Linear Programming

Math 308, Winter 2003

LAI, Chin Man Eric
38378006

Definition

Active Set: Active set for a feasible point x is the set of inequalities with zero slack.

Cone: A cone spanned by a set of vectors is the set of all possible positive linear combination of them.

Gradient: A gradient of a line or a plane is the normal vector of its equation.

Here is a basic setting of a 2 variables Linear Program:

Objective Function:

Max: $c_1 x_1 + c_2 x_2$

Constraints:	Slack variable:
$a_{11} x_1 + a_{12} x_2 <= b_1$	$x_3 = b_1 - a_{11} x_1 - a_{12} x_2$
$a_{21} x_1 + a_{22} x_2 <= b_2$	$x_4 = b_2 - a_{21} x_1 - a_{22} x_2$

a_{ij}, x_i are real number.

$x_i >= 0$

Introduction

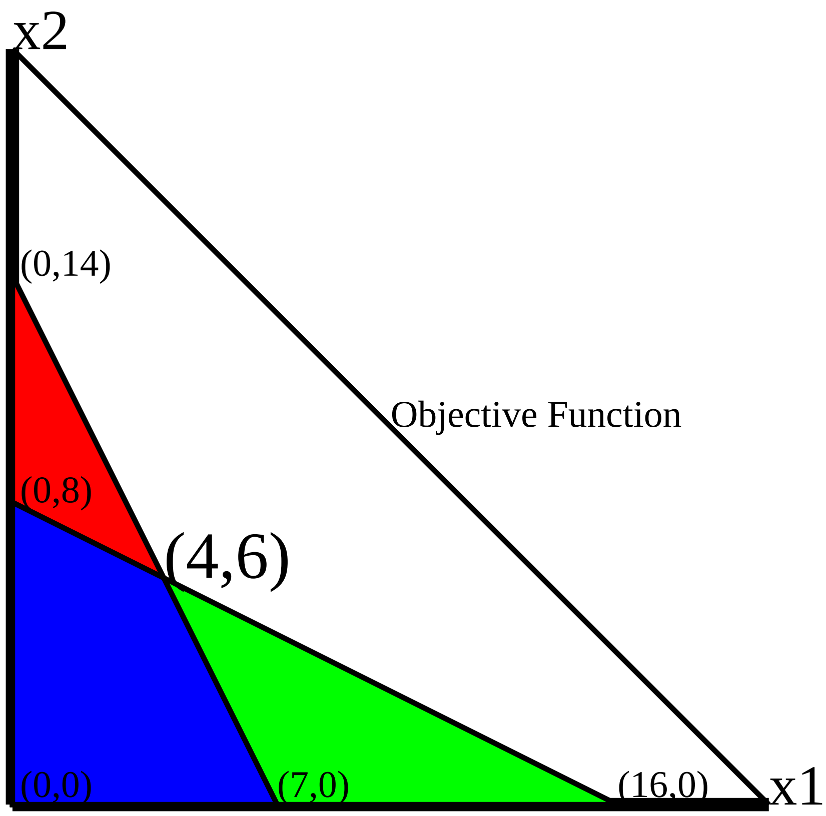
This project aims to graphically illustrate the principle of Dictionary method in Linear Programming (LP) in both 2-D and 3-D situation. Besides that, this postscript file also introduces how one can obtain the vertex, at which the optimal solution is located, by the normal vector of the objective function and the constraints. The audiences of this project are expected to have basic understanding on (LP) and its terminology.

The principle of the dictionary method is to achieve the maximum value of the objective function steps by steps through a finite number of iterations. Each iteration saturates one of the stack variables while maintaining the feasibility of the solution. It is closely related to geometry because each iteration, the basic solution is shifting from vertex to vertex until it reaches the vertex holding the optimal solution.

The method of finding the vertex of the optimal solution by normal vector based on the following claim:

> A point P, which is feasible, is an optimal solution to the LP if and only if the gradient of the objective function lies in the cone spanned by the gradient of the active constraint at P.

As a result, if we test the gradient of the objective function with the cones of gradient of different active sets at different vertexes, we can easily obtain the optimal solution with no calculations.



Graphical Representation of a 2-D LP

This page shows the graphical representation of a 2-D LP

Objective Function:

Max: x1 + x2

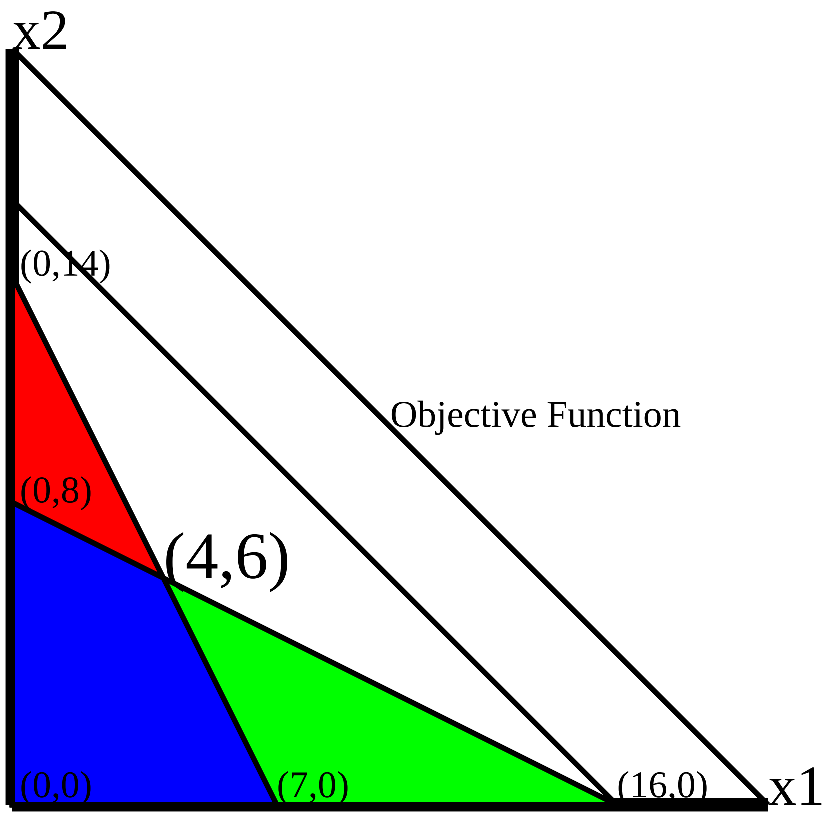
Constraints: Slack variable:

(1) 2 x1 + x2 <= 14 x3 = 14 - 2 x1 - x2 (red + blue)

(2) x1 + 2 x2 <= 16 x4 = 16 - x1 - 2 x2 (green + blue)

aij, xi are real number. x1, x2 >= 0

Optimal Solution (4,6).



Graphical Representation of a 2-D LP

This page shows the graphical representation of a 2-D LP

Objective Function:

Max: x1 + x2

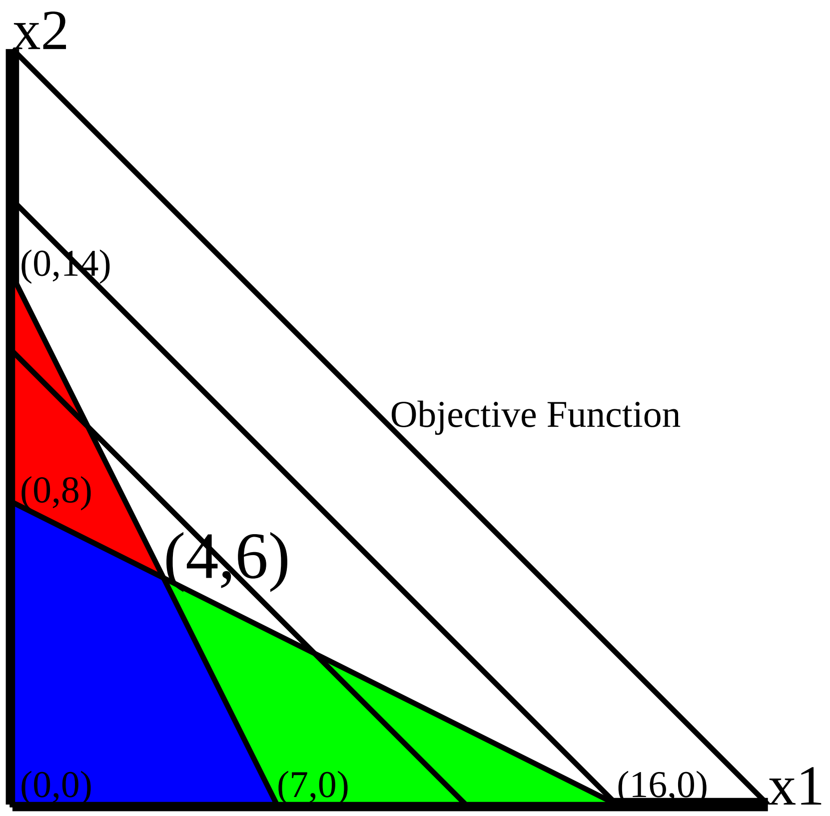
Constraints: Slack variable:

(1) 2 x1 + x2 <= 14 x3 = 14 - 2 x1 - x2 (red + blue)

(2) x1 + 2 x2 <= 16 x4 = 16 - x1 - 2 x2 (green + blue)

aij, xi are real number. x1, x2 >= 0

Optimal Solution (4,6).



Graphical Representation of a 2-D LP

This page shows the graphical representation of a 2-D LP

Objective Function:

Max: x1 + x2

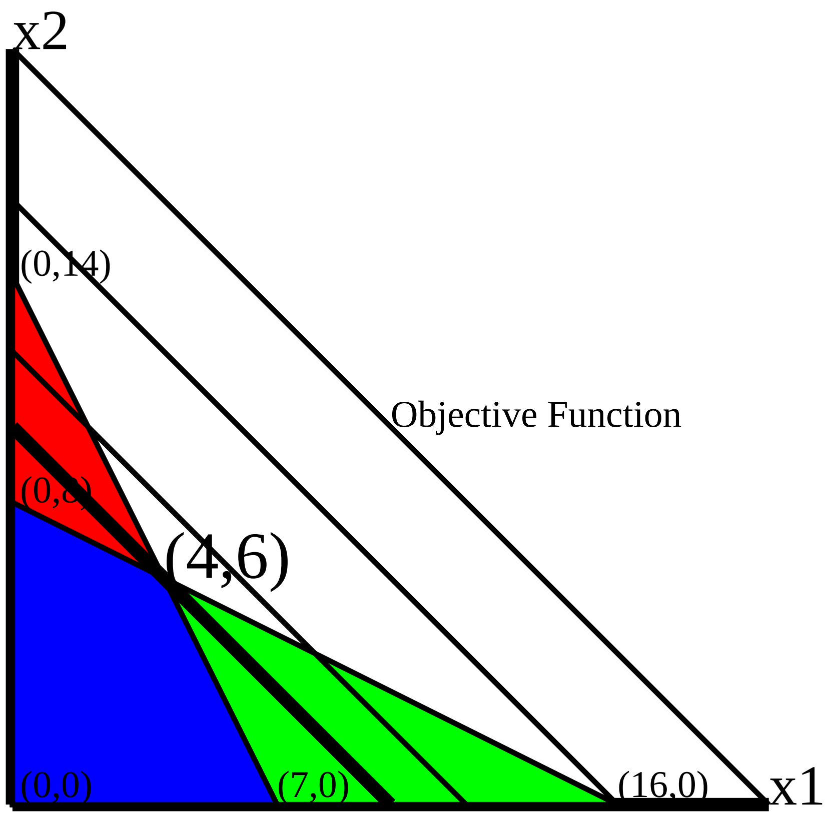
Constraints: Slack variable:

(1) 2 x1 + x2 <= 14 x3 = 14 - 2 x1 - x2 (red + blue)

(2) x1 + 2 x2 <= 16 x4 = 16 - x1 - 2 x2 (green + blue)

aij, xi are real number. x1, x2 >= 0

Optimal Solution (4,6).



Graphical Representation of a 2-D LP

This page shows the graphical representation of a 2-D LP

Objective Function:

Max: x1 + x2

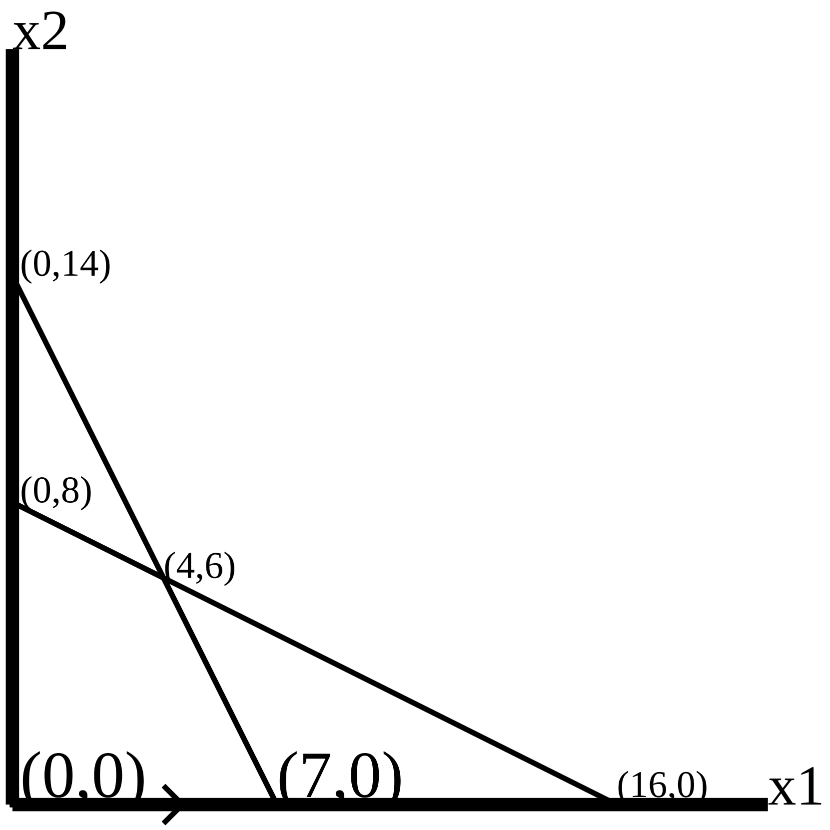
Constraints: Slack variable:

(1) 2 x1 + x2 <= 14 x3 = 14 - 2 x1 - x2 (red + blue)

(2) x1 + 2 x2 <= 16 x4 = 16 - x1 - 2 x2 (green + blue)

aij, xi are real number. x1, x2 >= 0

Optimal Solution (4,6).



Dictionary Method 2-D

Now, we try to solve the LP by Dictionary Method.

First iteration:

x3 = 14 - 2 x1 - x2

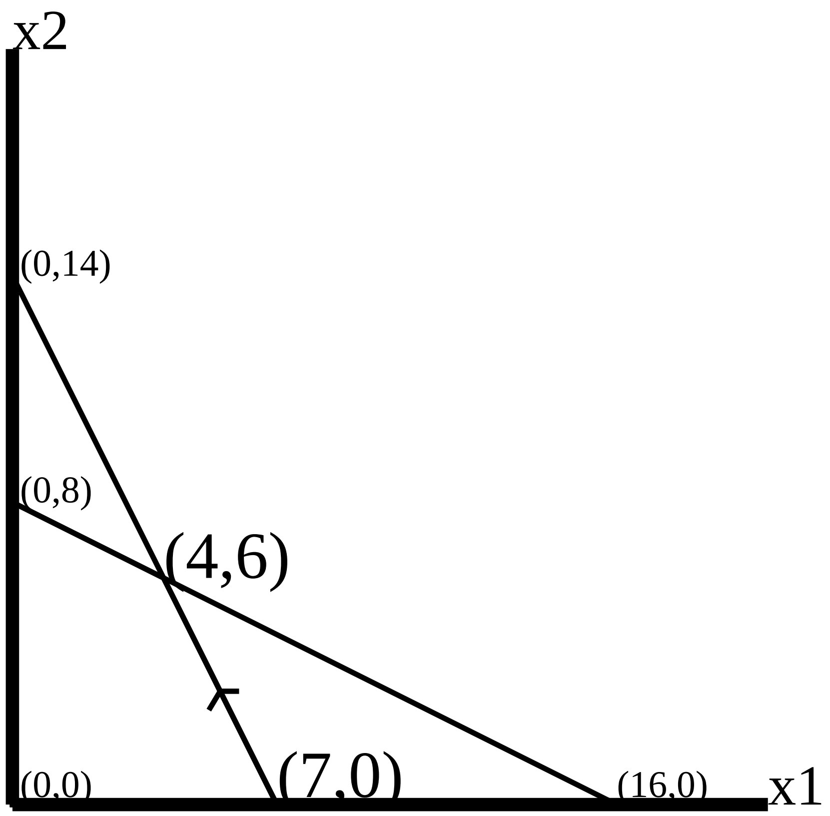
x4 = 16 - x1 - 2 x2

By putting x1 into the privot, we get:

x1 = 7 - 1/2 x2 - 1/2 x3

x4 = 9 - 3/2 x2 + 1/2 x3

Basic Solution changes from (0,0) to (7,0).



Dictionary Method 2-D

Second iteration:

x1 = 7 - 1/2 x2 - 1/2 x3

x4 = 9 - 3/2 x2 + 1/2 x3

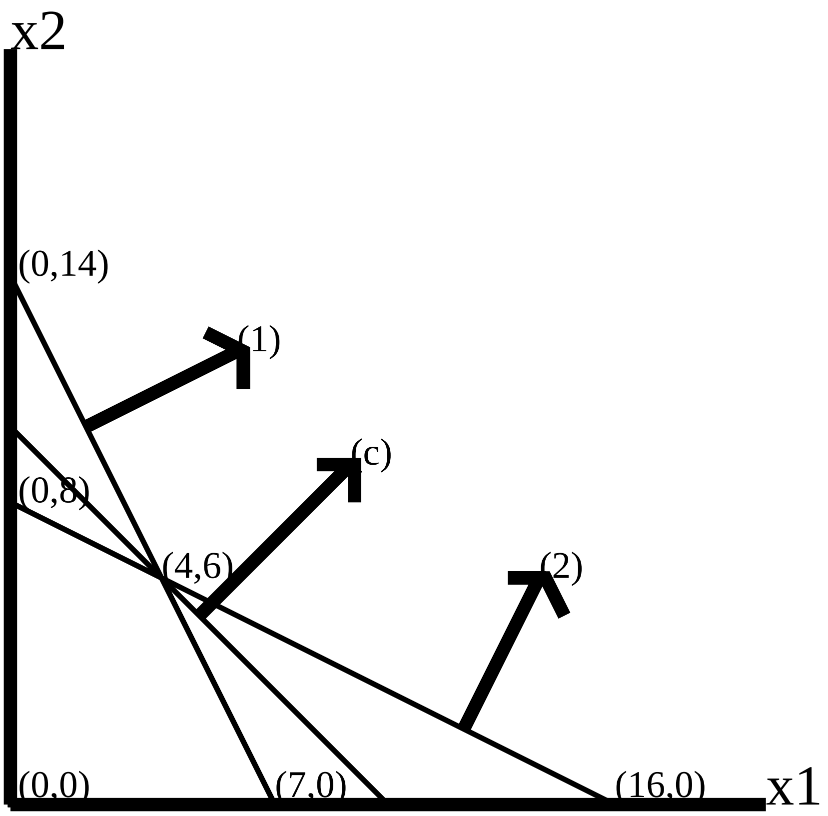
By putting x2 into the privot, we get:

x1 = 4 - 5/6 x3 + 2/3 x4

x2 = 6 + 1/3 x3 - 2/3 x4

Basic Solution changes from (7,0) to (4,6).

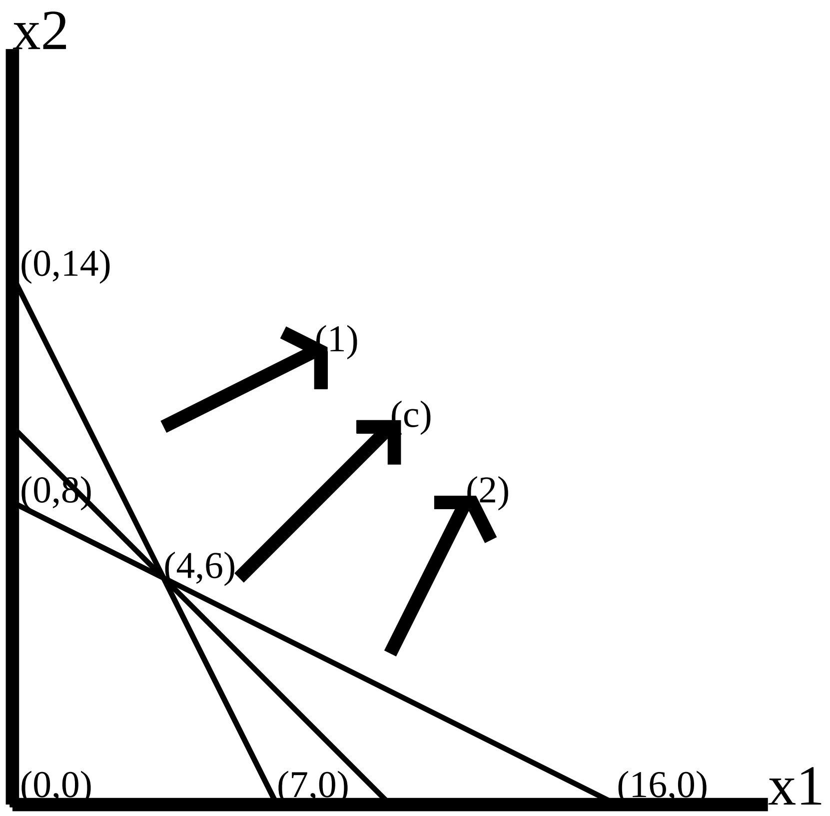
This is the optimal solution which yield an objective value of 10.



Gradient Interpretation in 2-D LP

After looking into the Dictionary Method, we are now going to confirm the optimal solution with the gradient of objective functions and constraints.

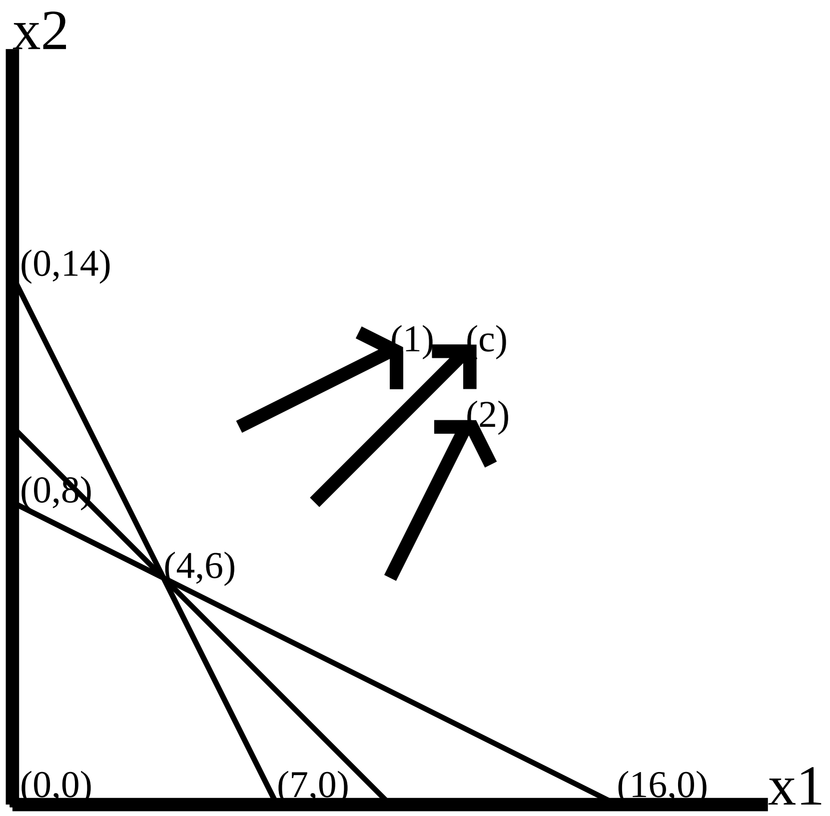
In this animation, we will put all the gradients at the same point to compare with the gradient of the objective function.



Gradient Interpretation in 2-D LP

After looking into the Dictionary Method, we are now going to confirm the optimal solution with the gradient of objective functions and constraints.

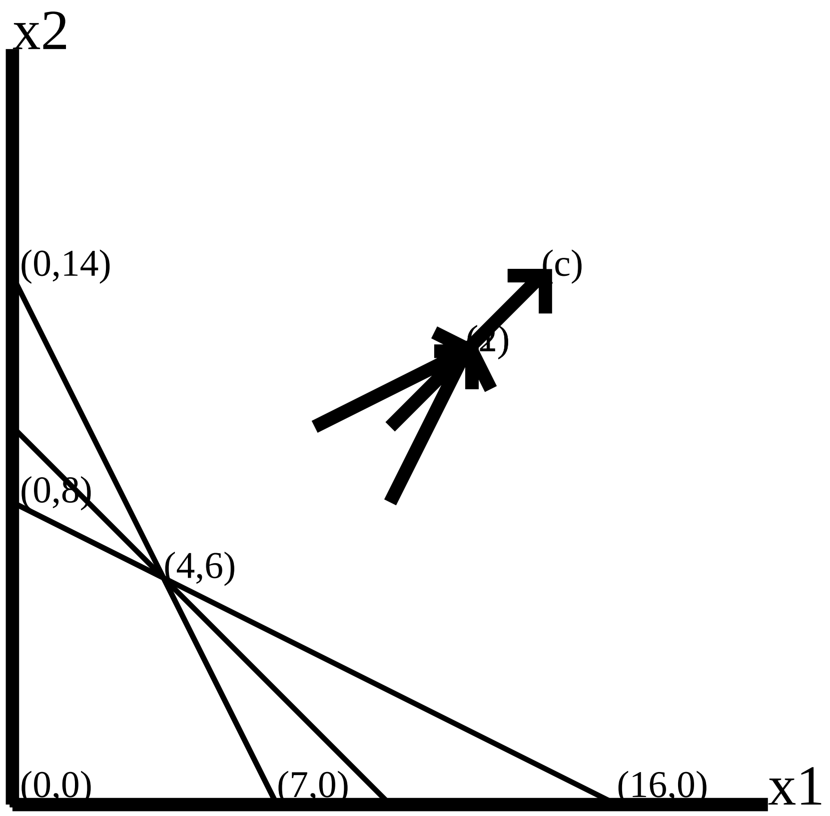
In this animation, we will put all the gradients at the same point to compare with the gradient of the objective function.



Gradient Interpretation in 2-D LP

After looking into the Dictionary Method, we are now going to confirm the optimal solution with the gradient of objective functions and constraints.

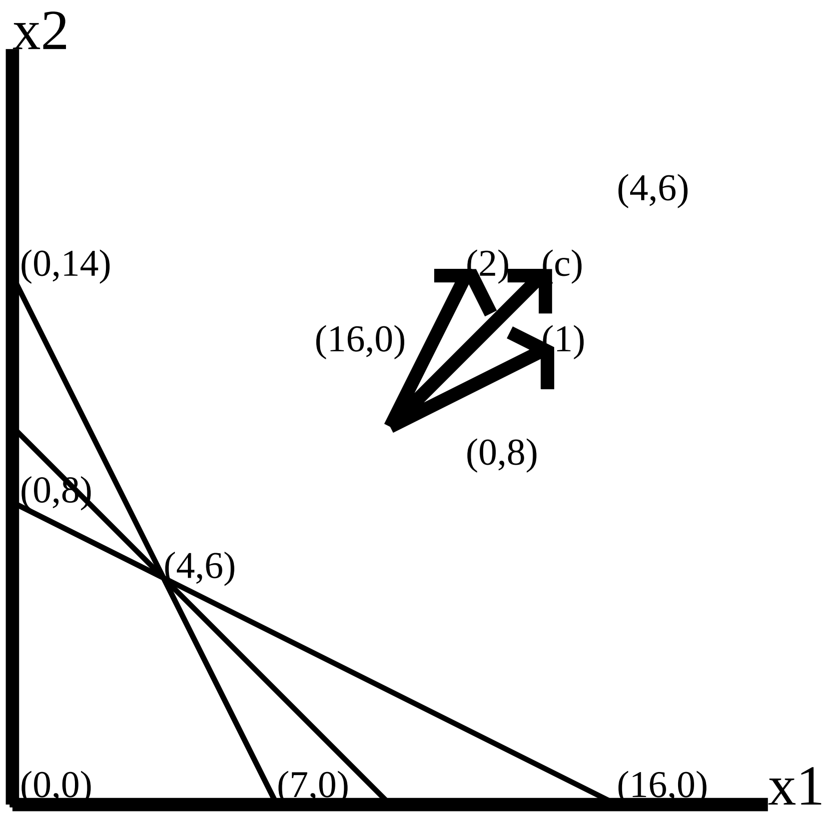
In this animation, we will put all the gradients at the same point to compare with the gradient of the objective function.



Gradient Interpretation in 2-D LP

After looking into the Dictionary Method, we are now going to confirm the optimal solution with the gradient of objective functions and constraints.

In this animation, we will put all the gradients at the same point to compare with the gradient of the objective function.

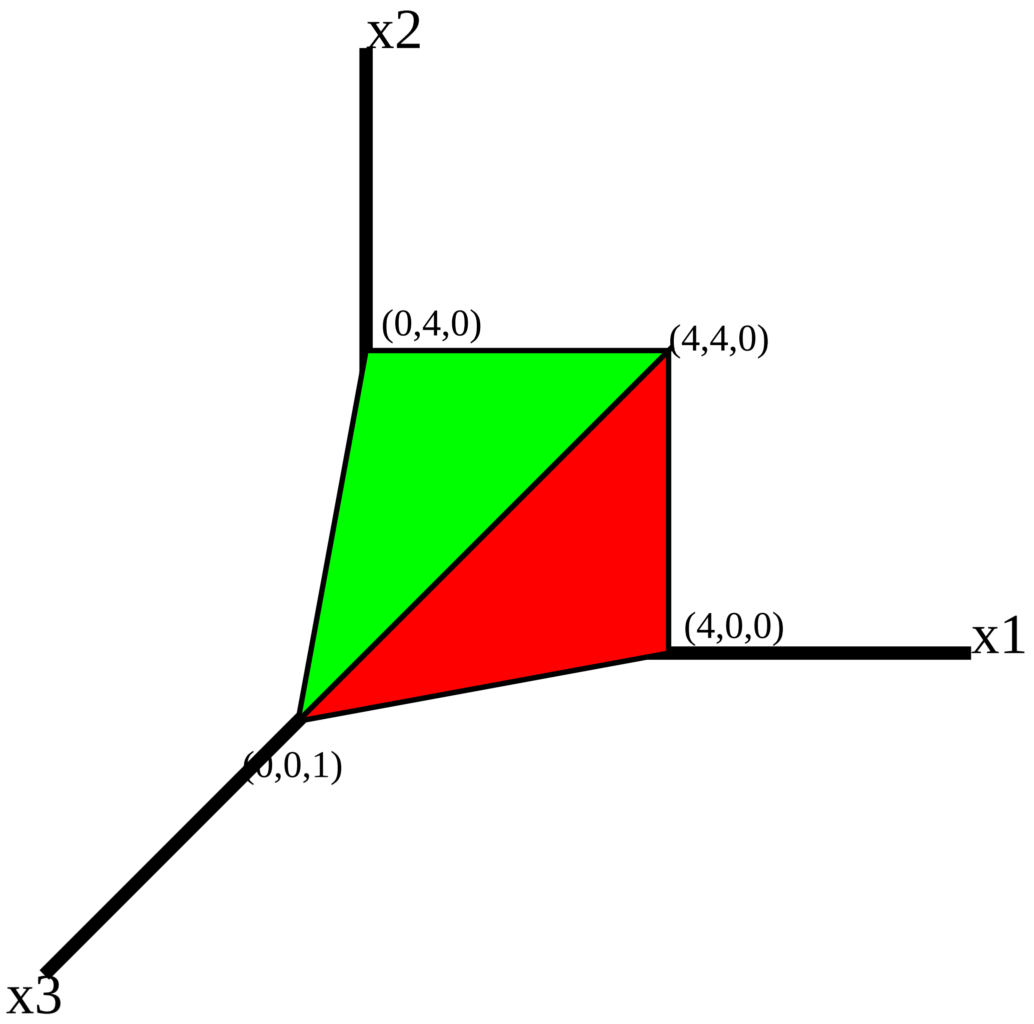


Gradient Interpretation in 2-D LP

After looking into the Dictionary Method, we now confirm the optimal solution with the gradient of objective functions and constraints.

In this animation, we will put all the gradients at the same point to compare with the gradient of the objective function.

It turns out that the gradient of the objective function is location in the cone of the active set of (4,6).
As a result, it is confirmed that (4,6) is the optimal solution.



Graphical Representation of a 3-D LP

In this section, we will show an example of a 3-D LP

Objective Function:

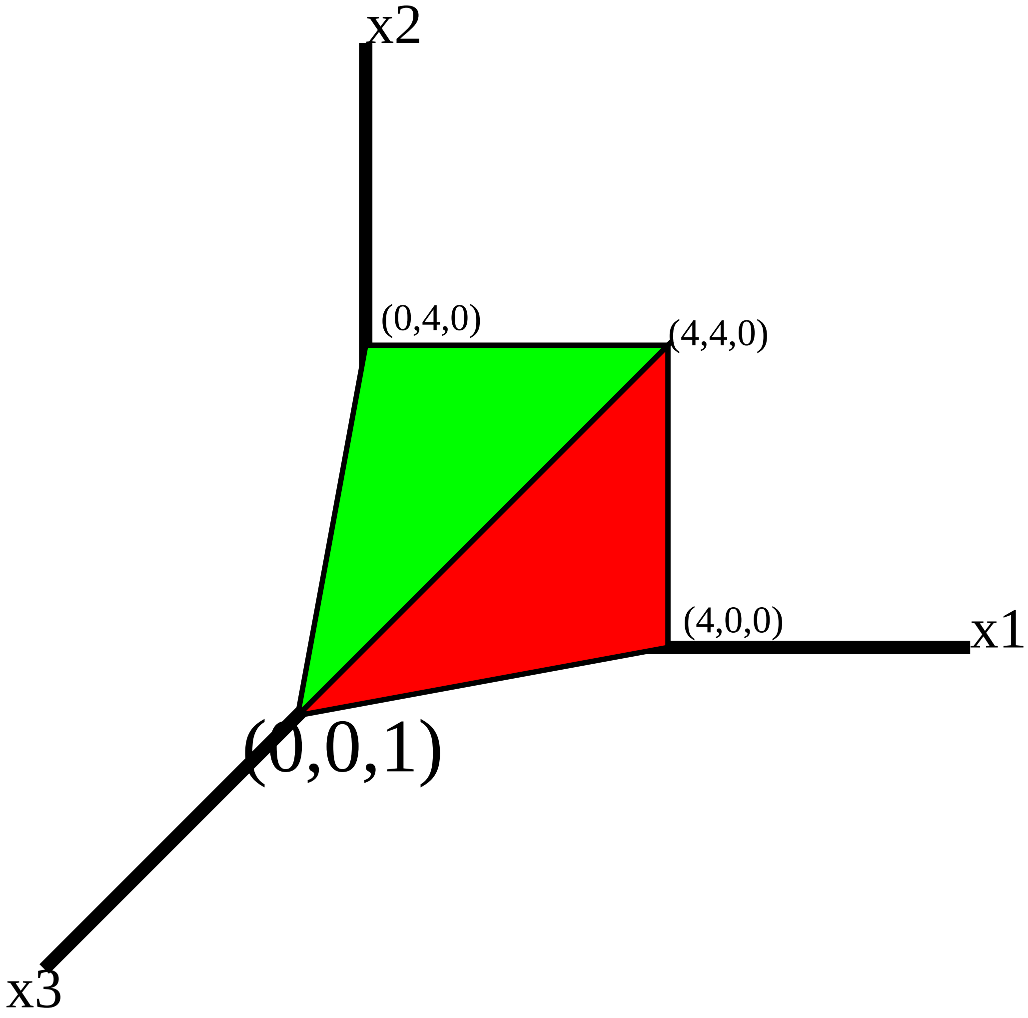
Max: x1 + x2 + 4 x3

Constraints:

(1) x1 + + 4 x3 <= 4 Red

(2) + x2 + 4 x3 <= 4 Green

aij, xi are real number. x1, x2, x3 >= 0



Dictionary Method in 3-D LP

We will now use dictionary method to find the optimal solution

Max: x1 + x2 + 4 x3

Slack Variable:

x4 = 4 - x1 - 4 x3

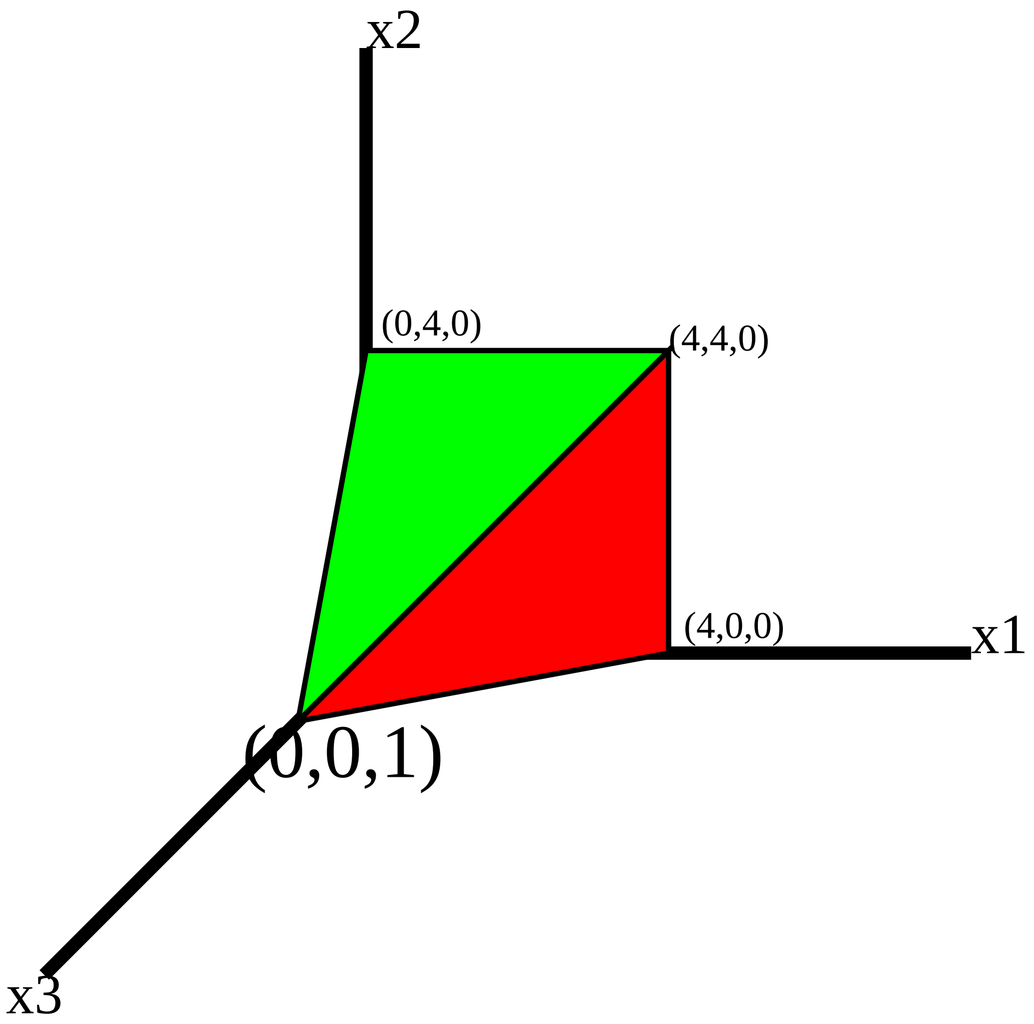
x5 = 4 - x2 - 4 x3

By putting x3 into the privot, we get:

x3 = 1 - 1/4 x1 - 1/4 x4

x5 = x1 - x2 + x4

Basic Solution (0,0,1).



Dictionary Method in 3-D LP

We will now use dictionary method to find the optimal solution

Max: x1 + x2 + 4 x3

Slack Variable:

x3 = 1 - 1/4 x1 - 1/4 x4

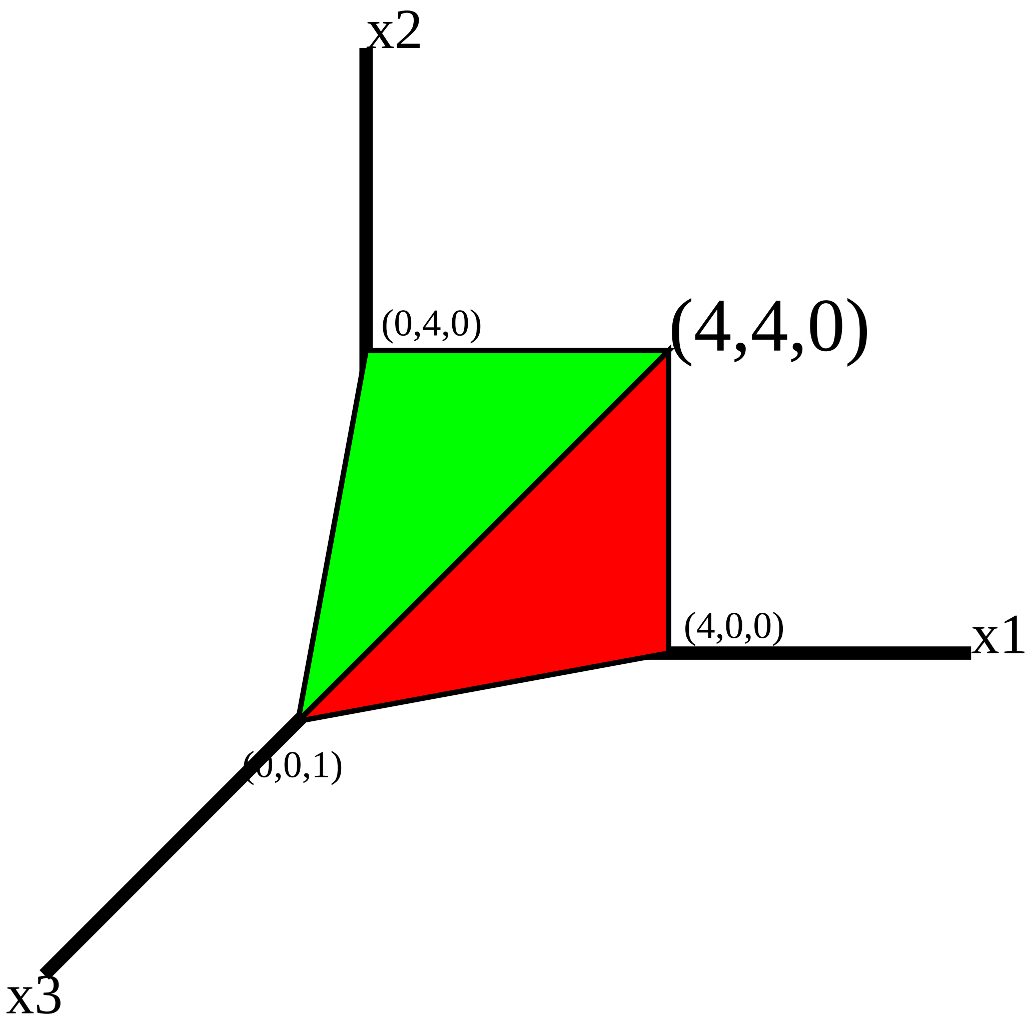
x5 = x1 - x2 + x4

By putting x2 into the privot, we get:

x3 = 1 - 1/4 x1 - 1/4 x4

x2 = x1 + x4 - x5

Basic Solution is still (0,0,1).



Dictionary Method in 3-D LP

We will now use dictionary method to find the optimal solution

Max: x1 + x2 + 4 x3

Slack Variable:

x3 = 1 - 1/4 x1 - 1/4 x4

x2 = x1 + x4 - x5

By putting x2 into the privot, we get:

x1 = 4 - 4 x3 - x4

x2 = 4 - 4 x3 - x5

Optimal Solution is (4,4,0).

Conclusion

In this project, we have seen how the dictionary method in Linear Programming is related to Geometry. By changing the basics and privot, we go alone the boundary of the feasible region to search for optimal solution. Through a finite number of iterations, we arrive the optimal solution by shifting from vertex to vertex.

Also, we have seen a way to find the vertex of the optimal solution by normal vector. Because if and only if the gradient of the objective function lies in the cone spanned by the gradient of the active constraint of a point, that point isthe optimal solution. As a result, if we test the gradient of the objective function with the cones of gradient of different active sets at different vertexes, we can easily obtain the optimal solution with no calculations.

Reference

Anstee. "UBC Math 340 Lecture note". Summer 2003.

Bennion. "Linear Programming and Game Theory". Bureau of Business and Economic Research College of Business and Public Service Michigan State University, 1960.

Dantzig. "Linear Programming & Extensions". Princeton University Press, 1963.

Minoux. "Mathematical Programming - Theory and Algorithms". John Wiley & Sons Ltd, 1986.

Mitrinovic. "Recent Adcances in Geometric Inequalities". Kluwer Academic Publishers, 1989.