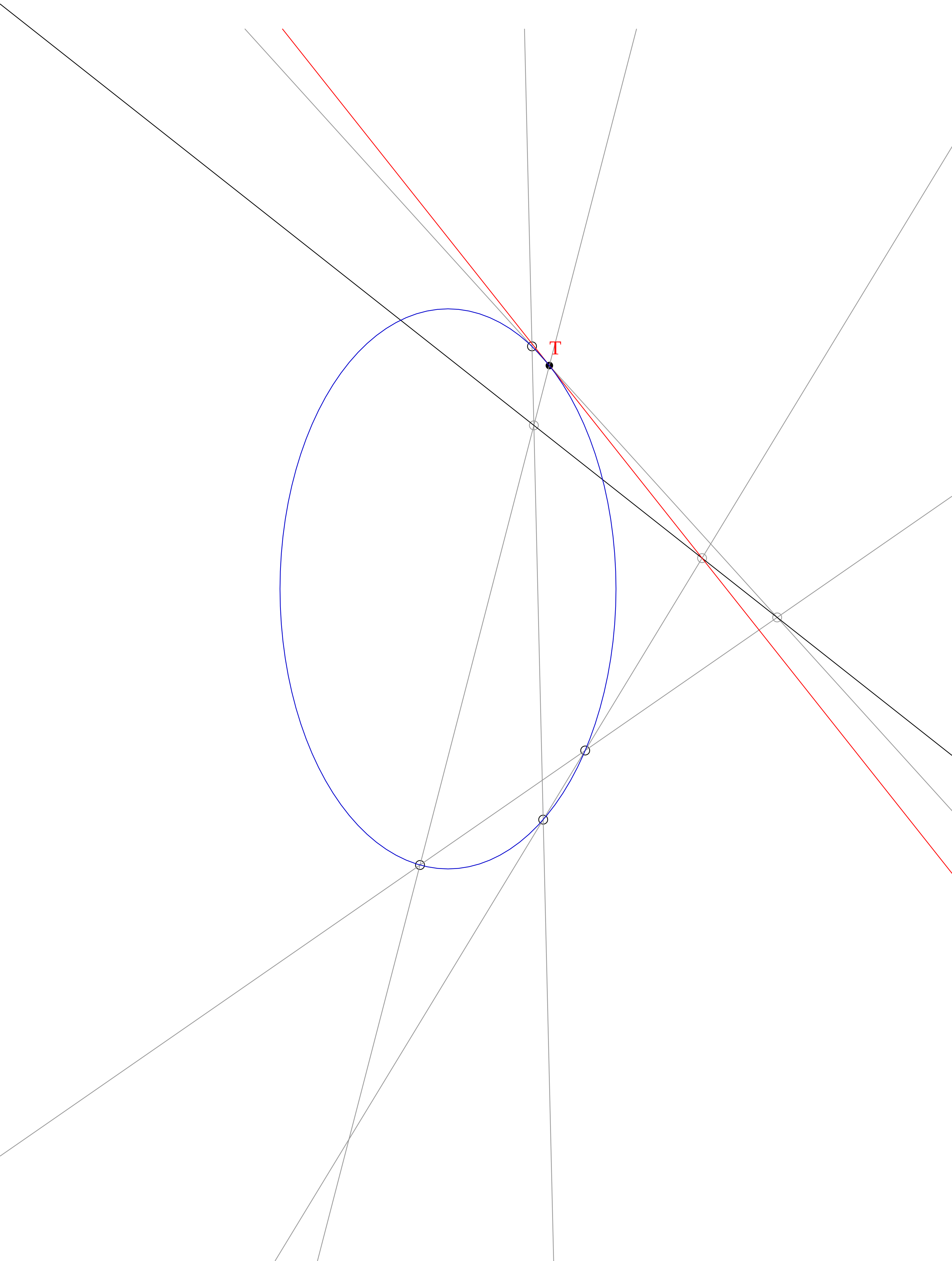
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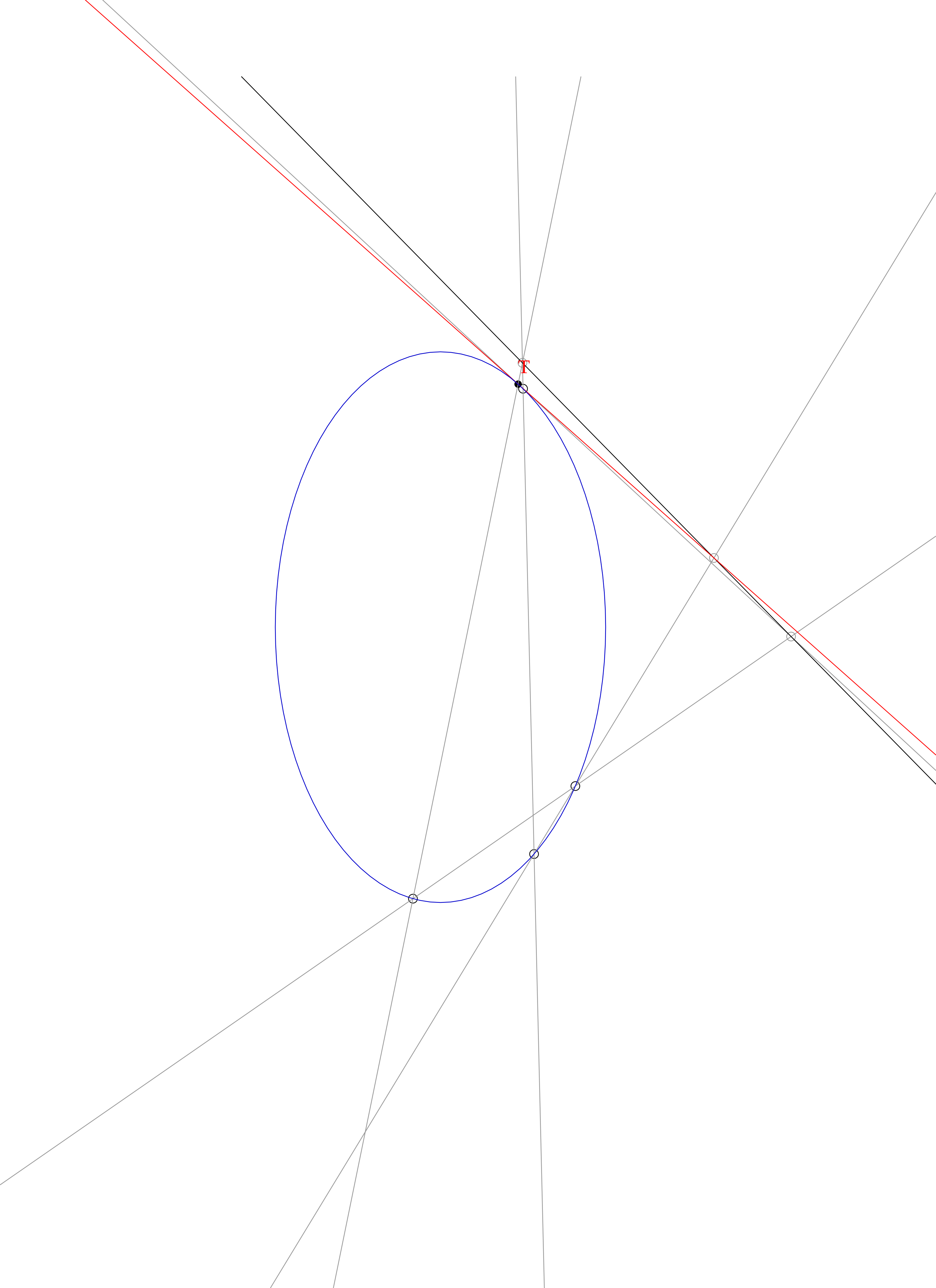
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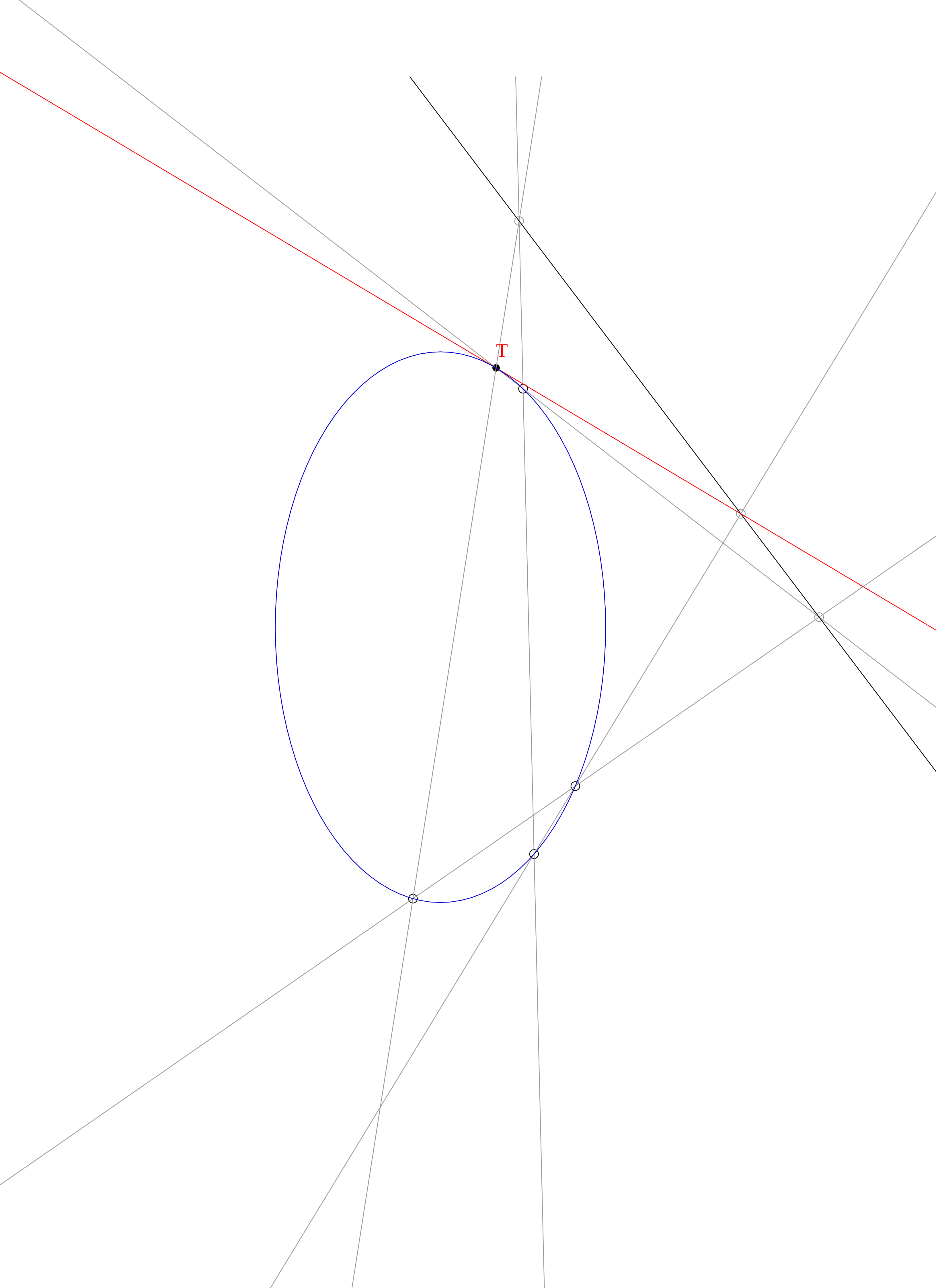
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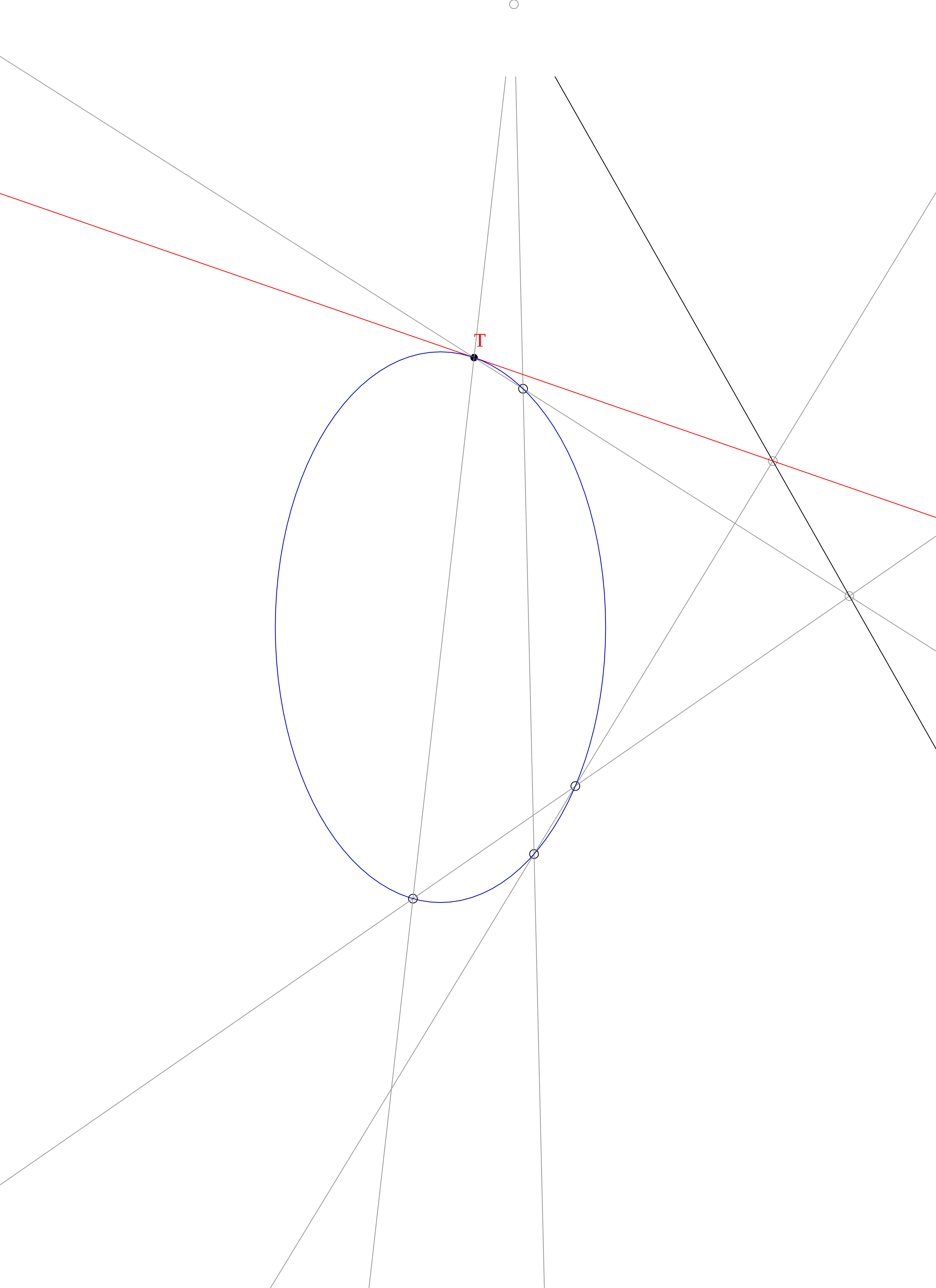
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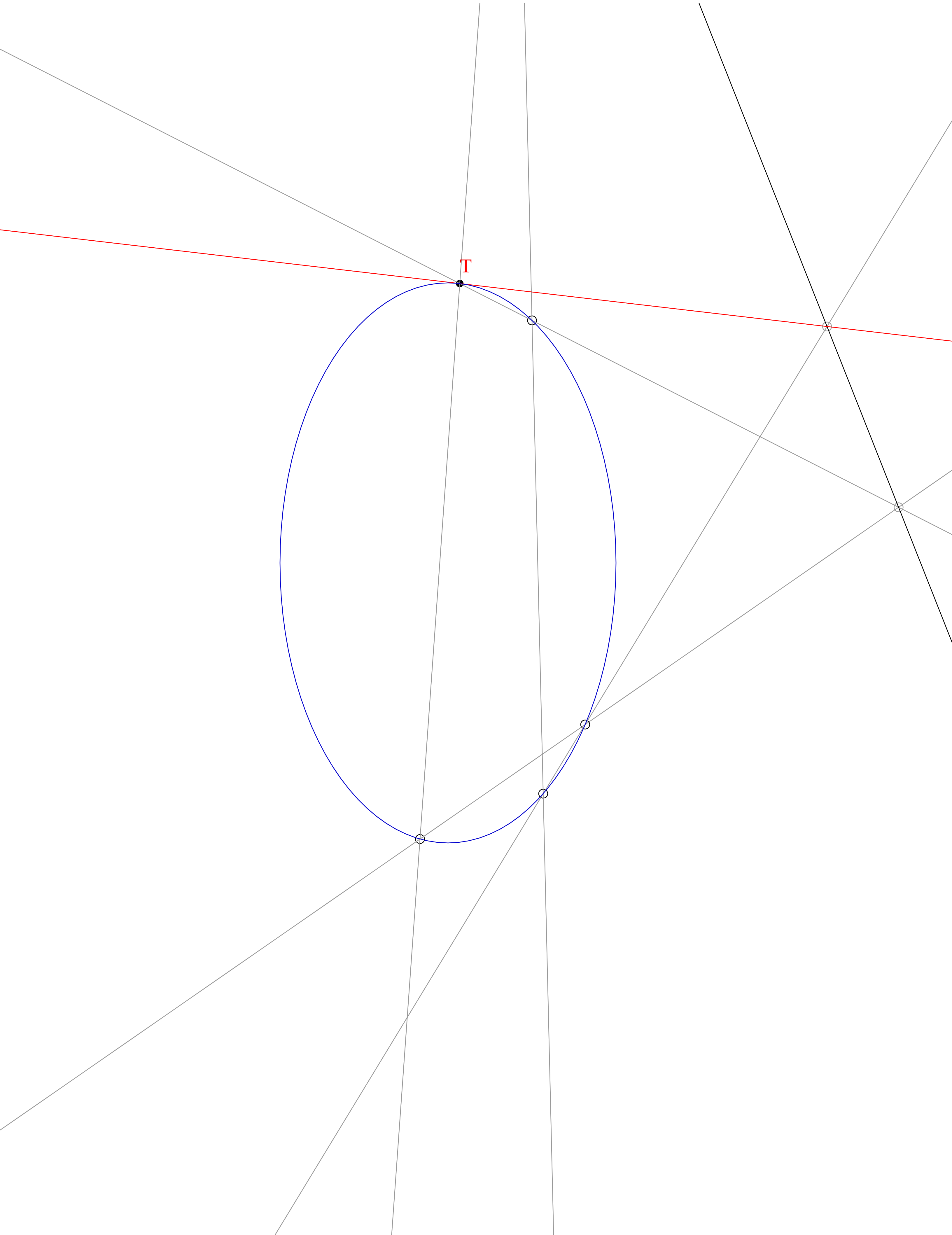
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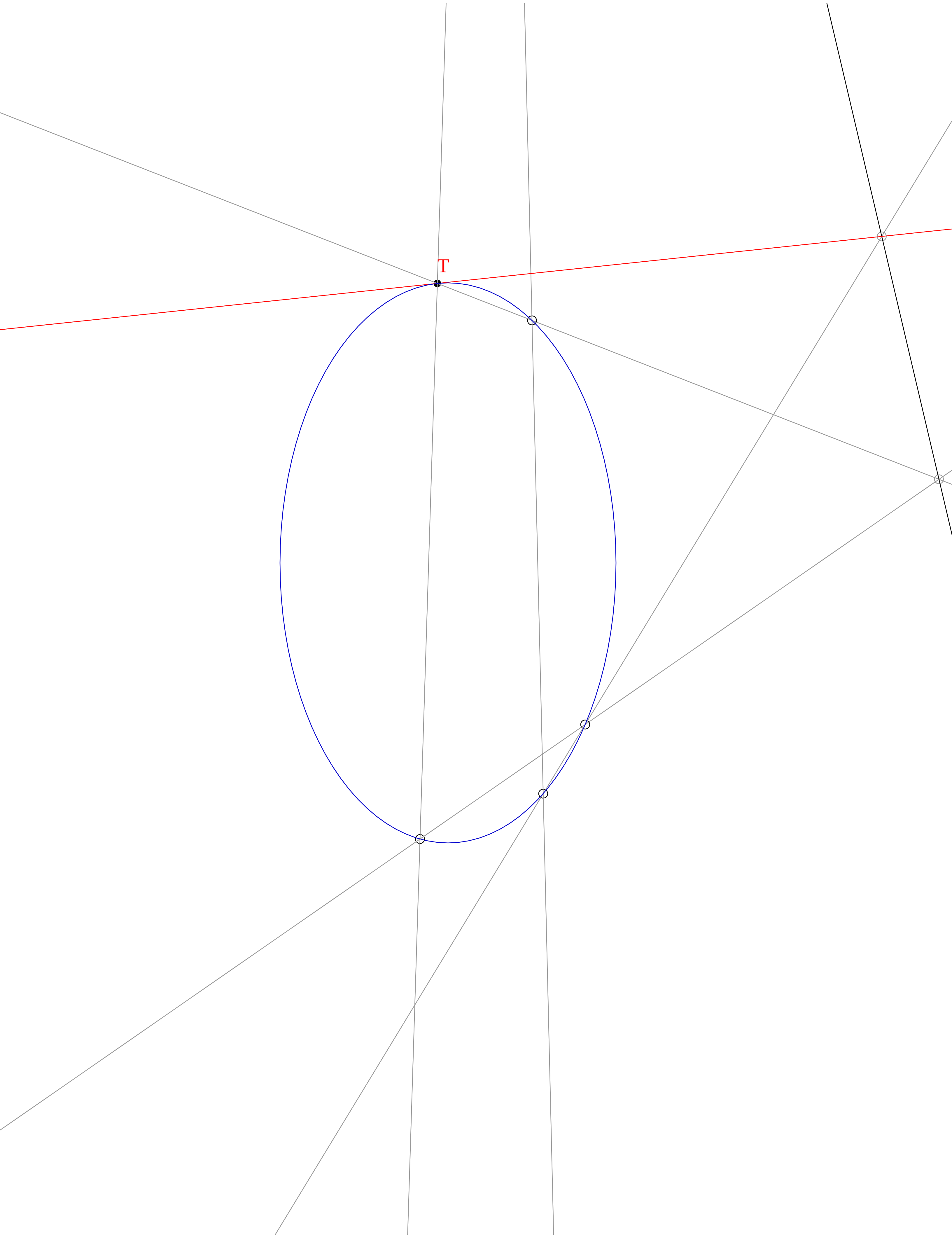
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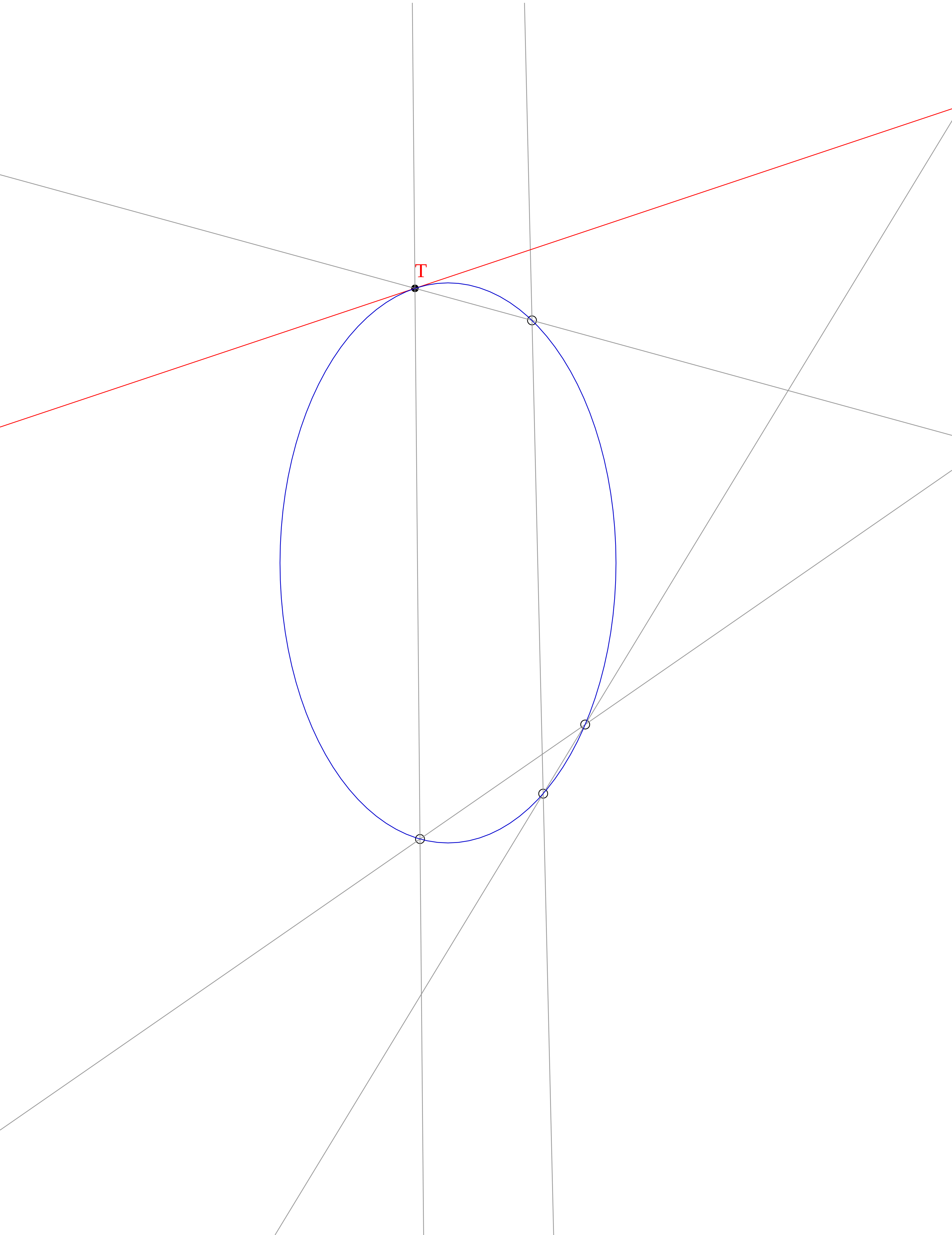
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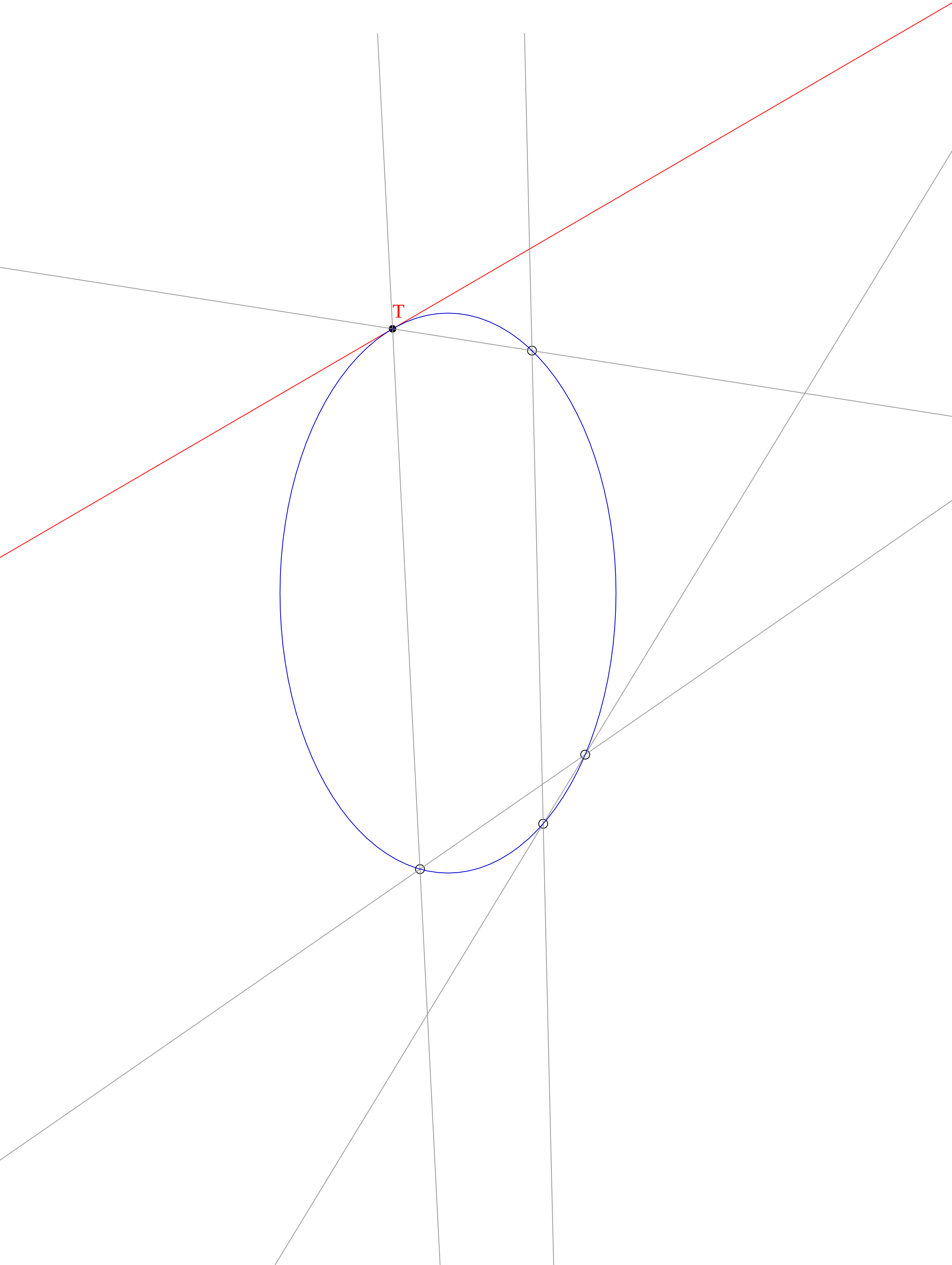
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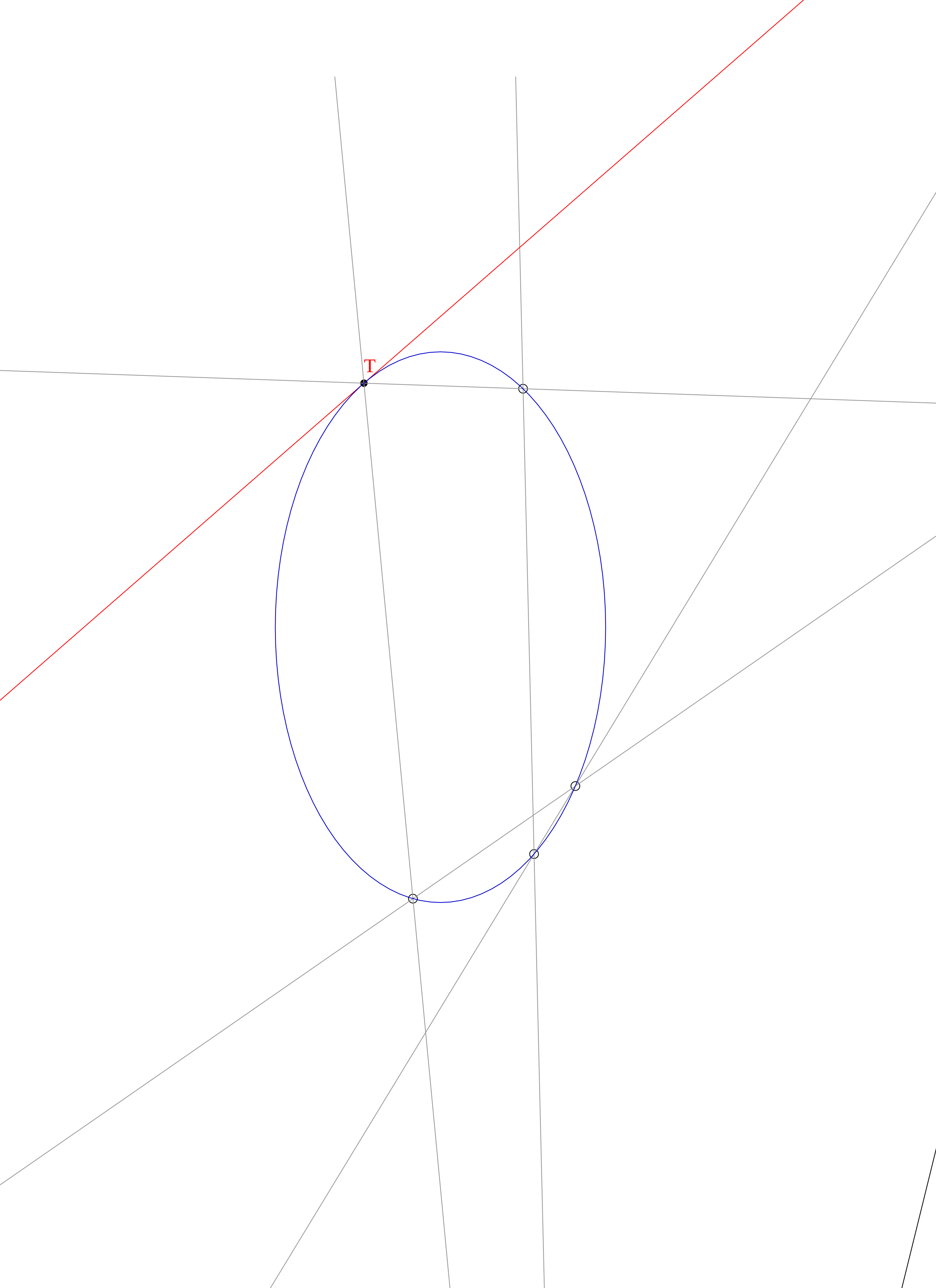
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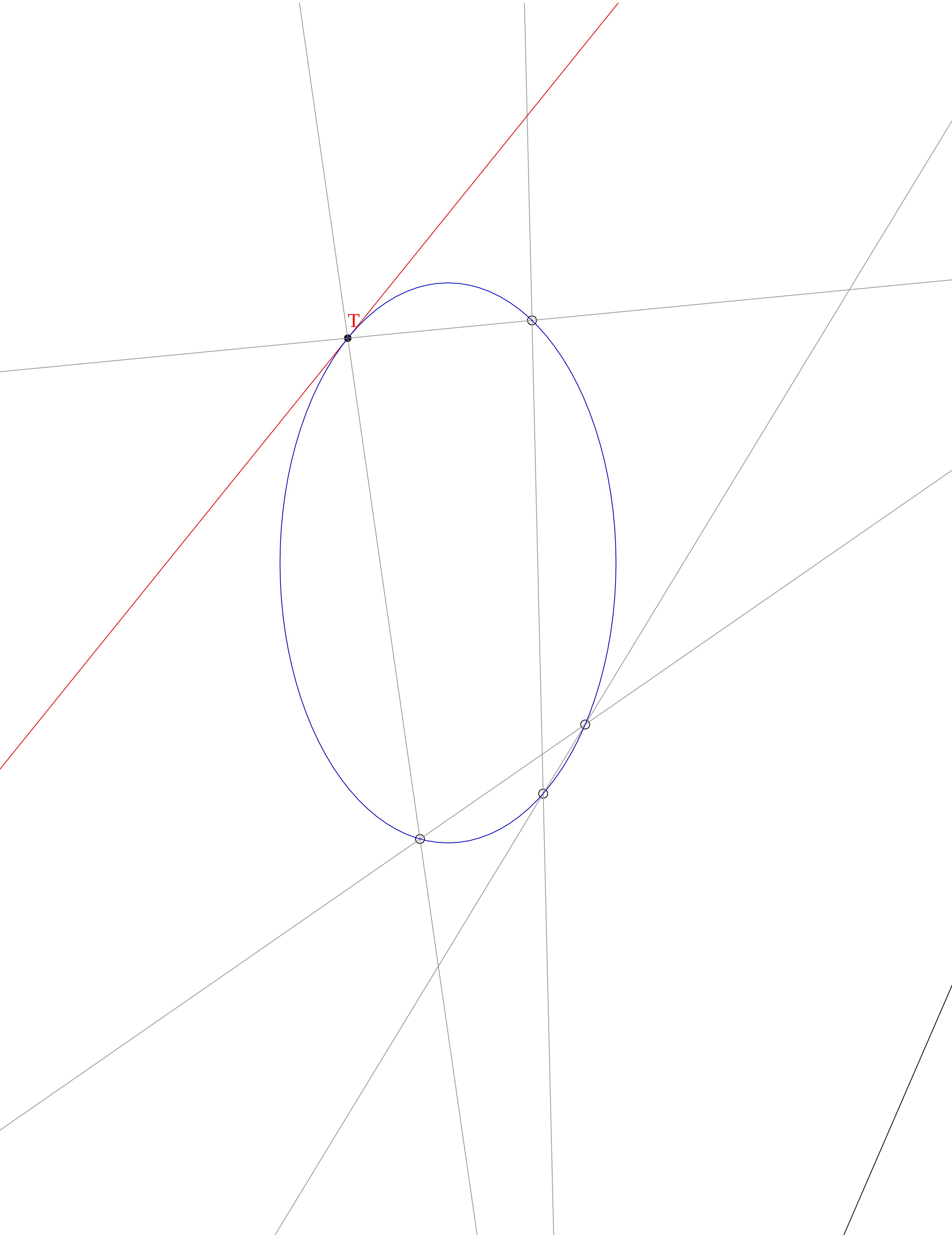
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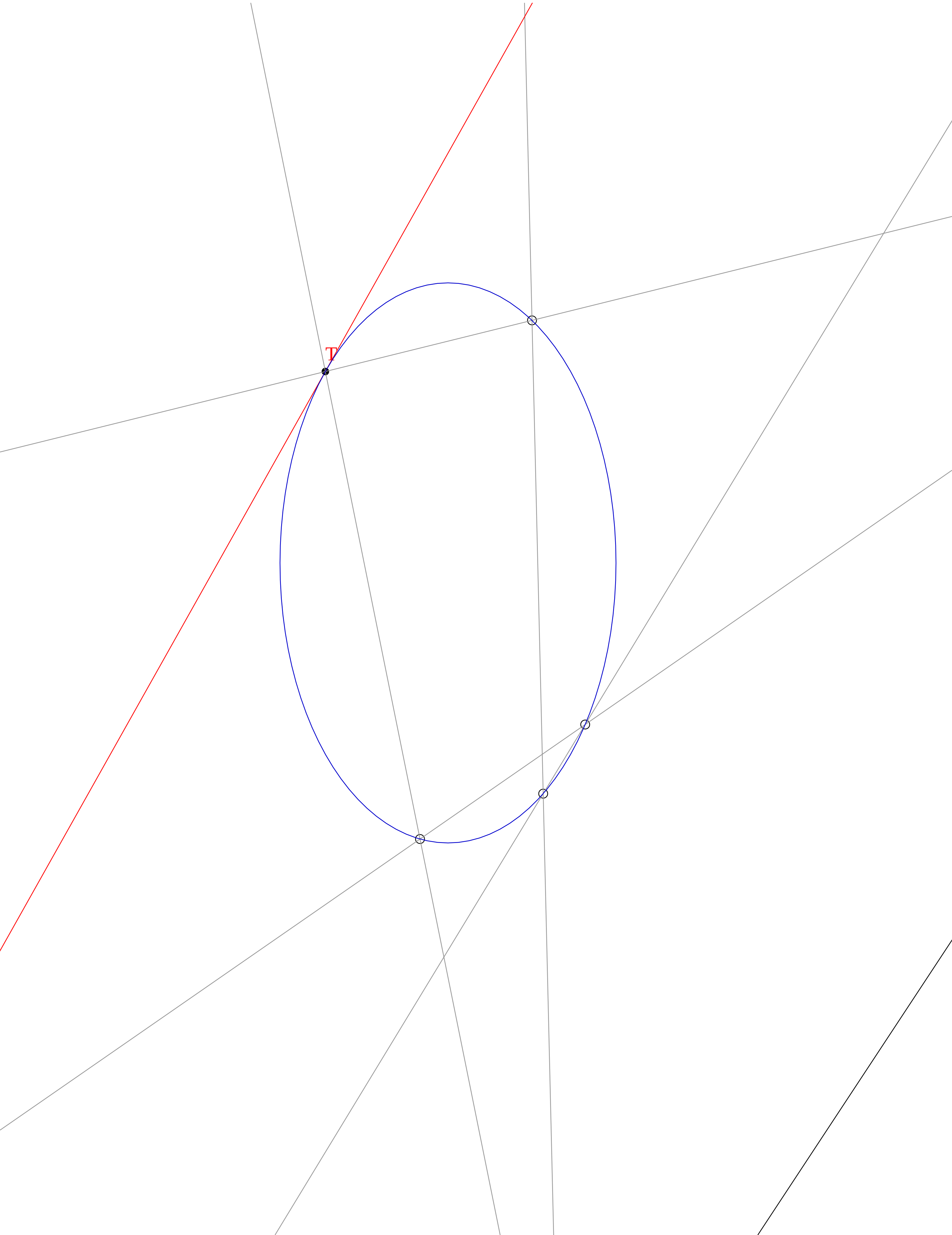
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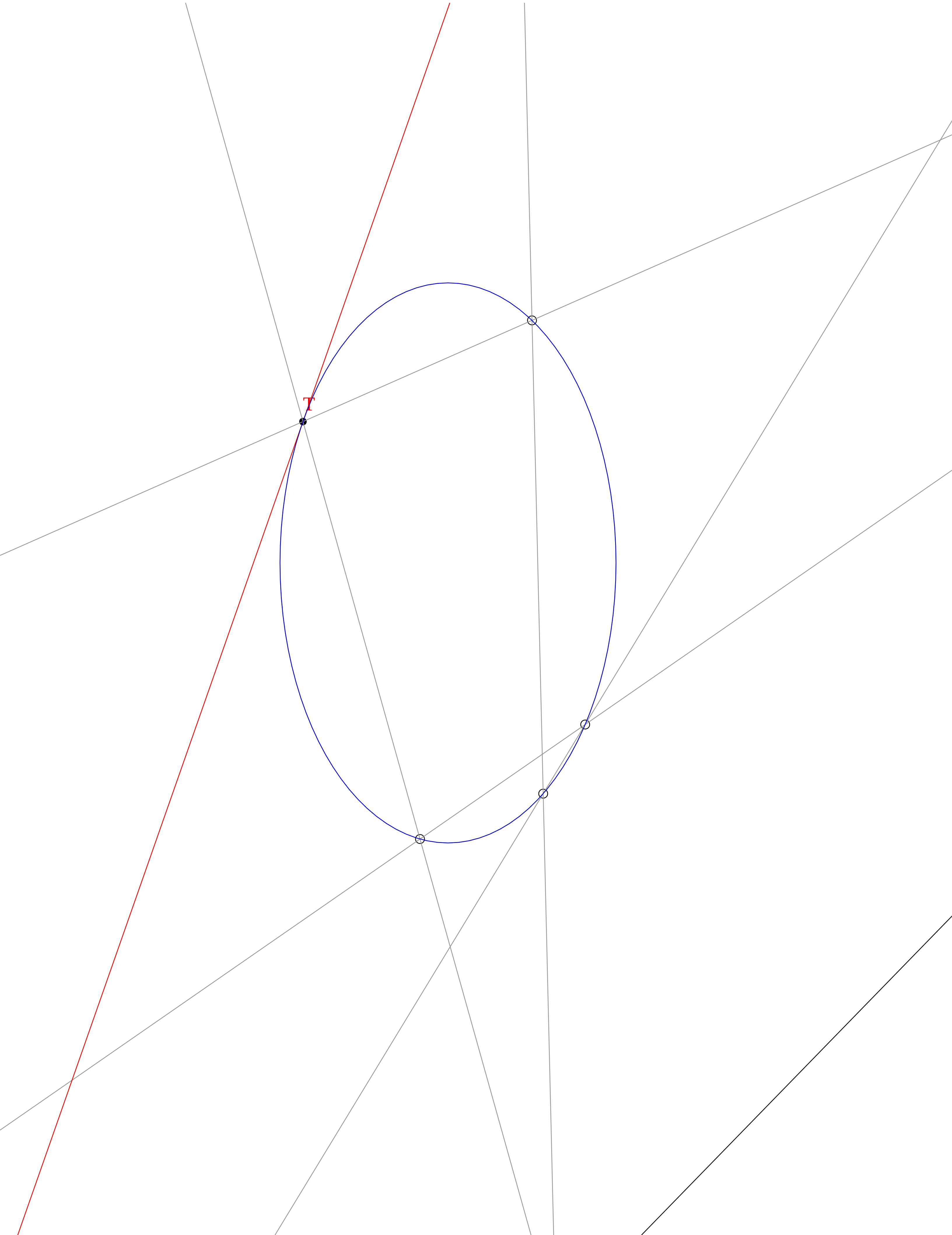
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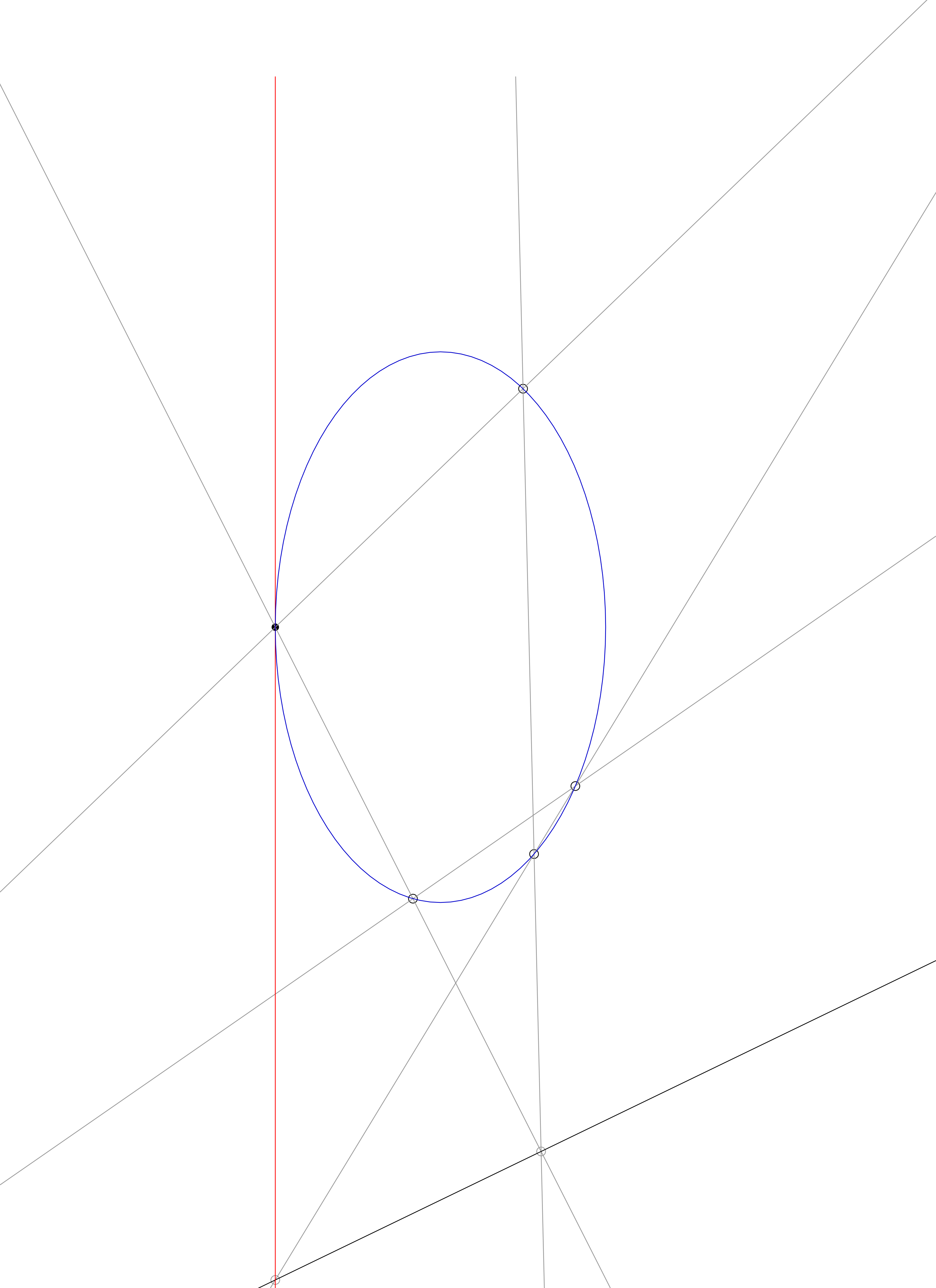

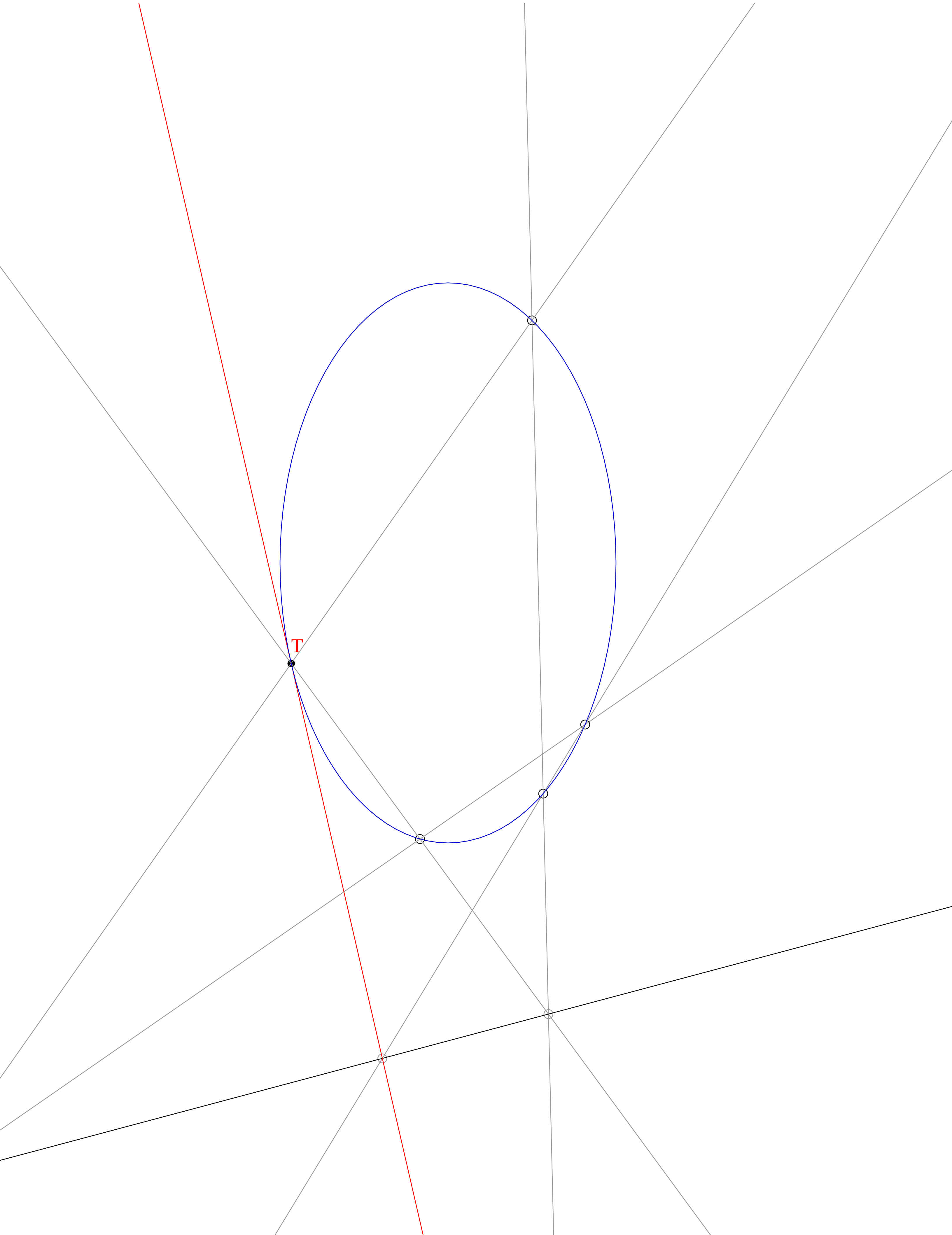
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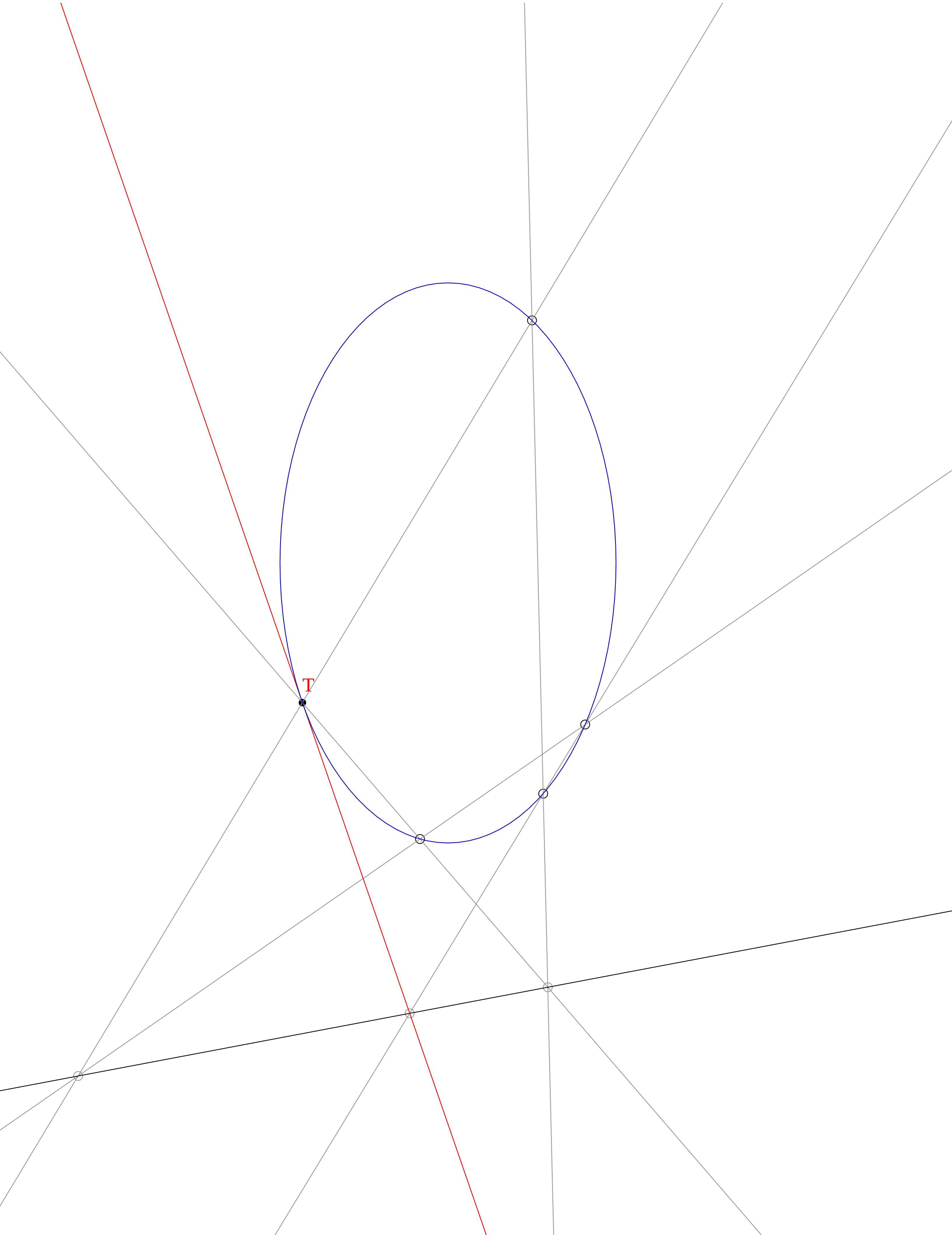
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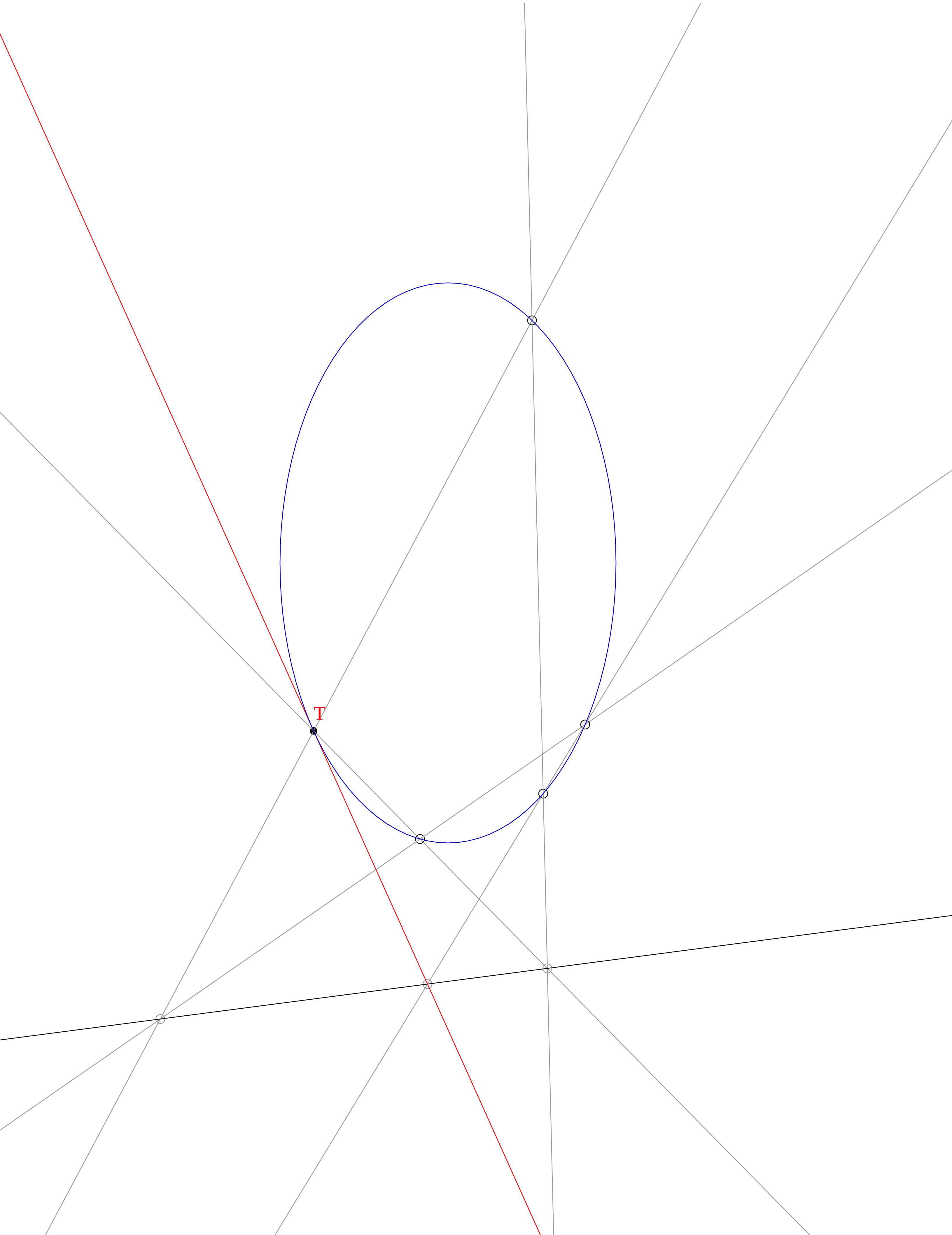
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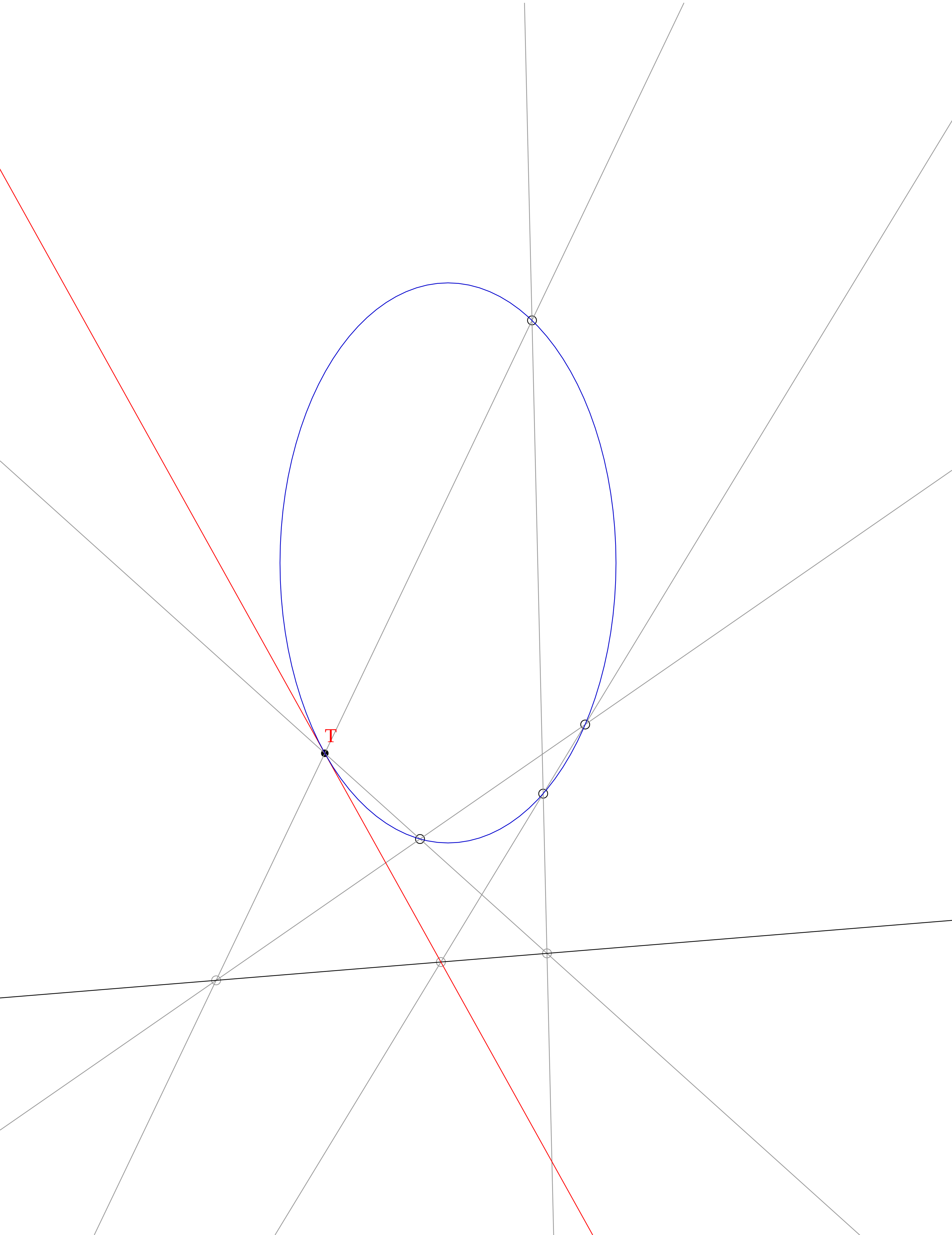
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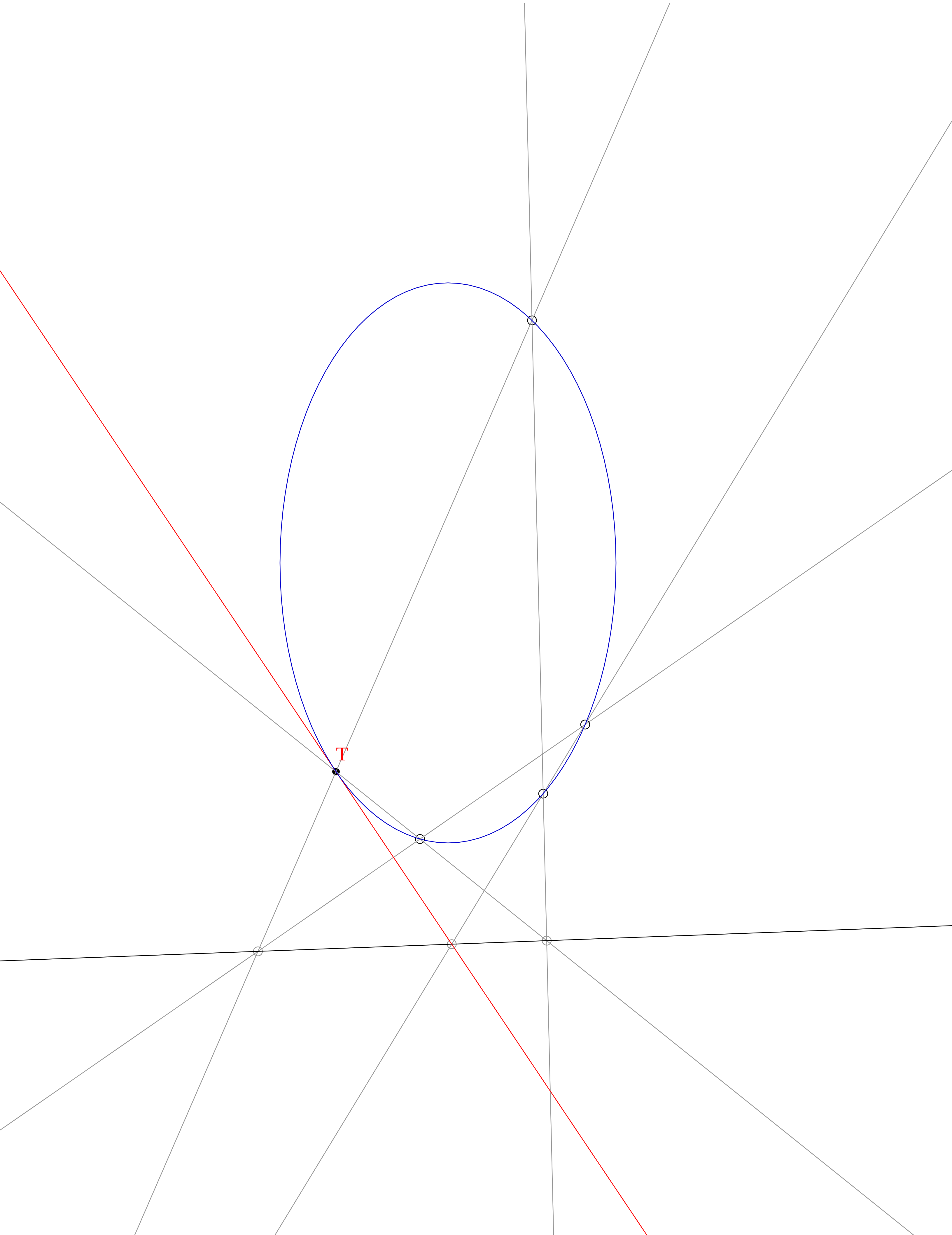
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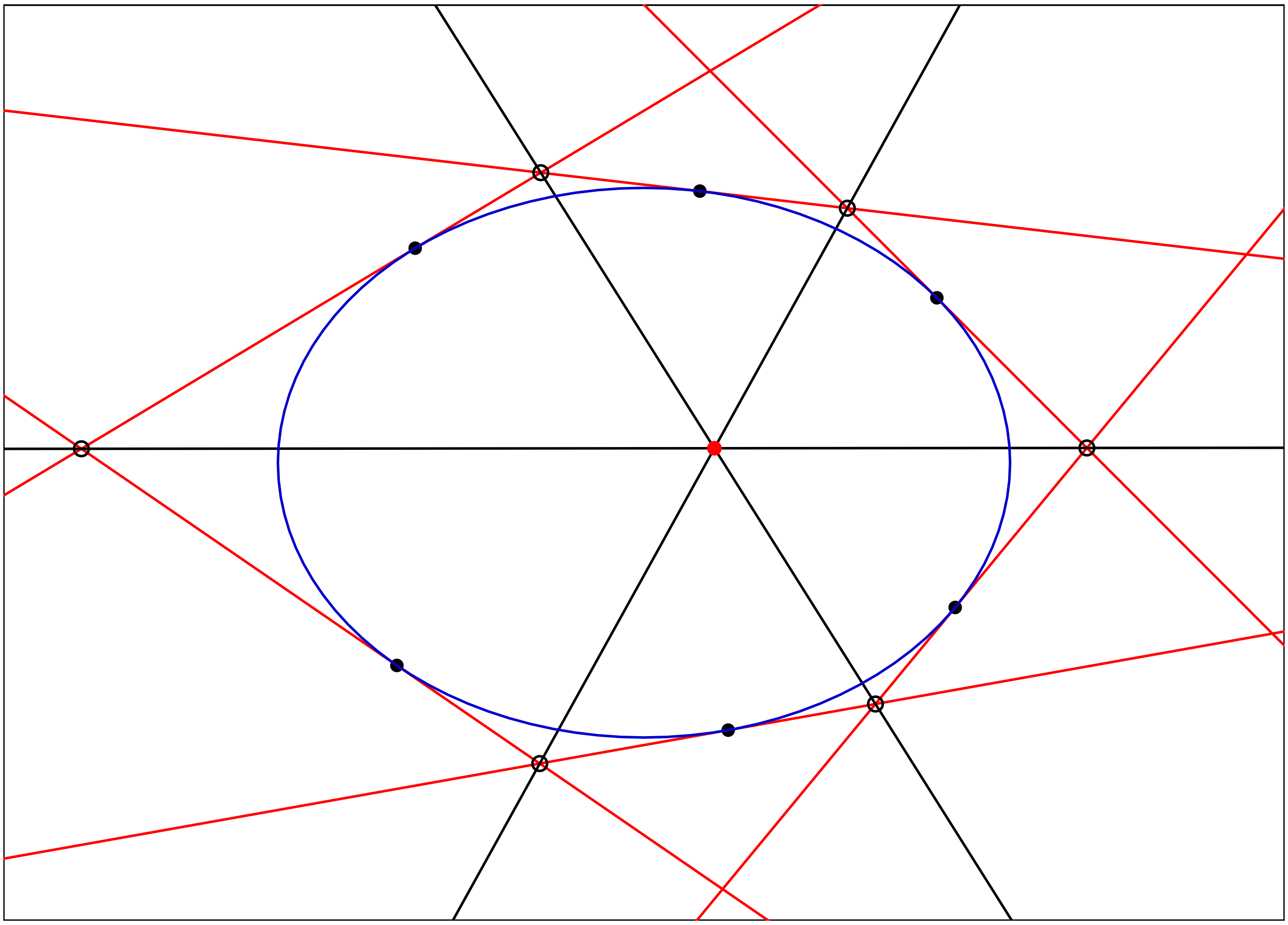
3.2 Brianchon’s Theorem and its applications

Brianchon’s Theorem ,the dual of Pascal’s theorem, states that if a hexagon is circumscribed to a conic, then the 3 lines that join the three pairs of opposite vertices are concurrent.

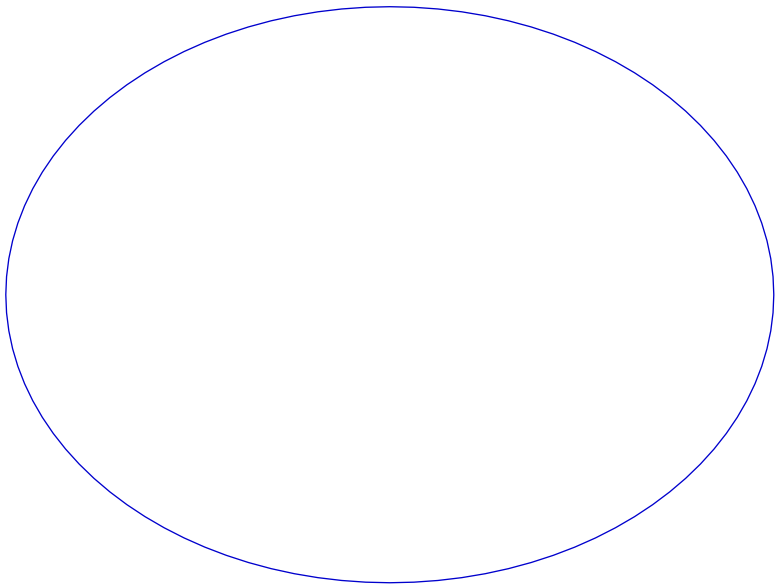
The following figure show the cases of the Brianchon’s theorem in eclipse.

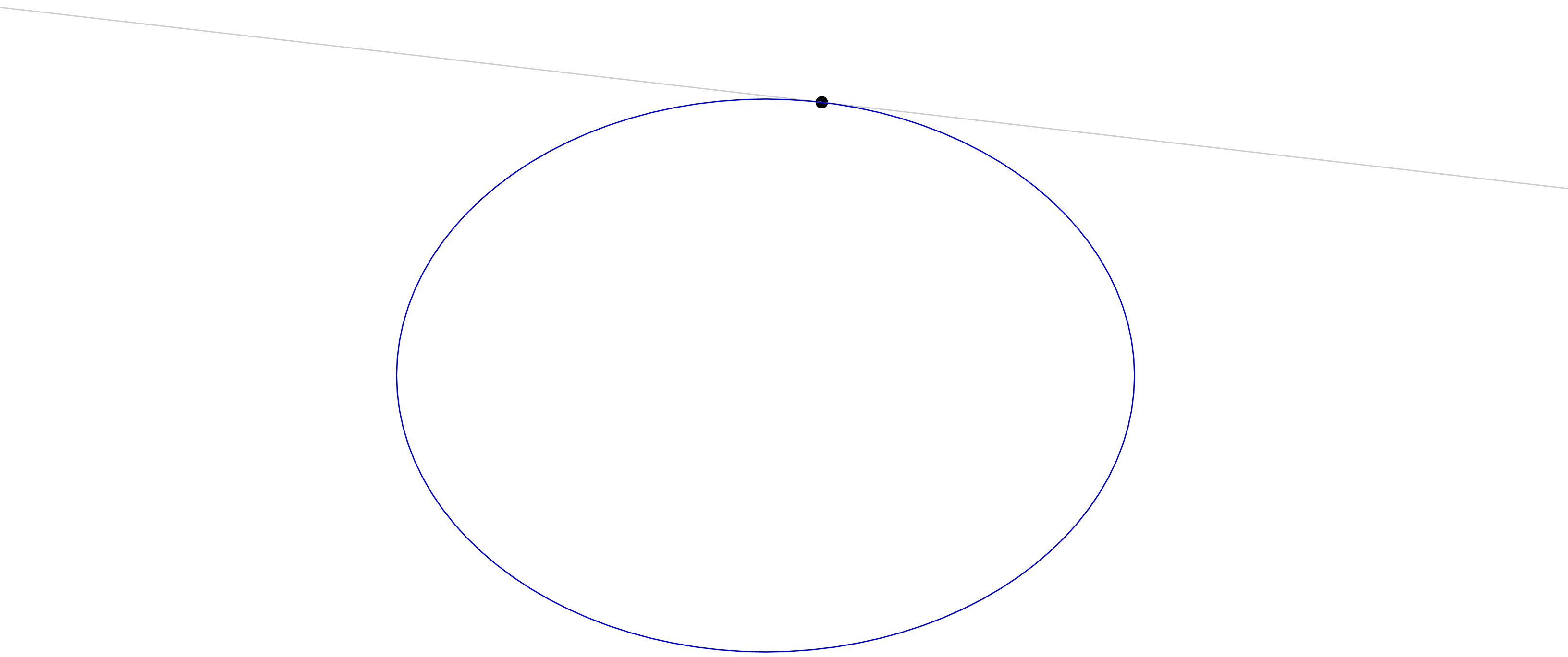


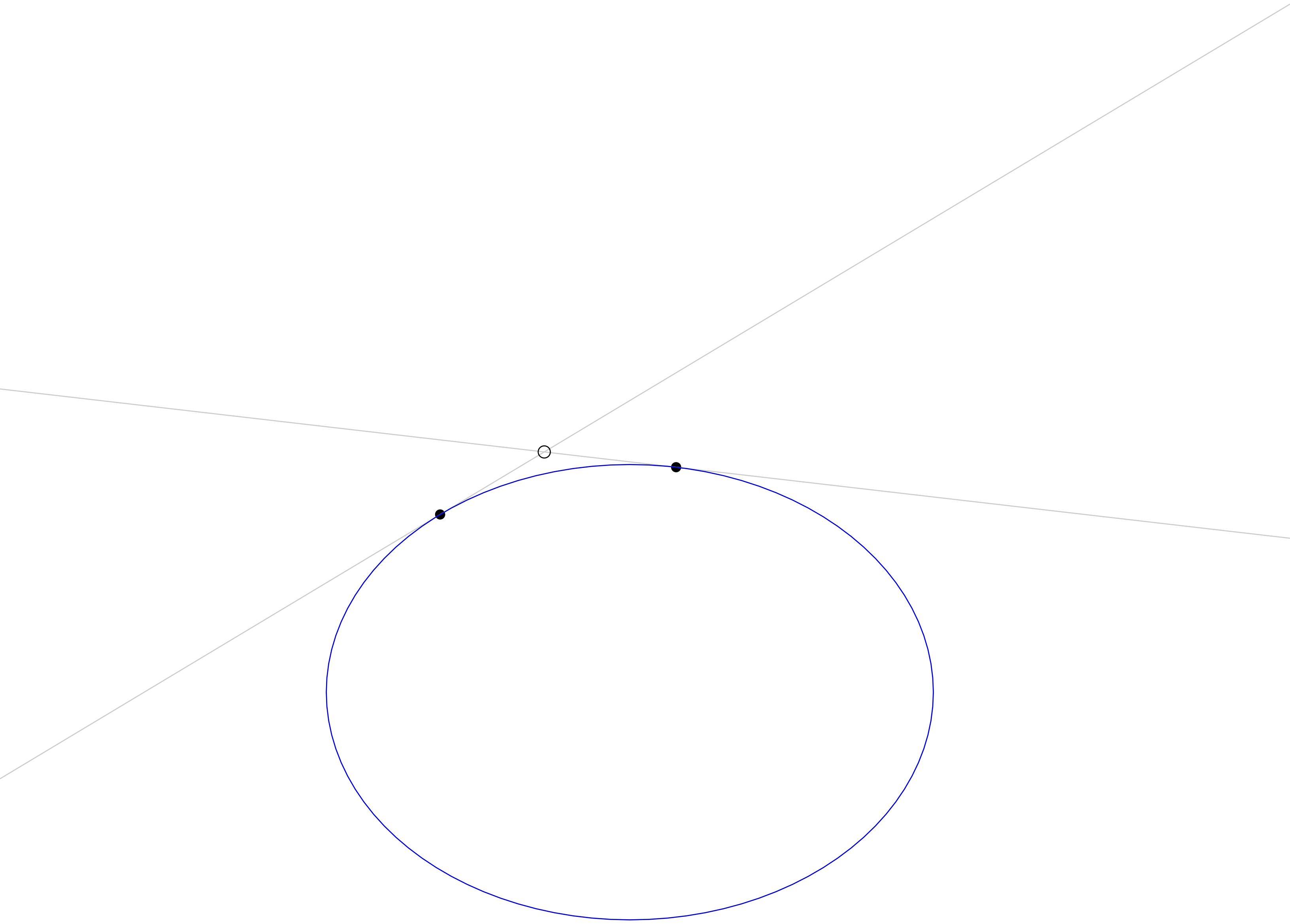
Applications of Brianchon's theorem

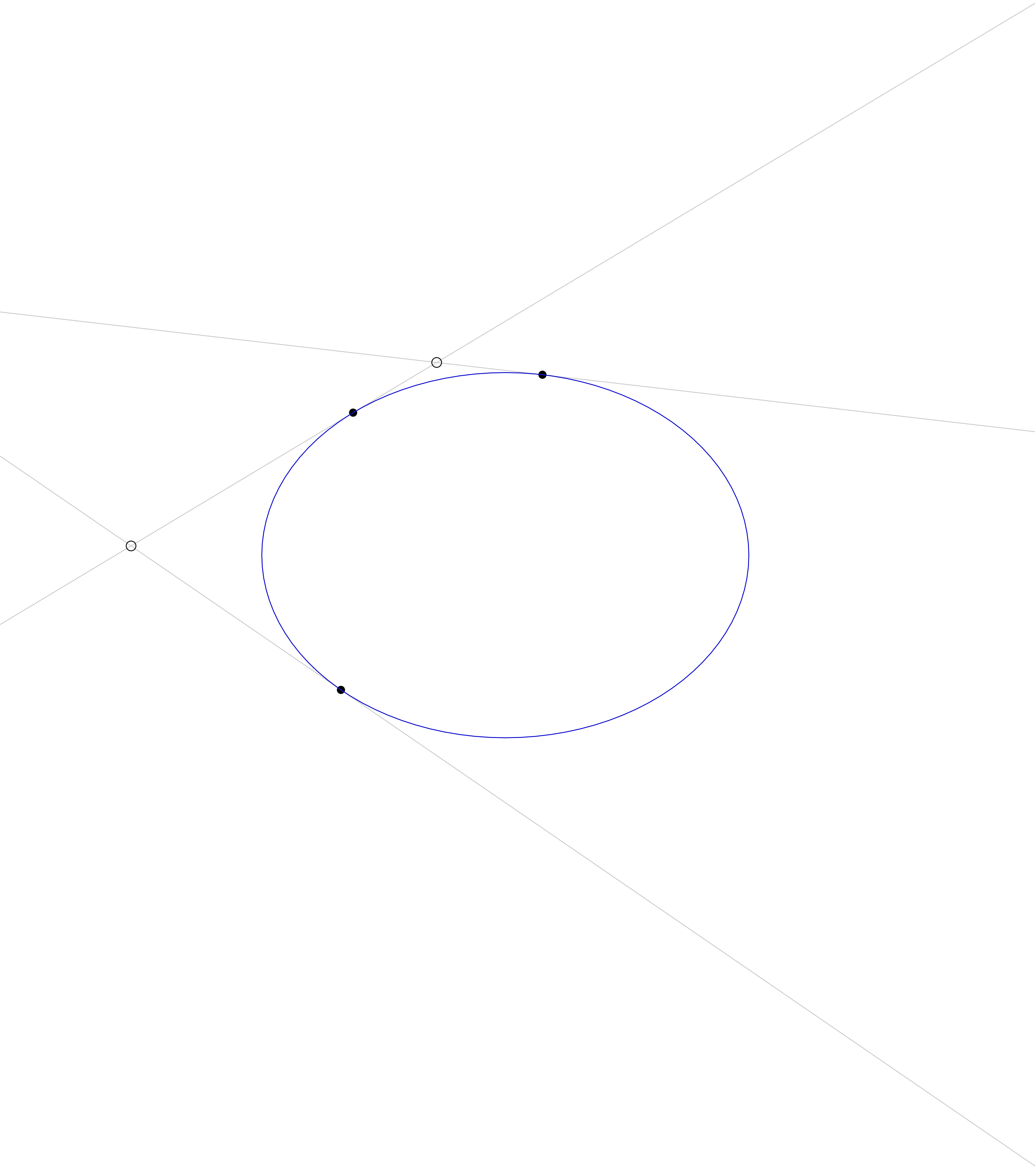
Brianchon's theorem has been proved most useful in finding tangent lines to a conic.

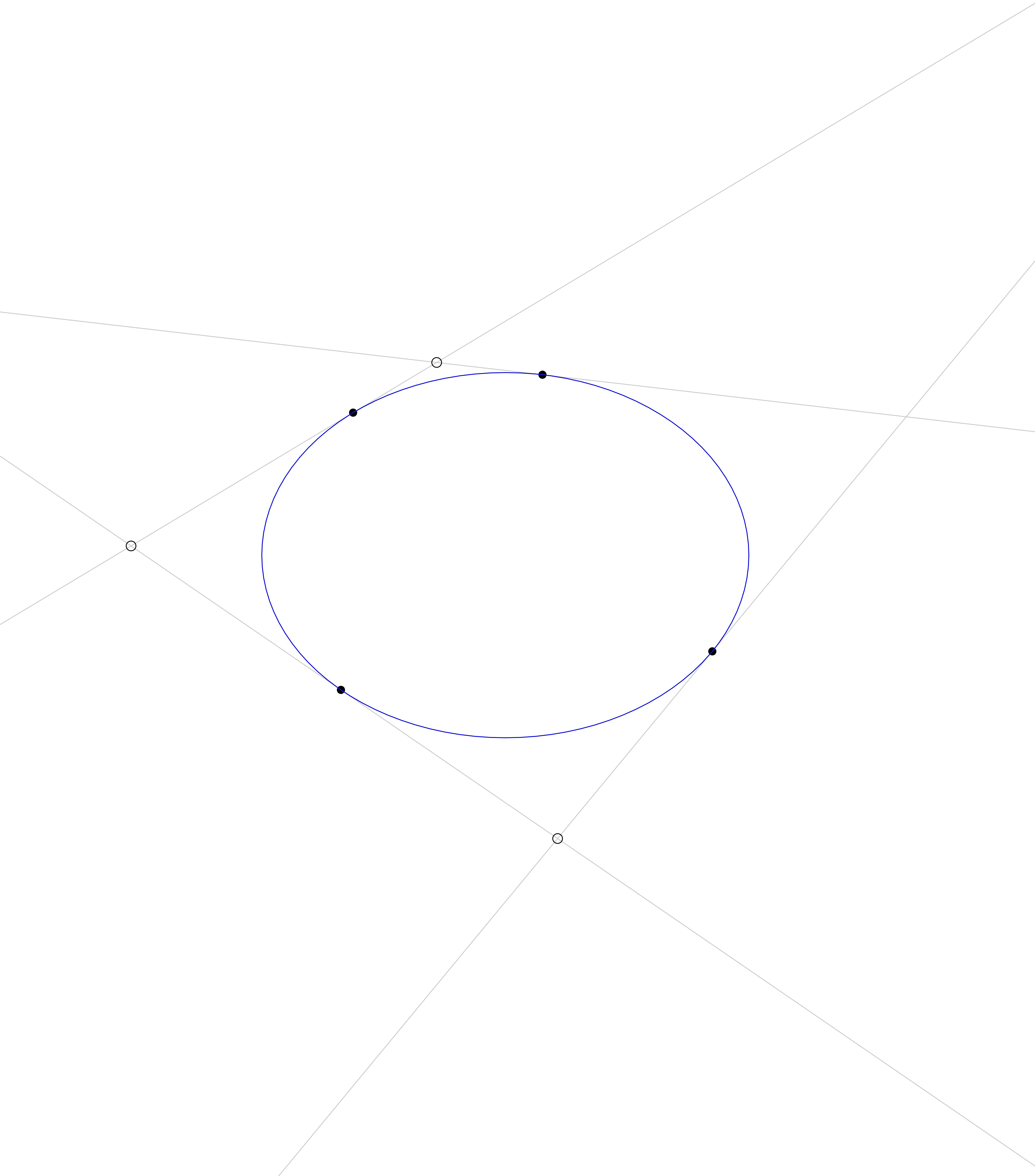
Application: Locate the sixth tangent line to a conic when five tangent lines on this conic are given. In other words, given five tangent lines to a conic, from a given point on one of the five lines, draw another tangent line to the conic.

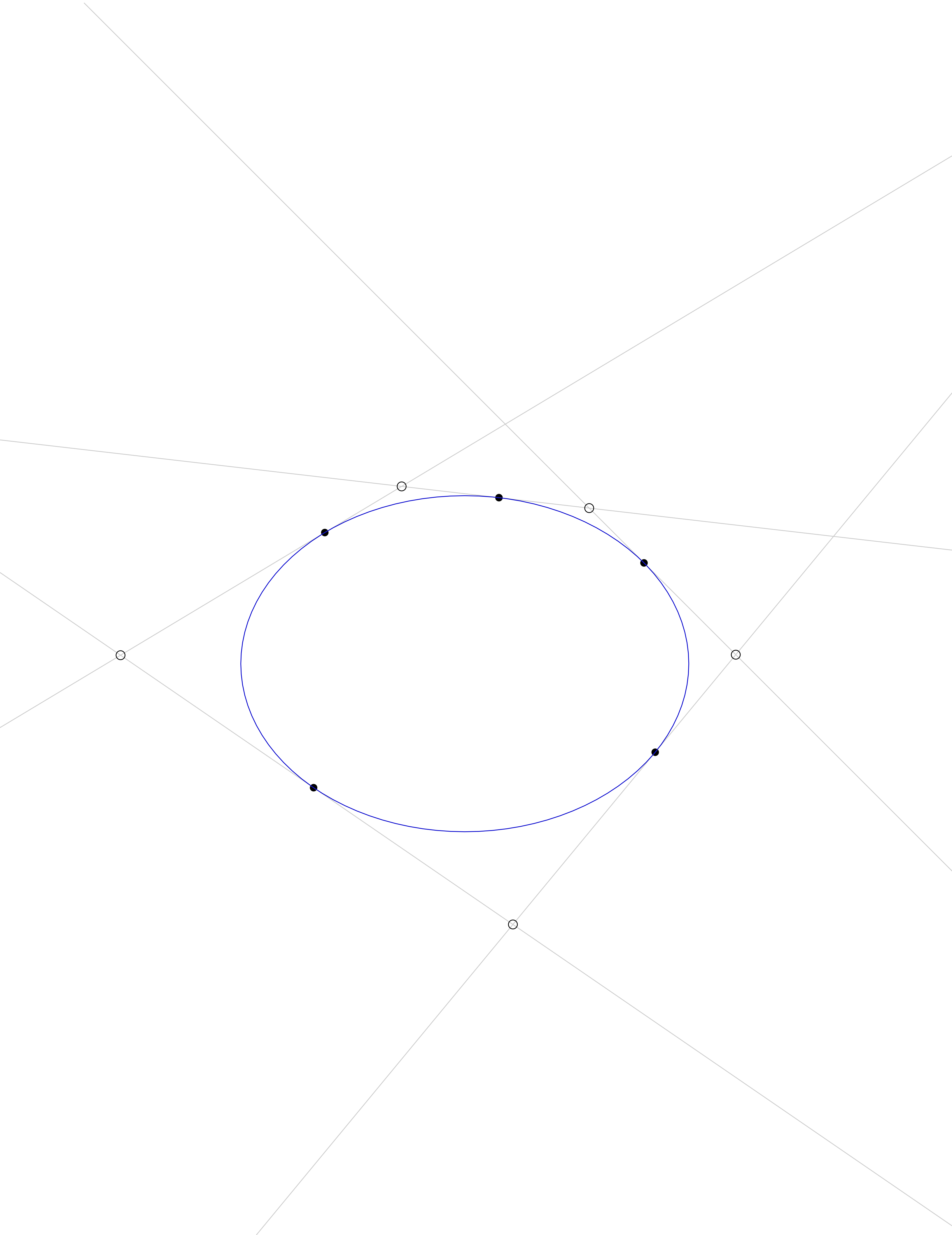


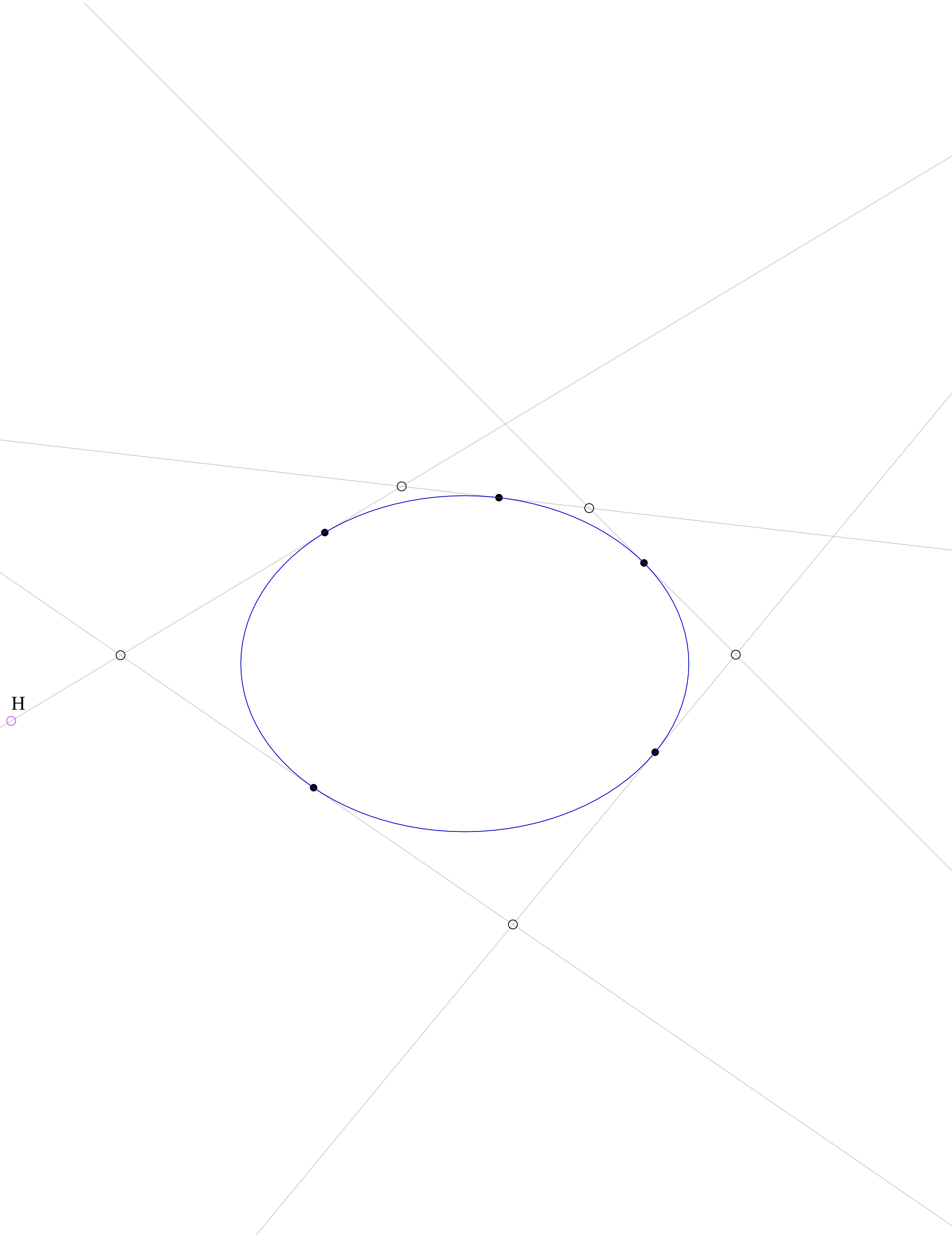








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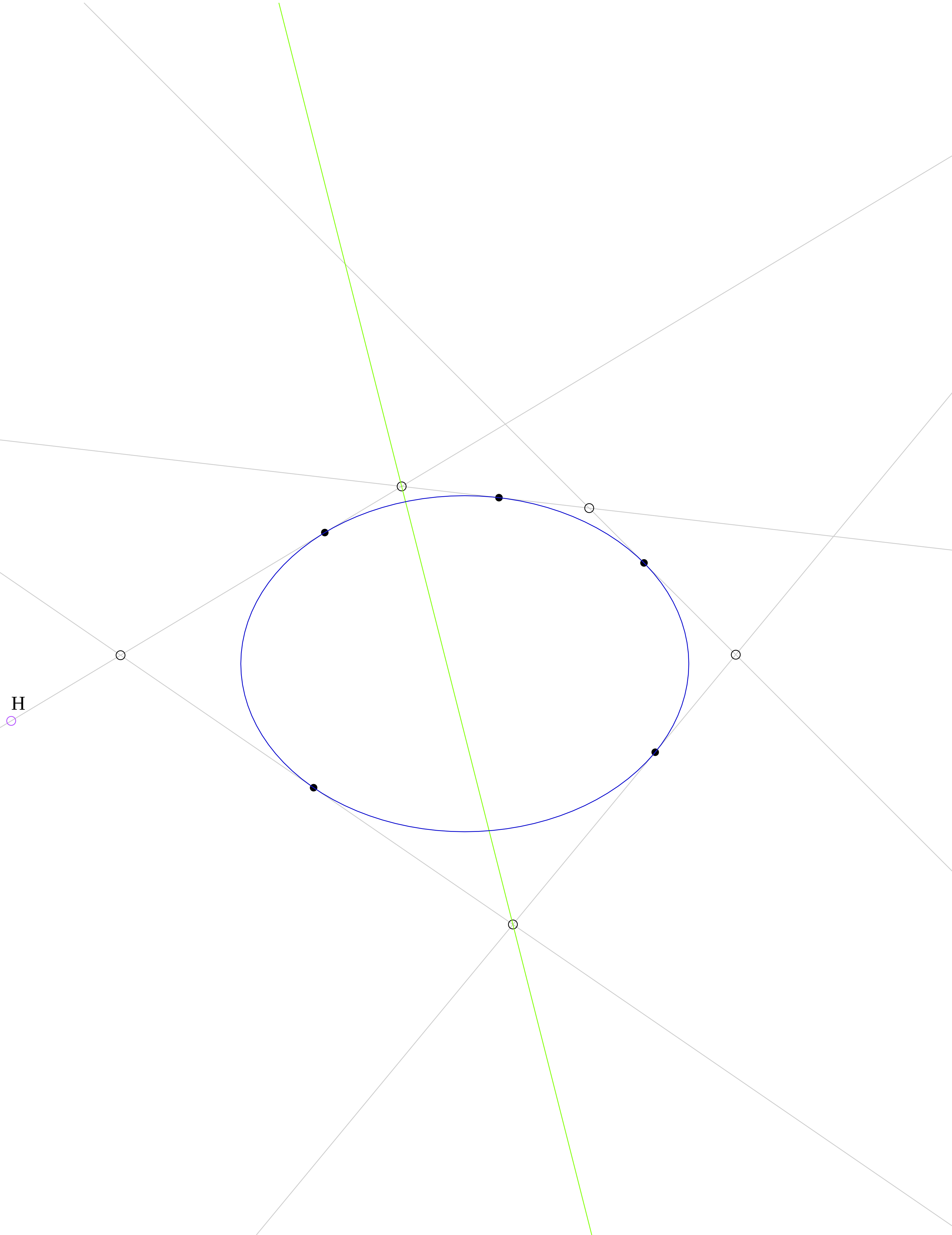
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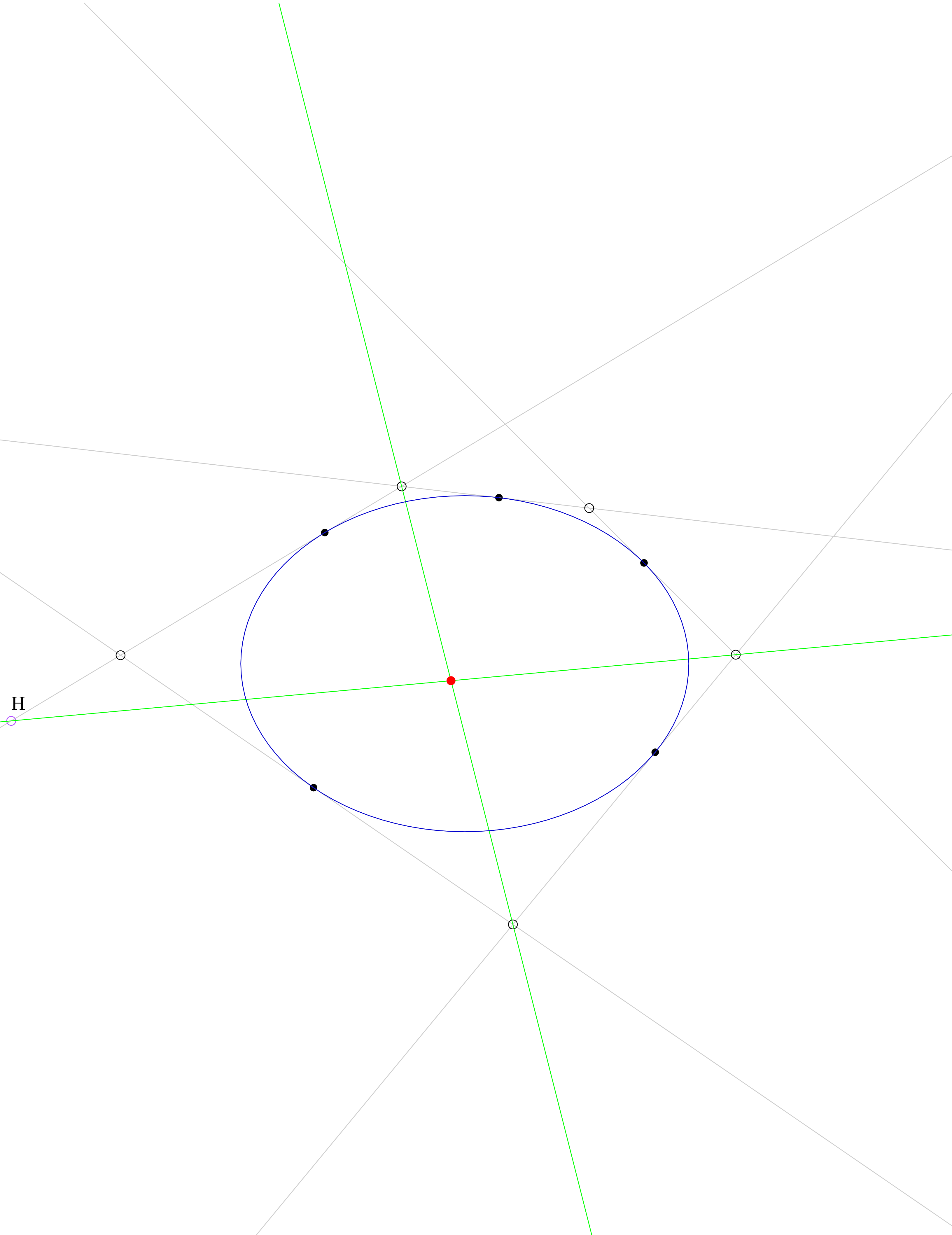
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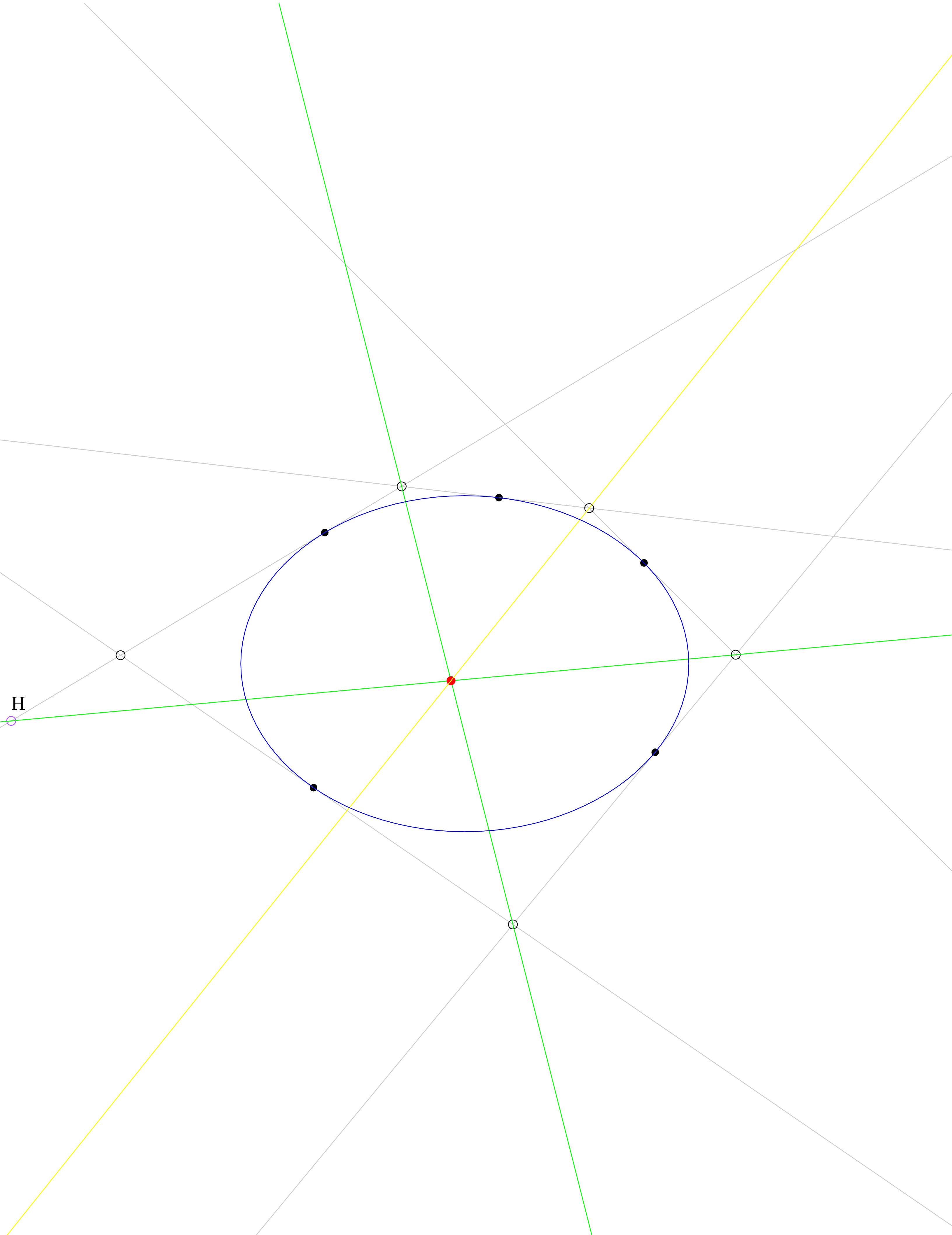
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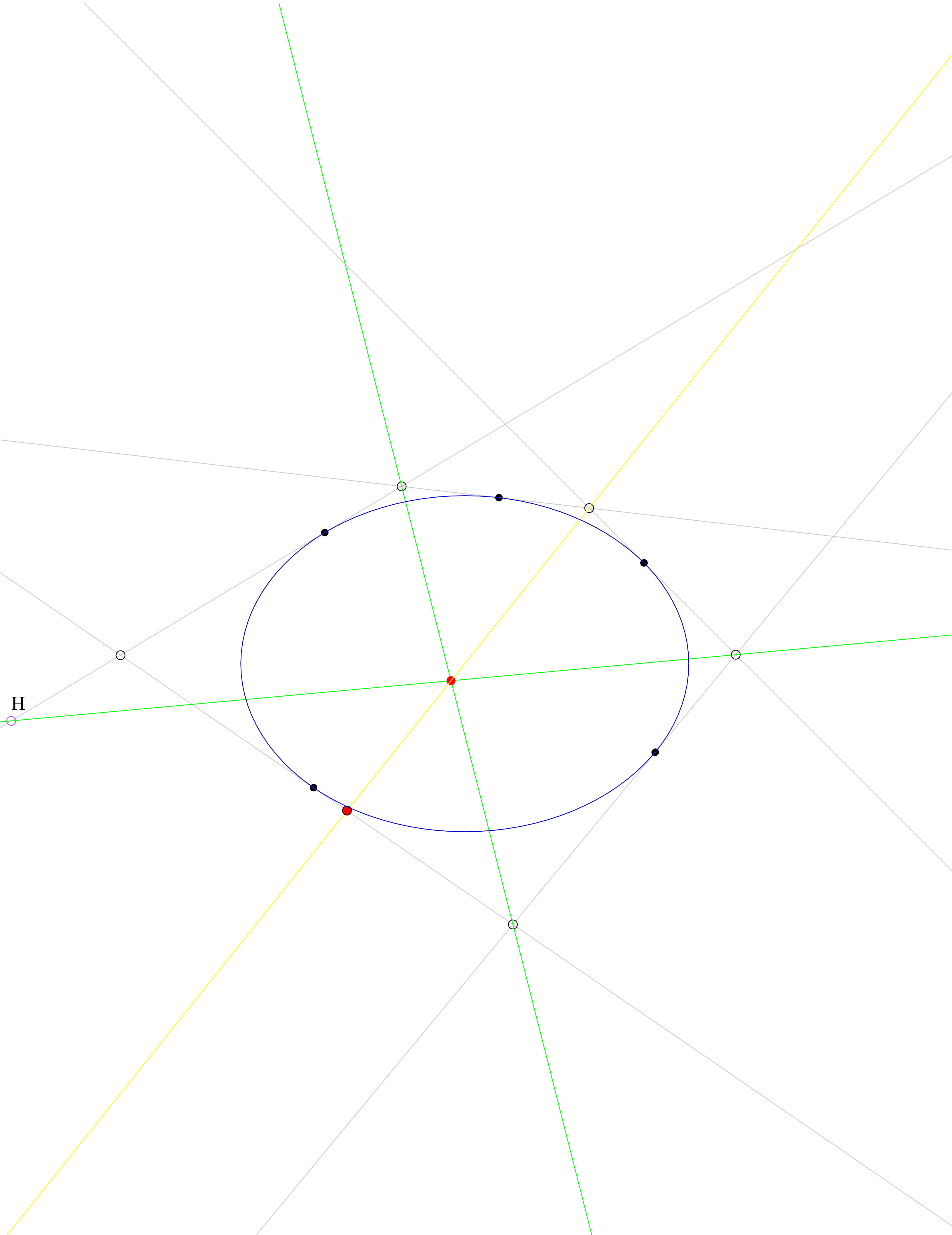
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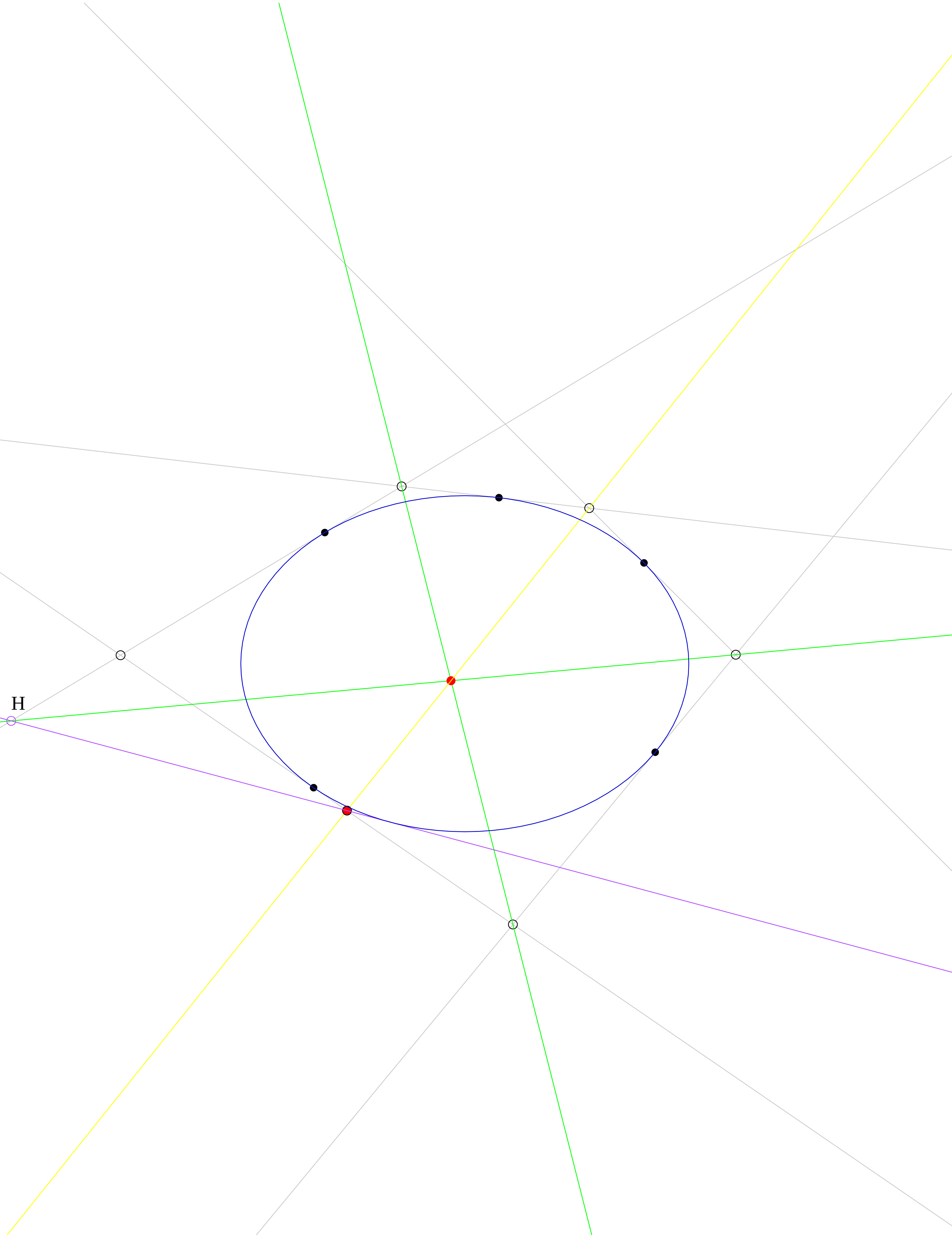
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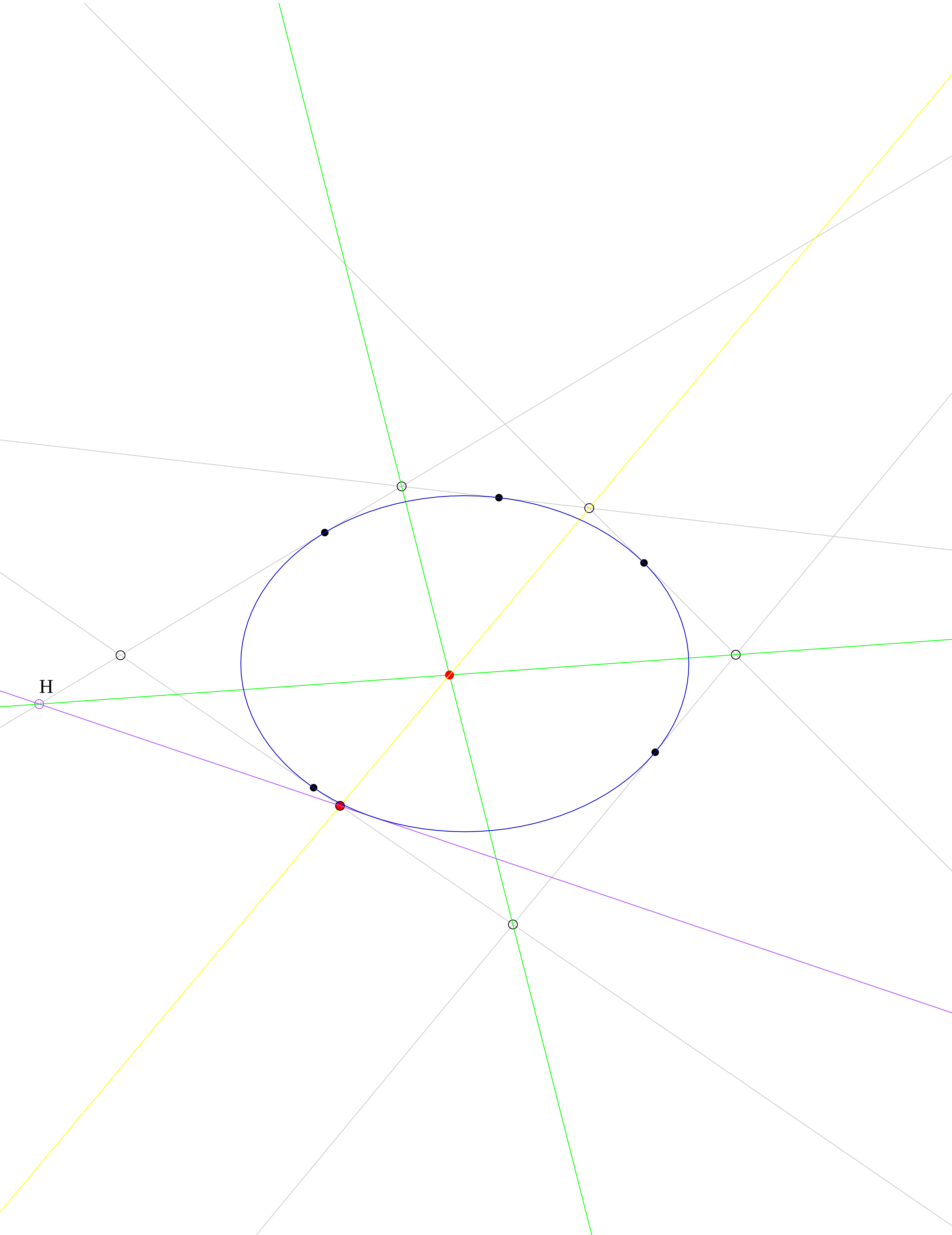
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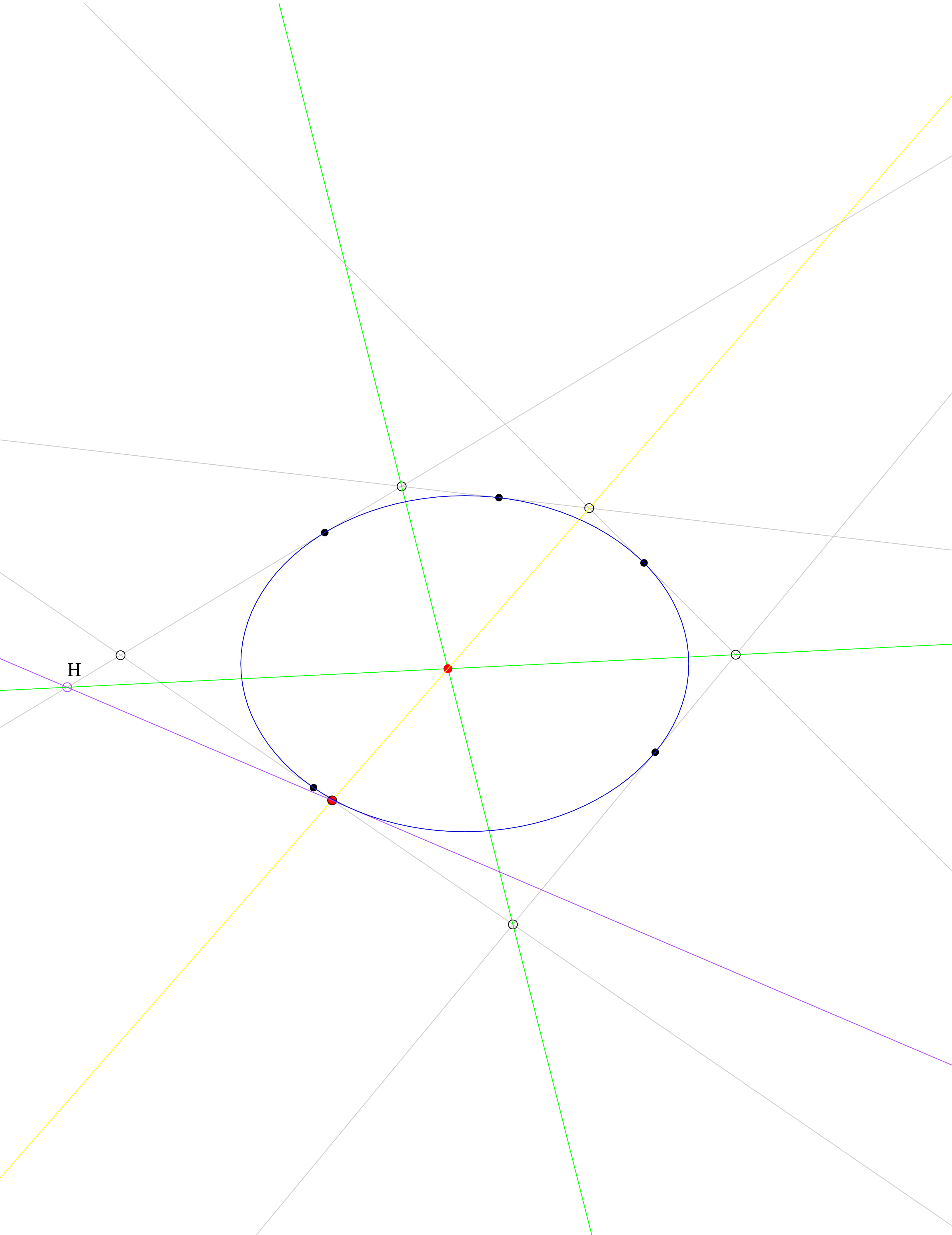
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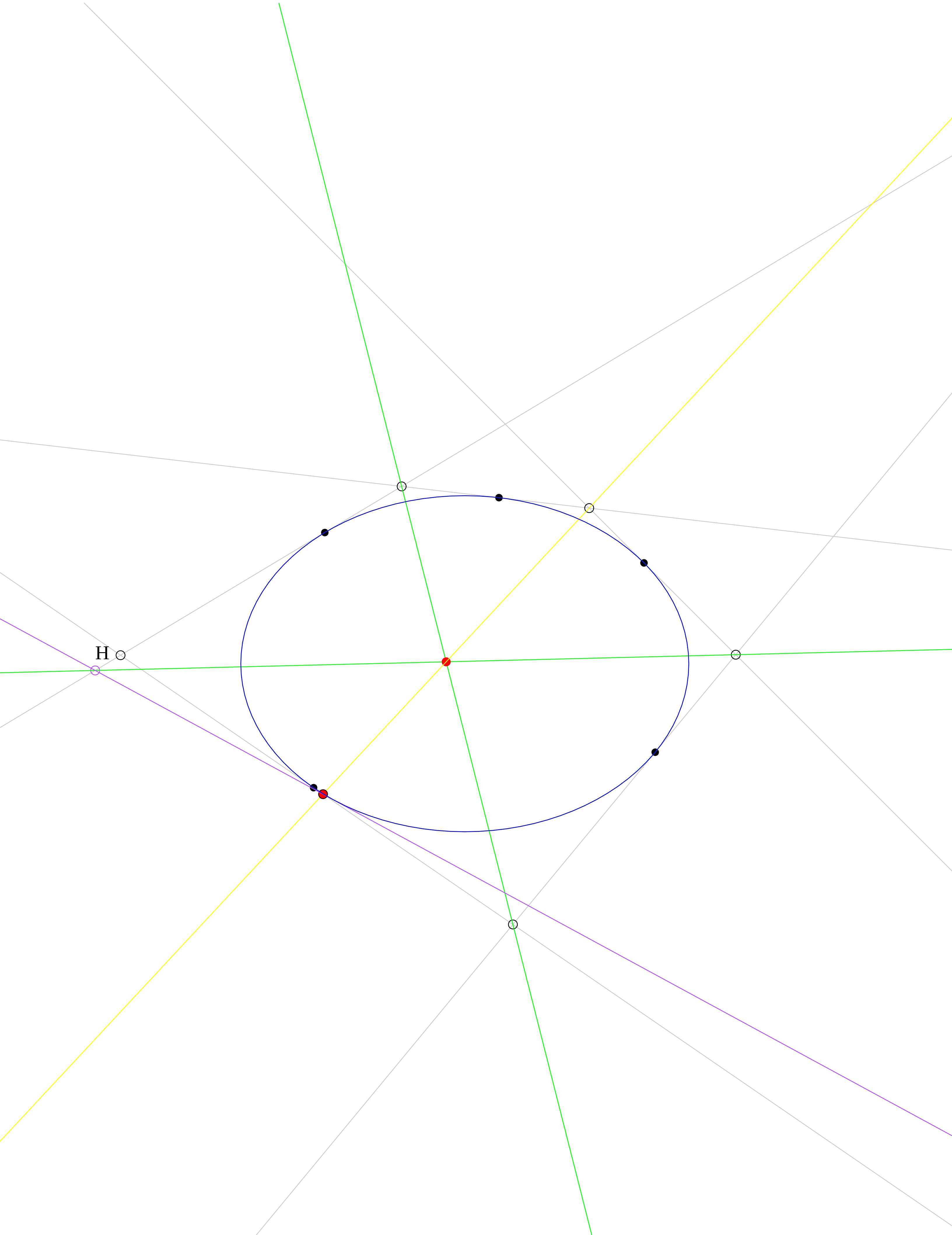
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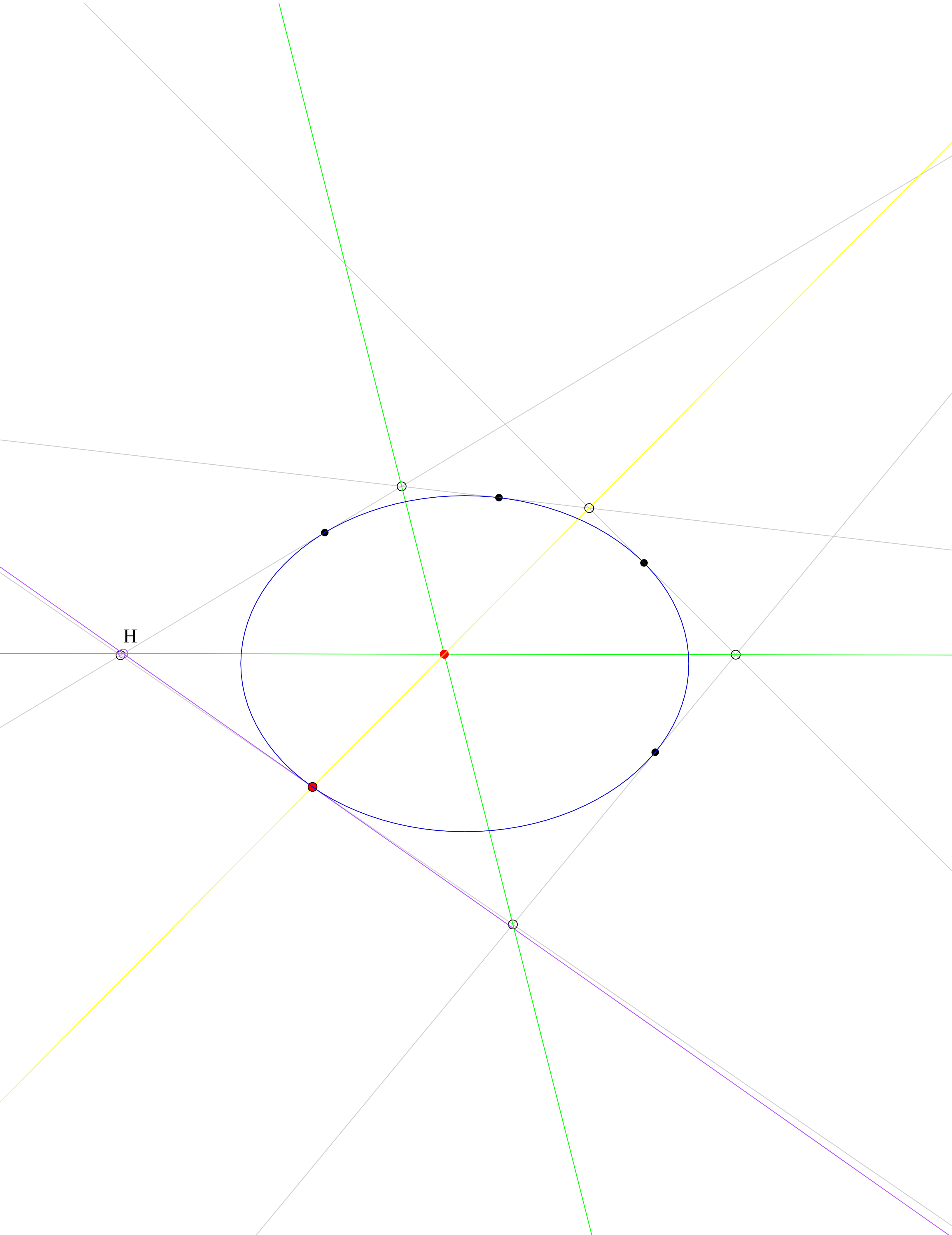
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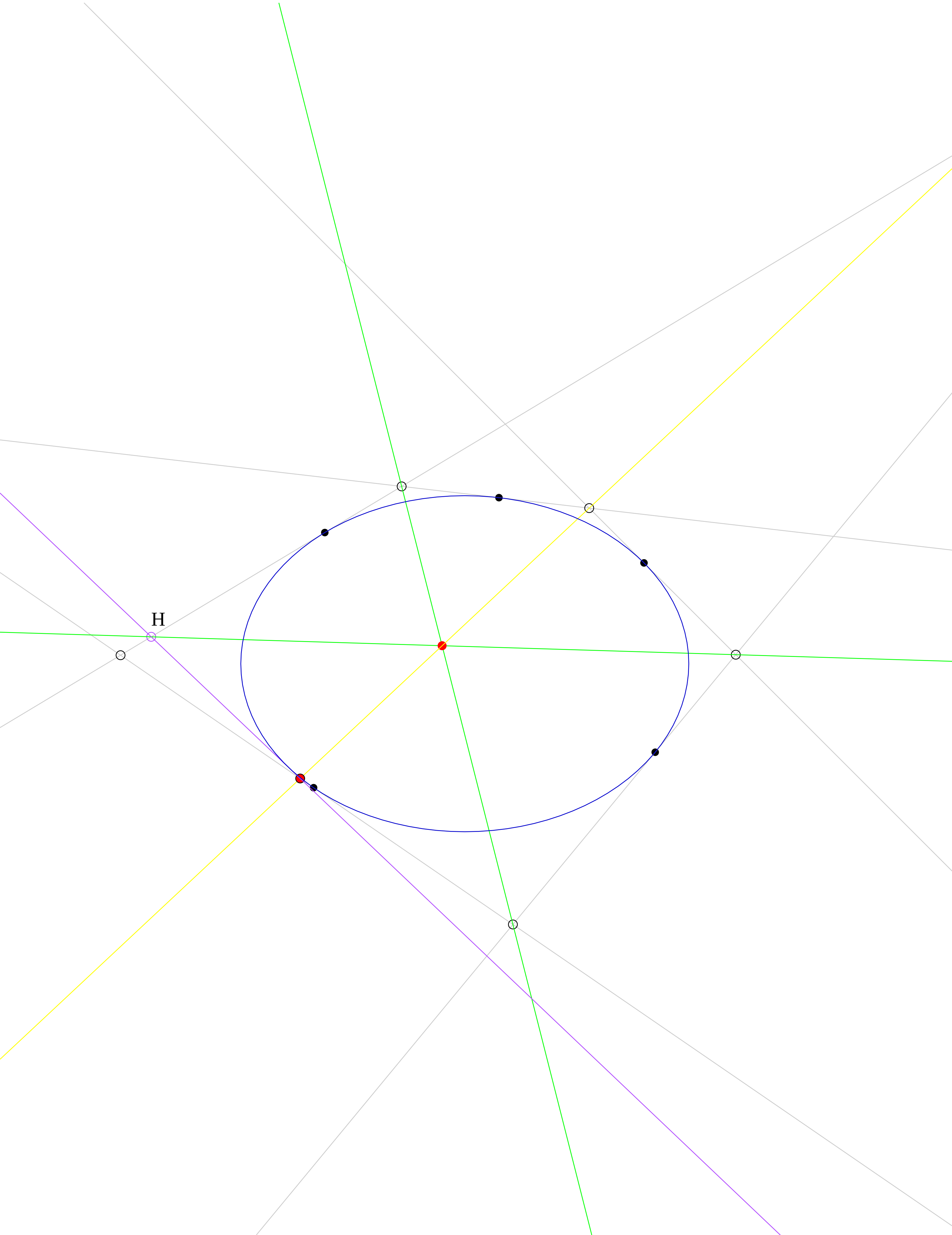
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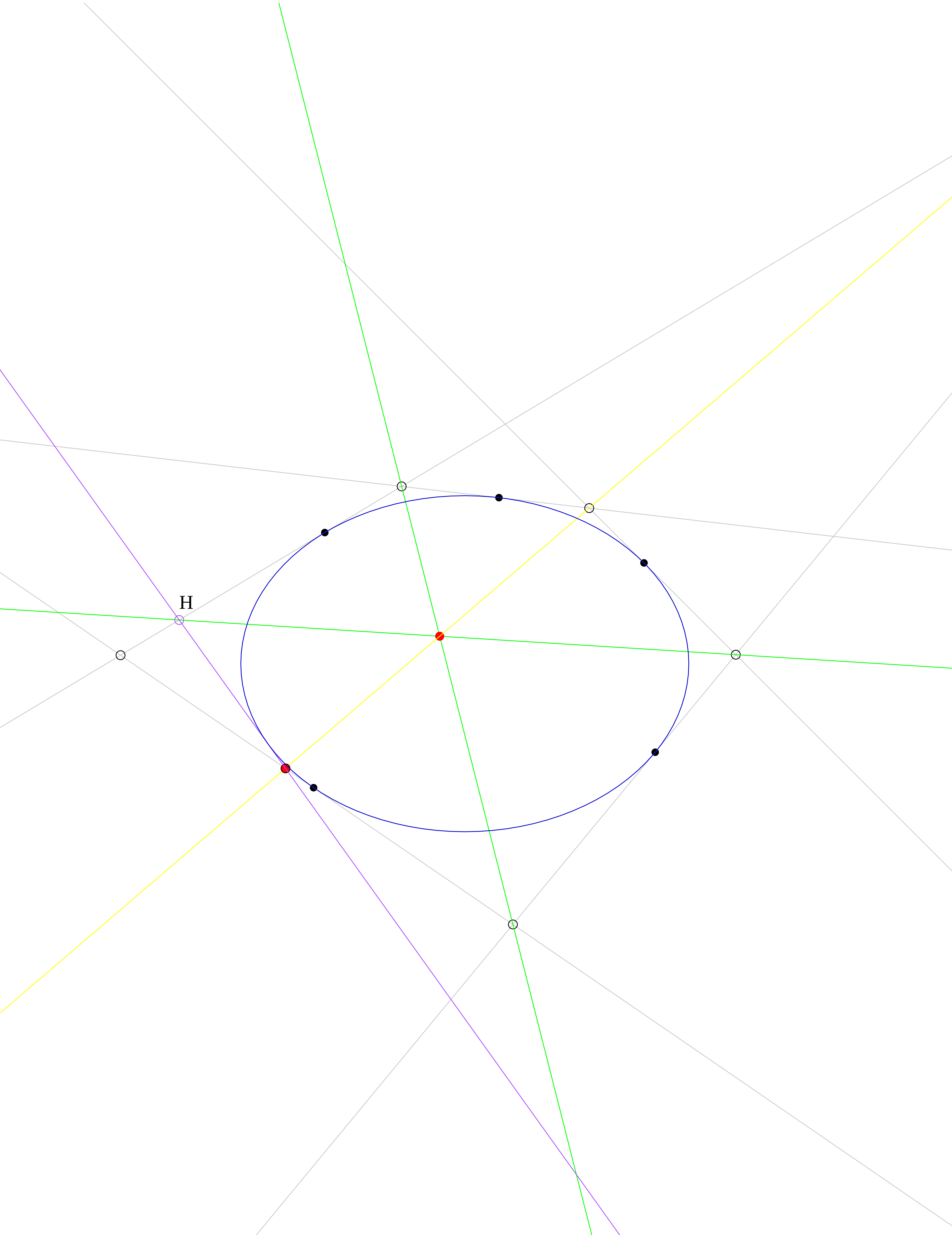
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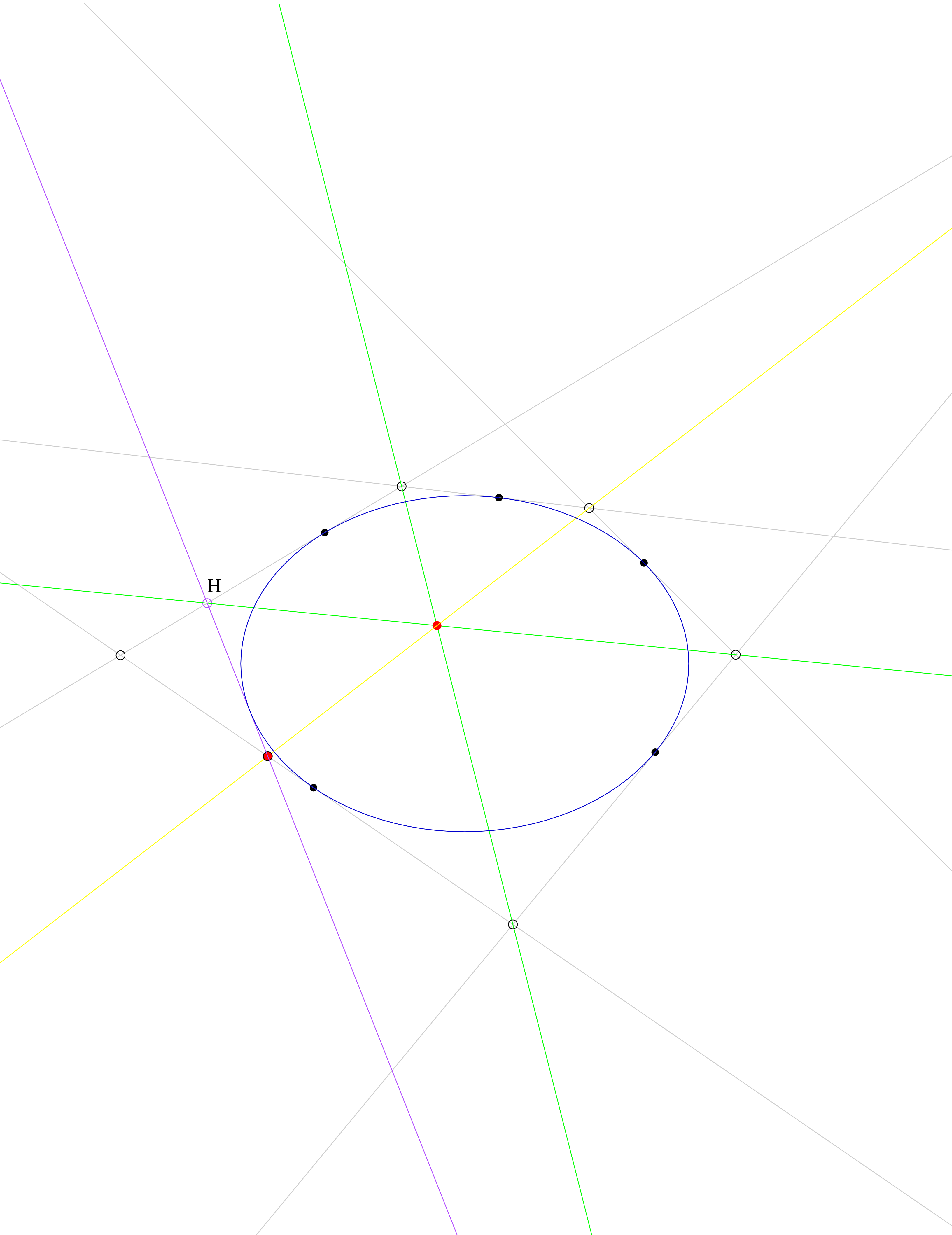
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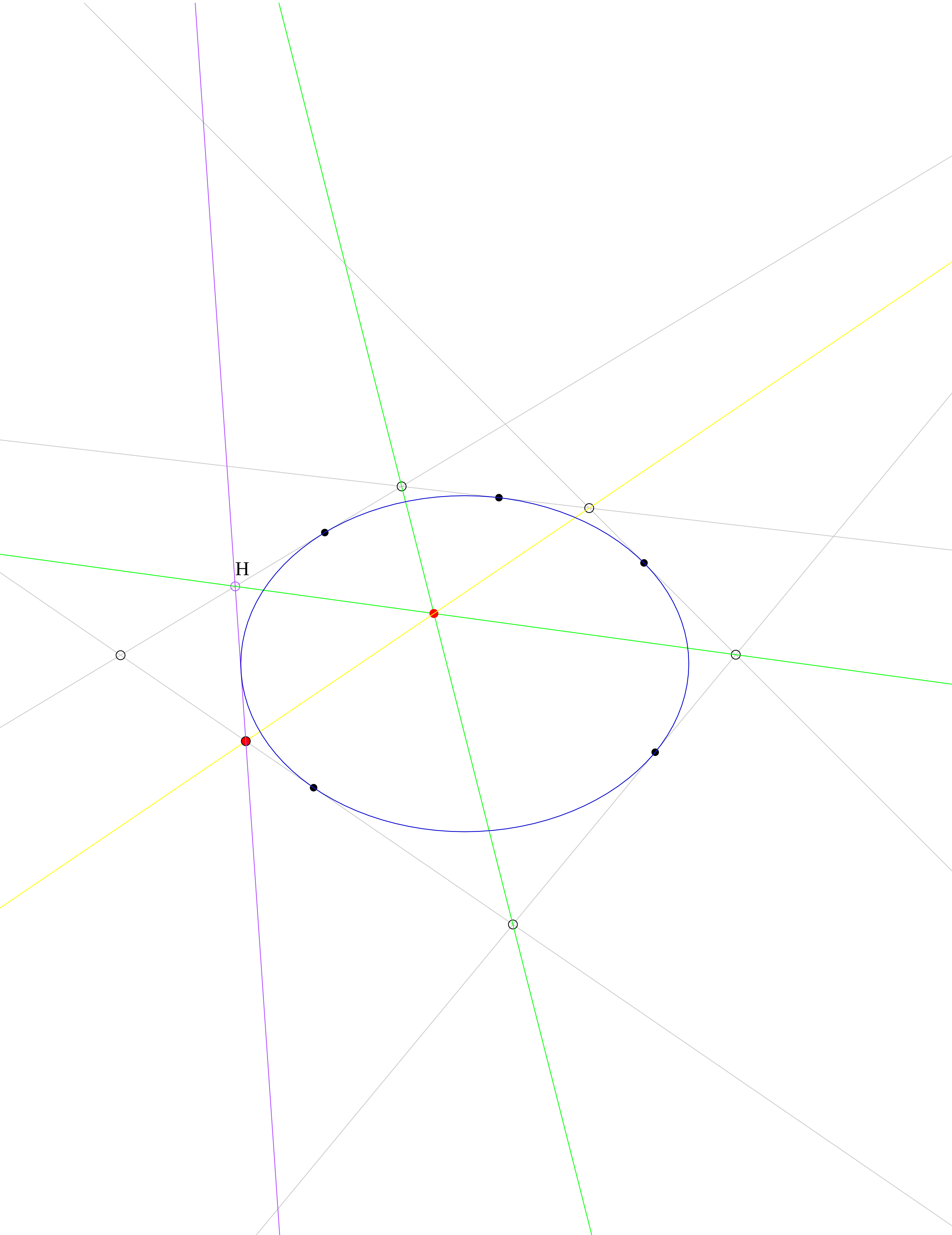
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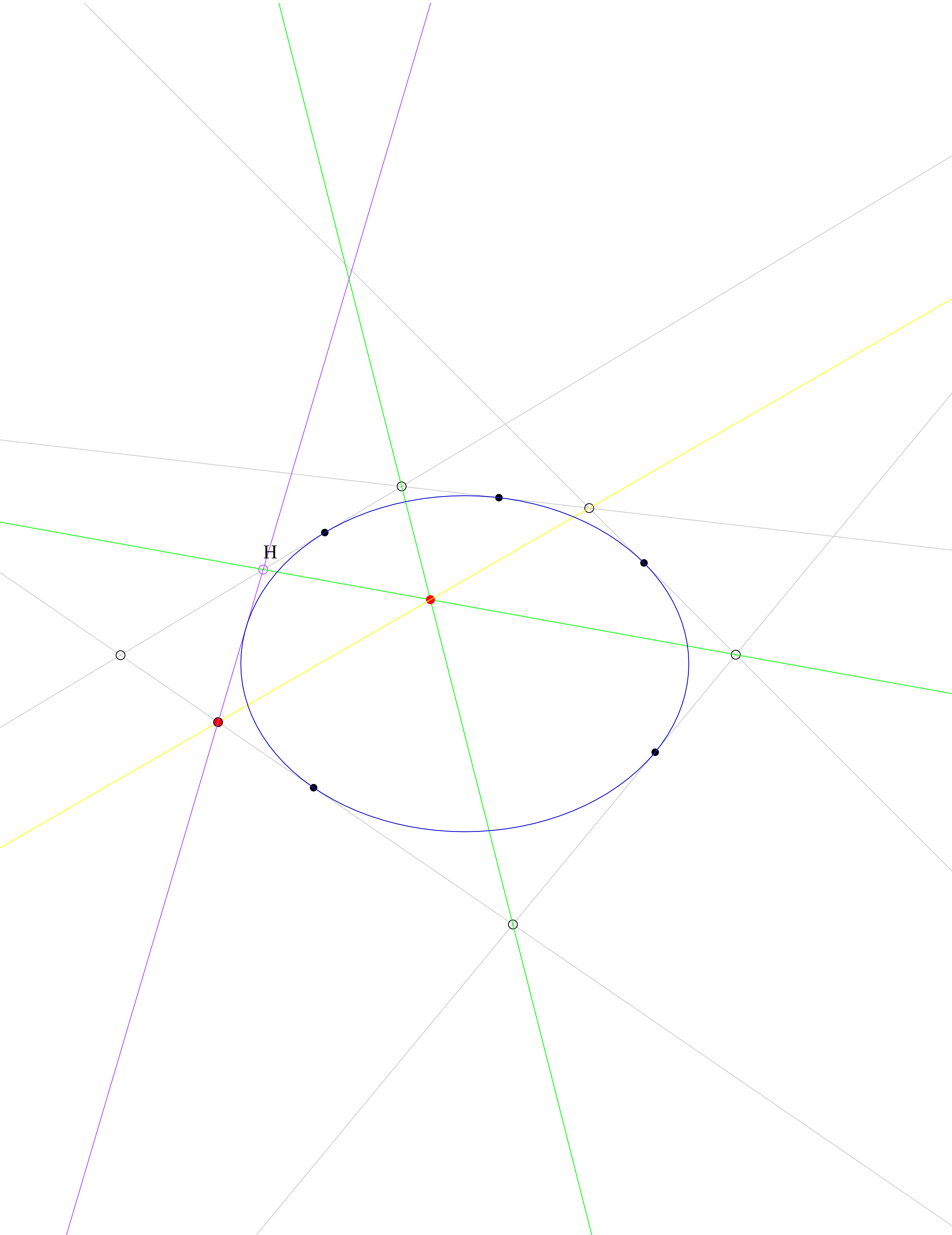
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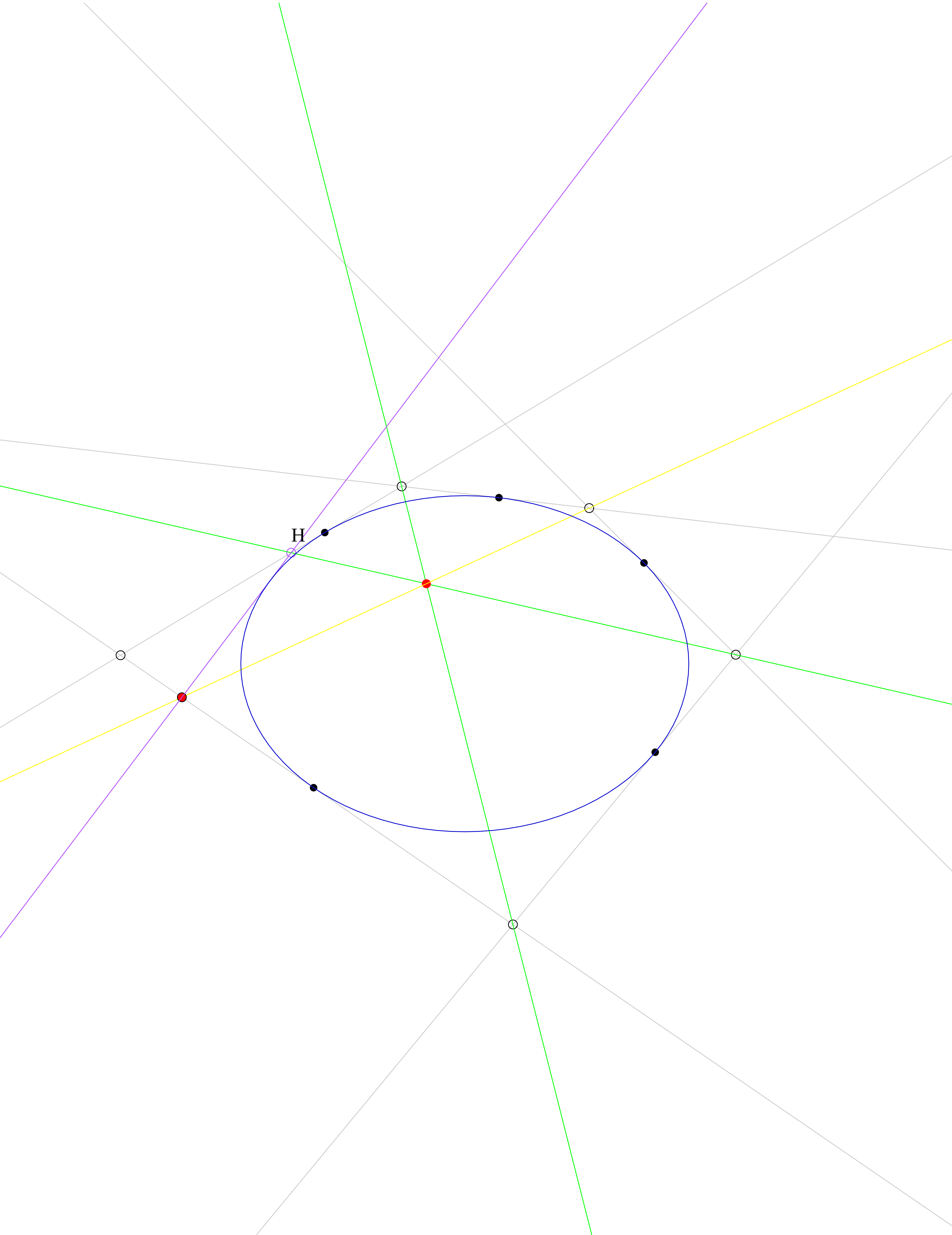
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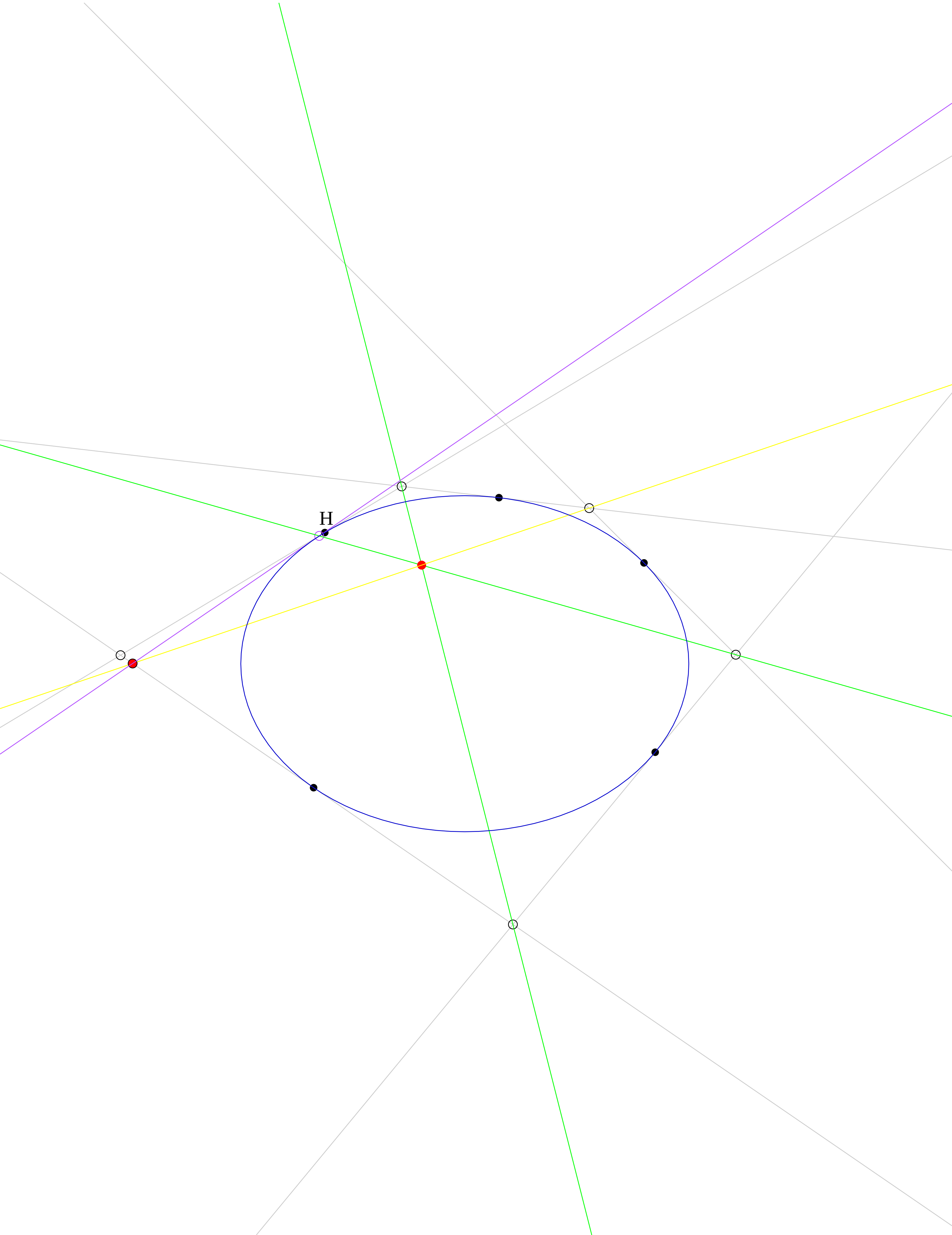
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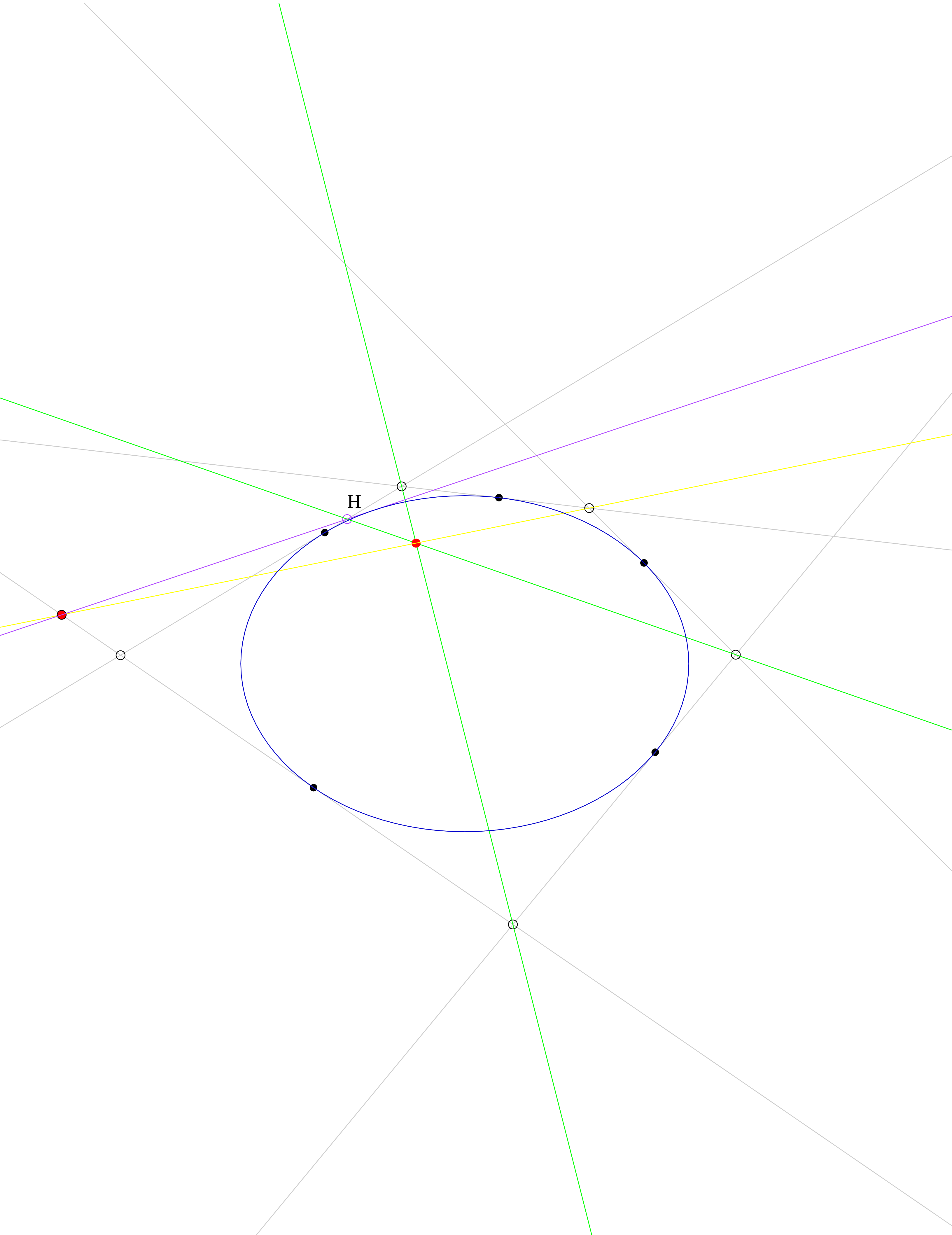
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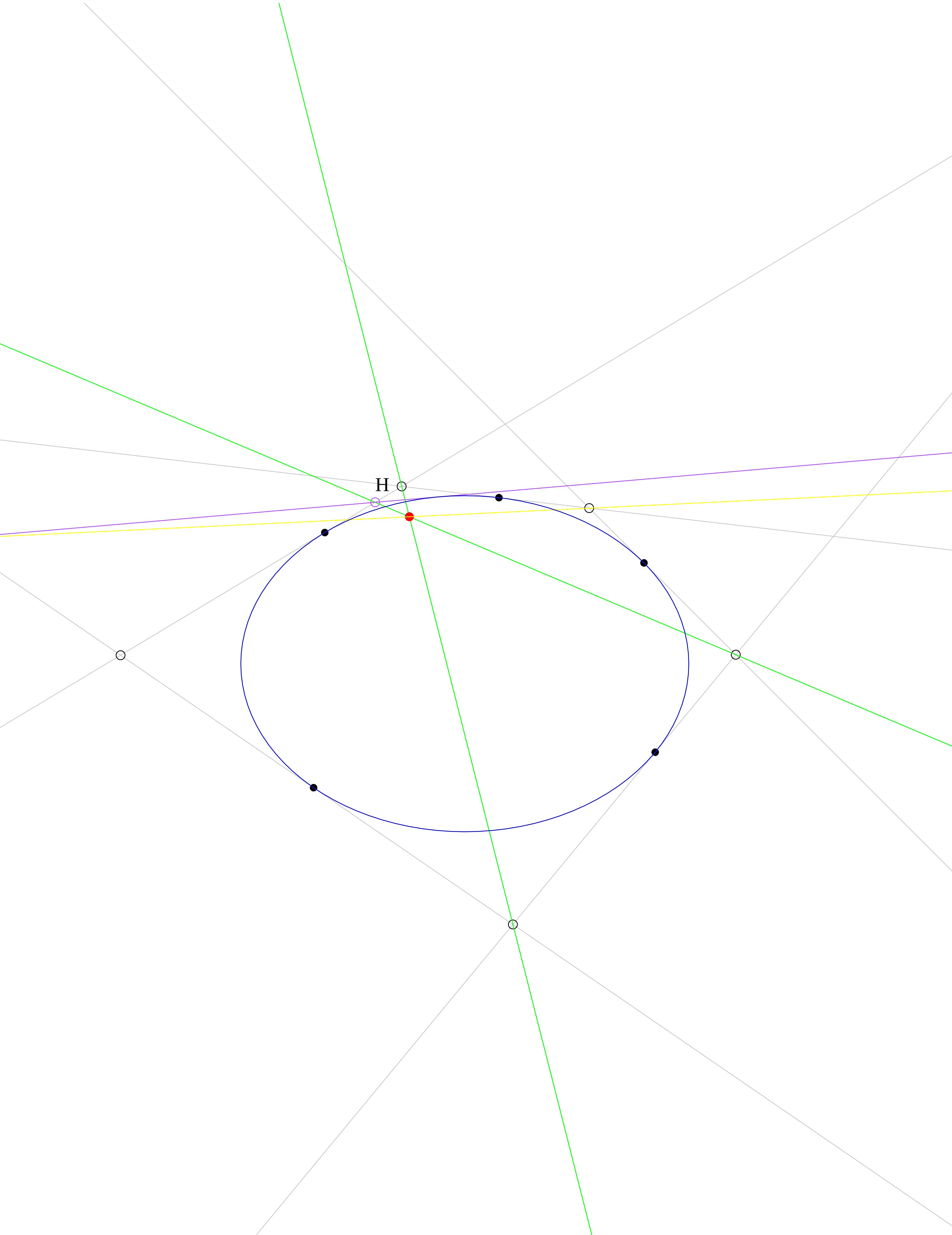
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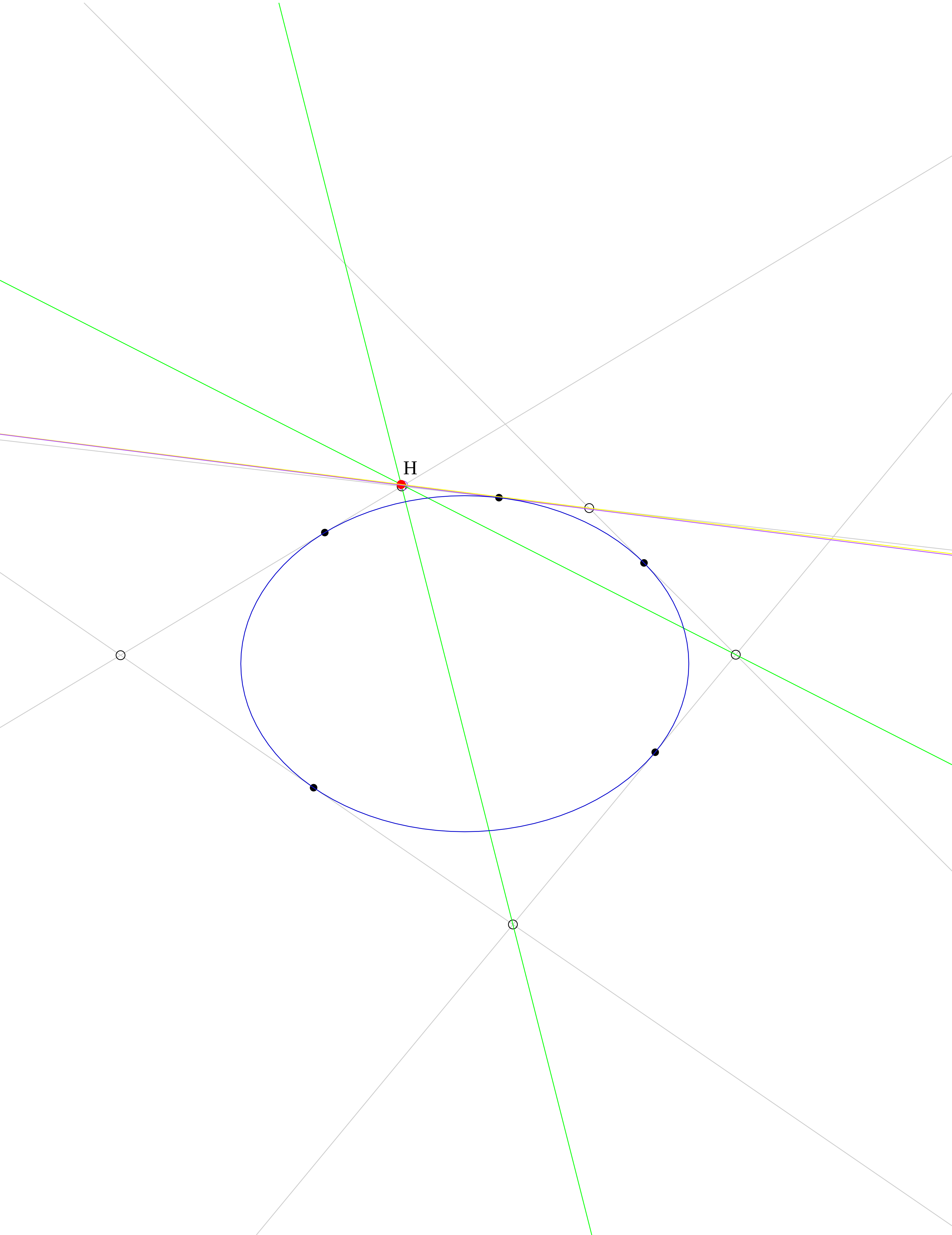
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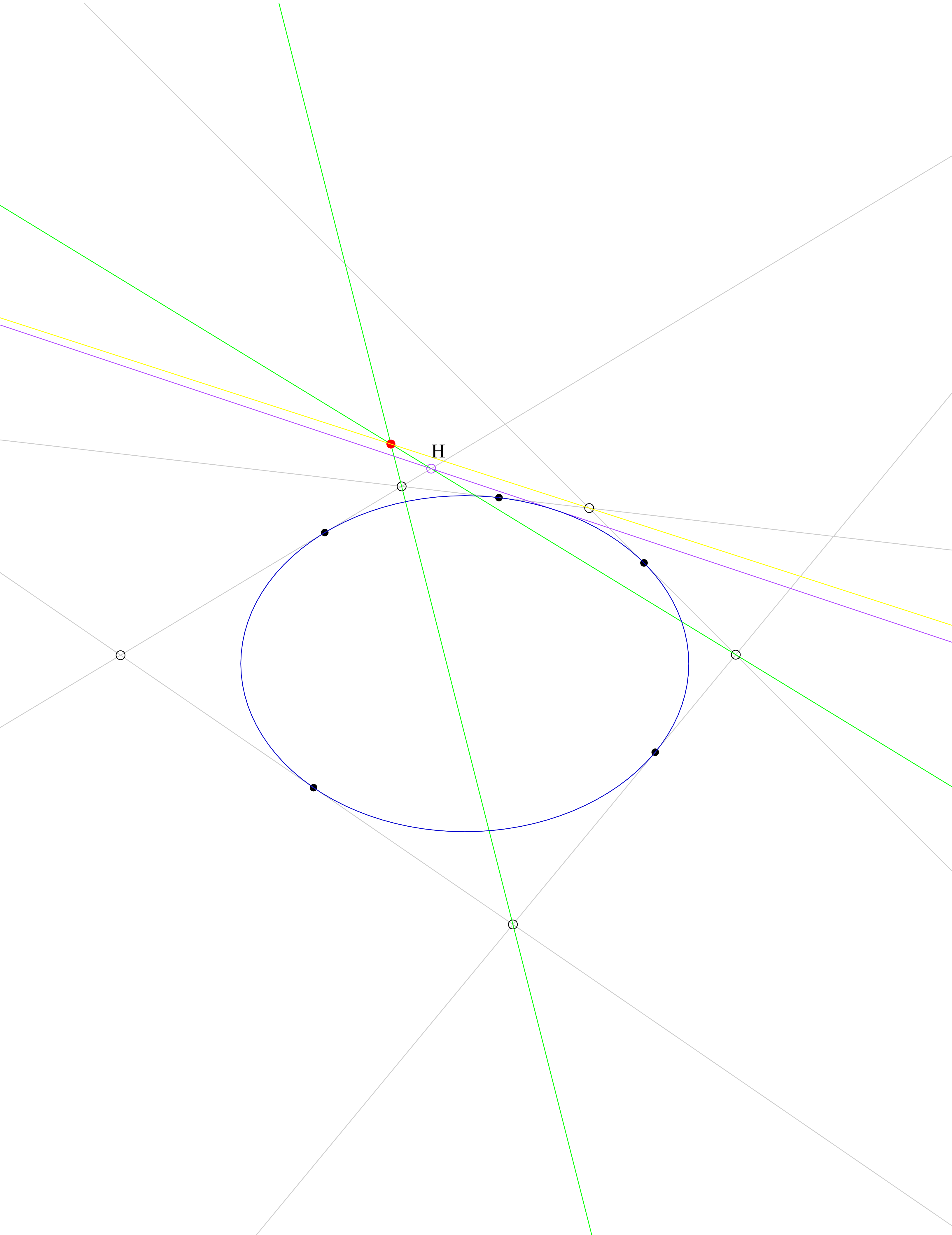
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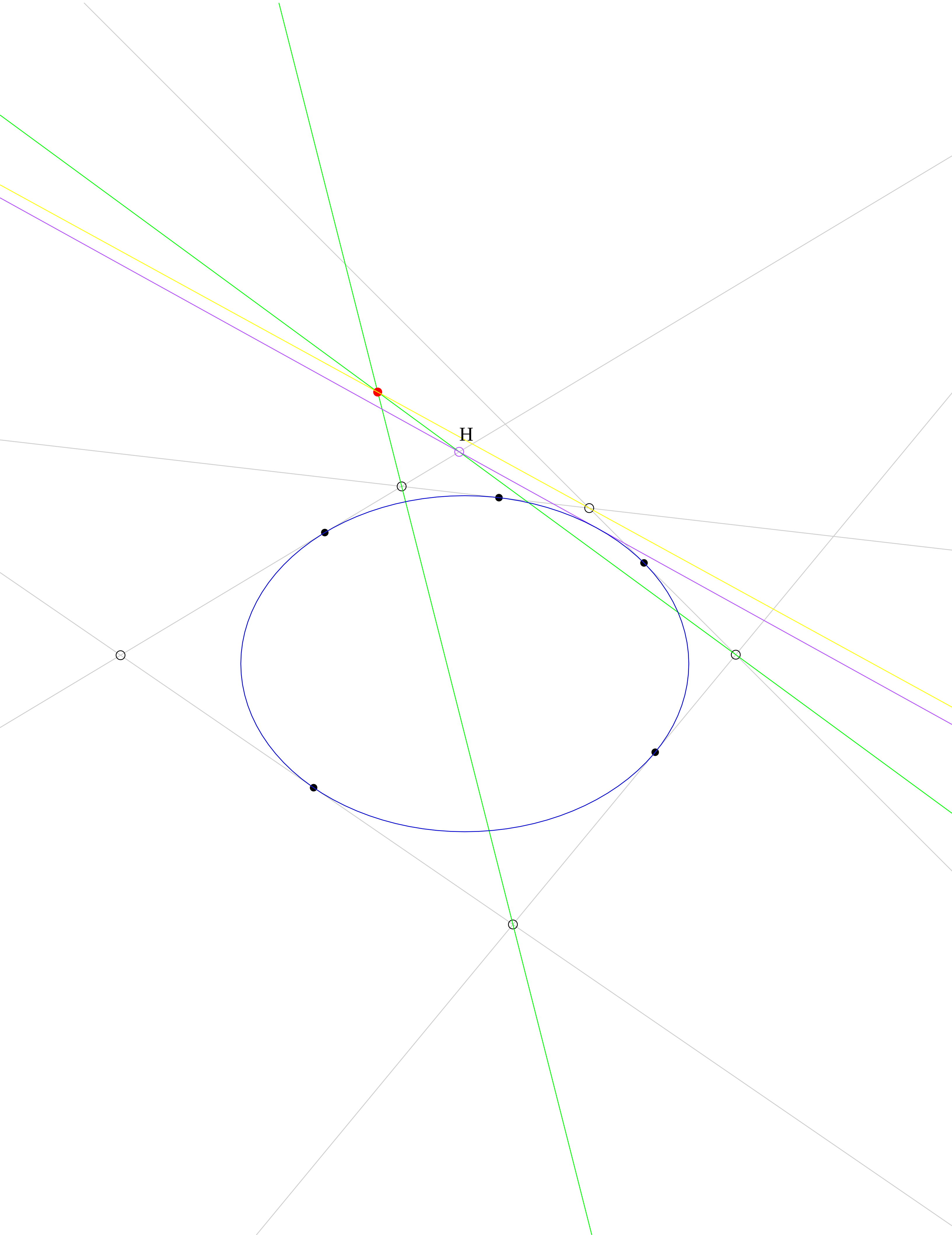
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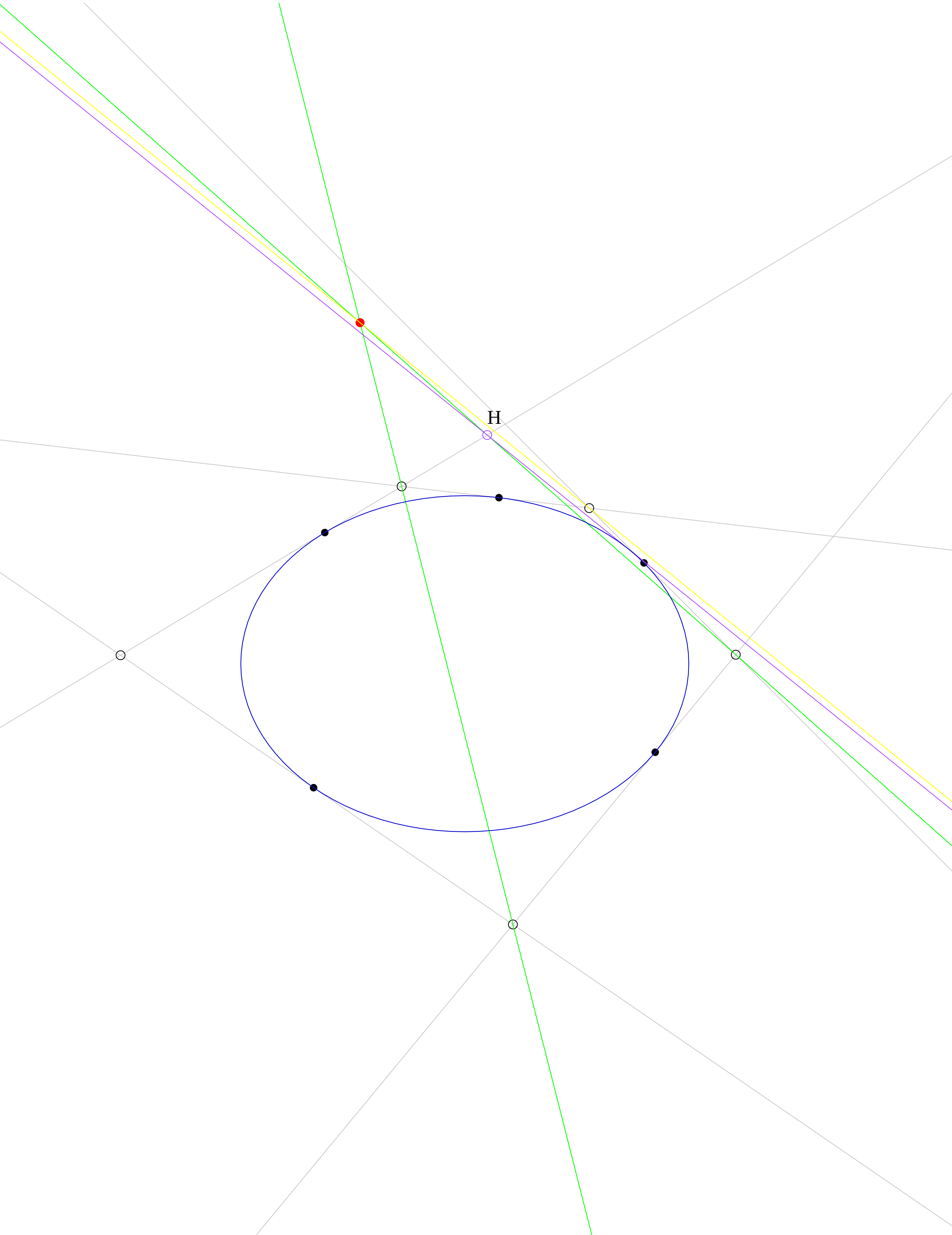
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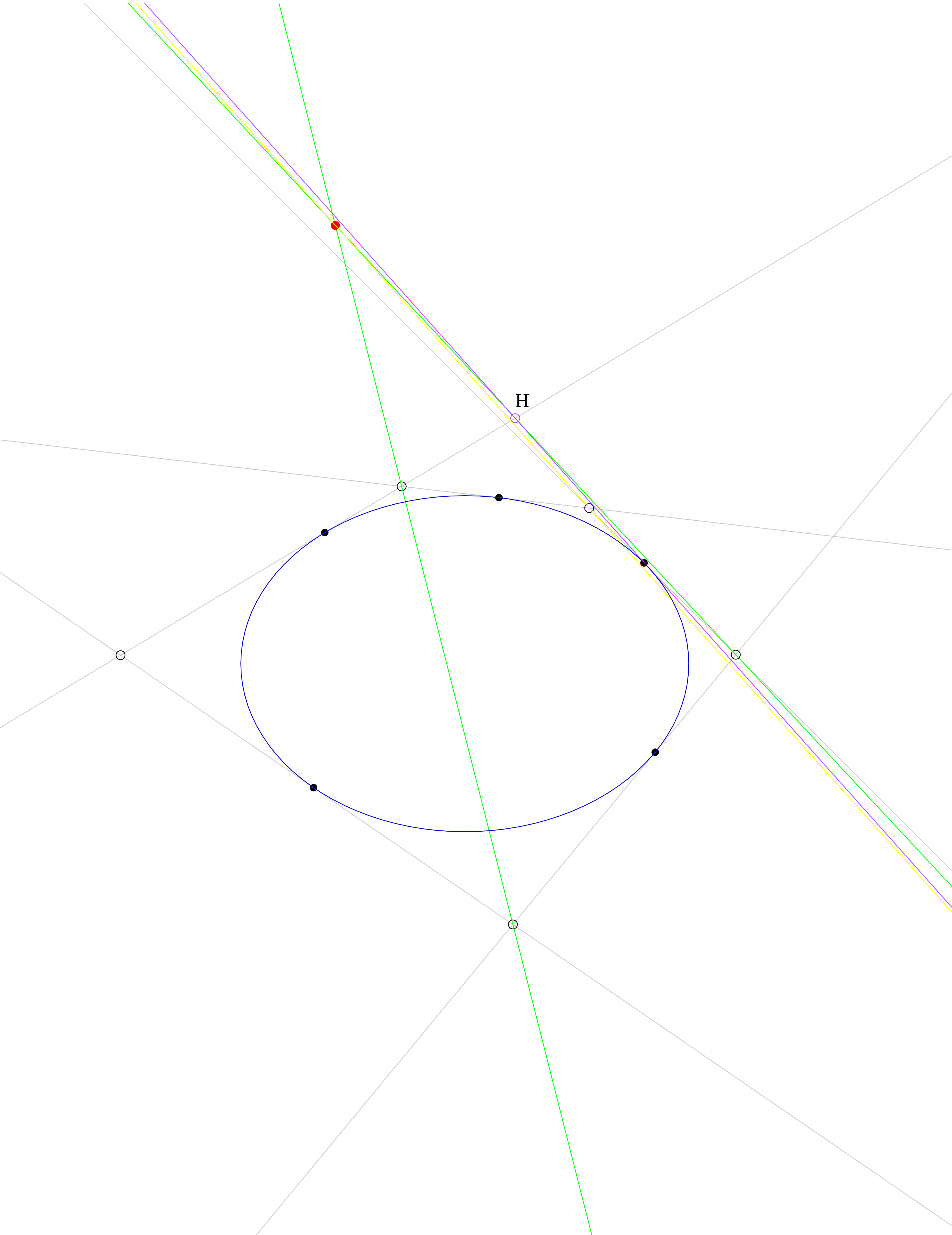
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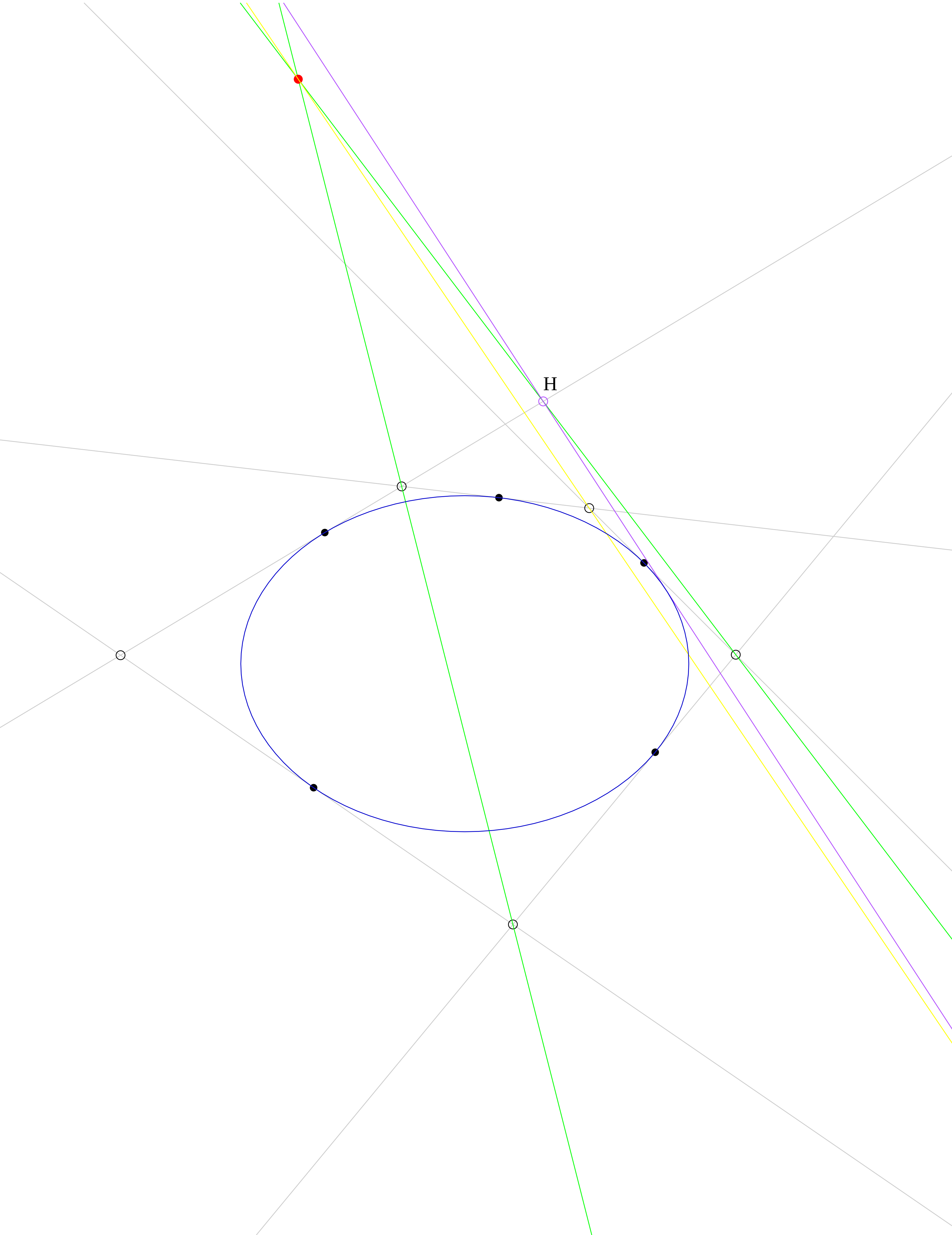
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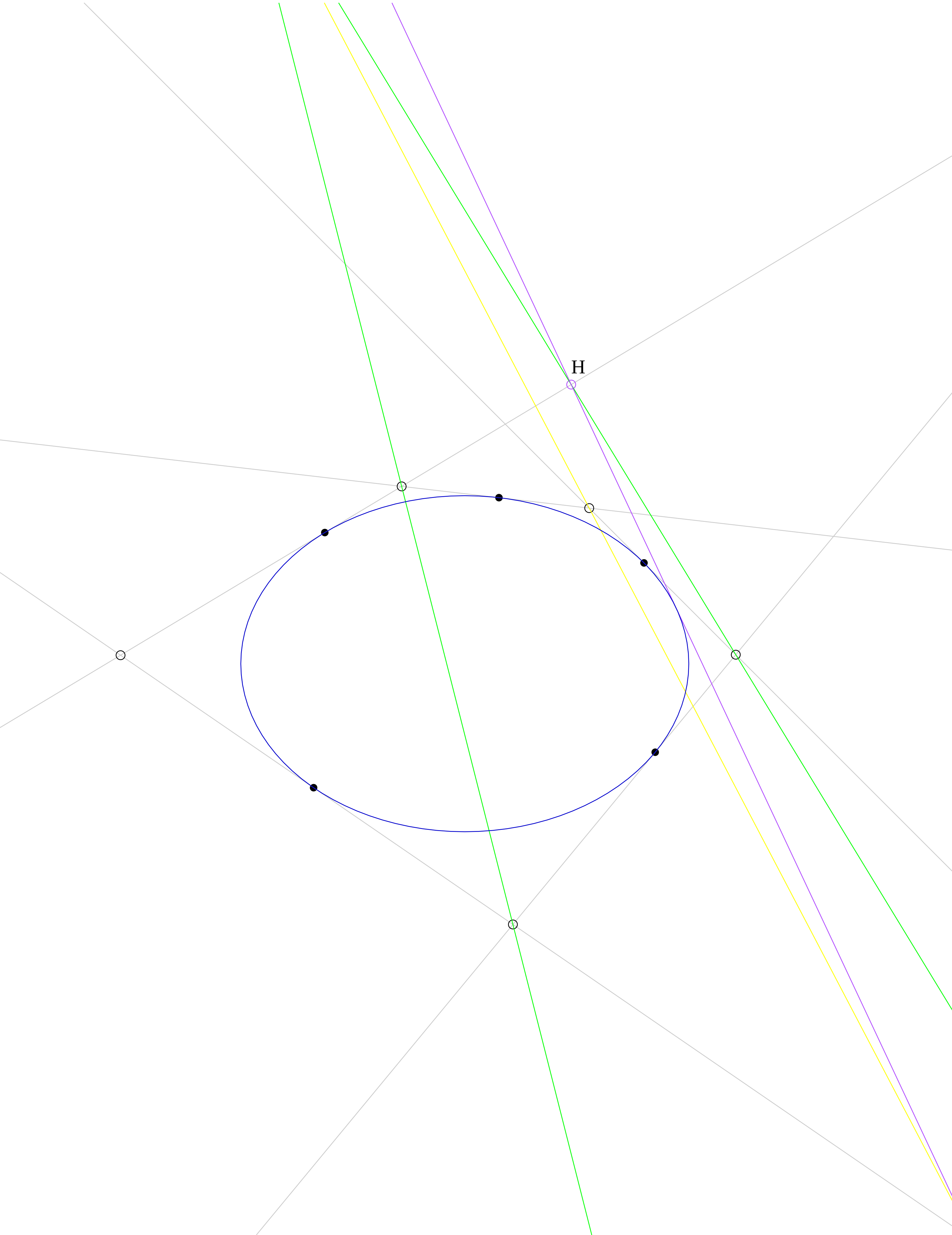
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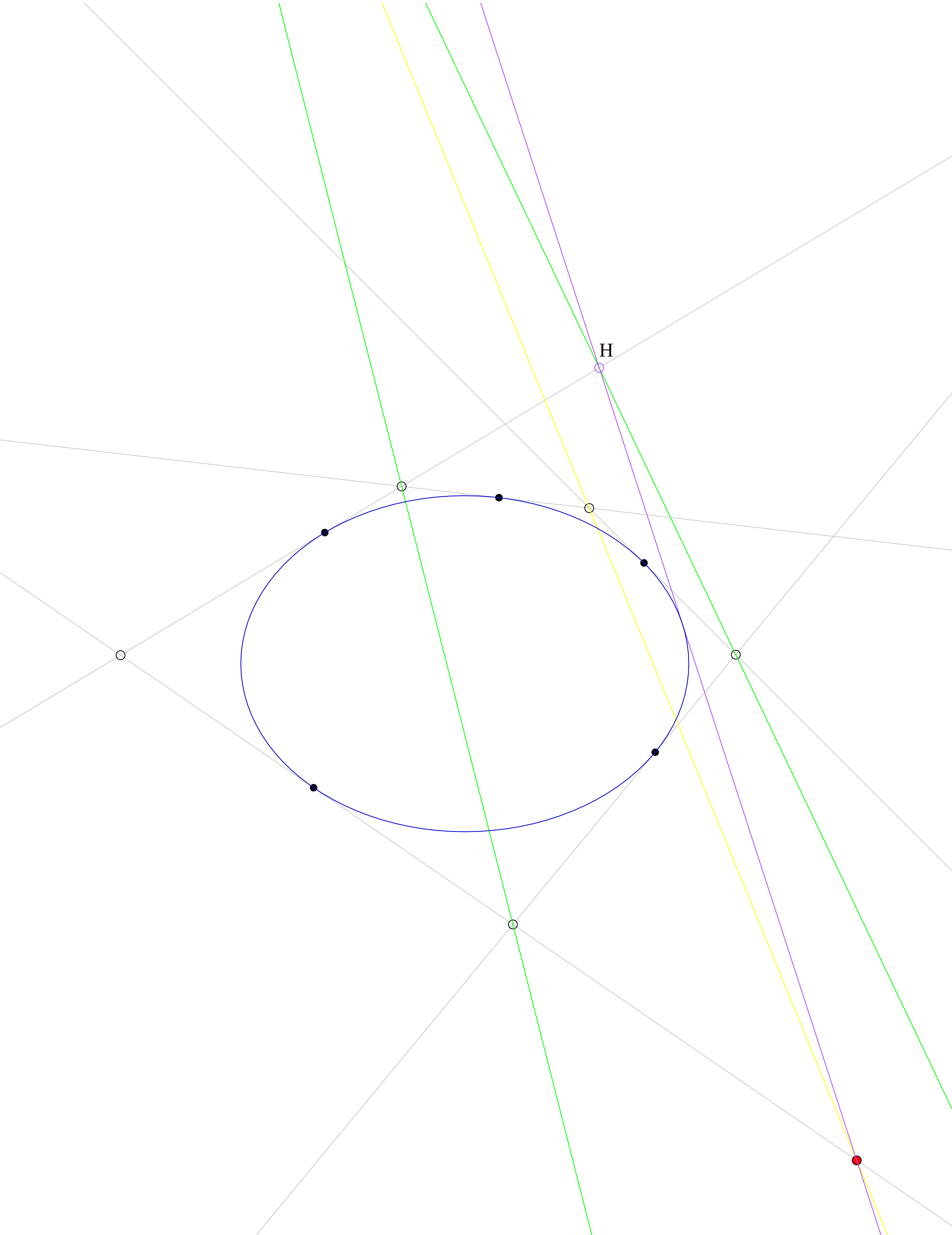
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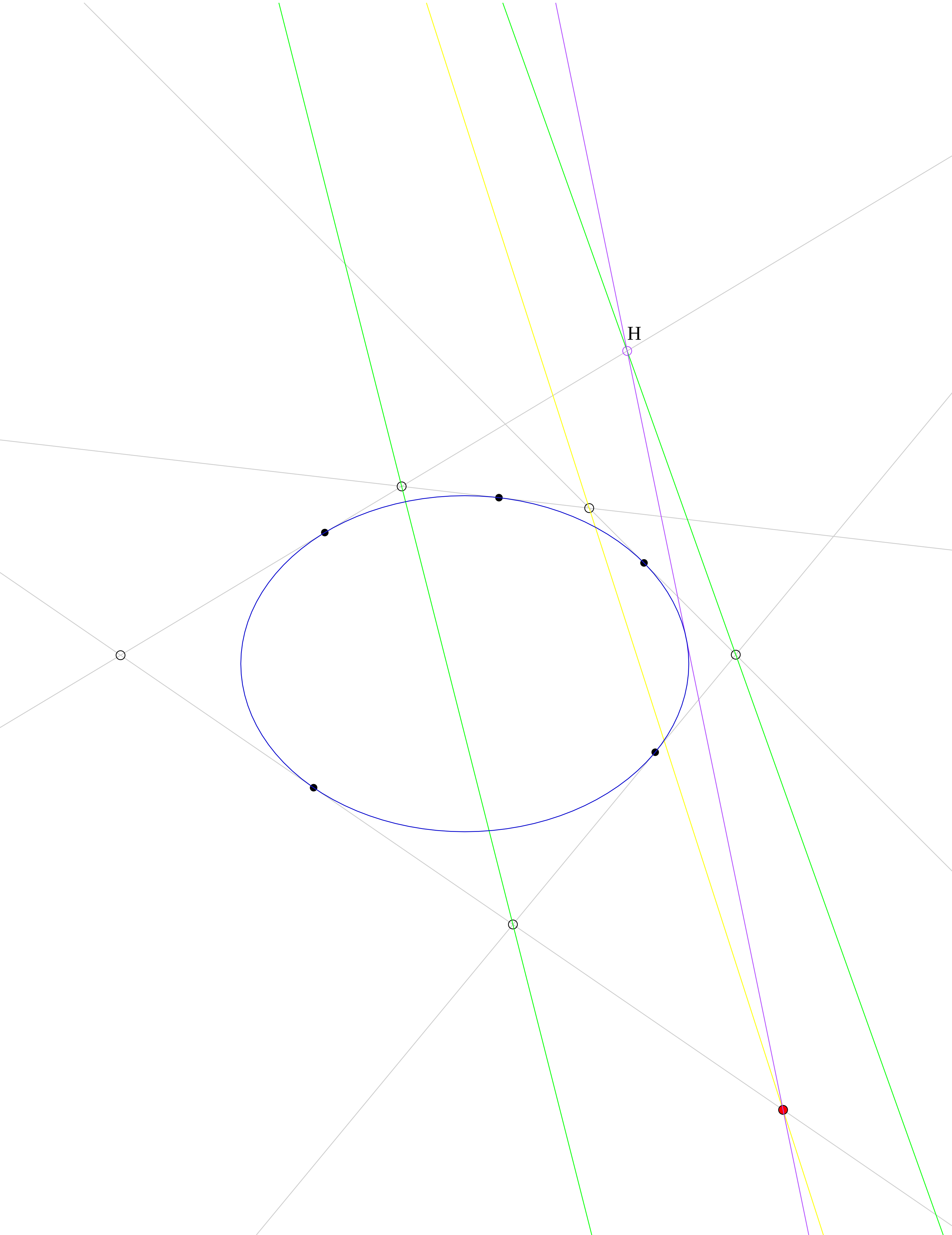
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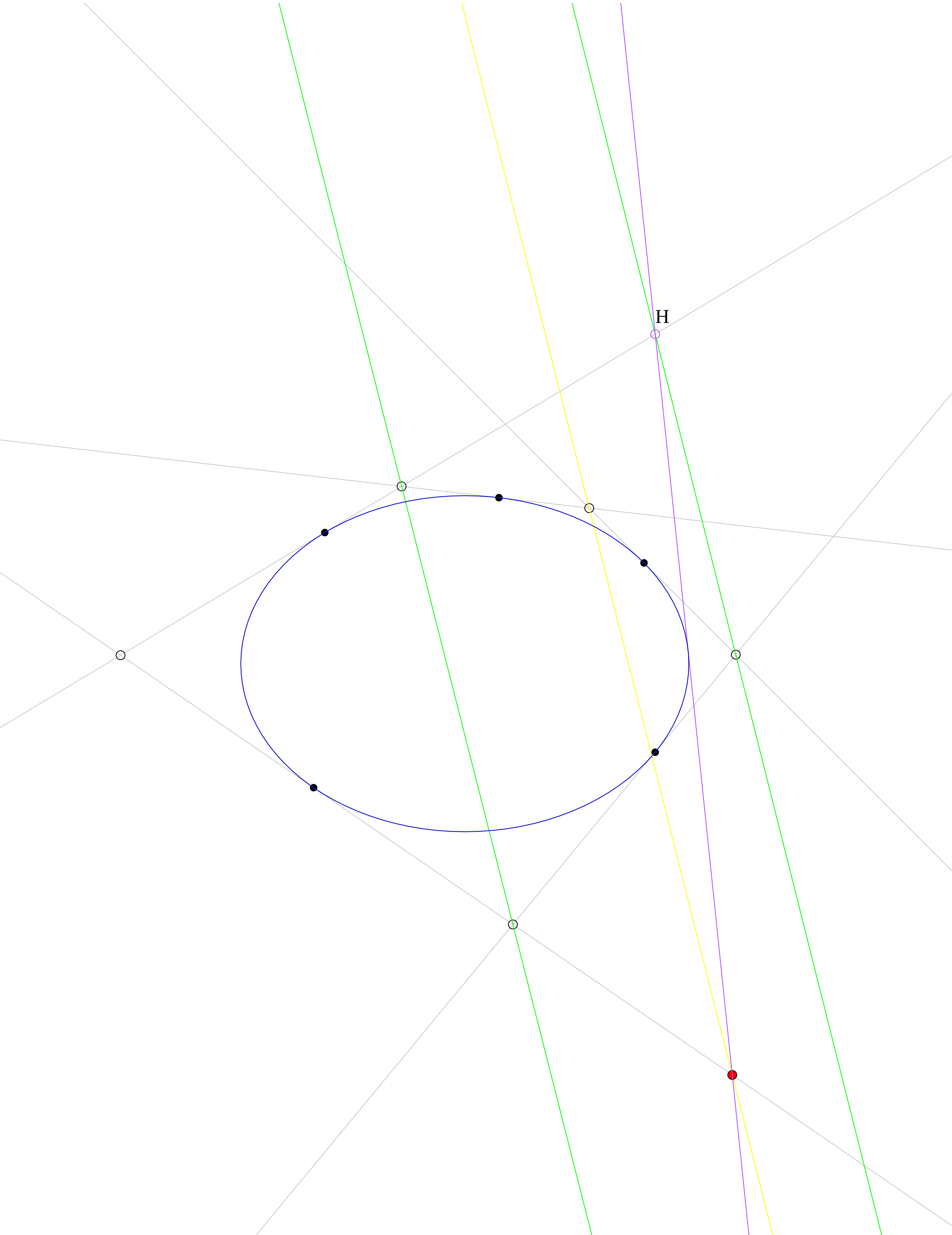
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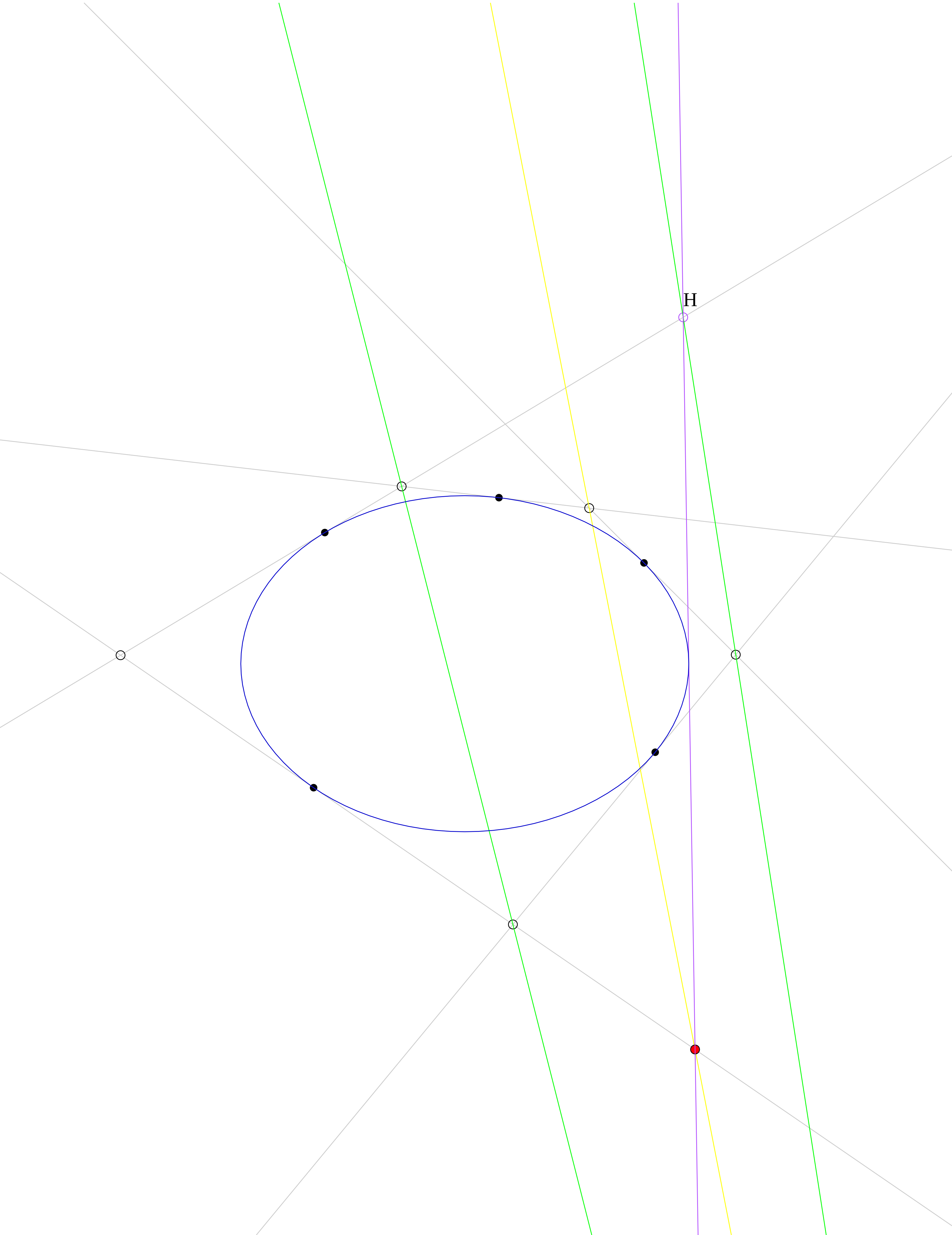
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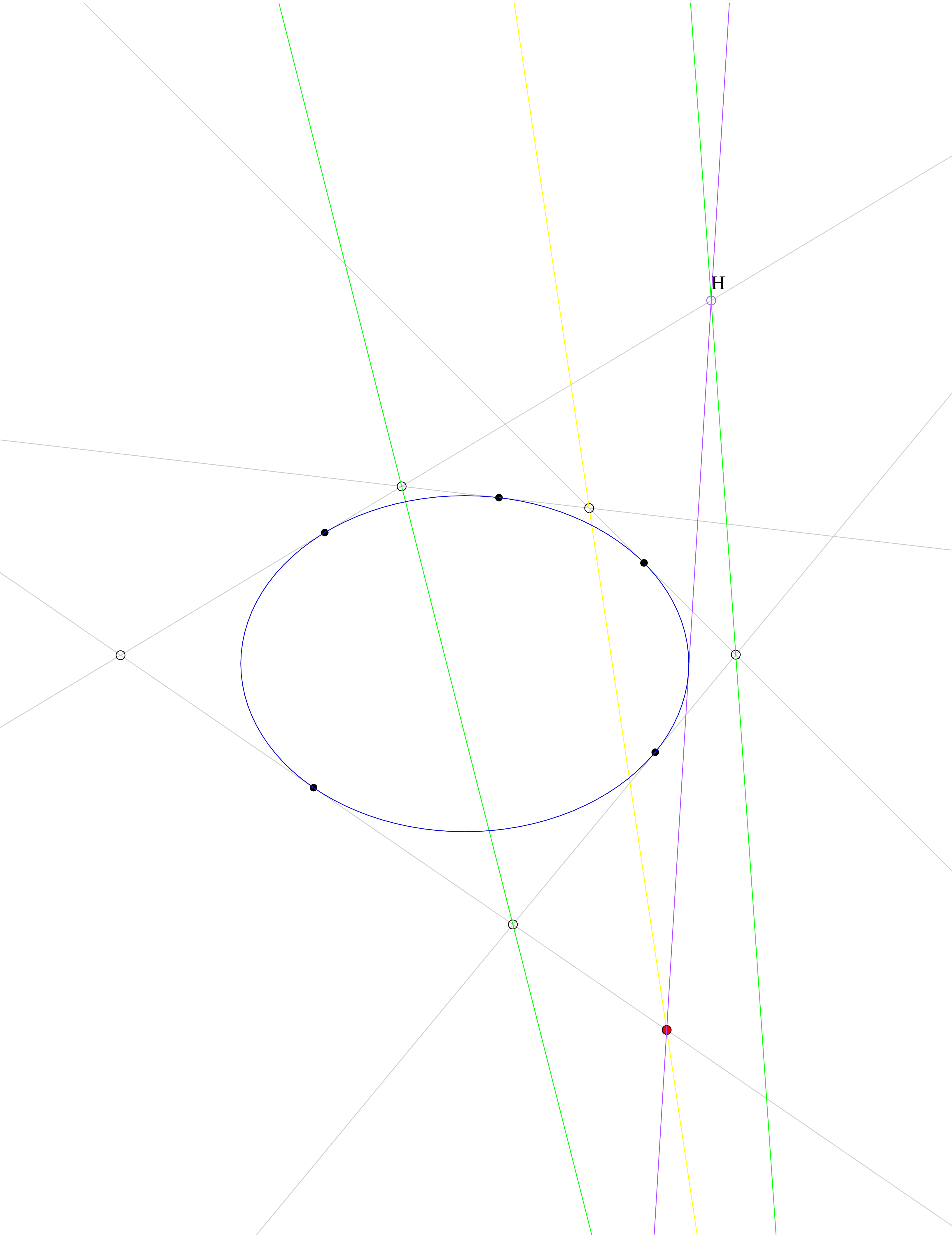
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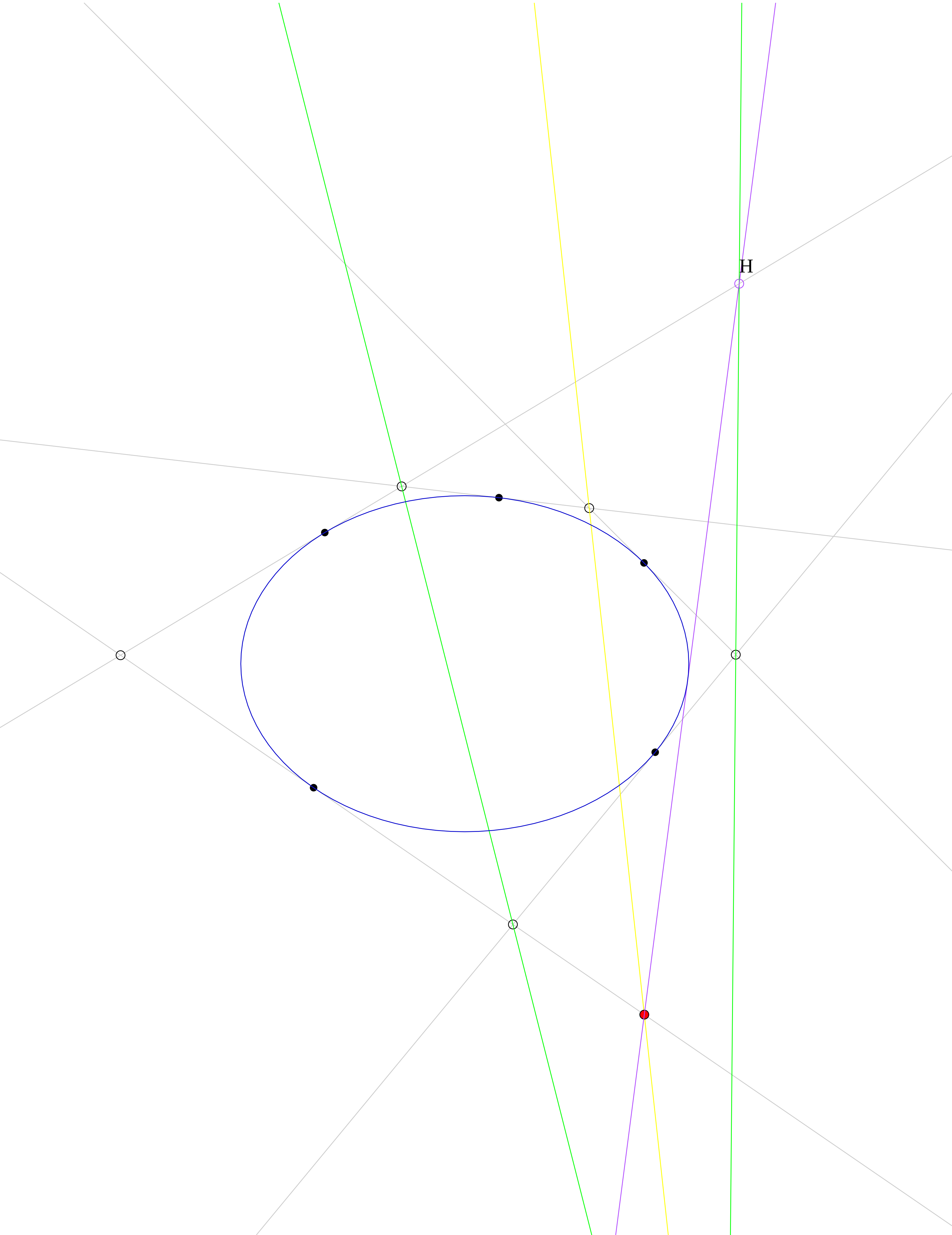
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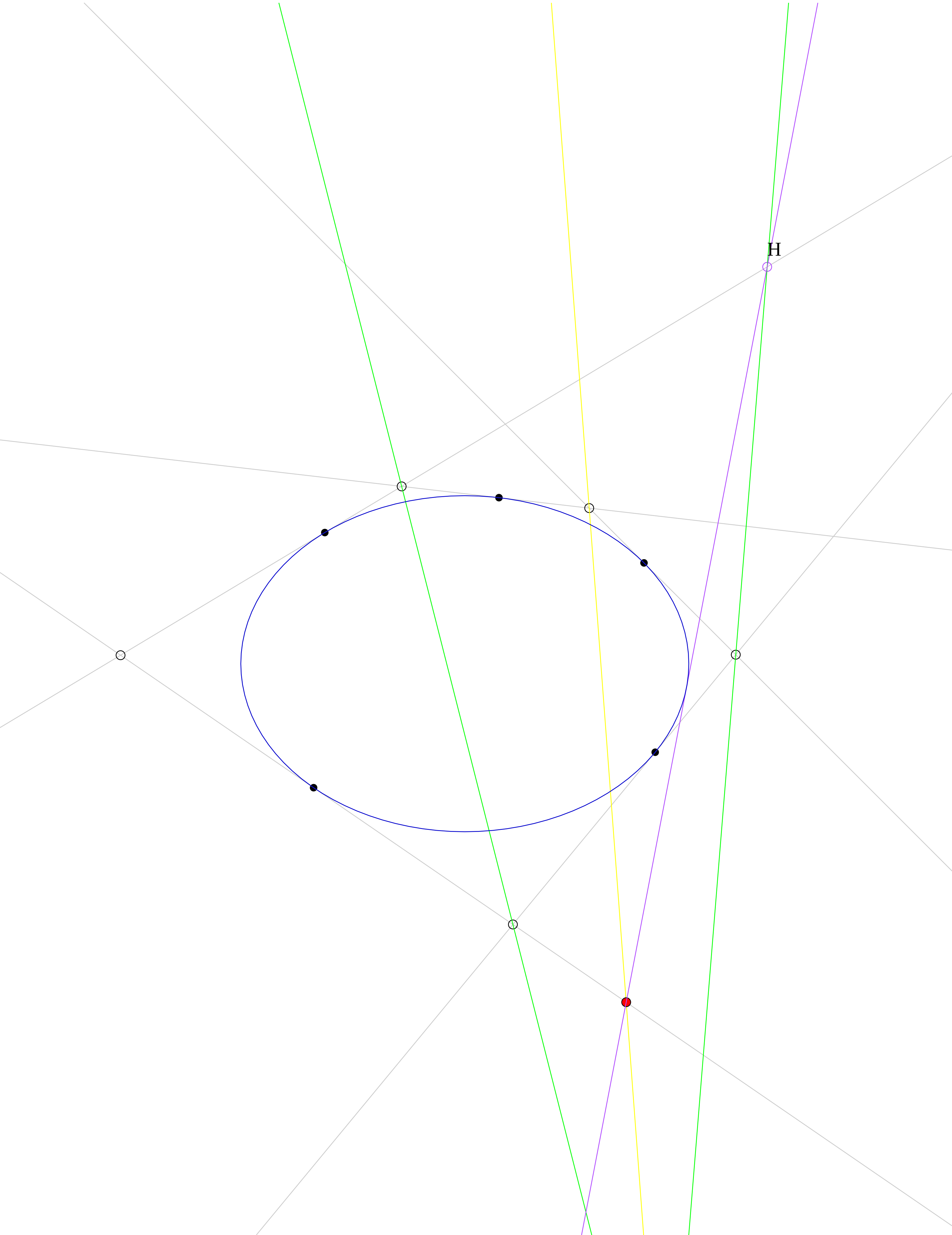
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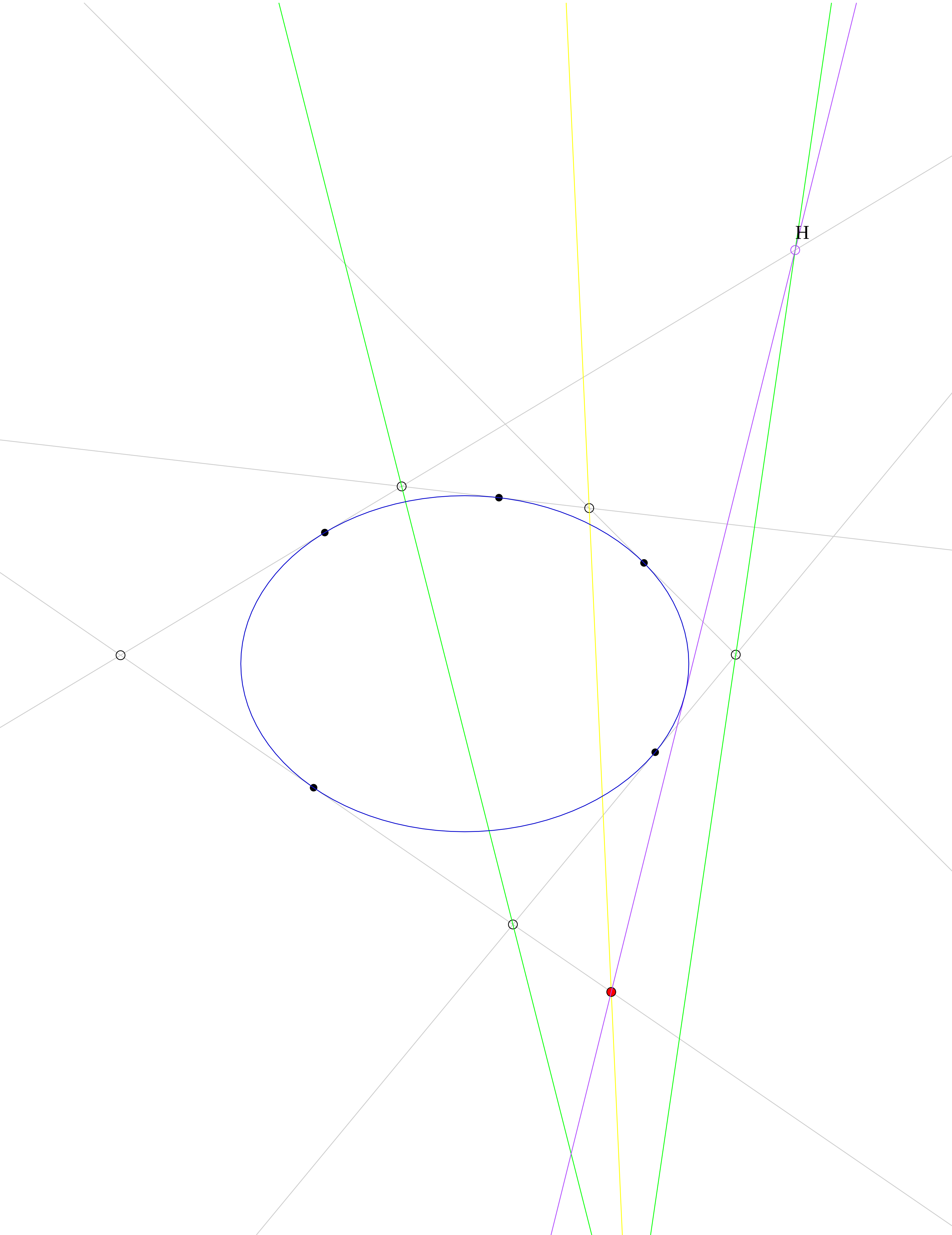
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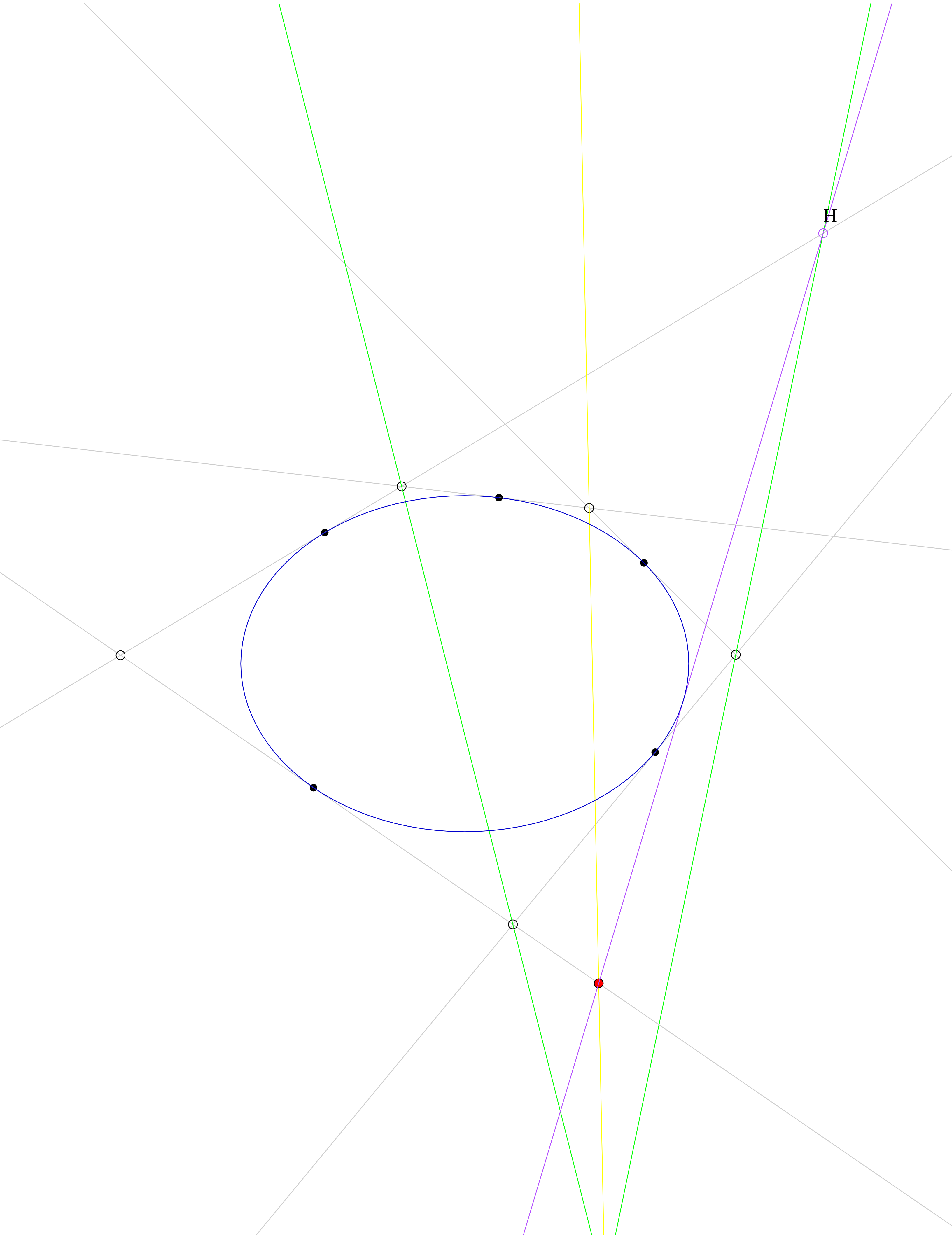
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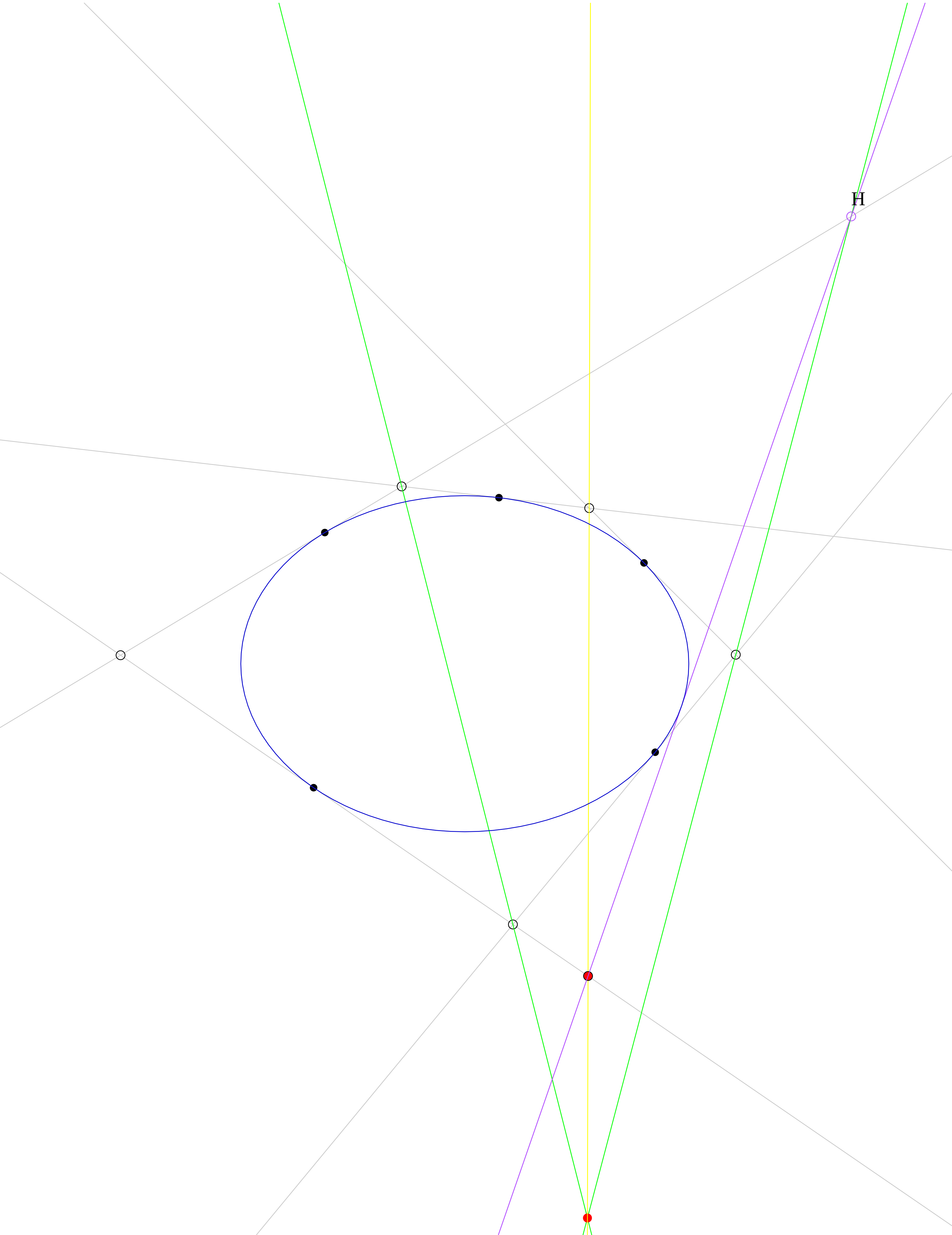
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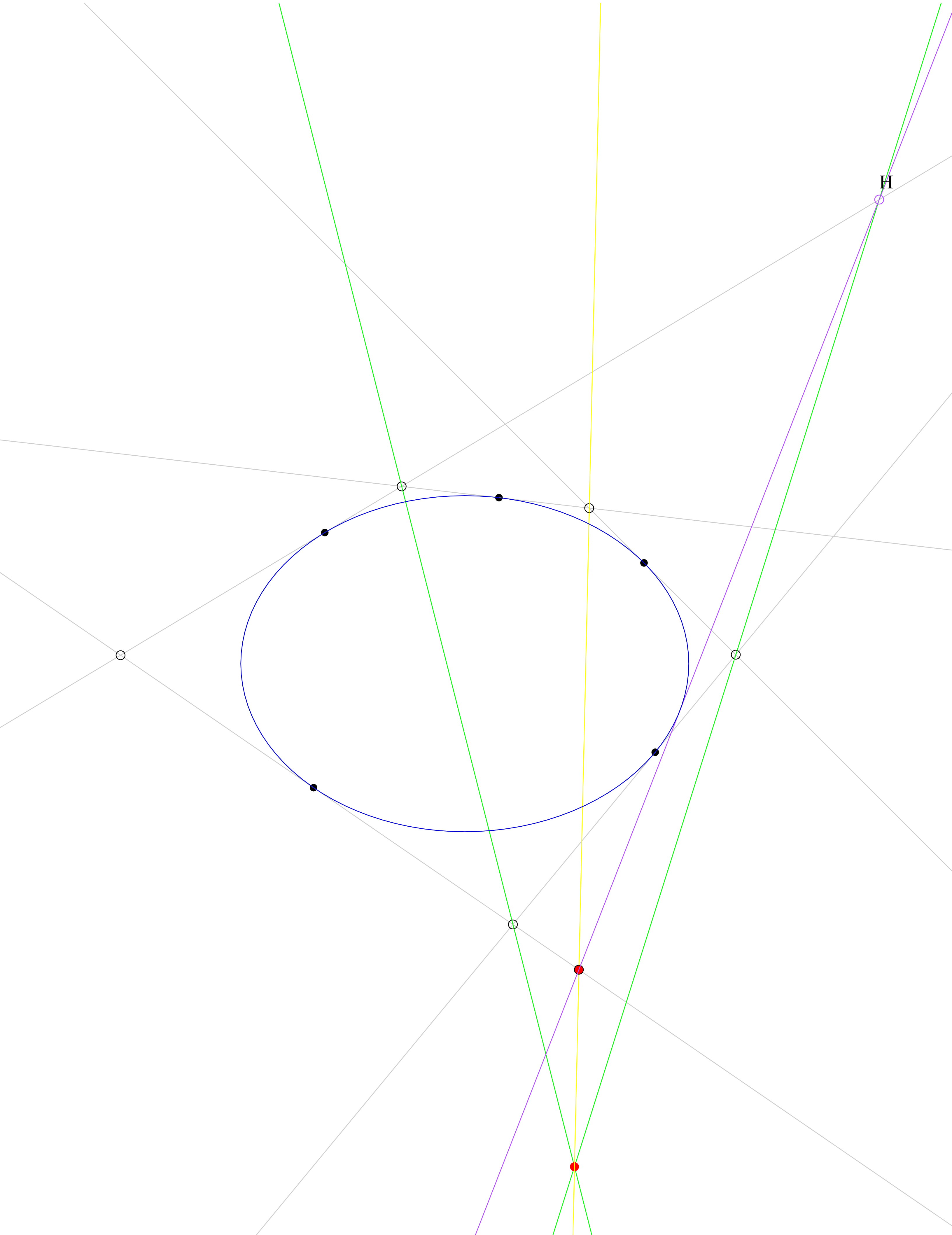
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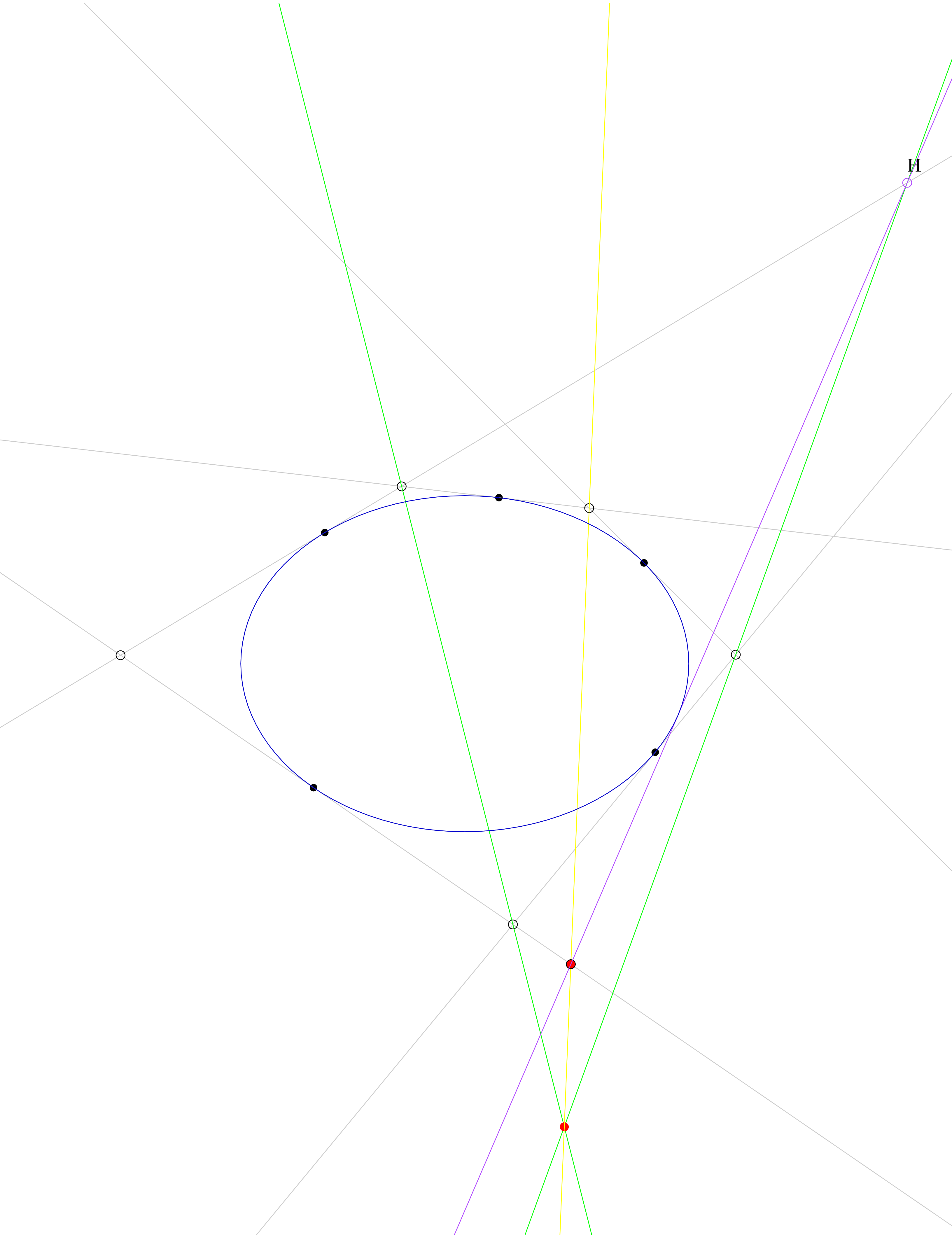
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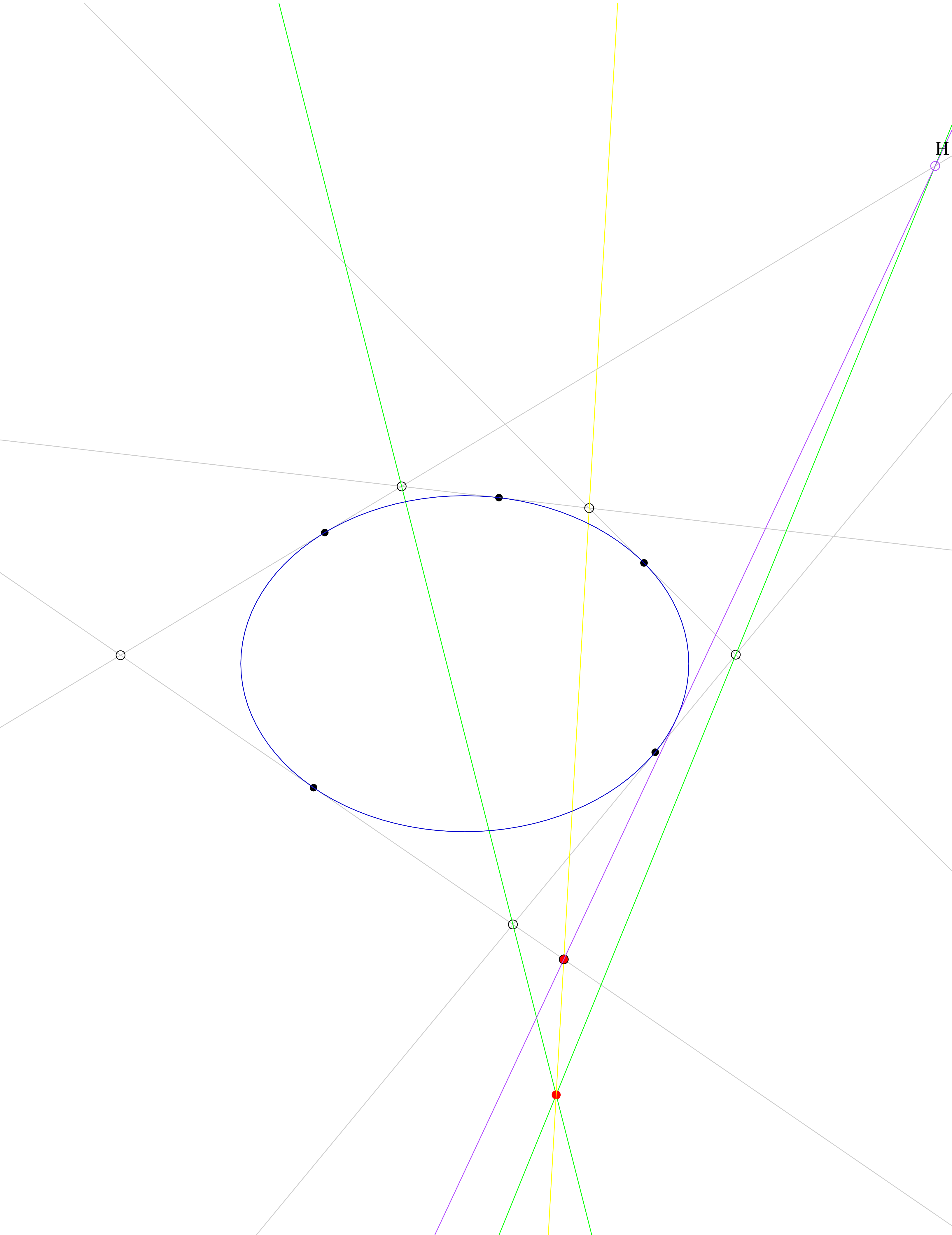
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