MATH 308

TERM PROJECT

Proof of Sperner's Lemma

By Jatinder Sidhu

I. Proof of Index Lemma

A. Definitions

i. Triangulation

ii. Triangle Completeness

iii. Content

iv. Index

B. Proof

II. Proof of Sperner’s Lemma

A. Definitions

i. Sperner’s Labeling

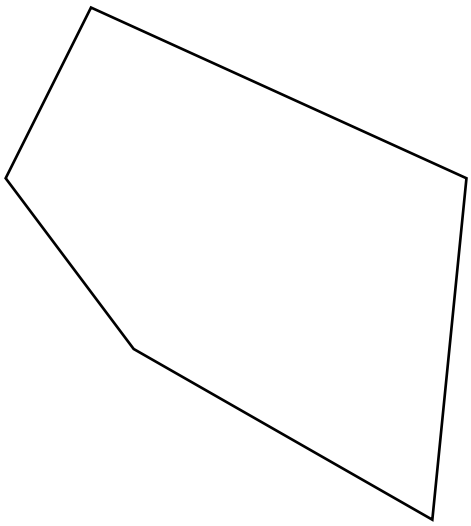
ii. Complete Segments

B. Proof

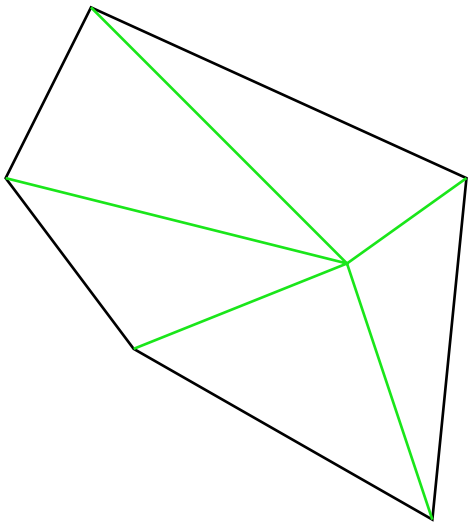
Index Lemma

A. Definitions

Def: Triangulation

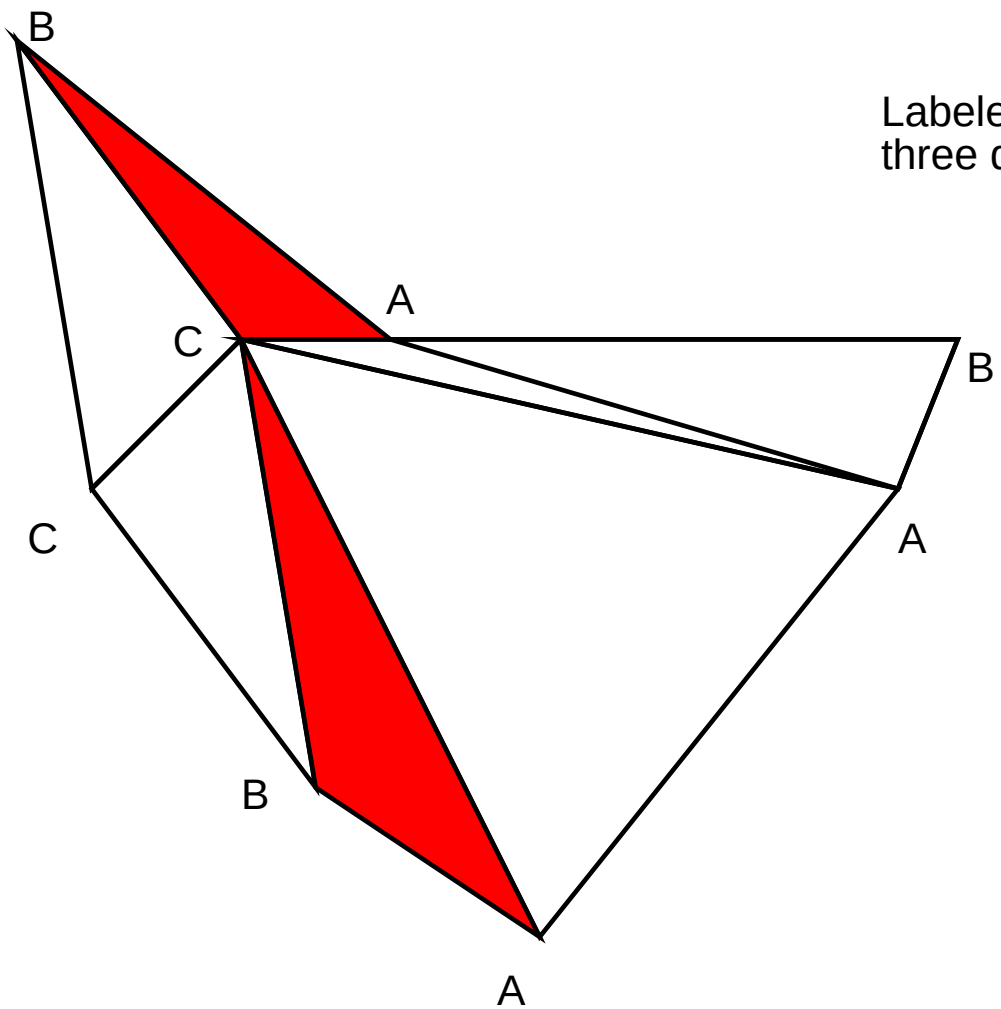


Simple (non-intersecting)
Polygon D



Division of D into a finite
number of triangles

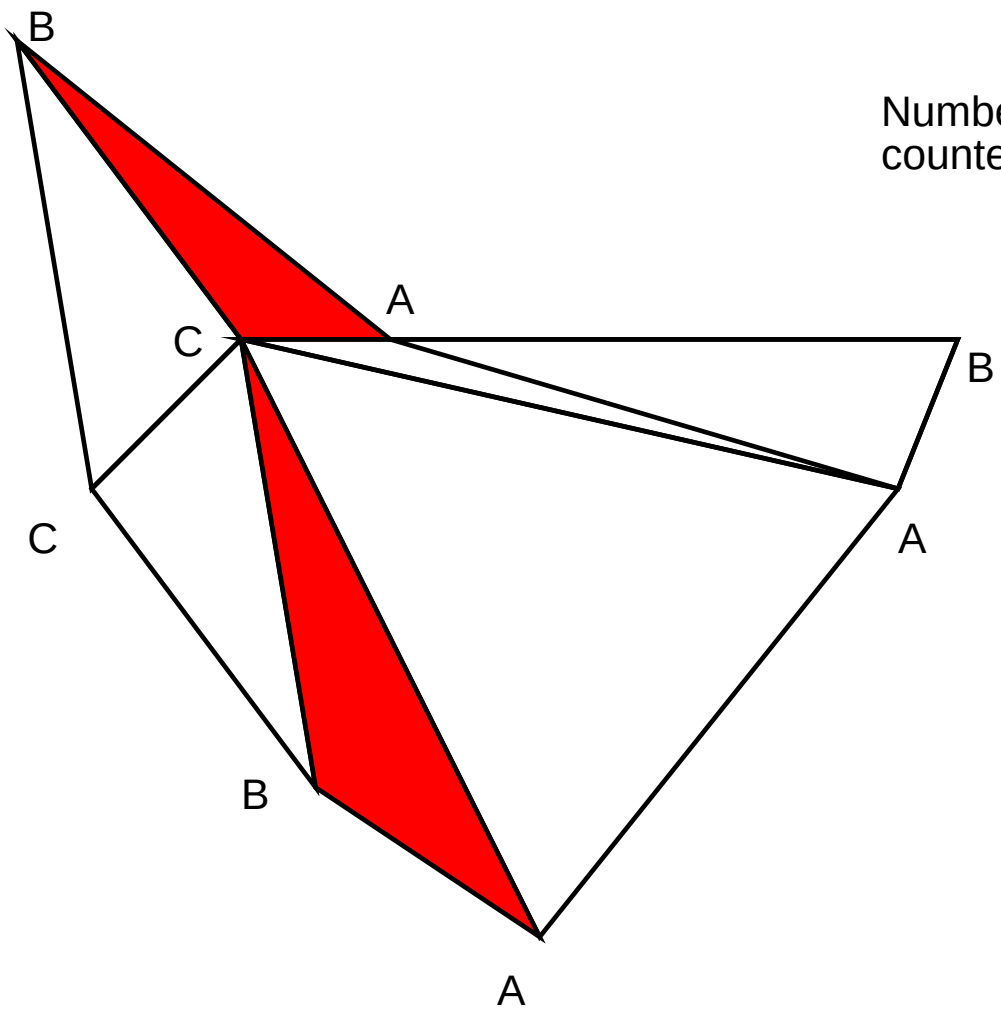
Def: Triangle Completeness



Labeled Triangles with
three different verticies

Complete Triangles -

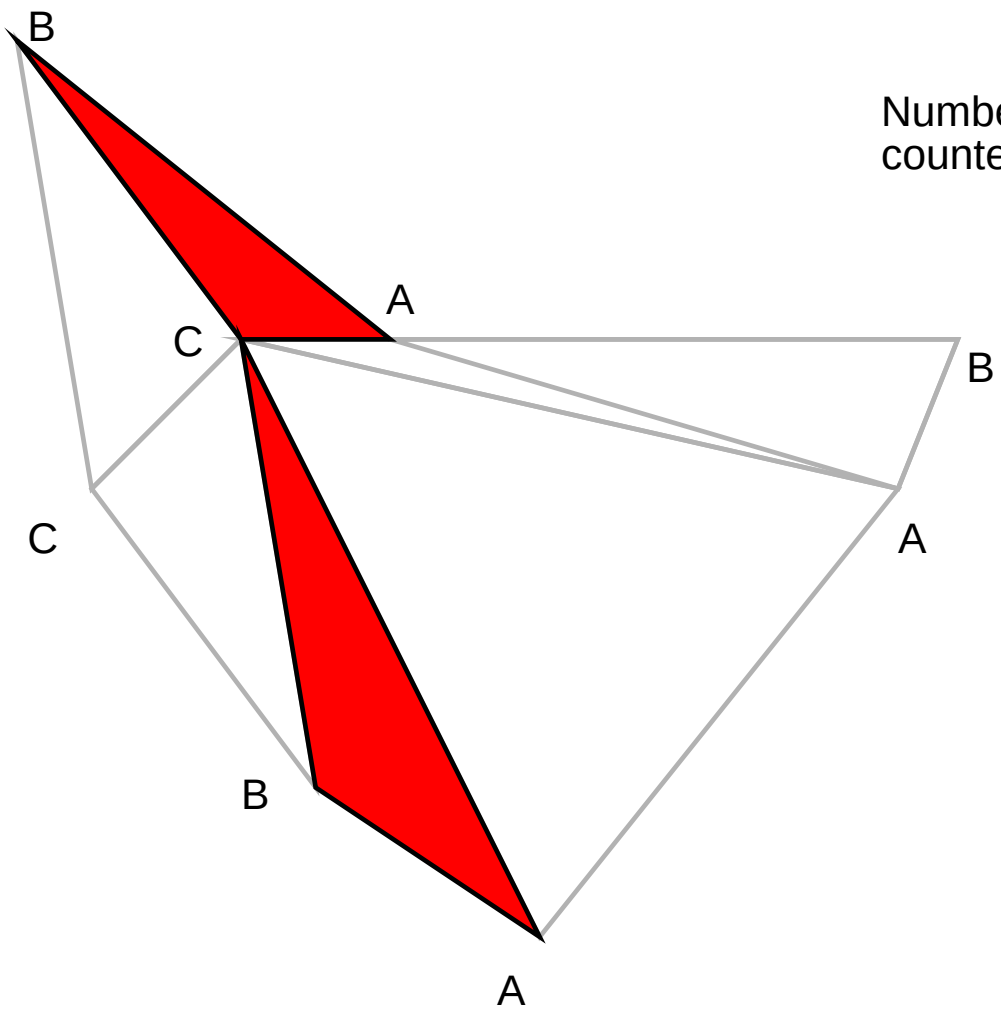
Def: Content



Number of complete triangles counted by orientation

Complete Triangles -

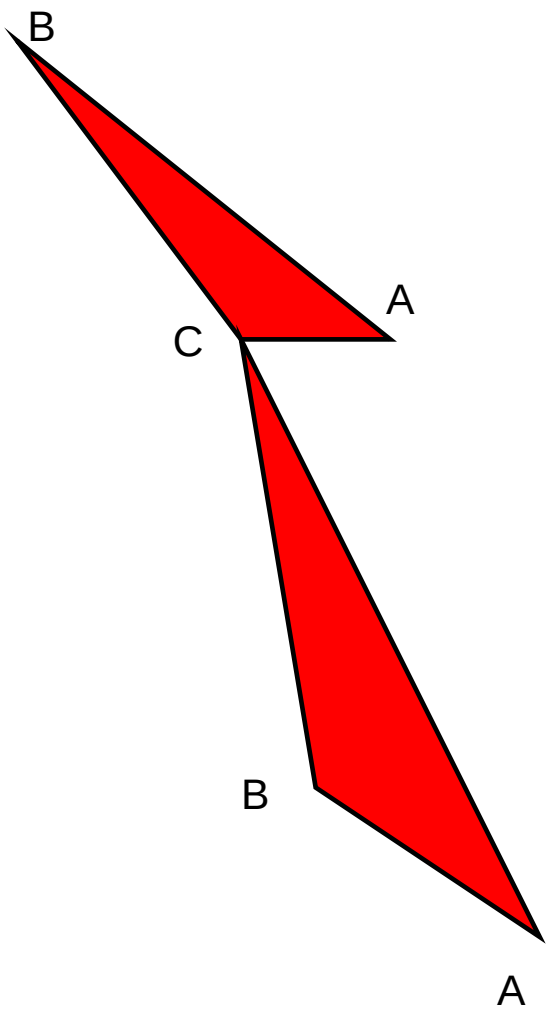
Def: Content continued...



Number of complete triangles counted by orientation

Complete Triangles -

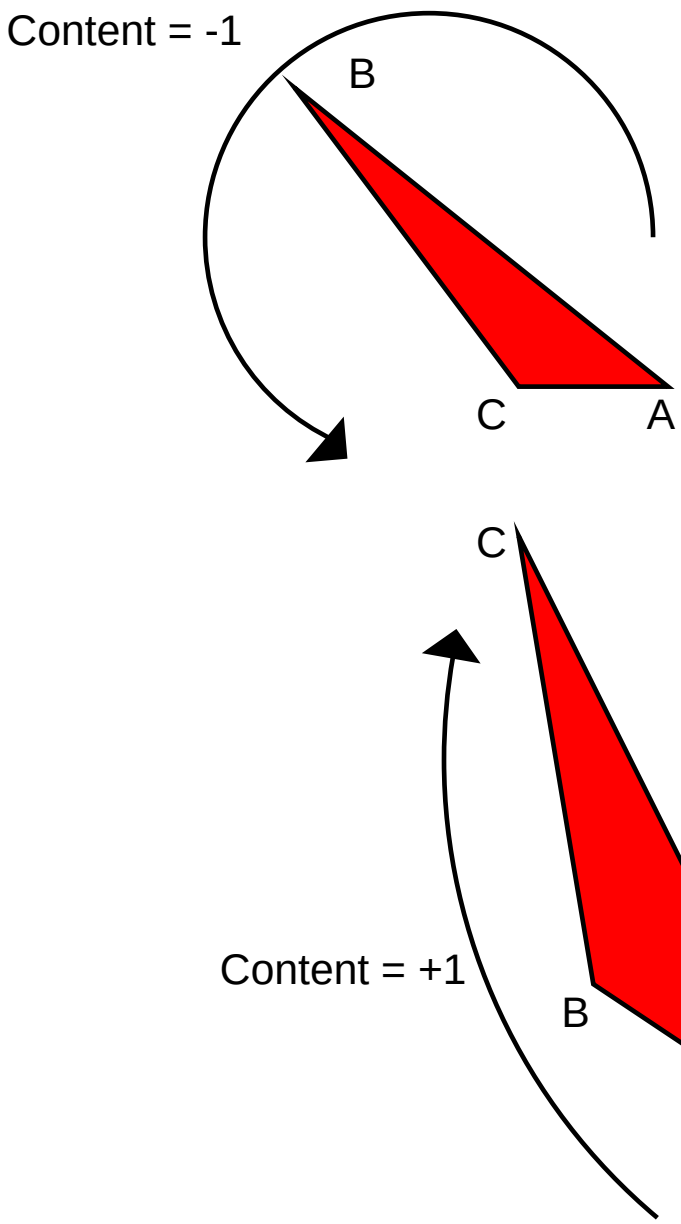
Def: Content continued...



Number of complete triangles counted by orientation

Complete Triangles -

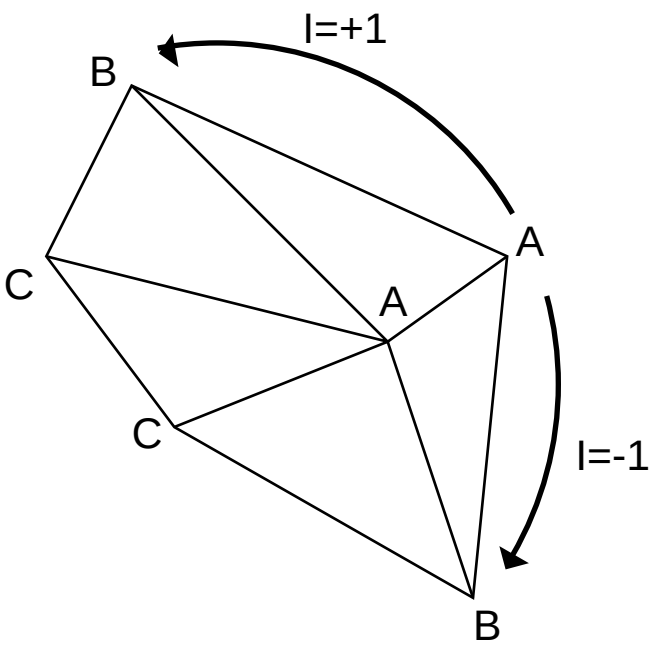
Def: Content



Number of complete triangles counted by orientation

Complete Triangles -

Def: Index



Number AB edges around polygon boundary: counted by orientation

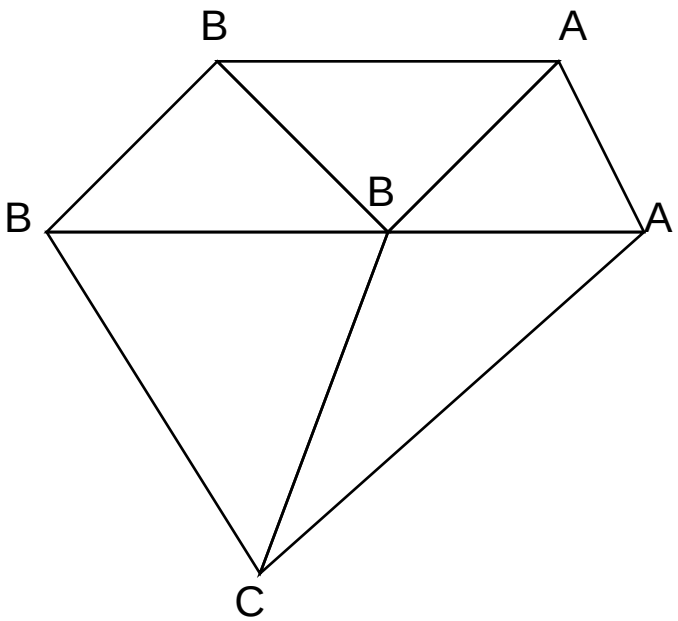
B. Index Lemma Proof

Index Lemma: Index equals Content

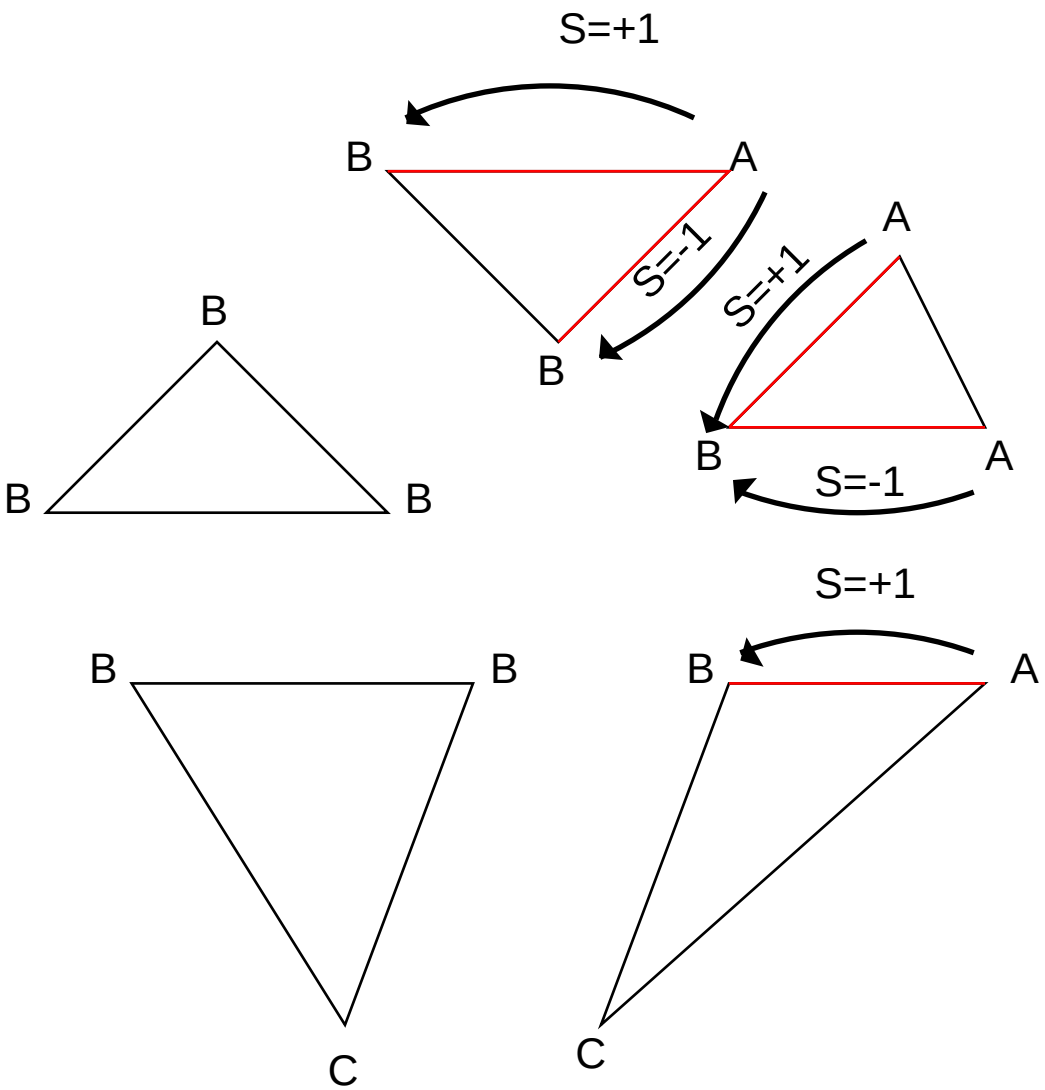
Index Lemma Proof continued...

Let S equal total number AB

edges of polygon



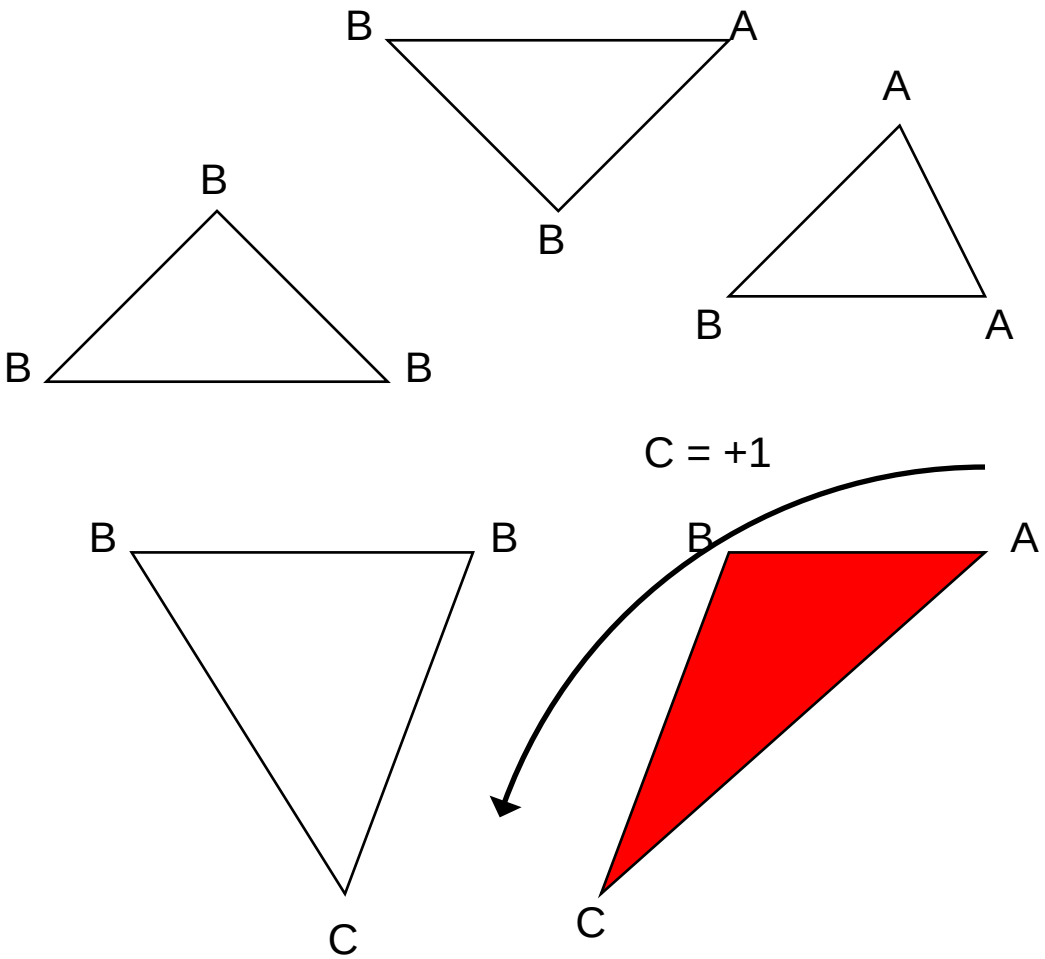
Index Lemma Proof continued...



$\therefore$ S = +1 -1 +1 -1 +1 = 1

Index Lemma Proof continued...

Let C = content of polygon



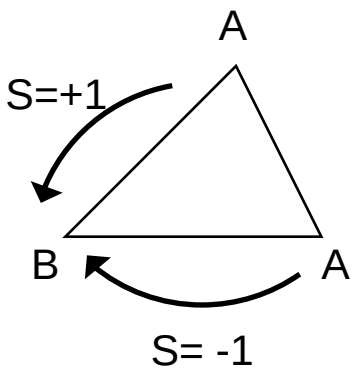
$\therefore$ C = +1 = 1

Index Lemma Proof continued...

Proof of C = S

Examine all cases affecting C and S

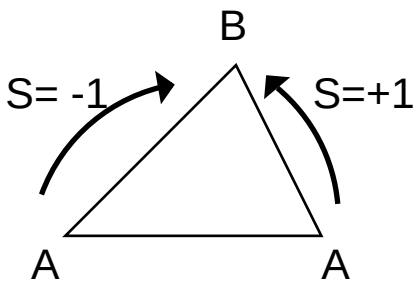
Case I:



S= +1 -1 = 0

C= 0

Case II:



S= +1 -1 = 0

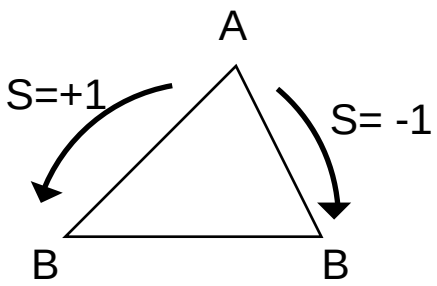
C= 0

Index Lemma Proof continued...

Proof of C = S

Examine all cases affecting C and S

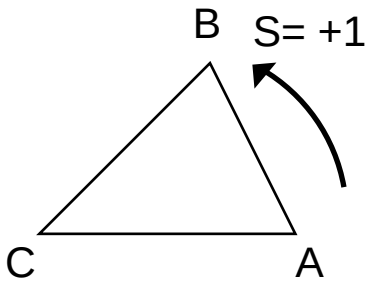
Case III:



S= +1 -1 = 0

C= 0

Case IV:



S= 1

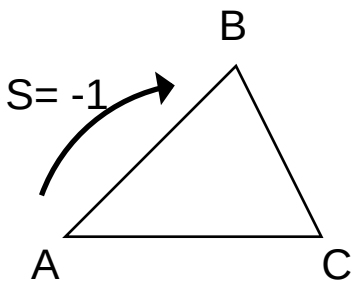
C= 1

Index Lemma Proof continued...

Proof of C = S

Examine all cases affecting C and S

Case V:



S= -1

C= -1

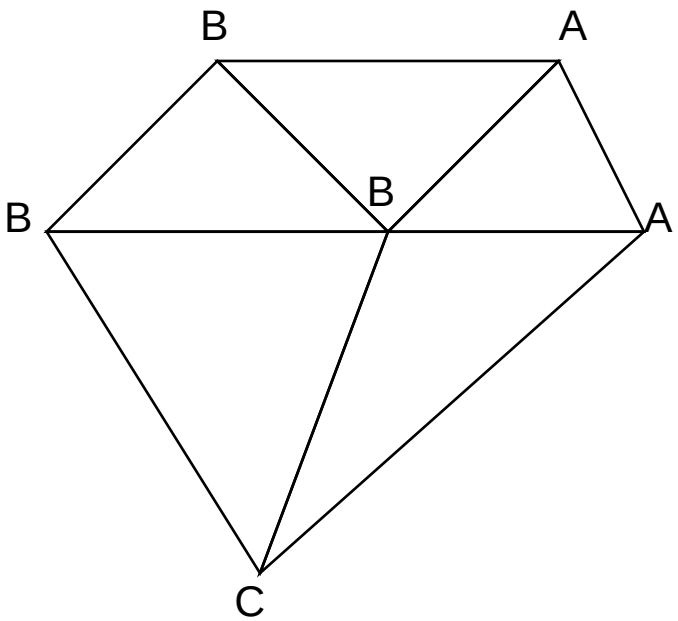
∴ C = S

Index Lemma Proof continued...

Let I = Index of polygon

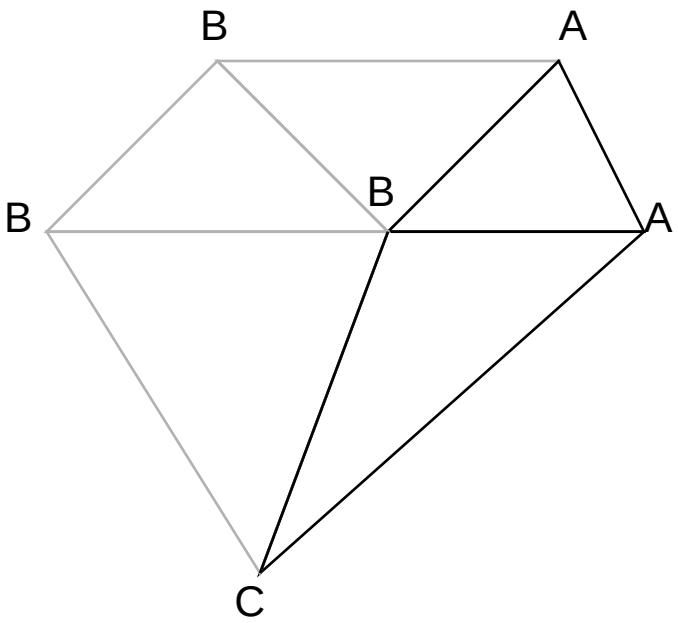
Proof of S = I

Consider an inside AB edge of any polygon



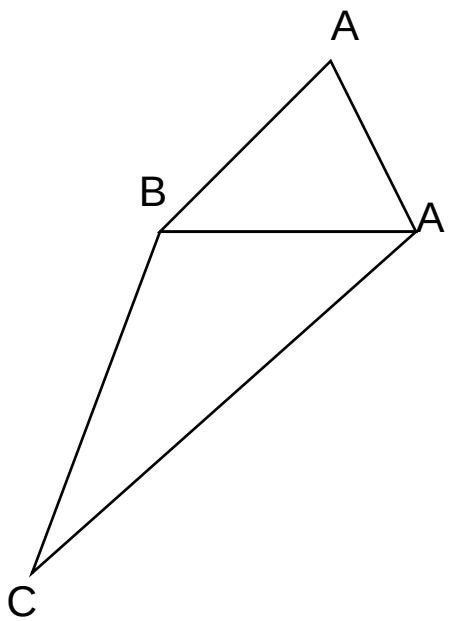
Index Lemma Proof continued...

Proof of S = I



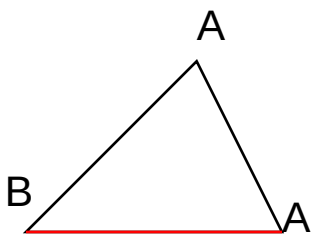
Index Lemma Proof continued...

Proof of S = I

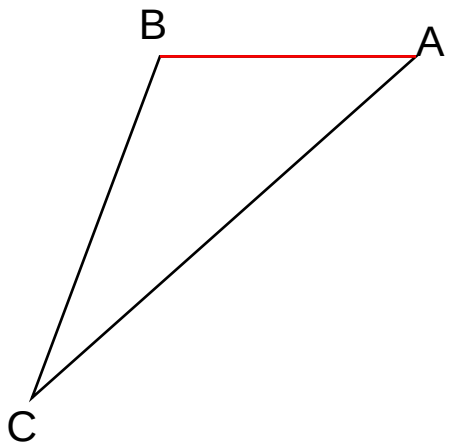


Index Lemma Proof continued...

Proof of S = I

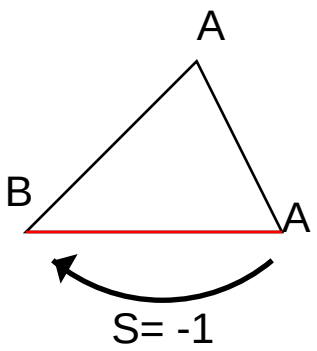


All inside edges are an edge of two triangles

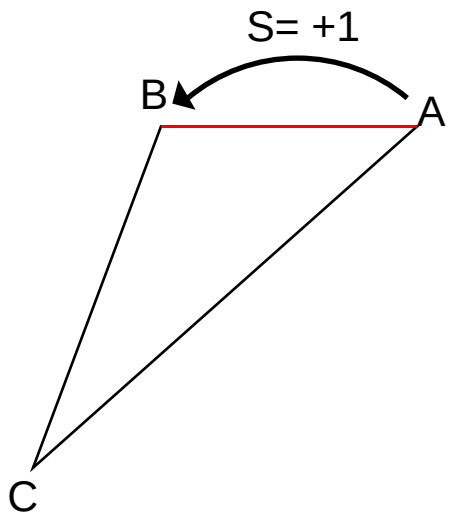


Index Lemma Proof

Proof of S = I



S= +1 -1 = 0



∴ S independent of inside edges
S = I
C = S
C = I ---> Index Lemma

Sperner’ s Lemma

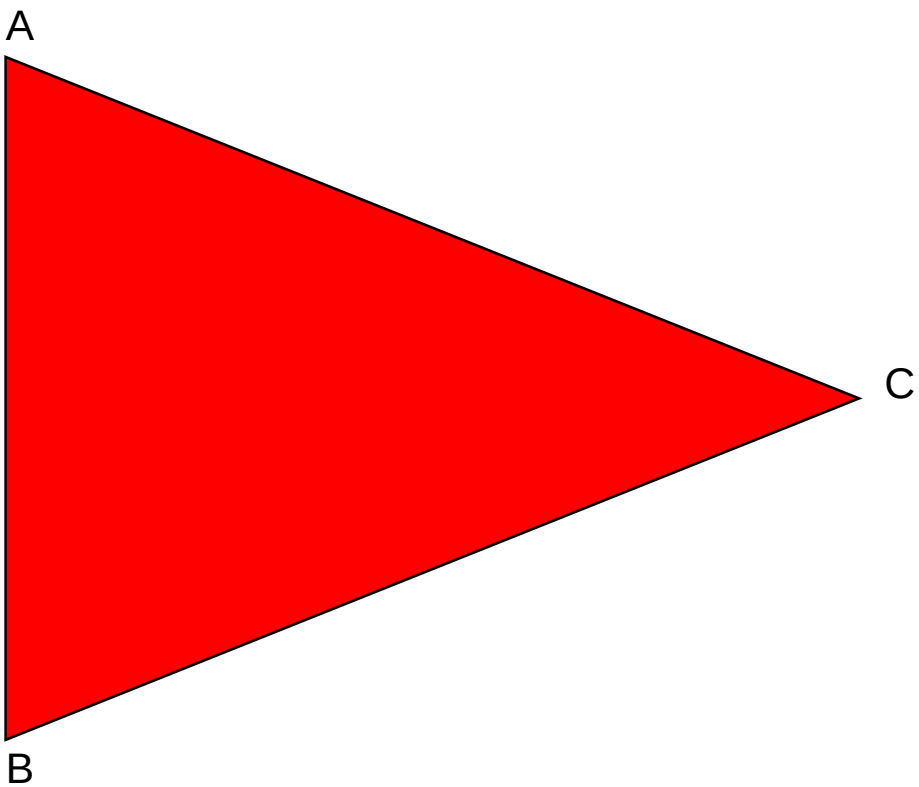
A. Definitions

Def: Sperner Labeling

Triangles triangulation following

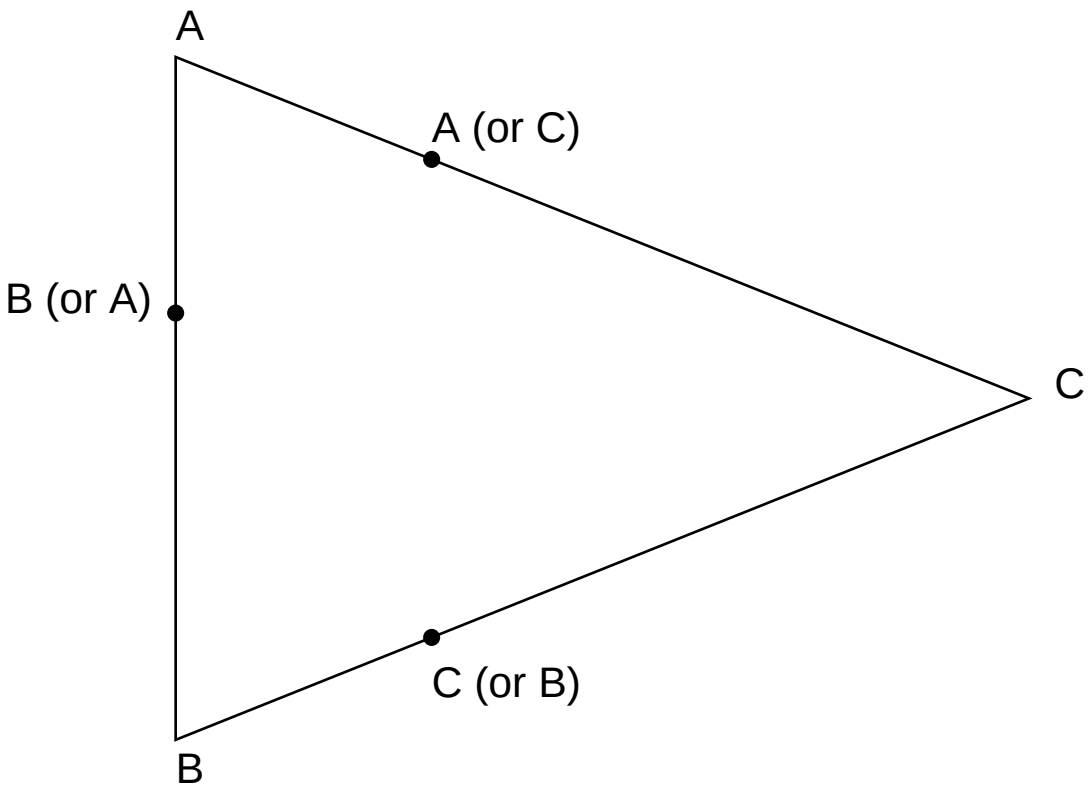
three conditions

I. Original Triangle Always Complete



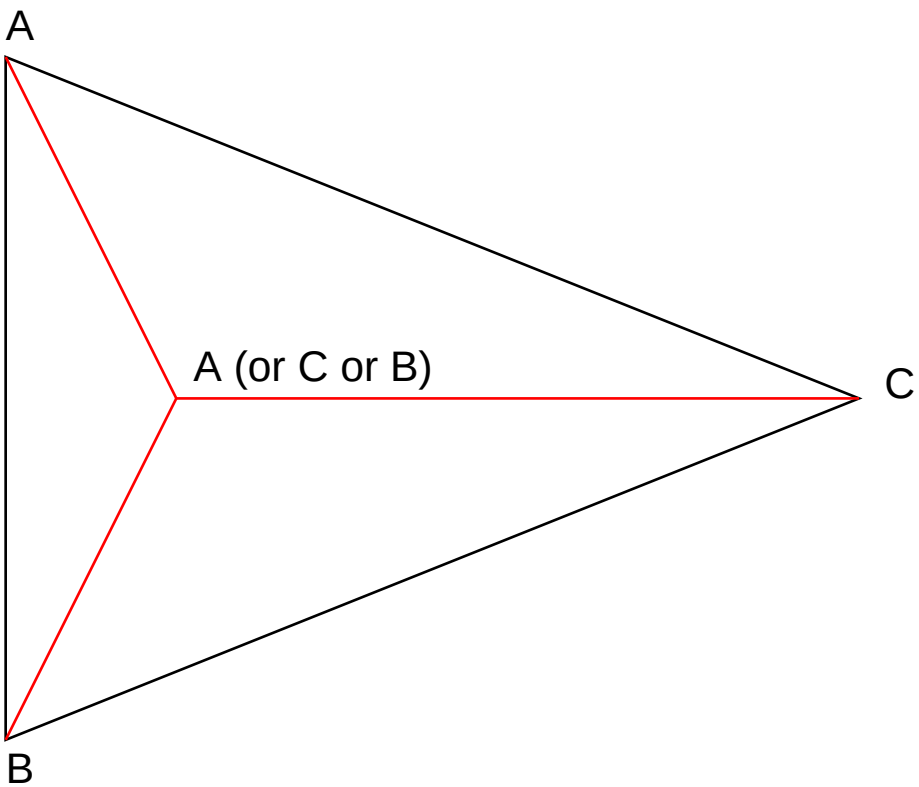
Def: Sperner Labeling Continued...

II. Triangulation Vertices on an edge are depedent on that Edge:



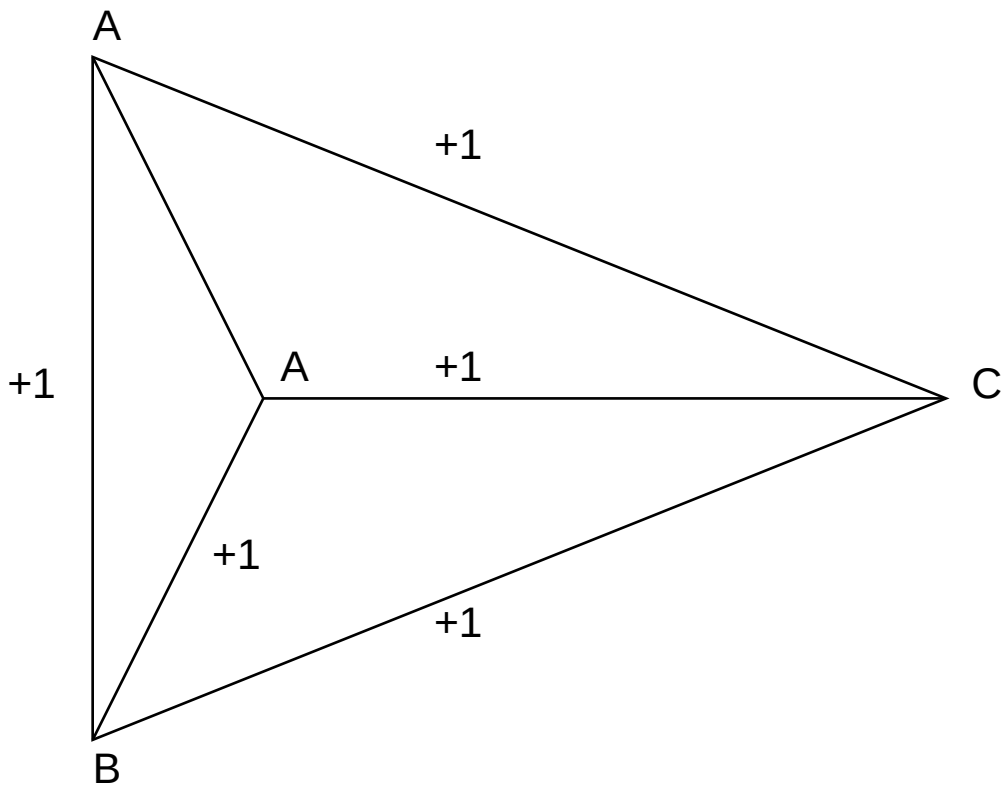
Def: Sperner Labeling Continued...

III. Inside Triangulation Vertices:



Def: Complete Segments

Edges with different labeled verticies



Total Complete Seg.: +1 +1 +1 +1 +1 = 5

B. Sperner’s Lemma Proof

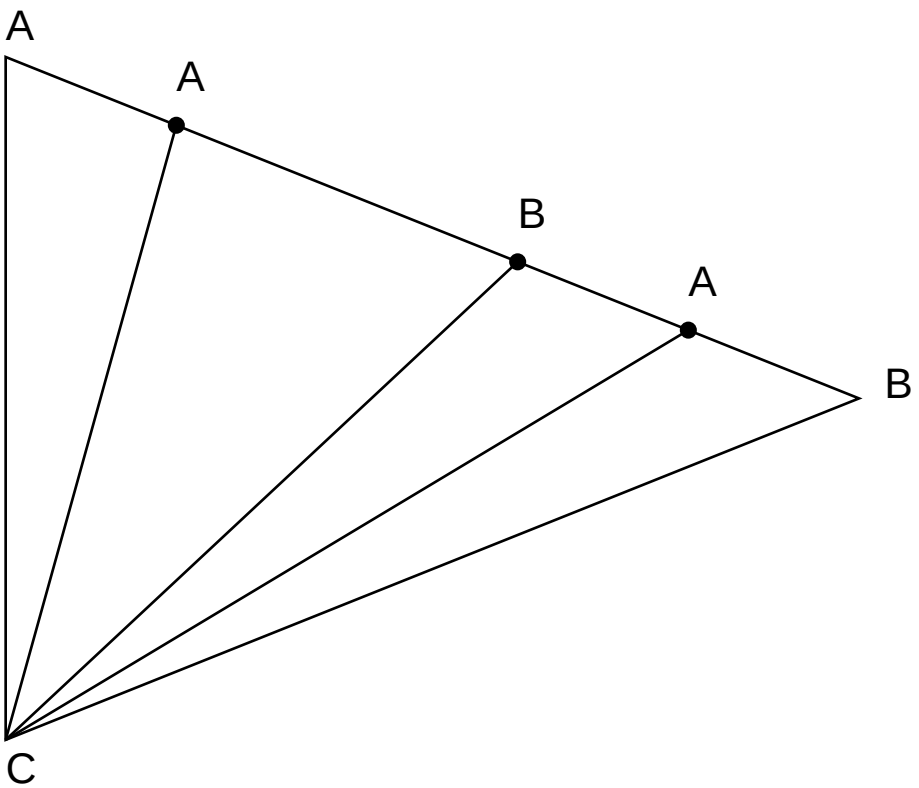
Sperner’s Lemma: Every Sperner Labeling contains at least one Complete Triangle

Sperner’s Lemma Proof

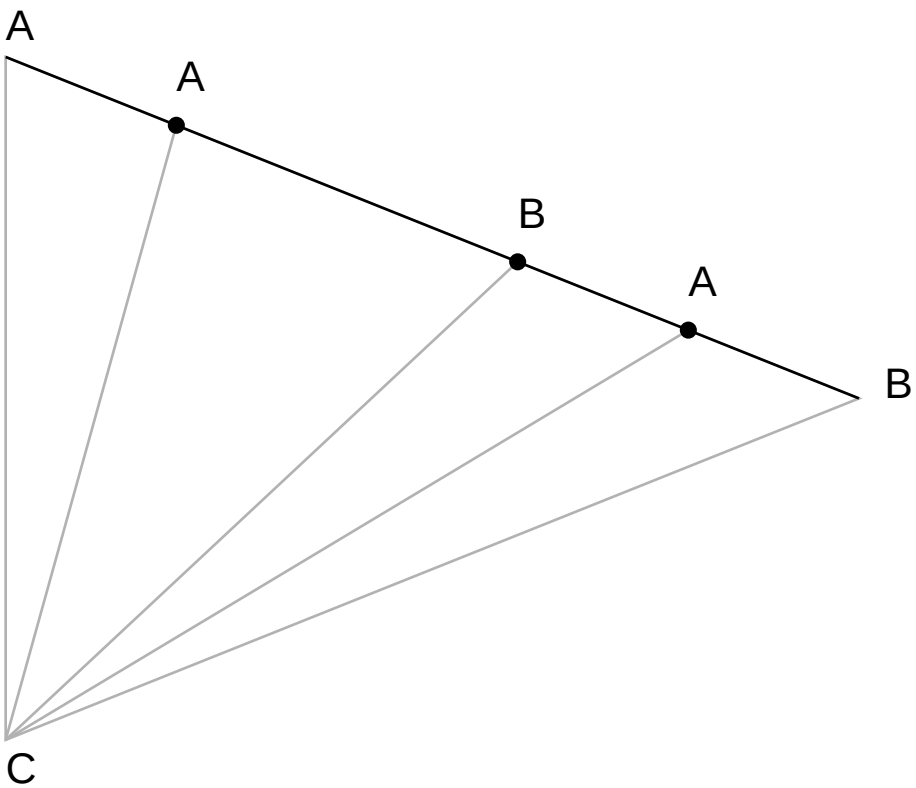
From Index Lemma I = C

∴ Content dependent only on outer AB edges

Consider following trinagulated triangle

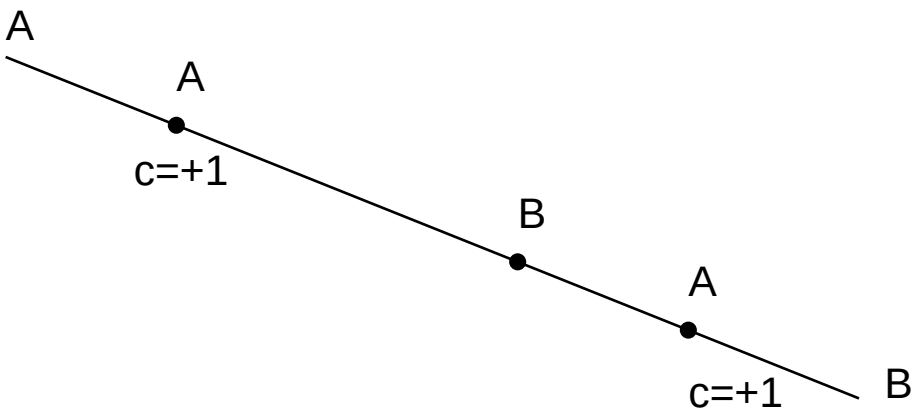


Sperner’s Lemma Proof



Sperner’s Lemma Proof

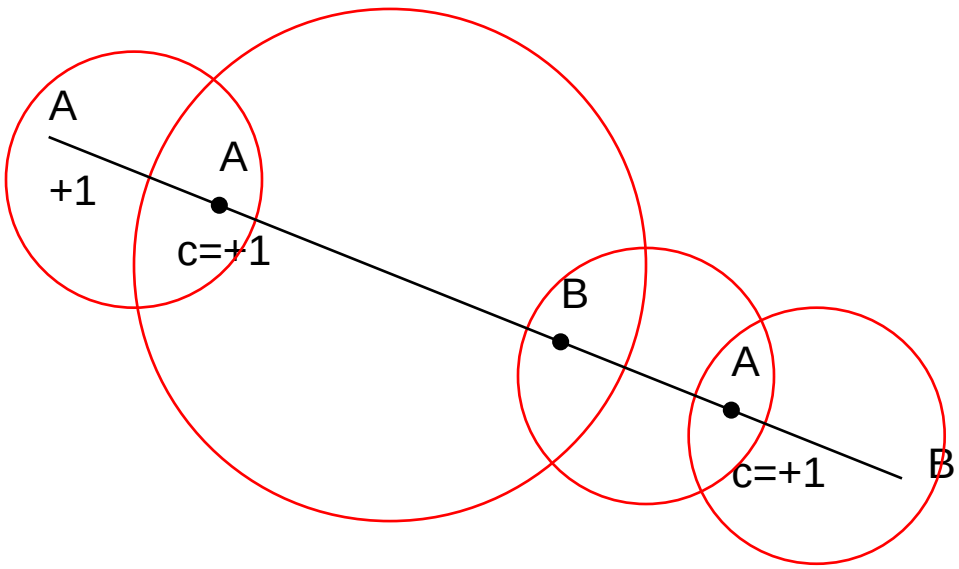
Let c equal internal A verticies of original edge AB



c=+1 +1 = 2

Sperner’s Lemma Proof

Internal A verticies share 2 segments

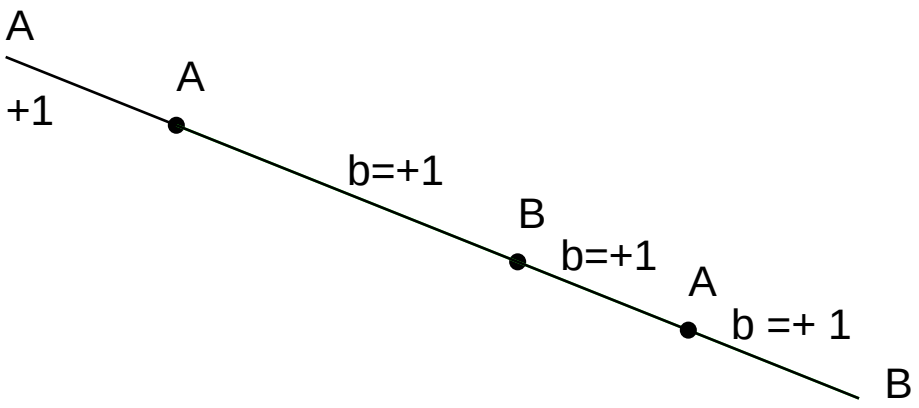


Total A verticies of all segments = 2c + 1

= 2*2 + 1 = 5

Sperner’s Lemma Proof

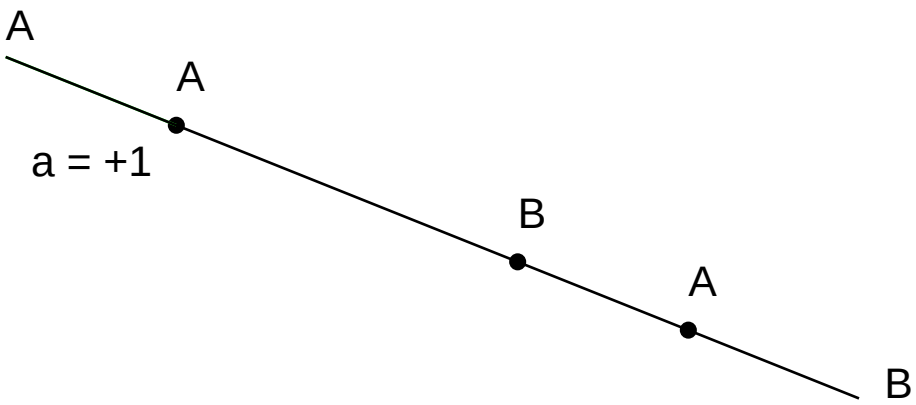
Let b equal Complete Segments on edge AB



Since b is on edge AB it is only dependent on AB segments. Index I is equal to zero if b equals zero and non-zero if b is non-zero by previous definition

Sperner’s Lemma Proof

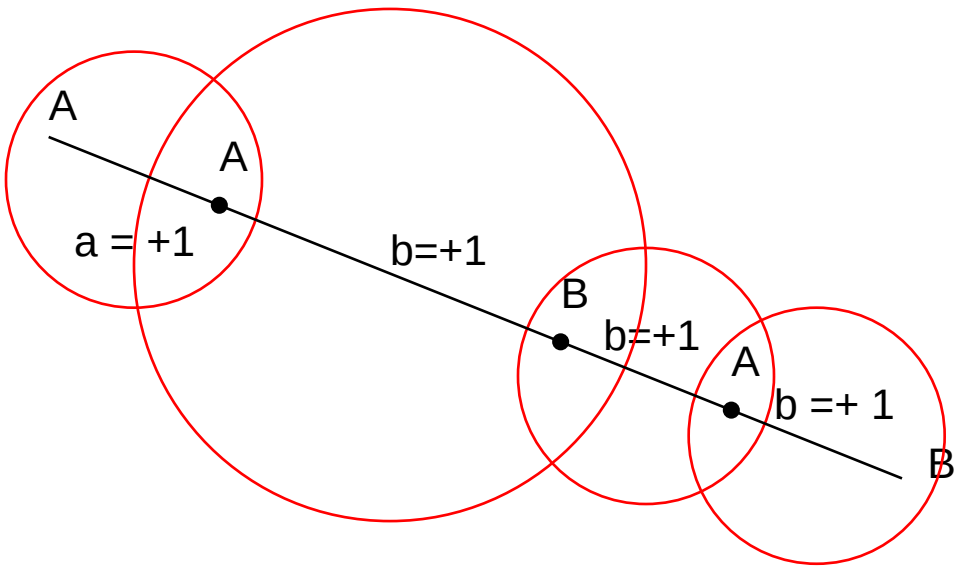
Let a equal AA segments on edge AB



Sperner's Lemma Proof

Each AA segment shares 2 A verticies

Each AB segment shares 1 A vertex



Total A verticies of all segments = +2 +1 +1 +1 = 5

= 2a + b

Sperner's Lemma Proof

∴ Total A verticies of all segments = 2c + 1

Total A verticies of all segments = 2a + b

2c + 1 = 2a + b

b = 2c + 1 - 2a

b is always odd and not equal to zero

End of Proof

Summary

Since b, the number of complete segments, is on edge AB it is dependent only on AB edges. This dictates I, the Index, which if not equal to zero imply's a complete triangle by the Index Lemma

References

Sperner’s Lemma, John Web, University of Cape Town

http://www.nrich.maths.org.uk/mathsf/journalf/nov00/art2/

Sperner’s Lemma, Alaxander Bogomolny, www.cut-the-knot.org

http://www.cut-the-knot.org/do_you_know/poincare.shtml#spernerlemma