Math 308
Euclidean Geometry
2003-04 term1

Final project

Triangle centers

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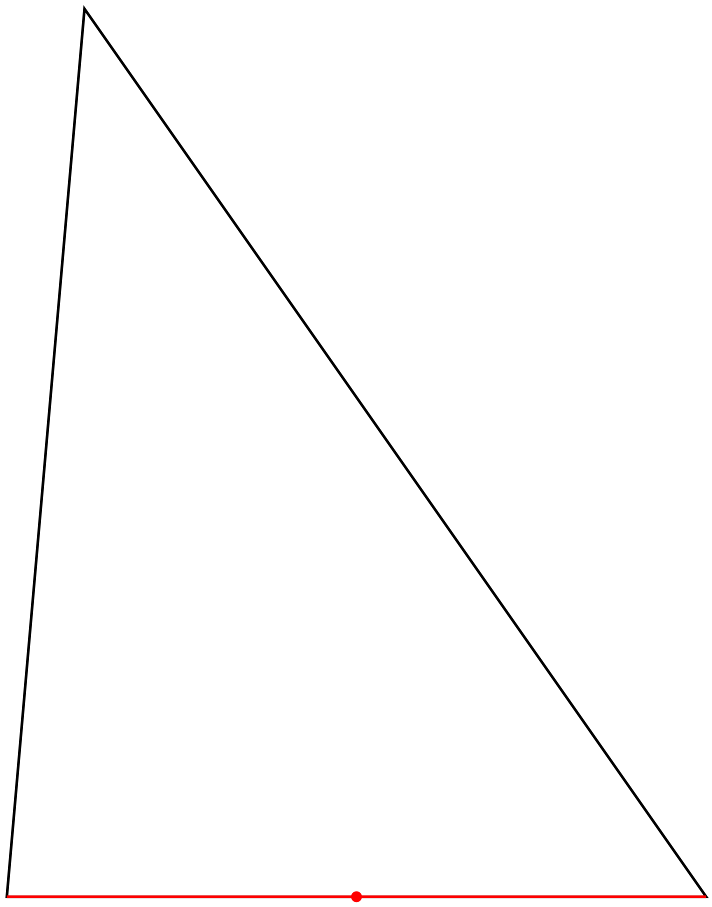
Content

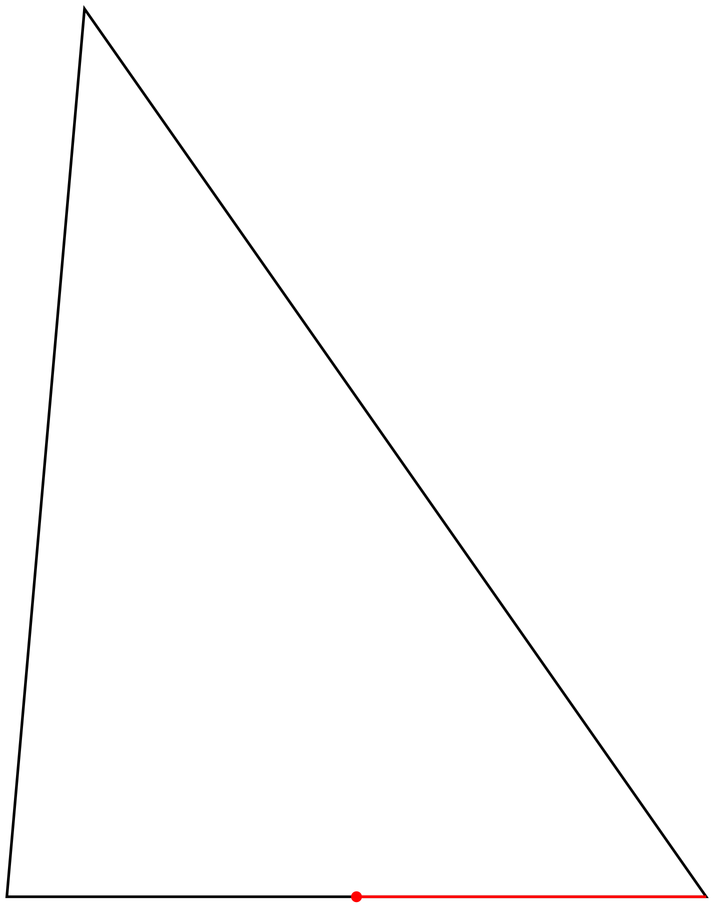
1. The Cirumcenter (O)

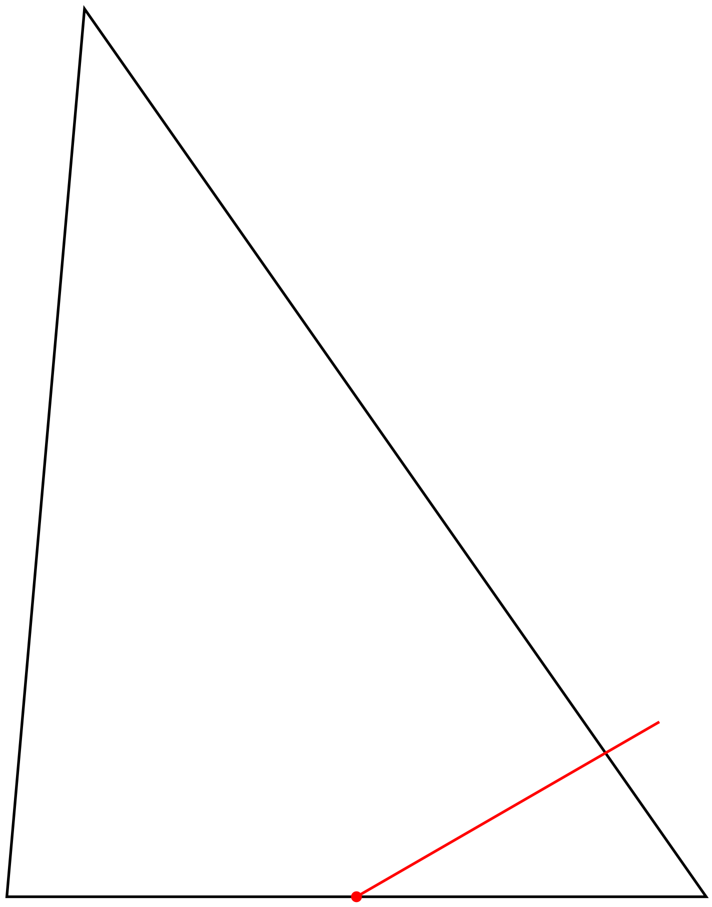
1.1 The Cirumcenter (O)

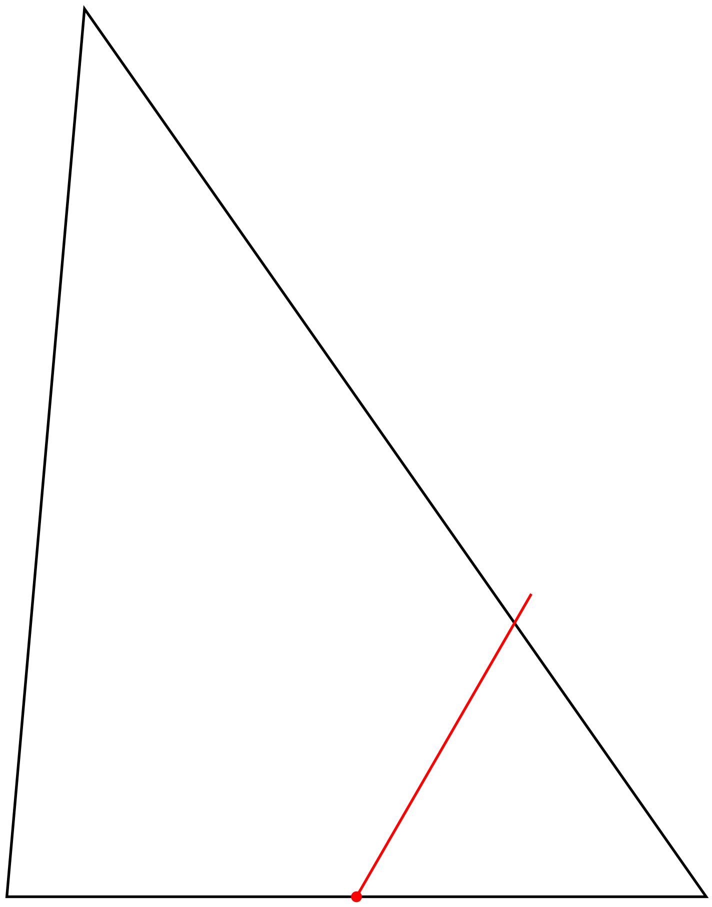
- the formation of O

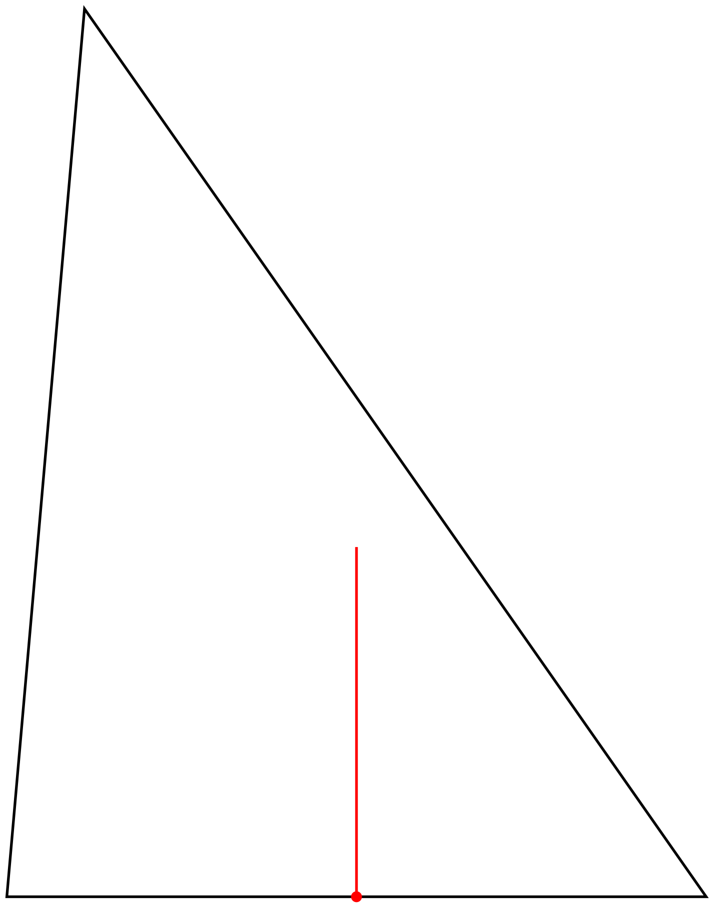


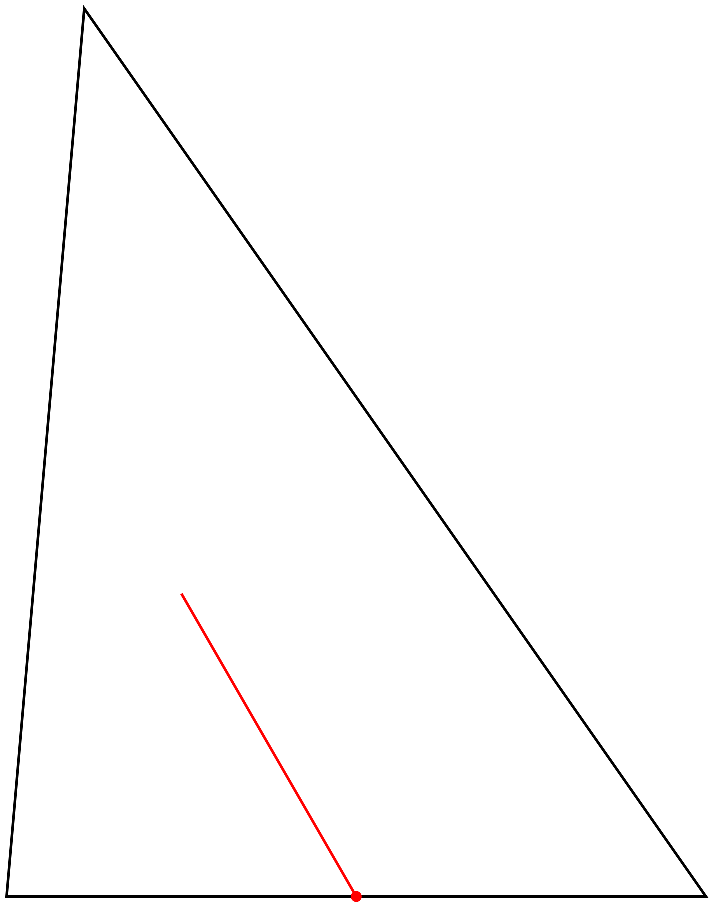
A perpendicular bisector of a side is constructed first by finding the mid-point of that side,

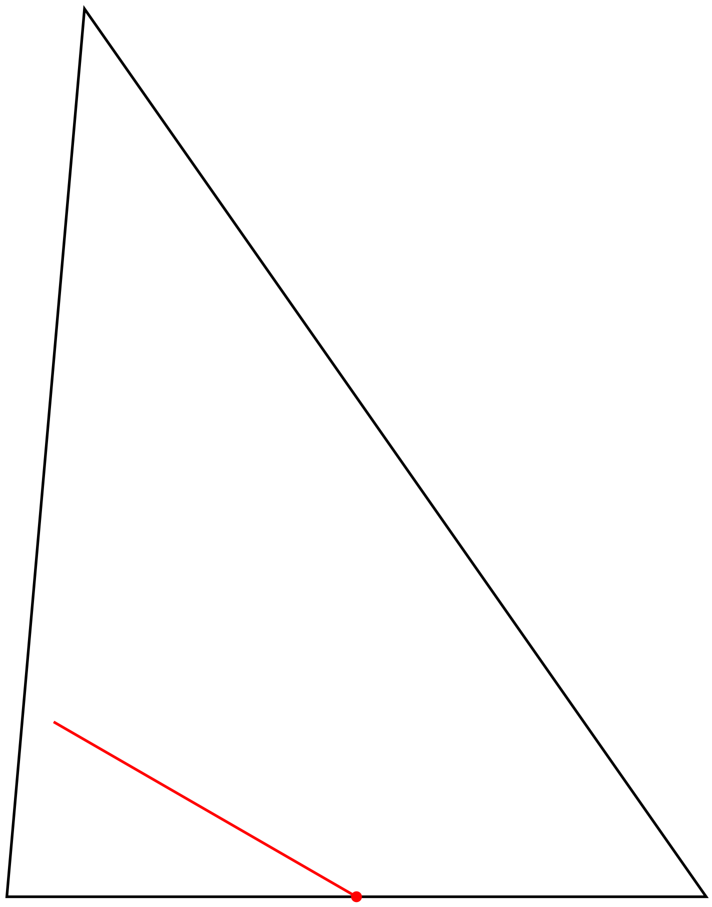


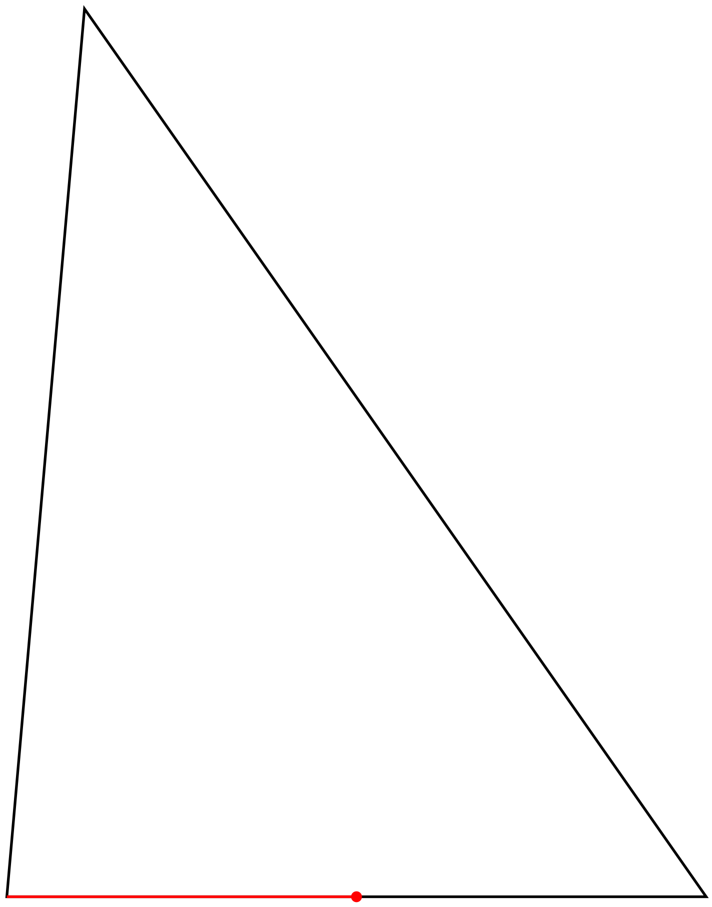


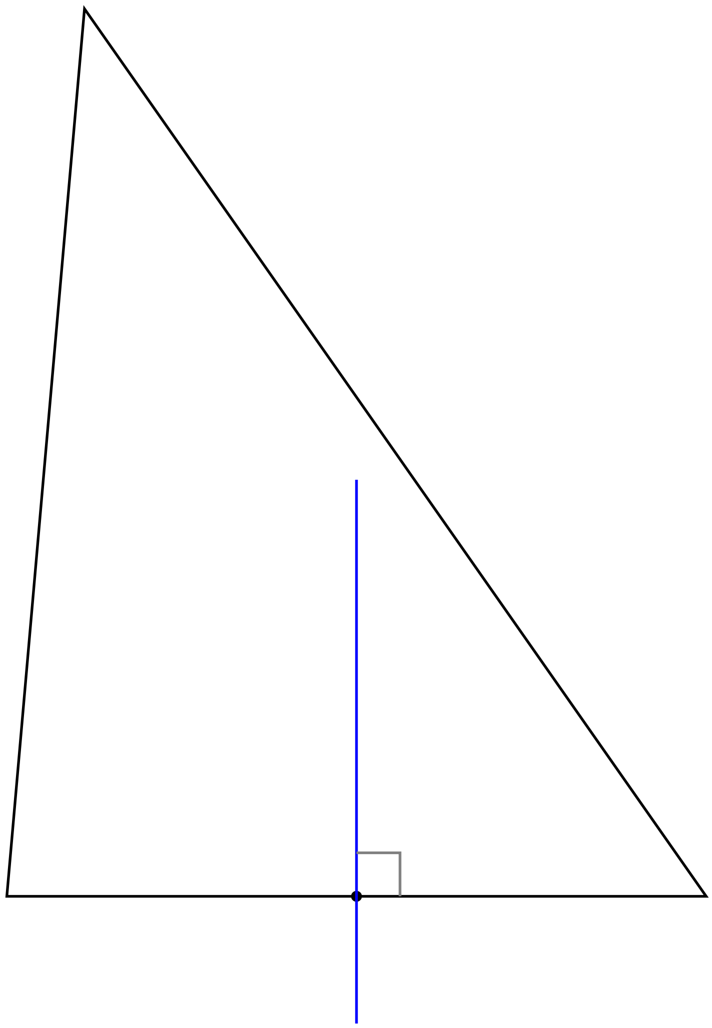




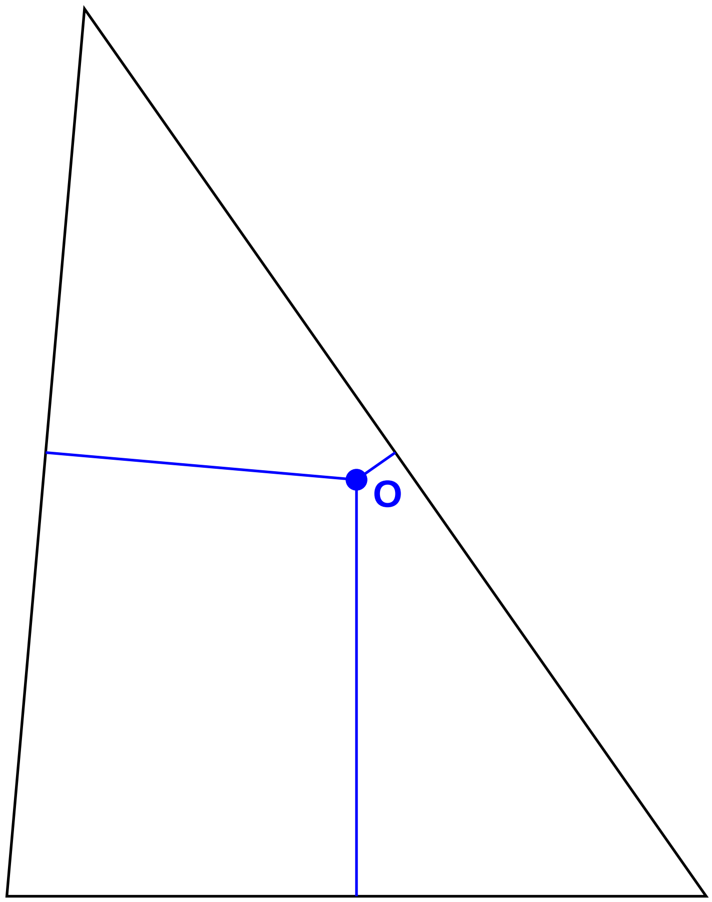








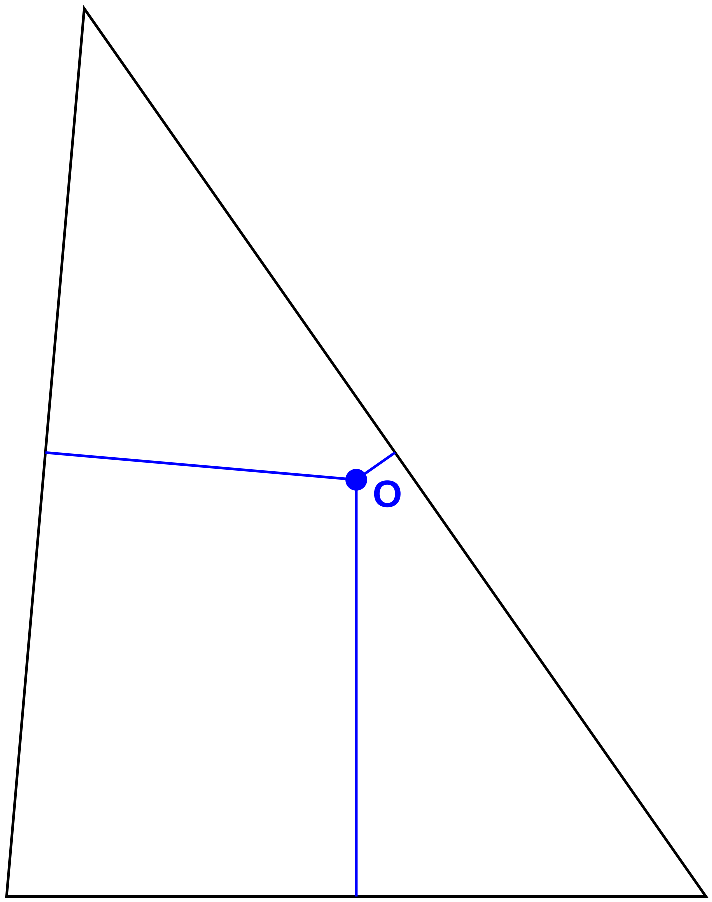
then a line perpendicular to this side is to be constructed.



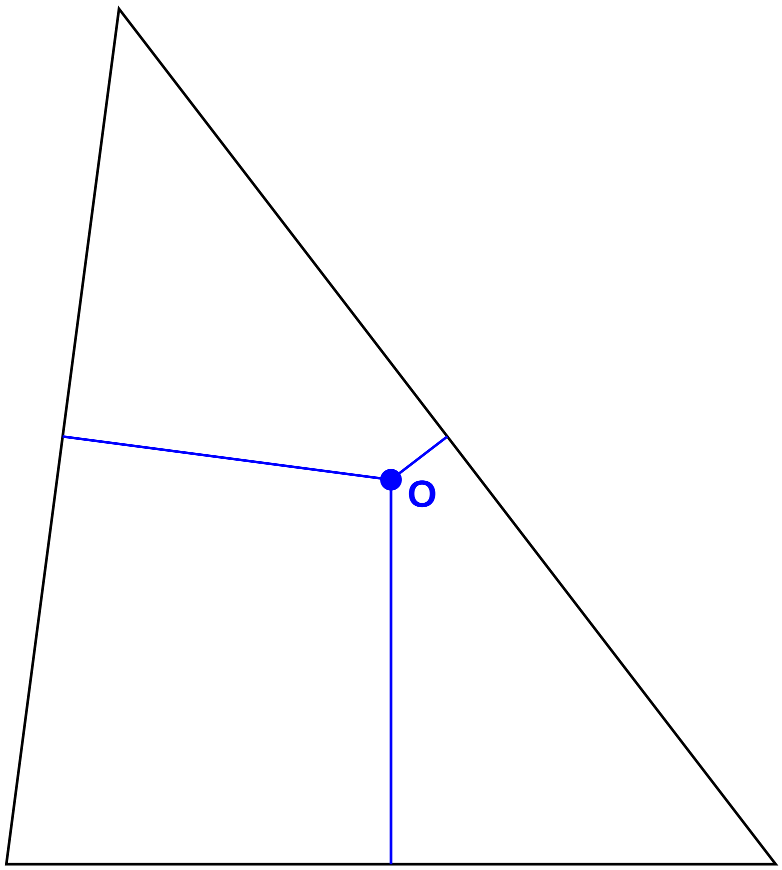
The circumcenter, O, is where the perpendicular bisectors of all three sides meet.

1.2 The Cirumcenter (O)

- the position of O



With acute triangles, the circumcenter, O, is always within the triangle itself.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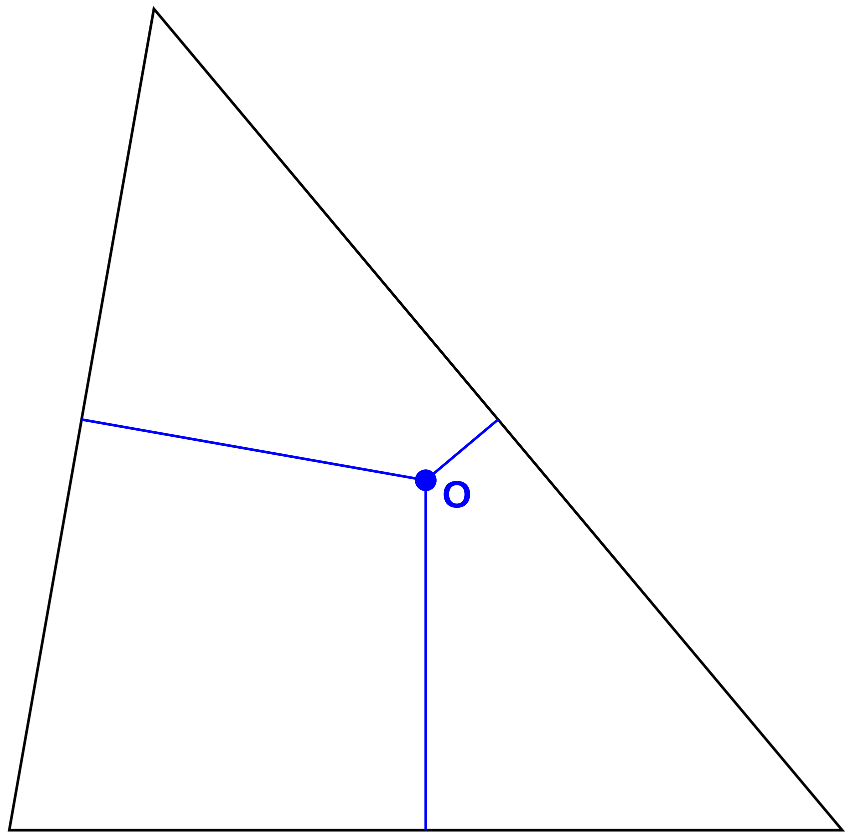
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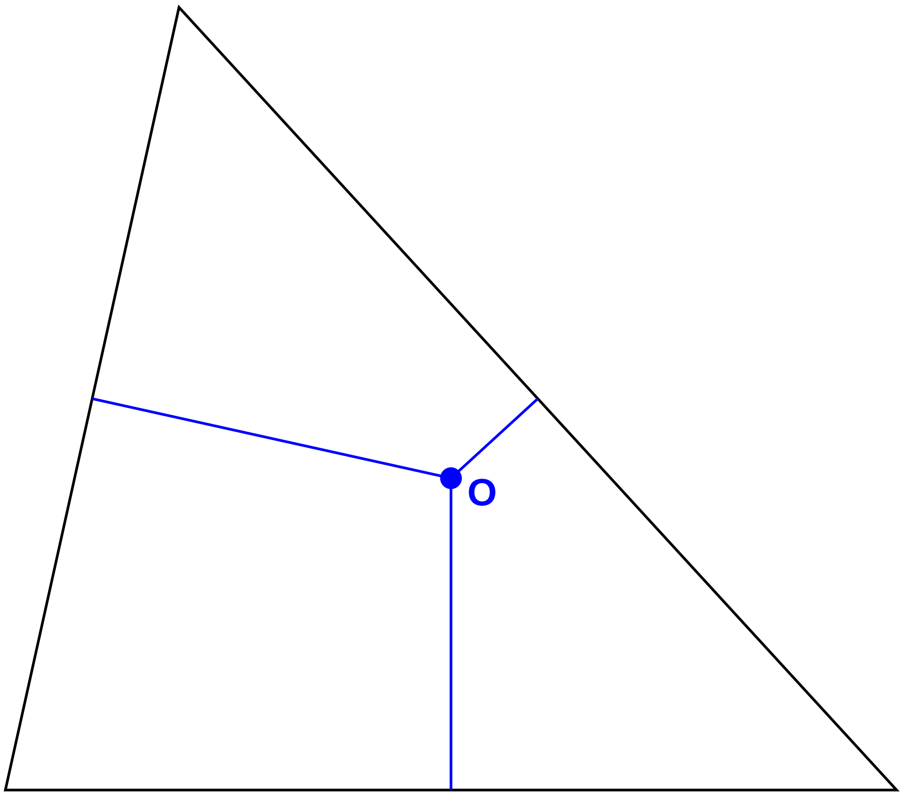
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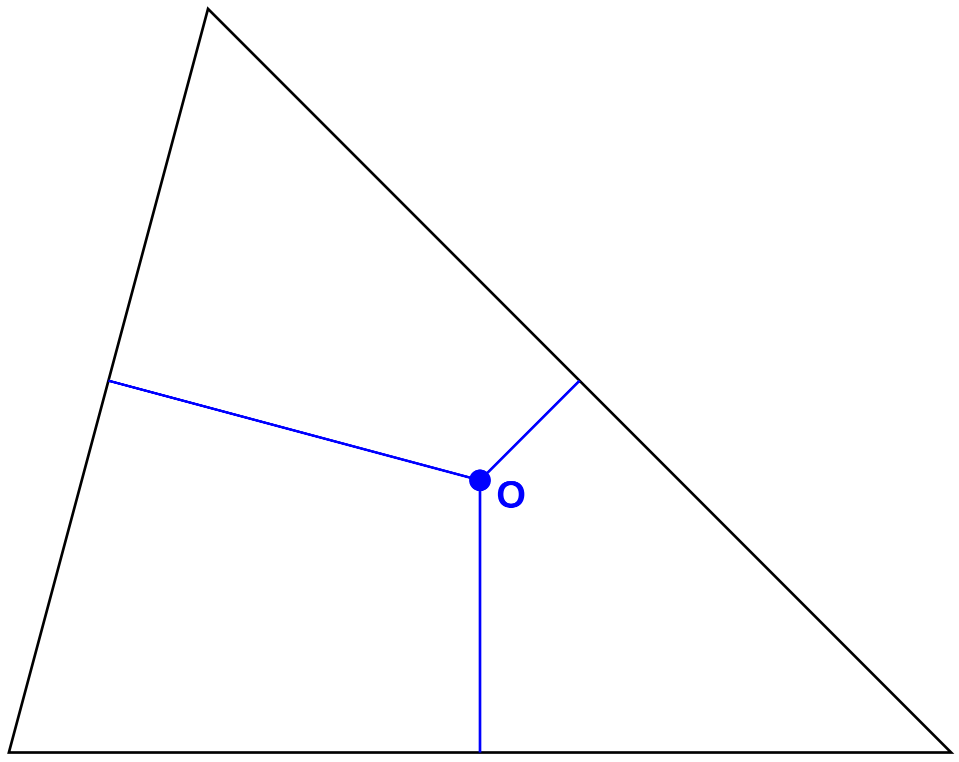
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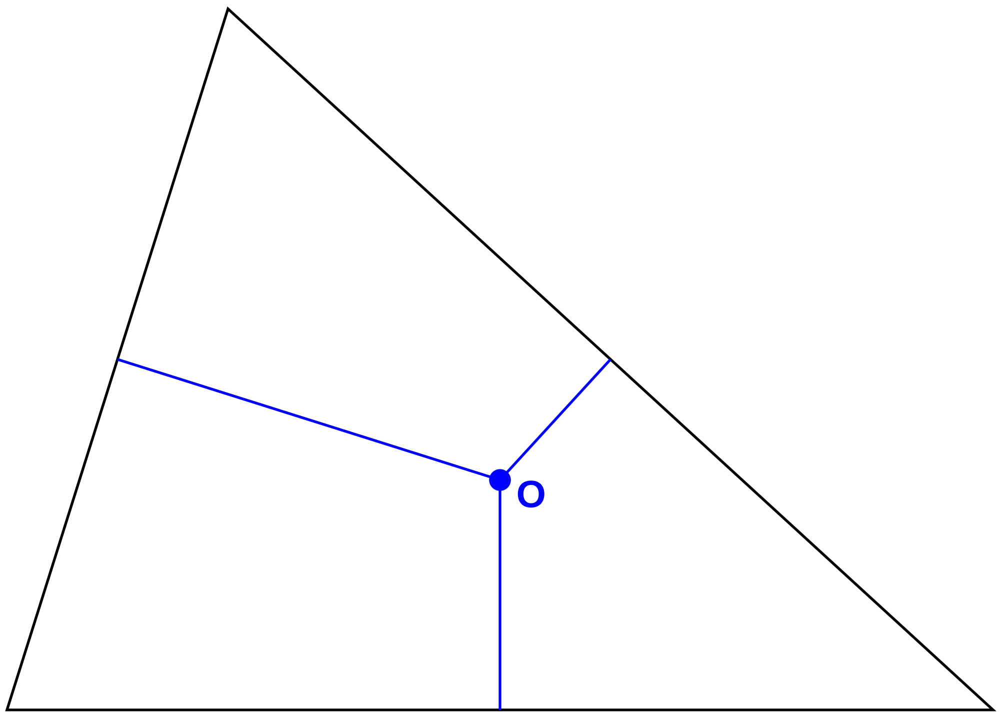
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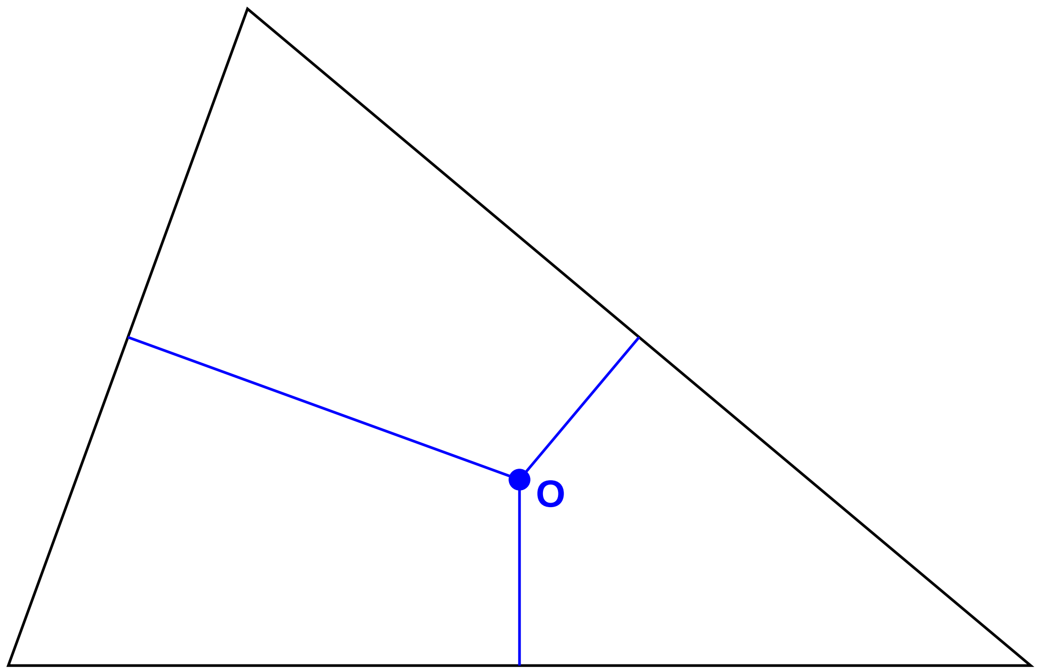
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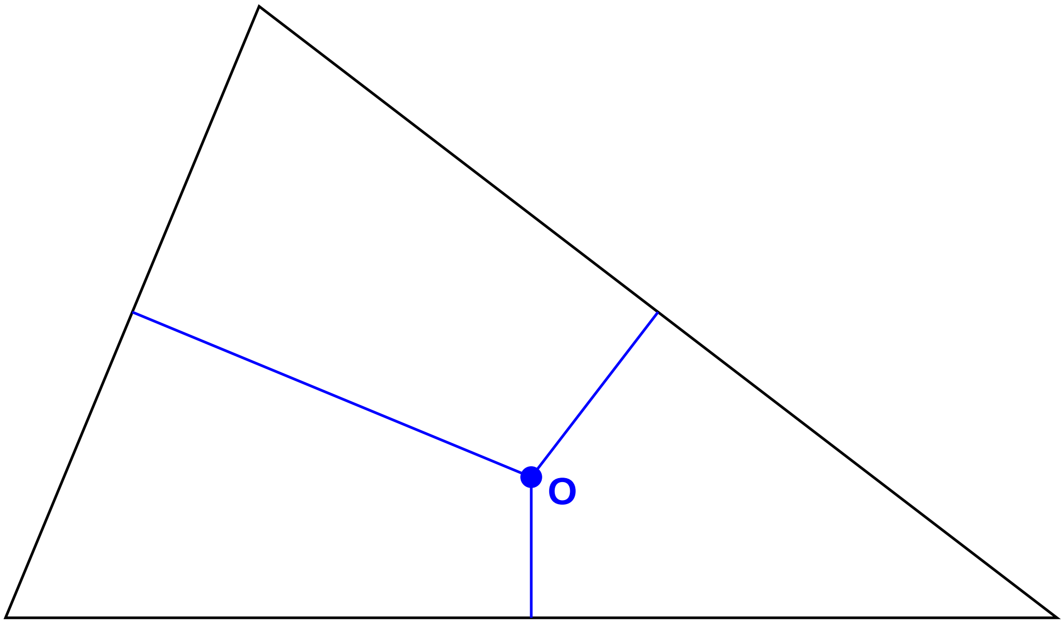
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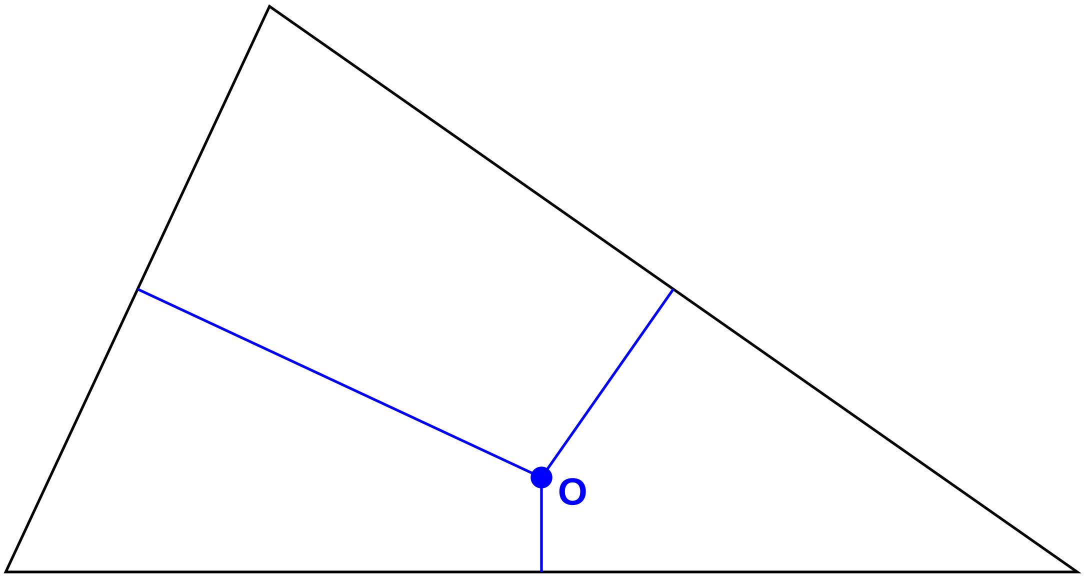
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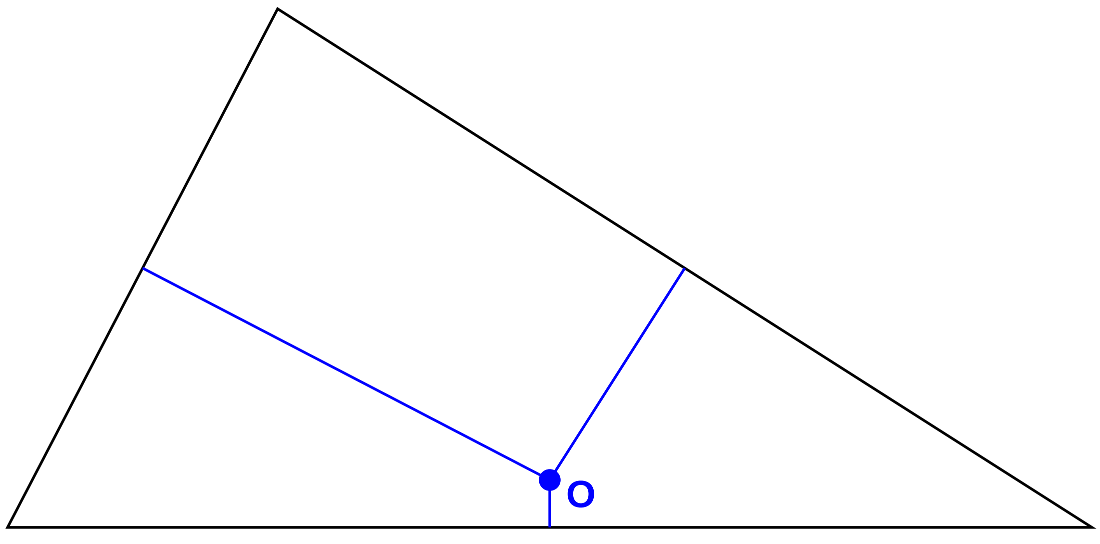
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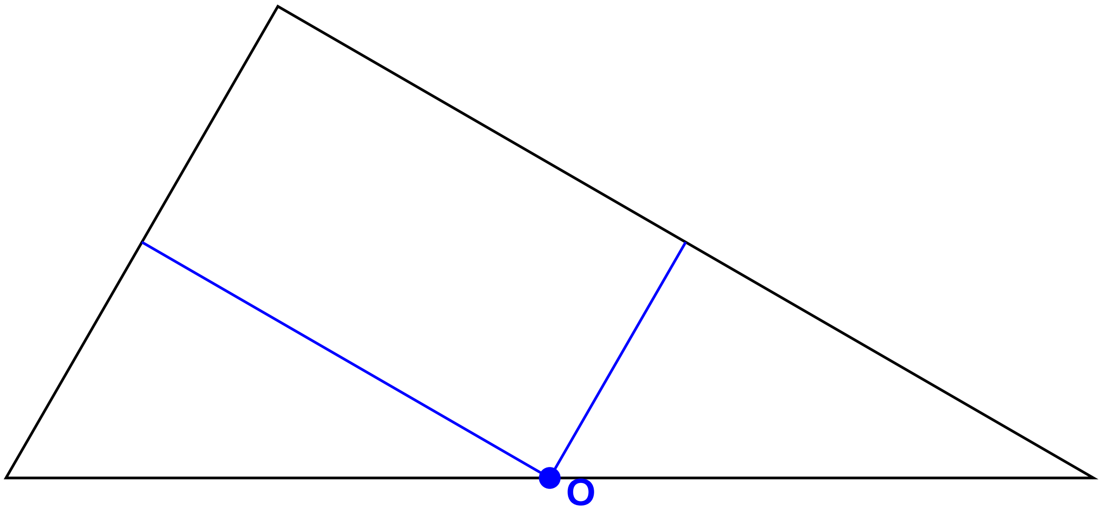
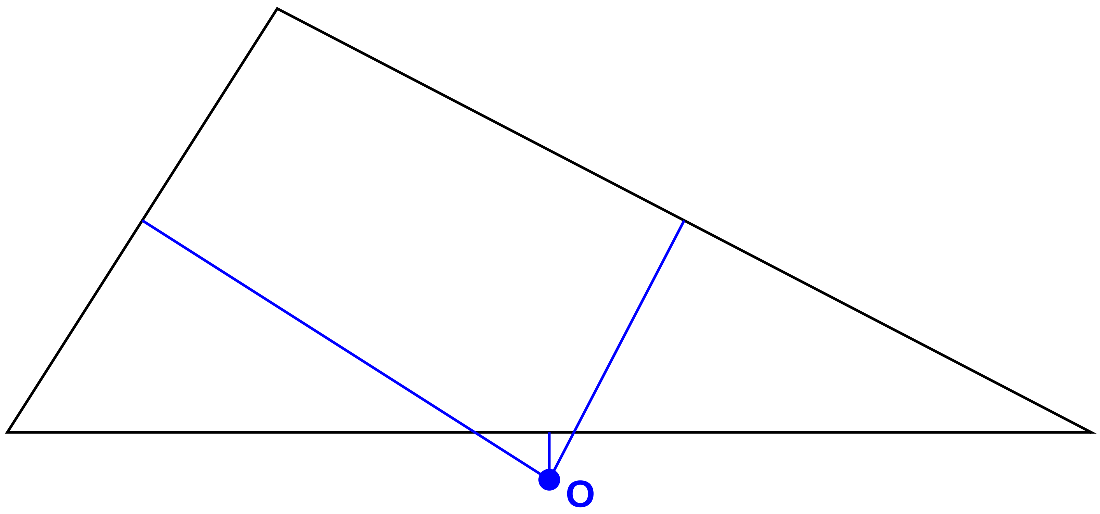
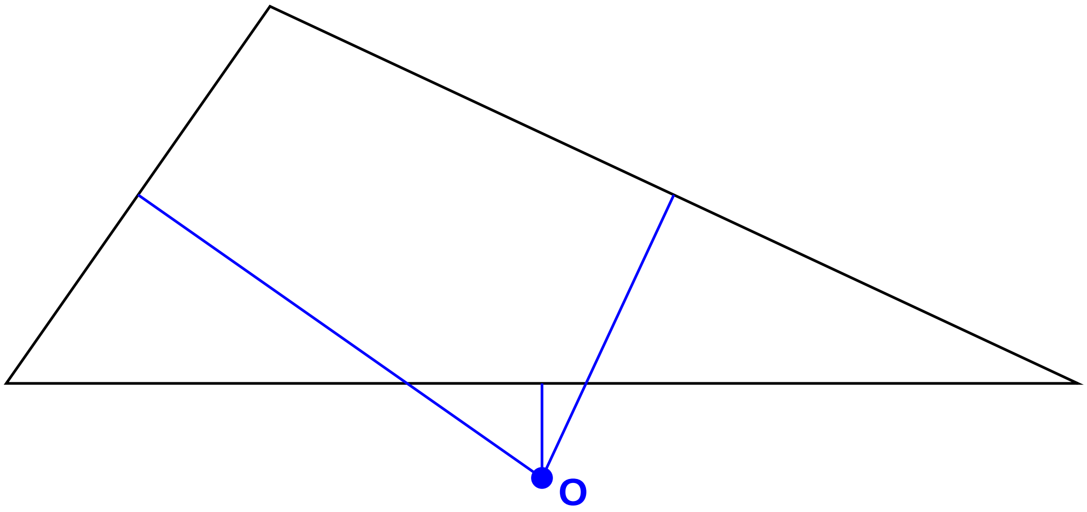


However, in the case of a right triangle, the circumcenter is at the midpoint of the hypotenuse.



And with obtuse triangles, their circumcenters are outside of the triangles on the extension of the perpendicular bisector opposite to the obtuse angle.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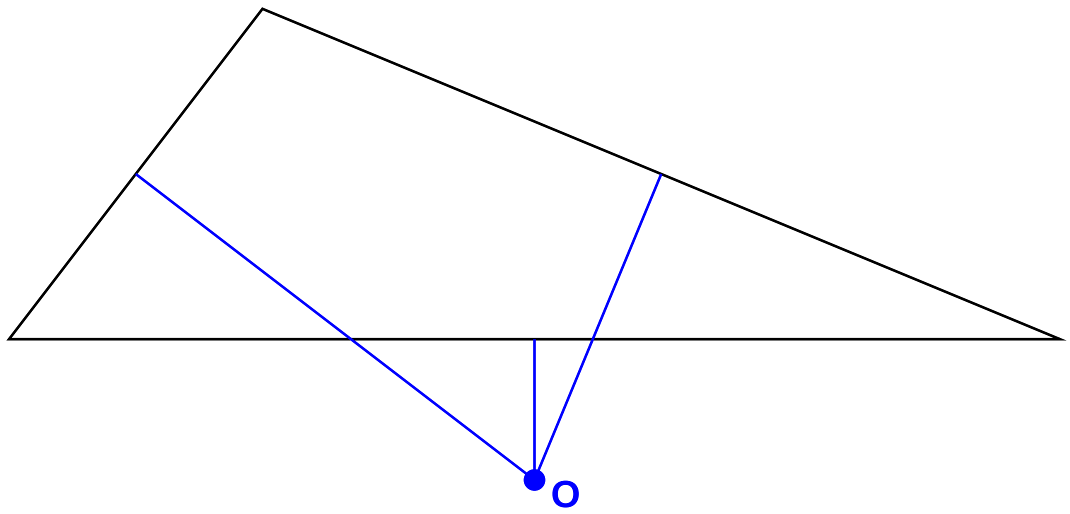
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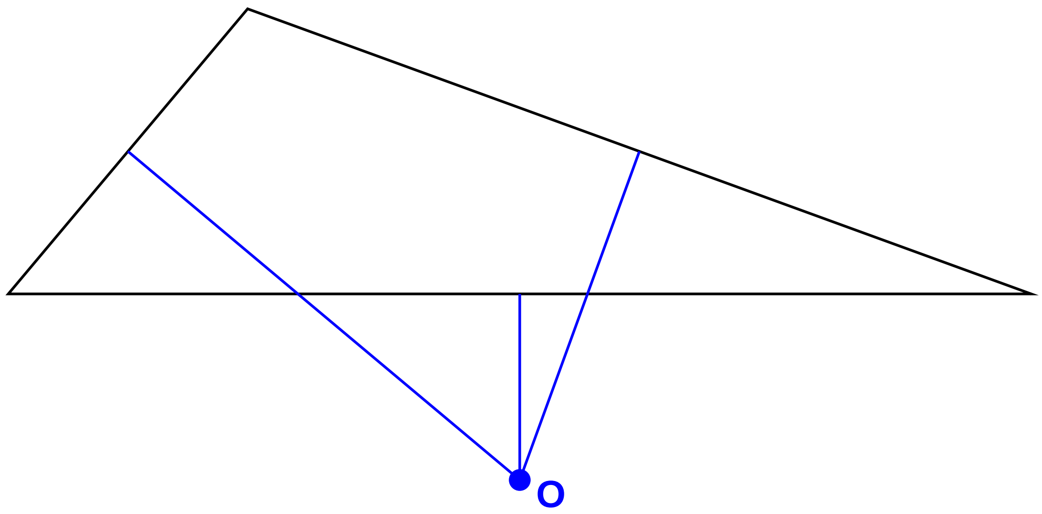
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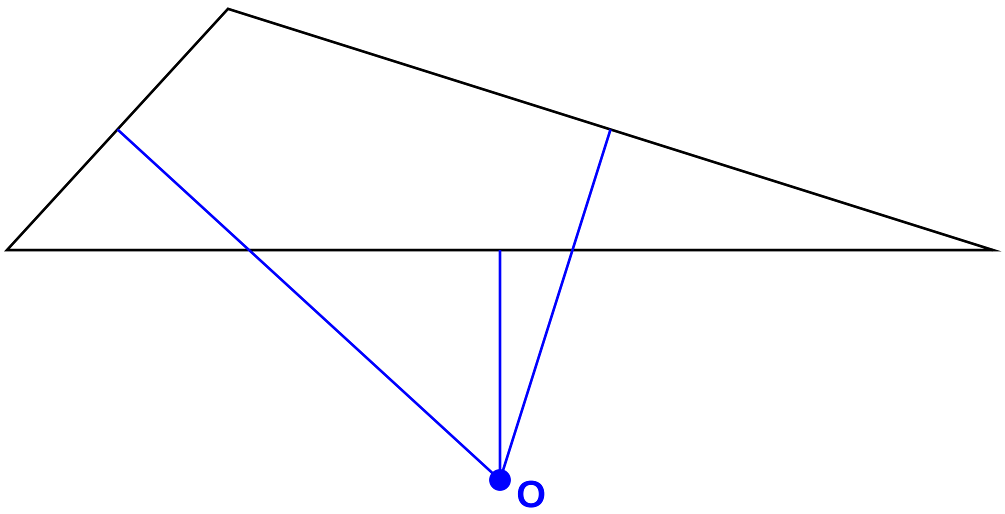
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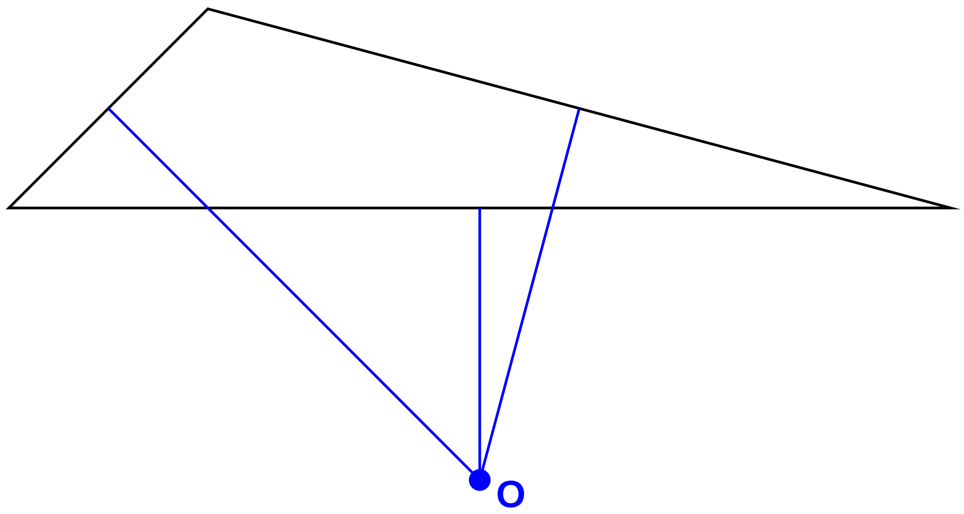
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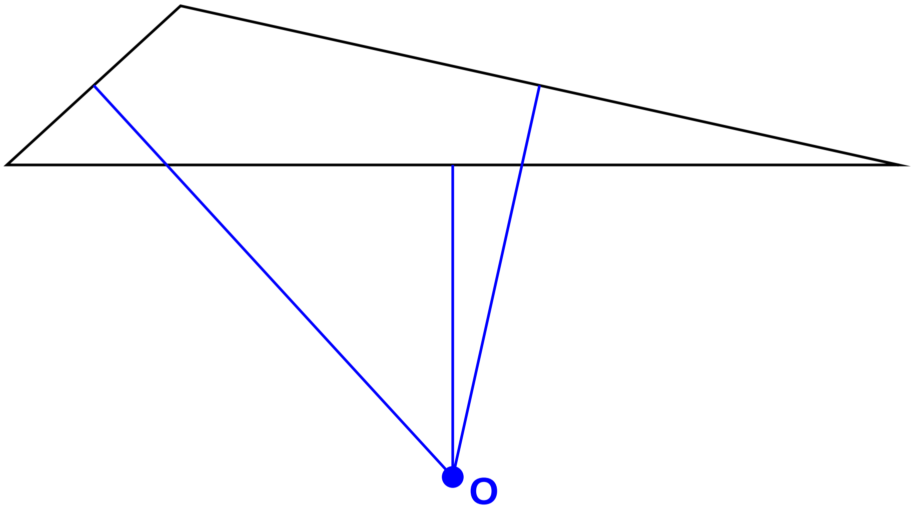
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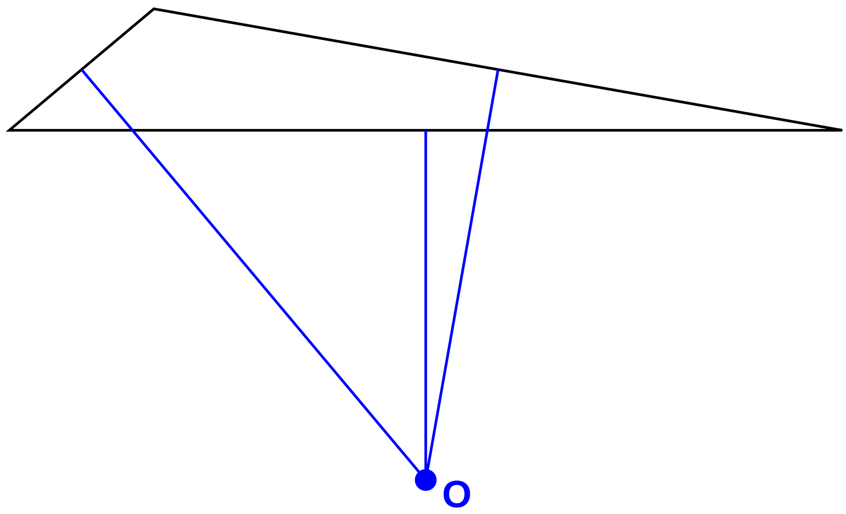
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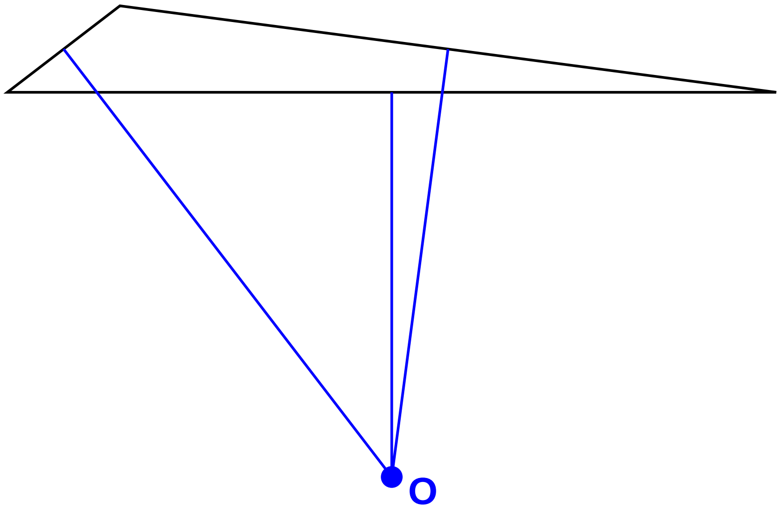
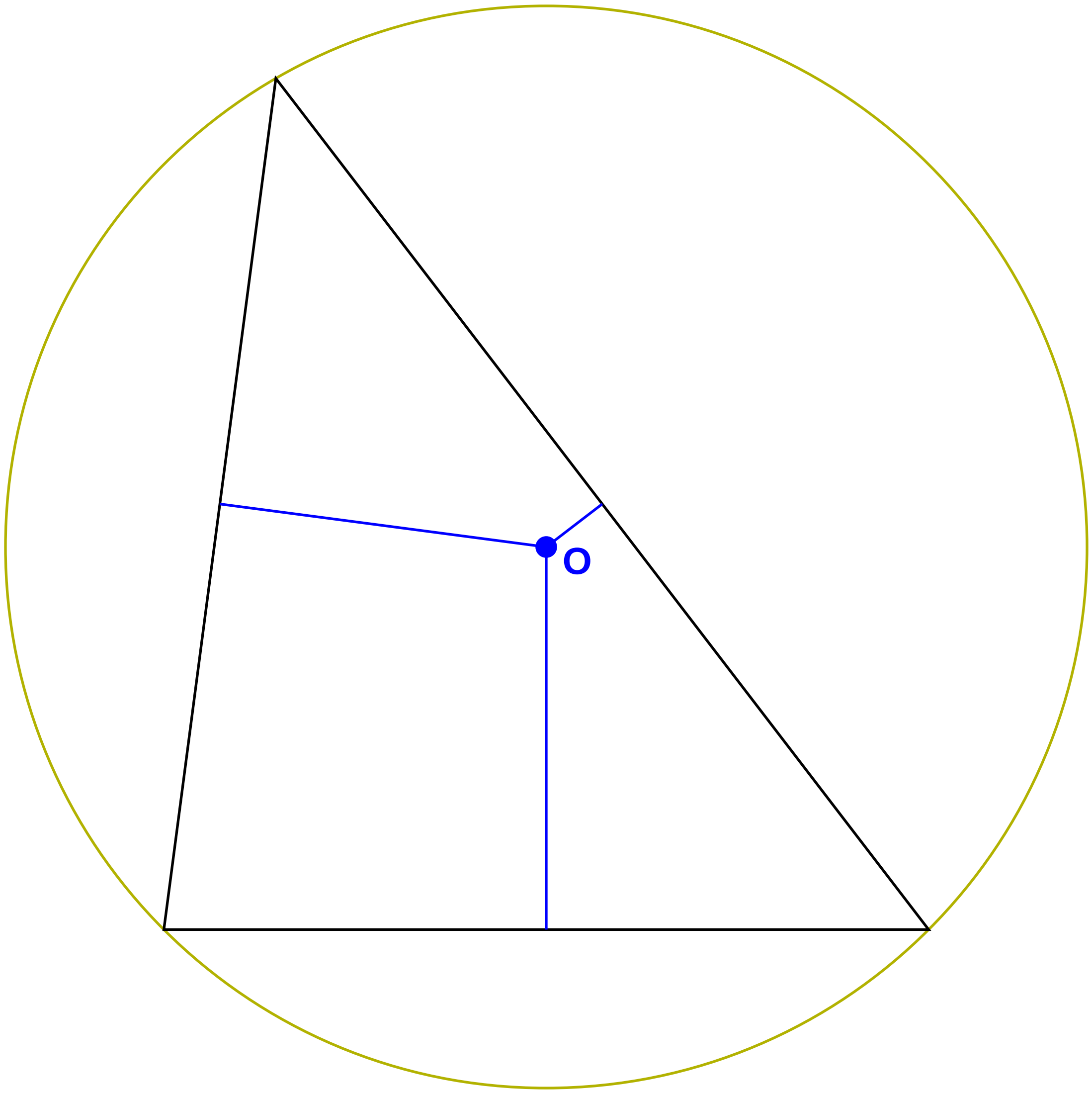
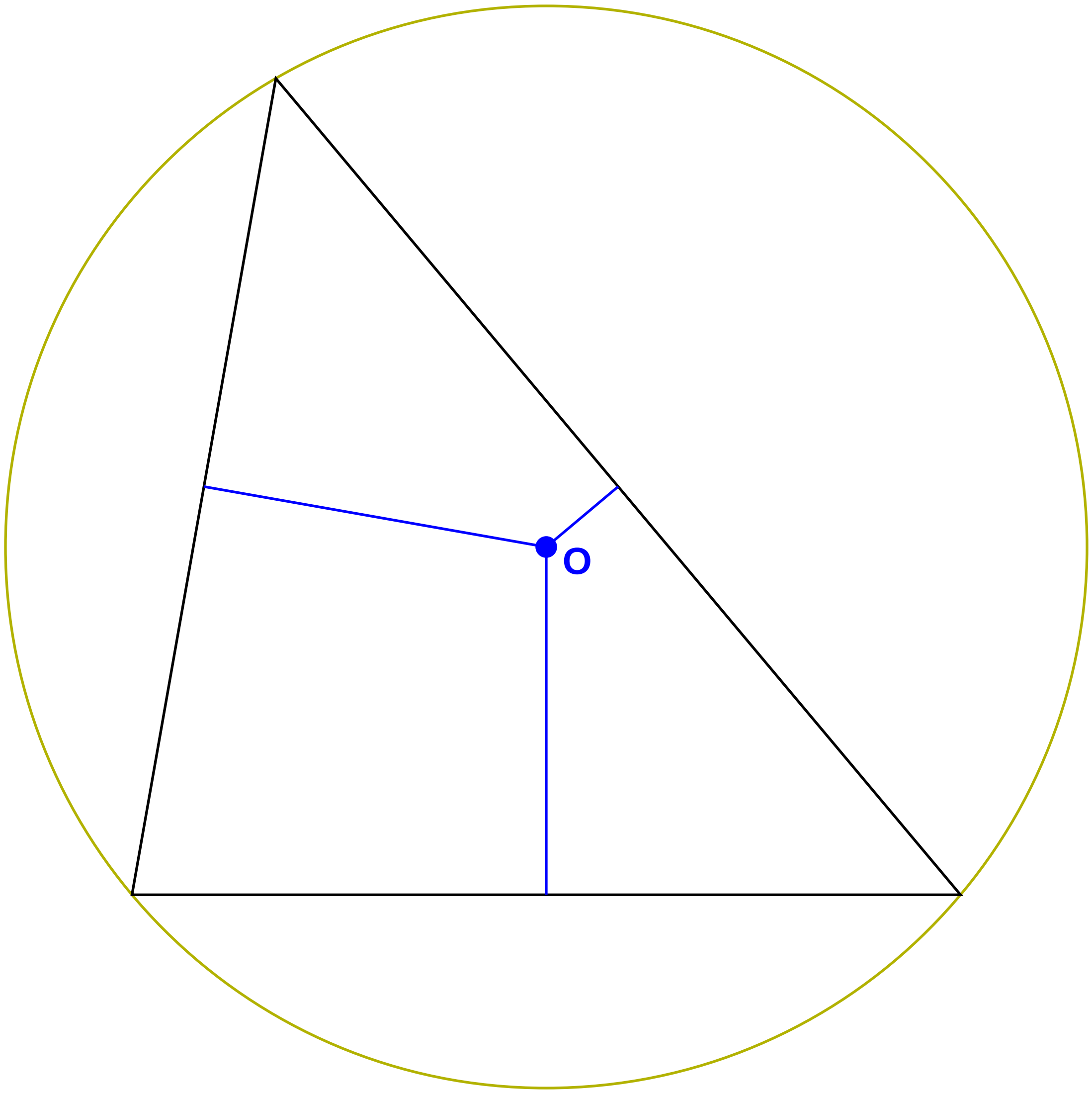

O

1.3 The Cirumcenter (O)

-center of circumscribed circle



Simultaneously, the circumcenter is also the center of the circumscribed circle - the circle made of the vertices of the original triangle.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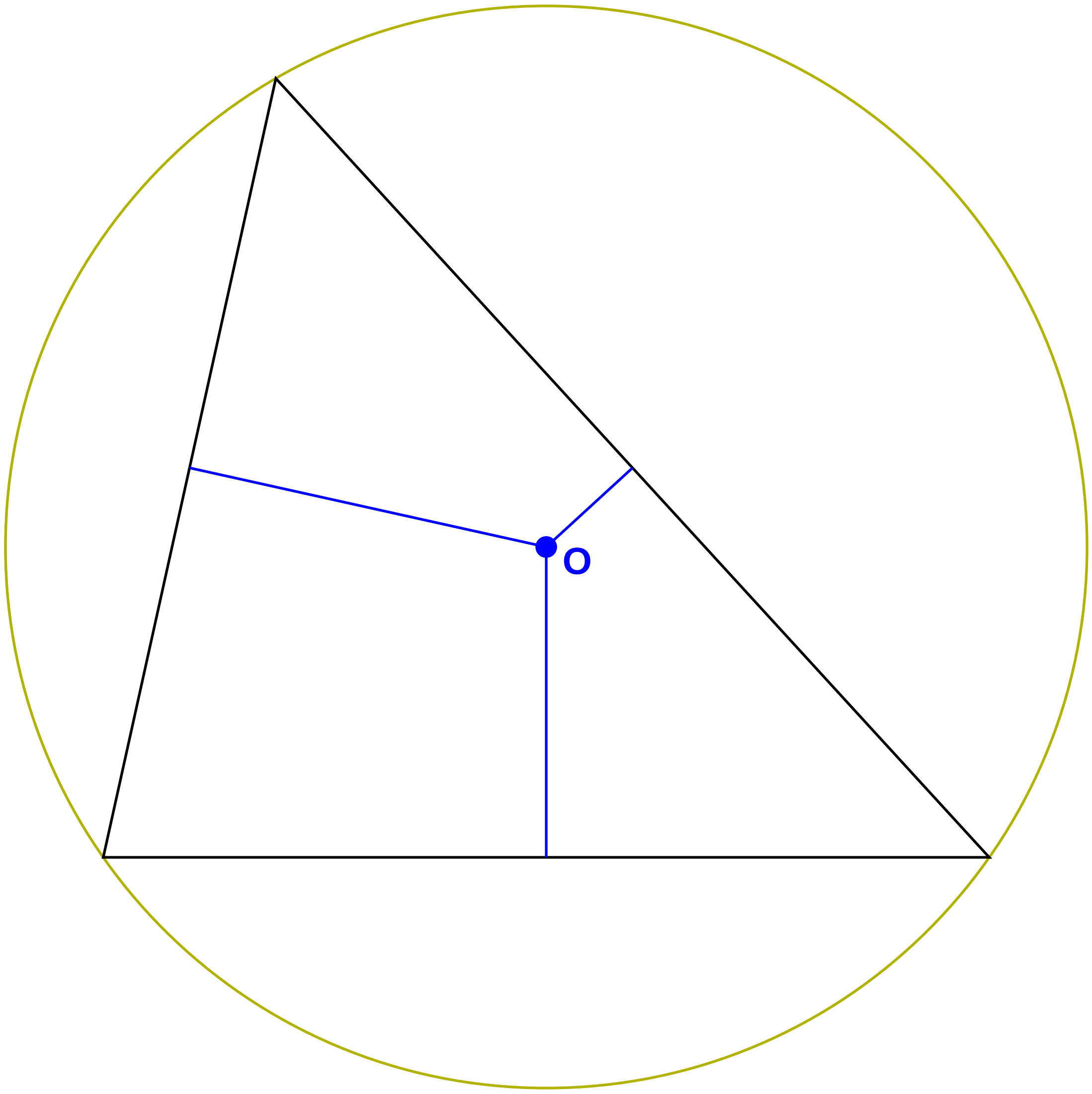
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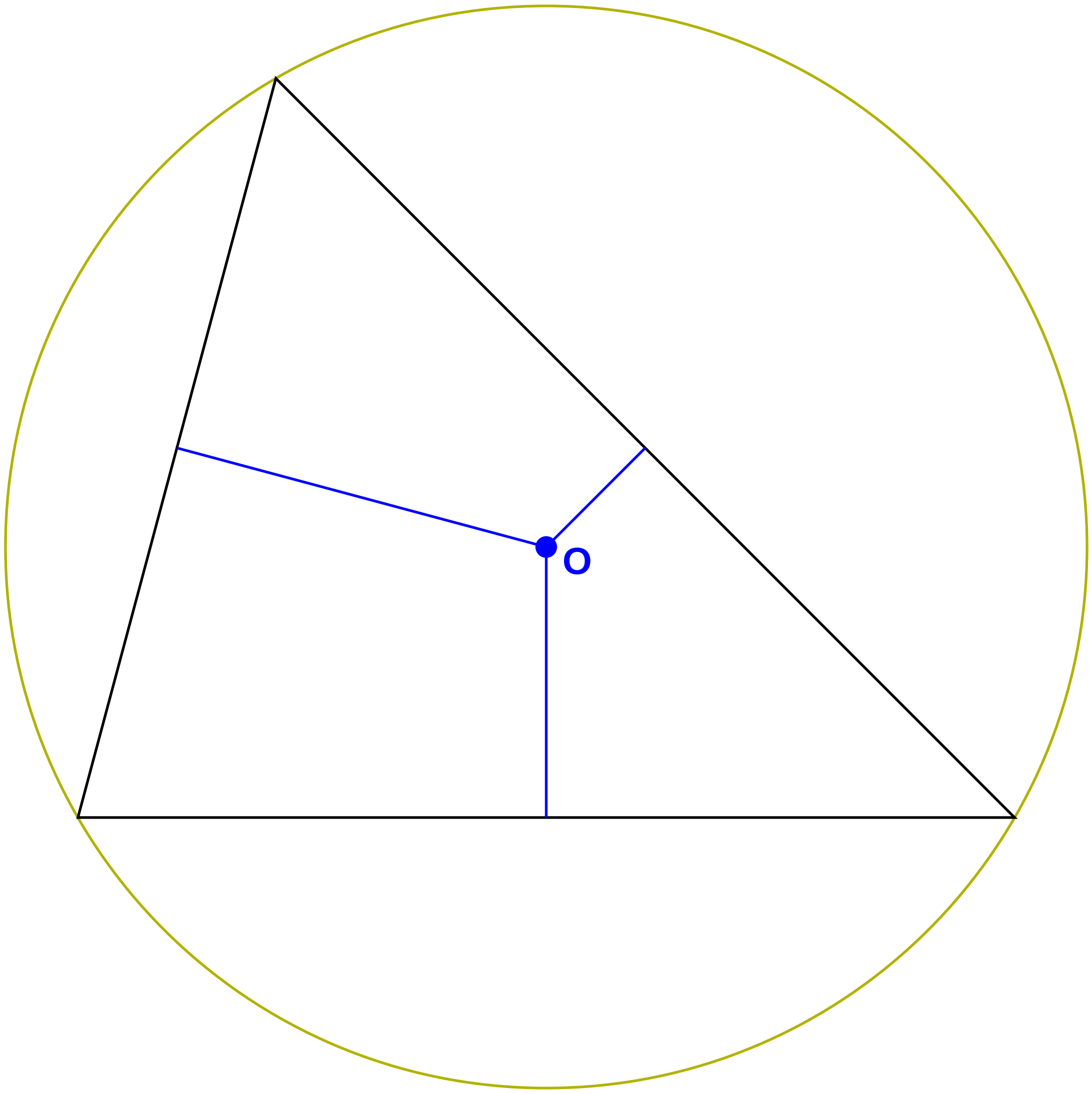
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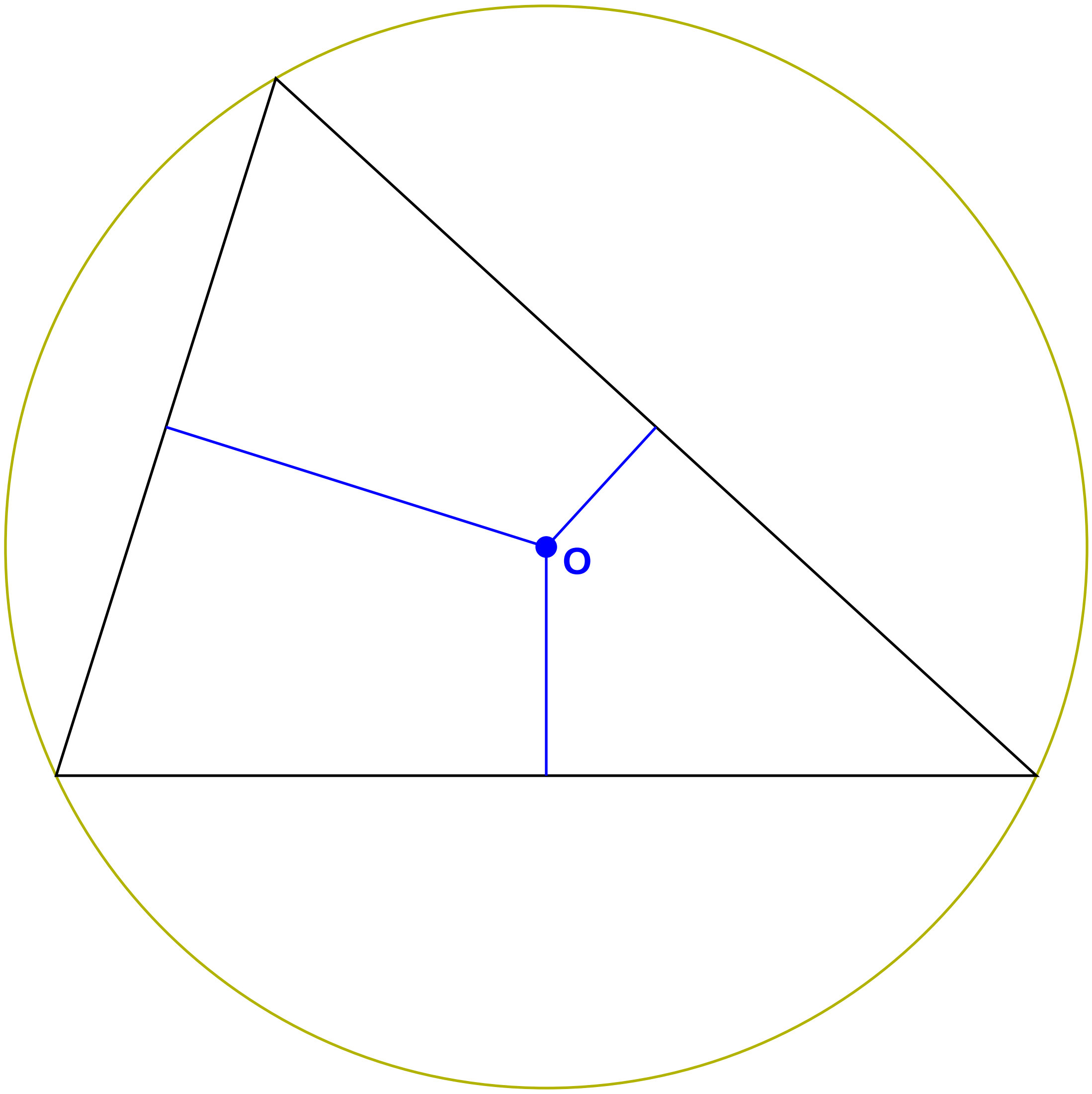
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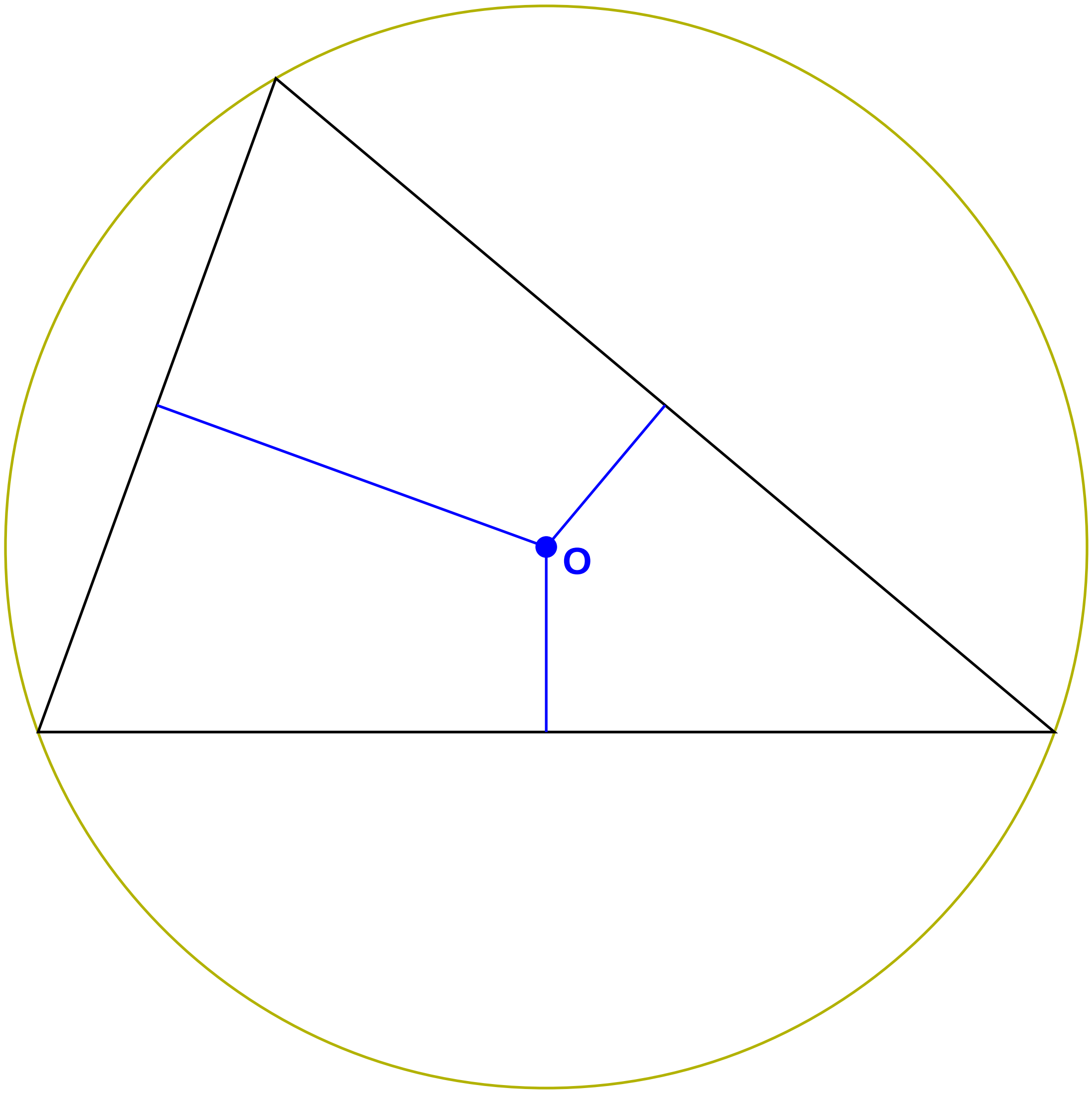
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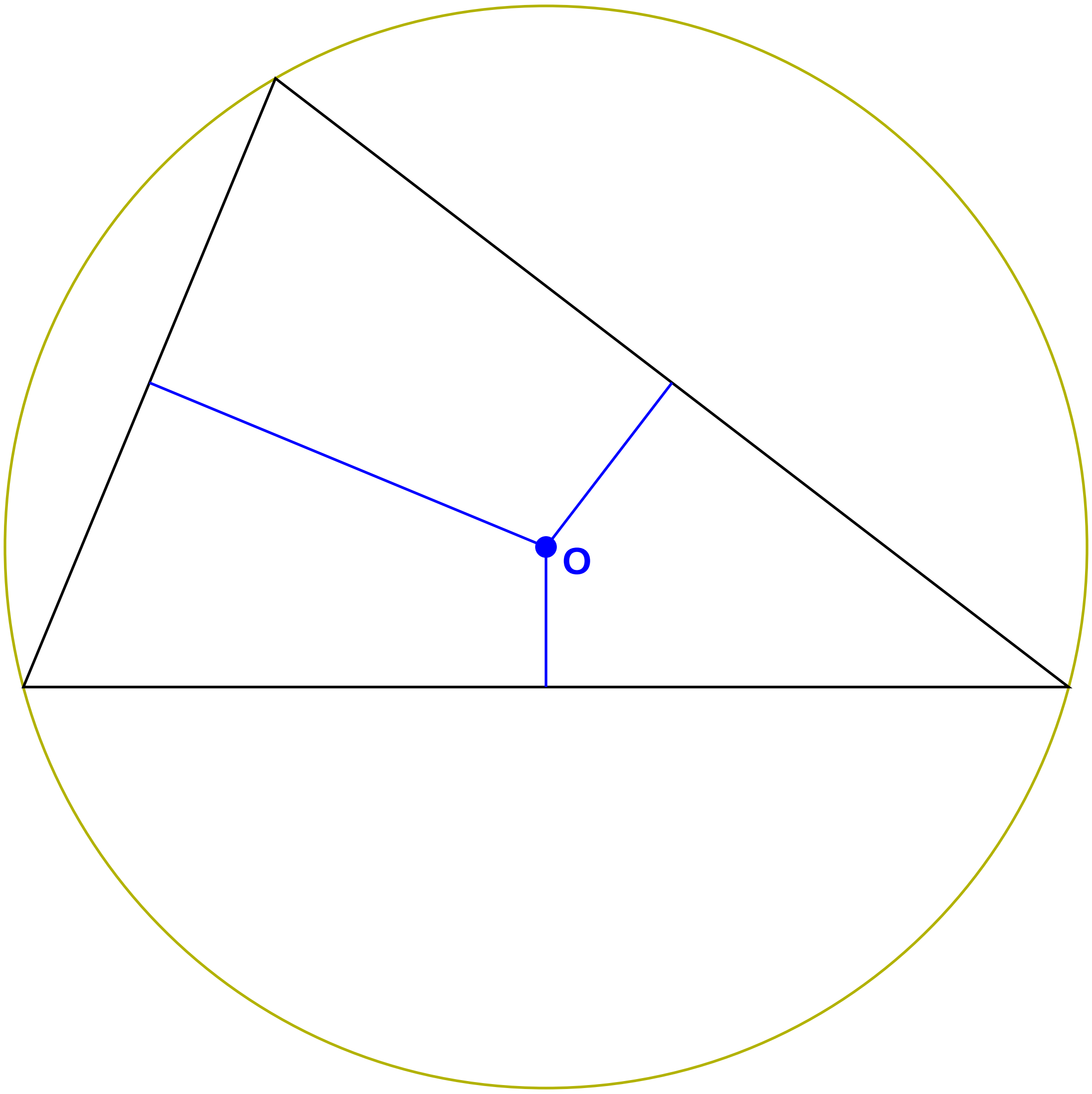
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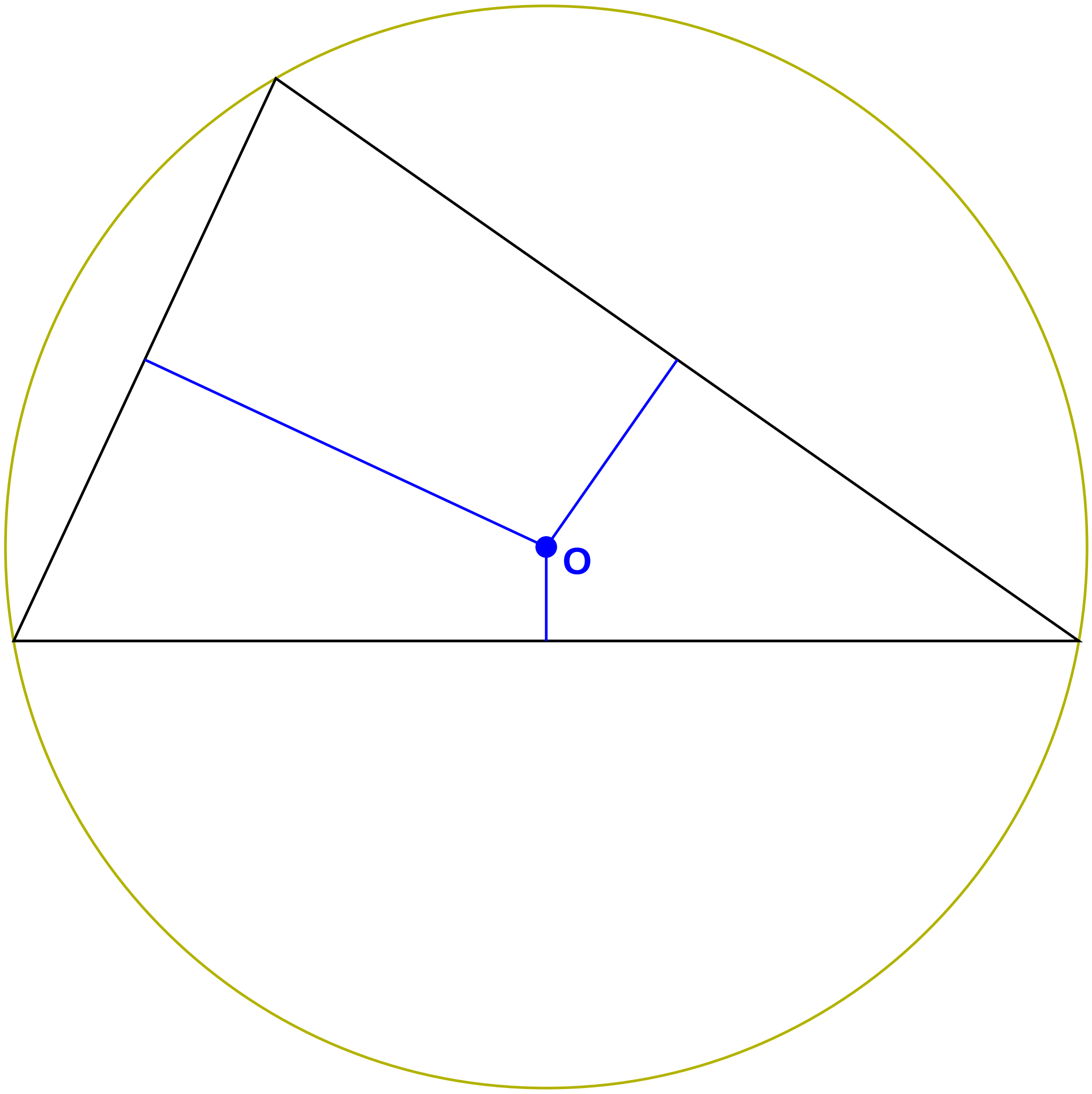
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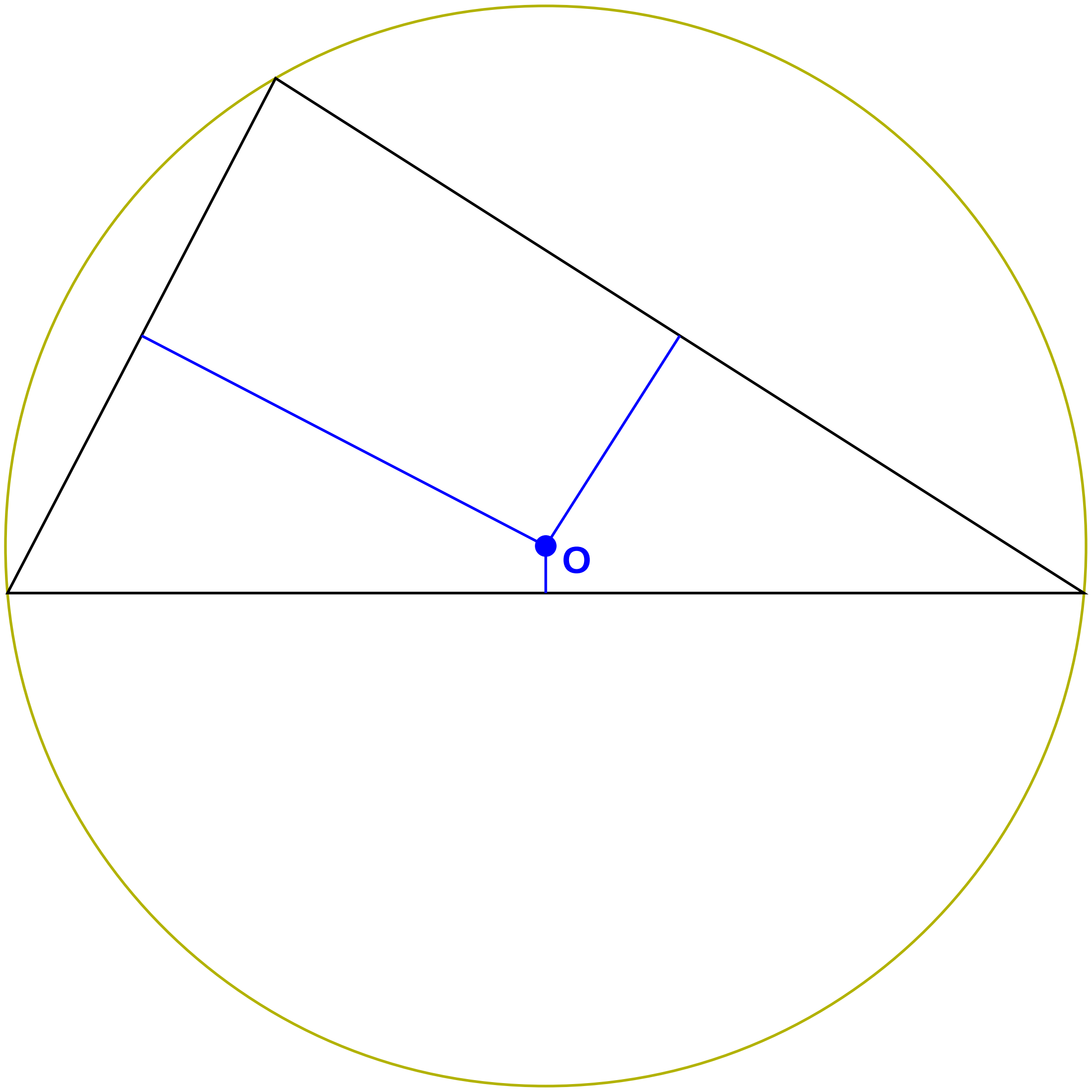
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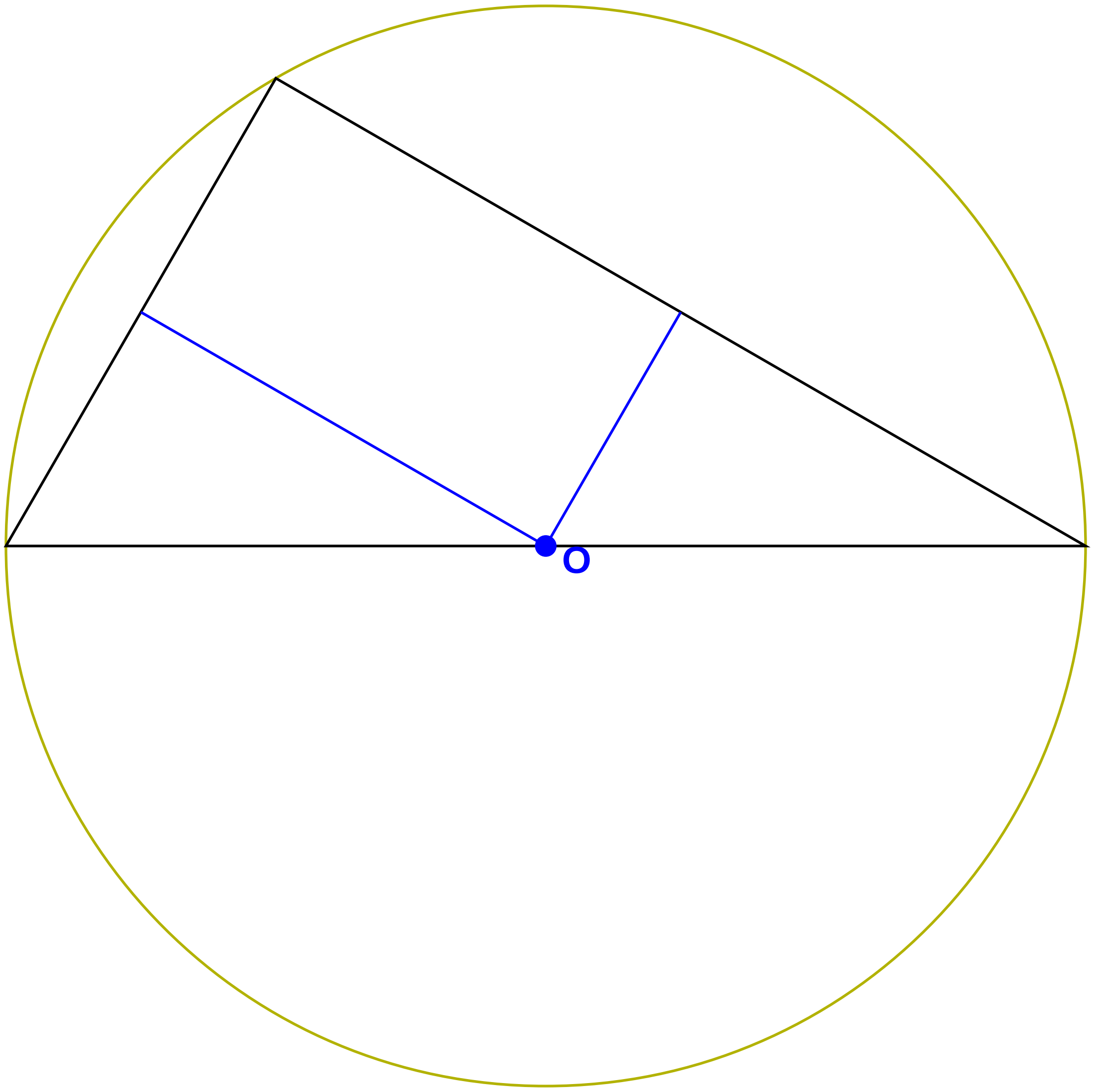
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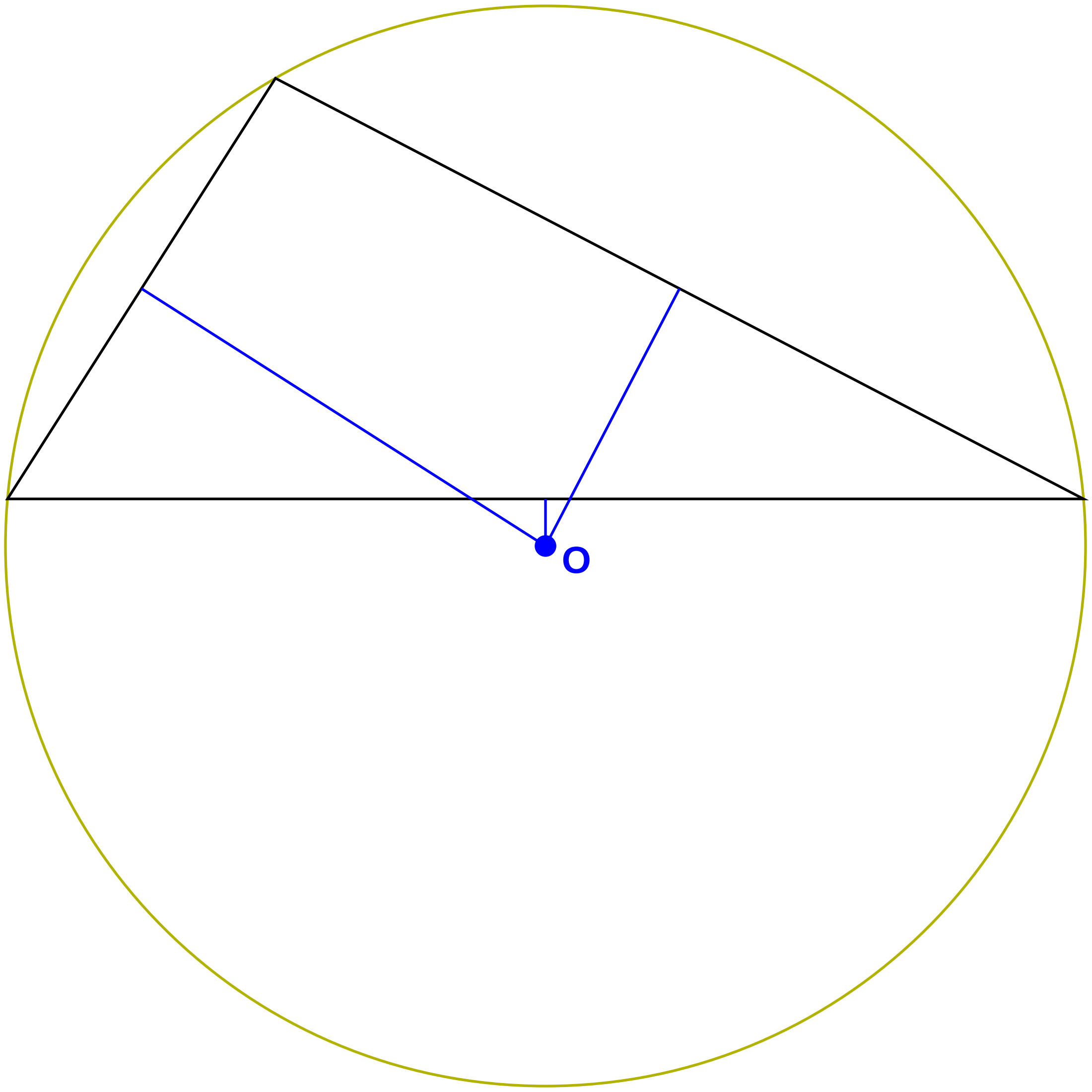
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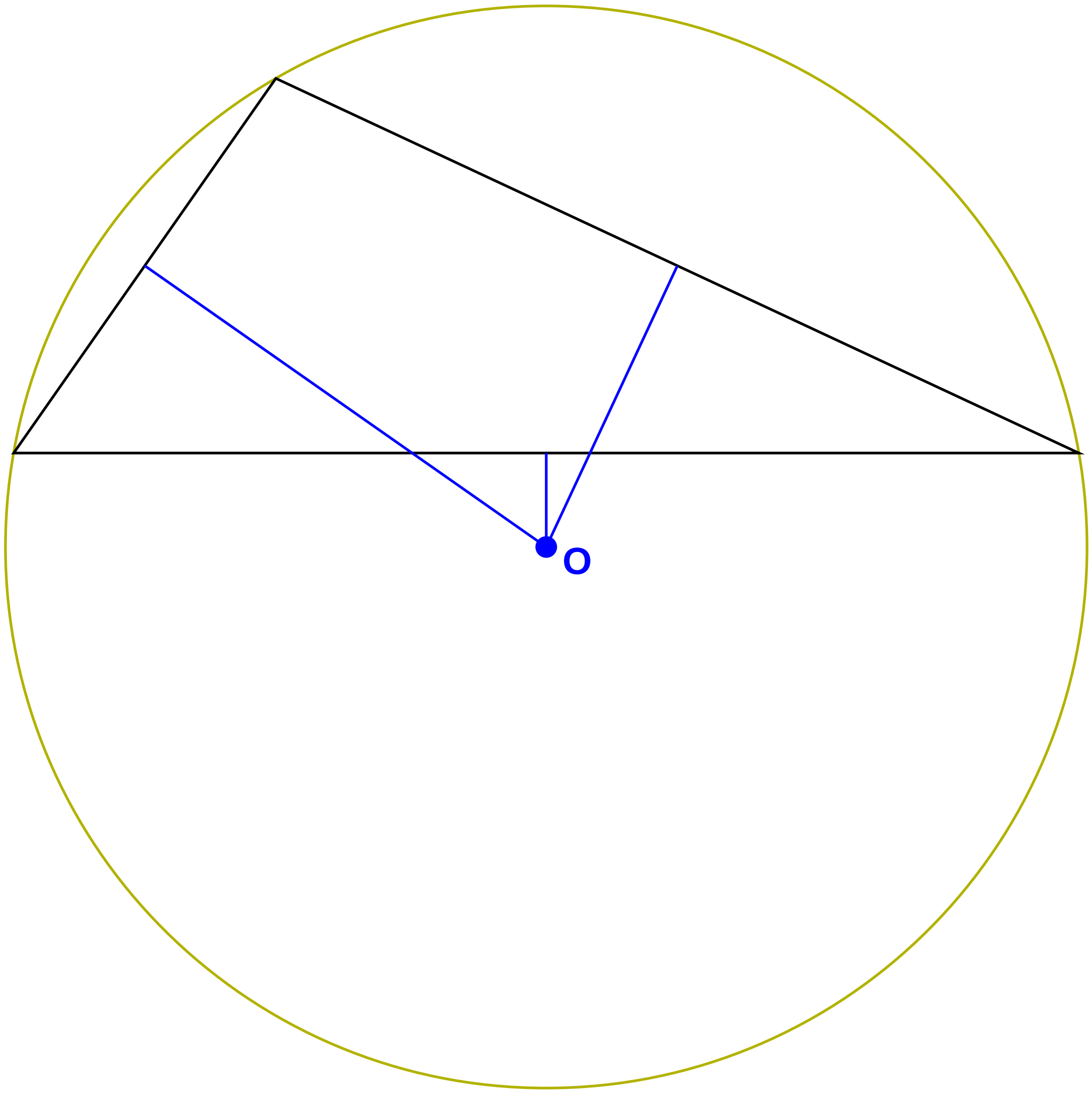
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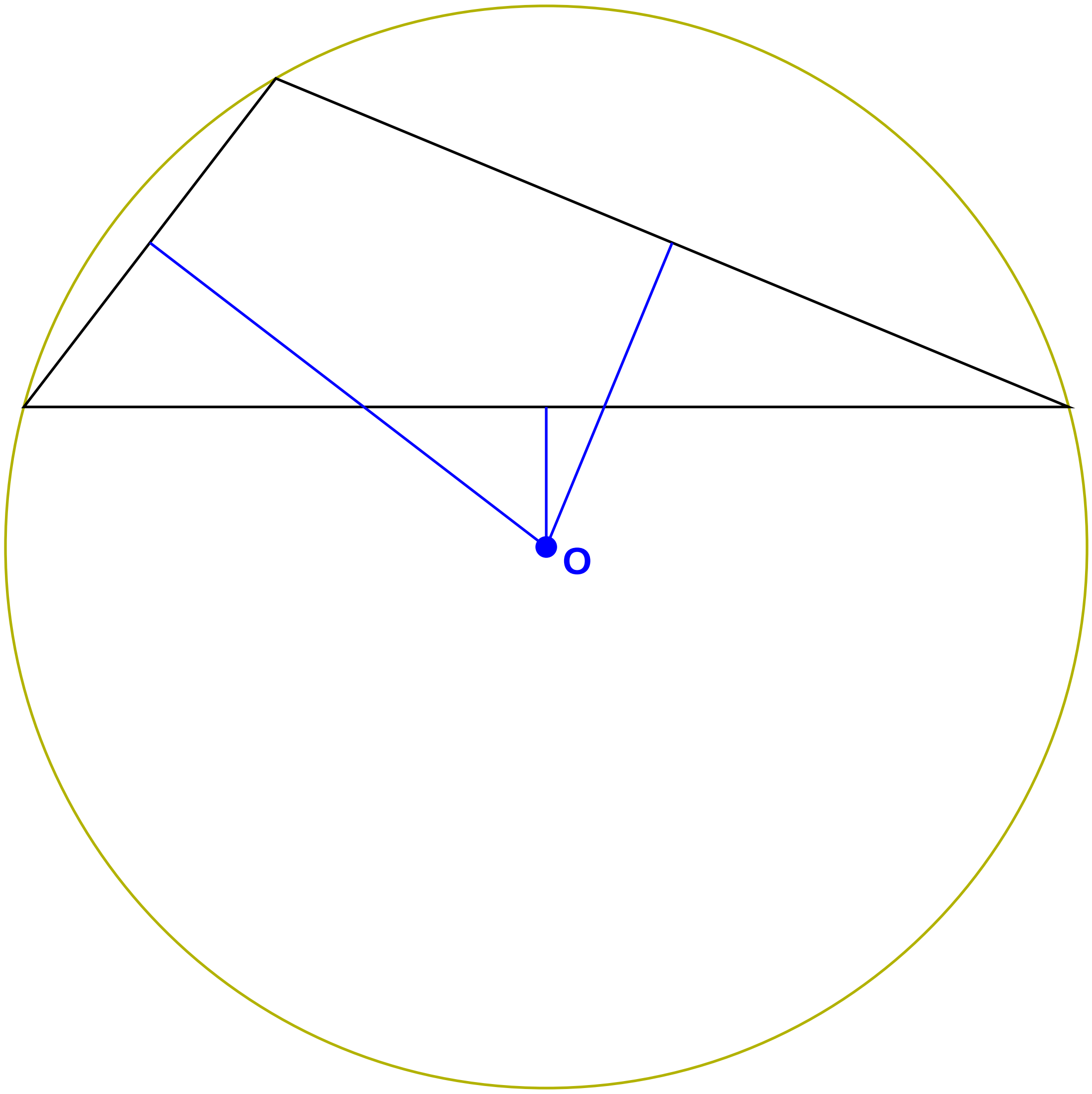
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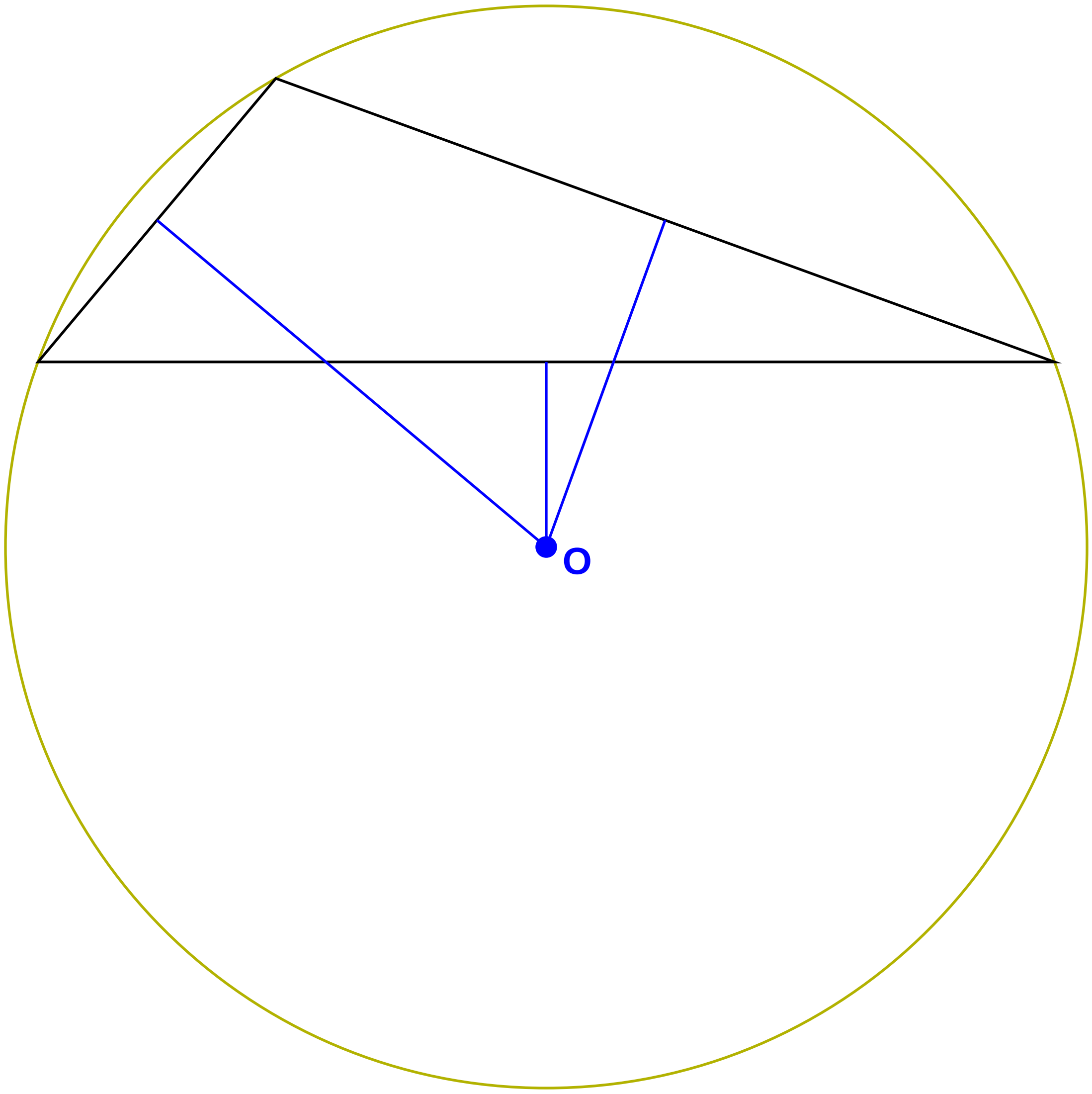
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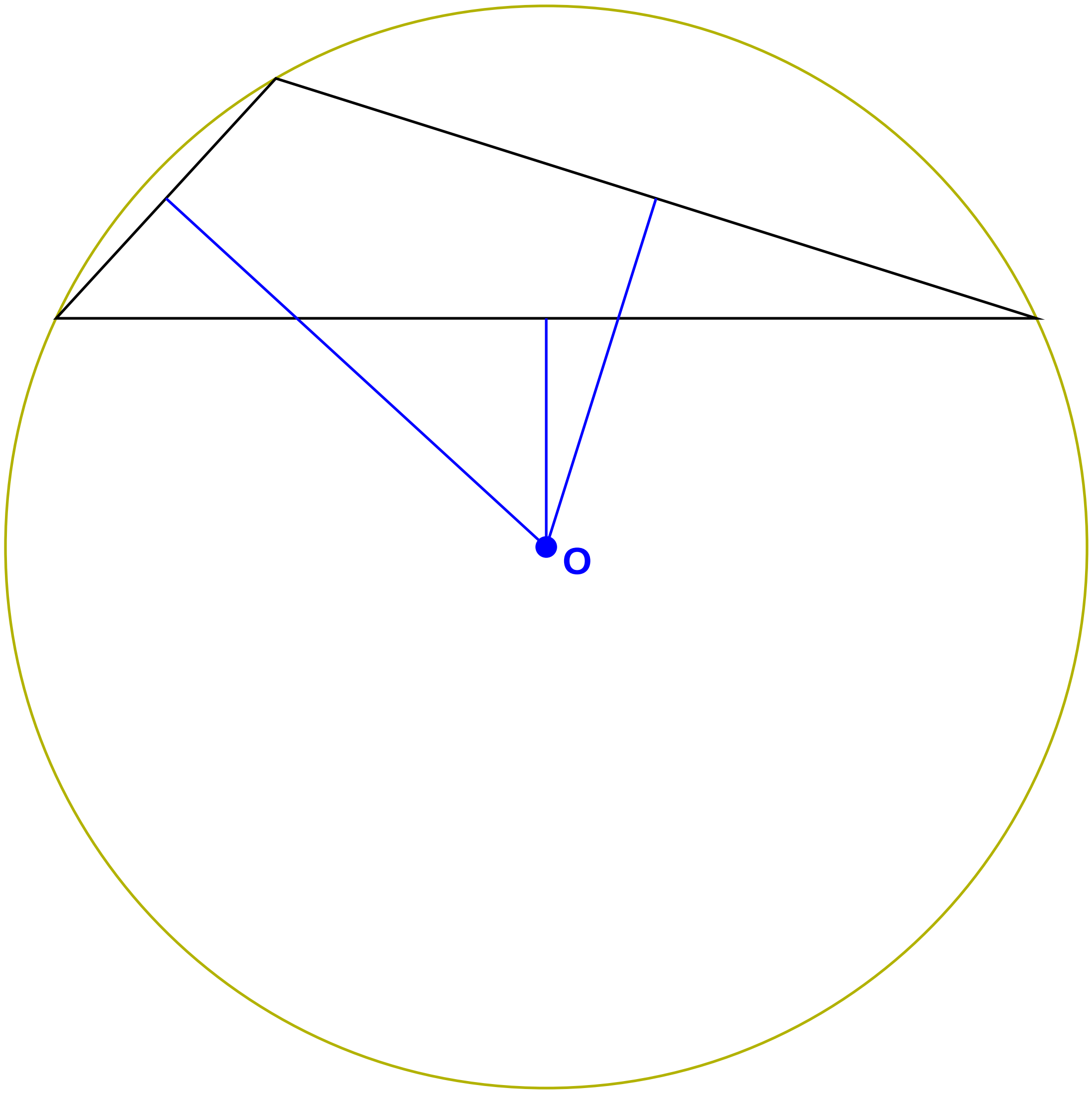
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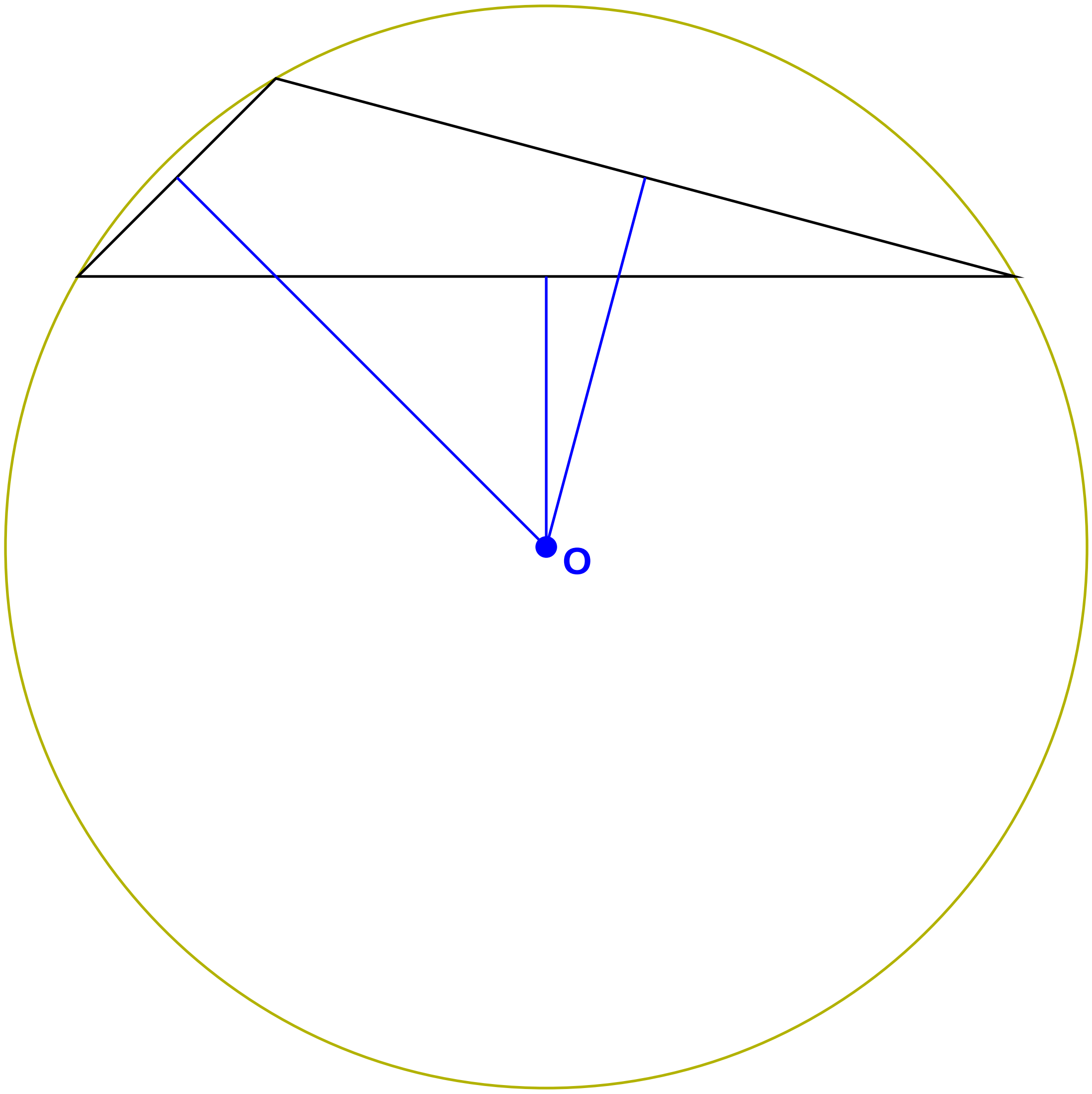

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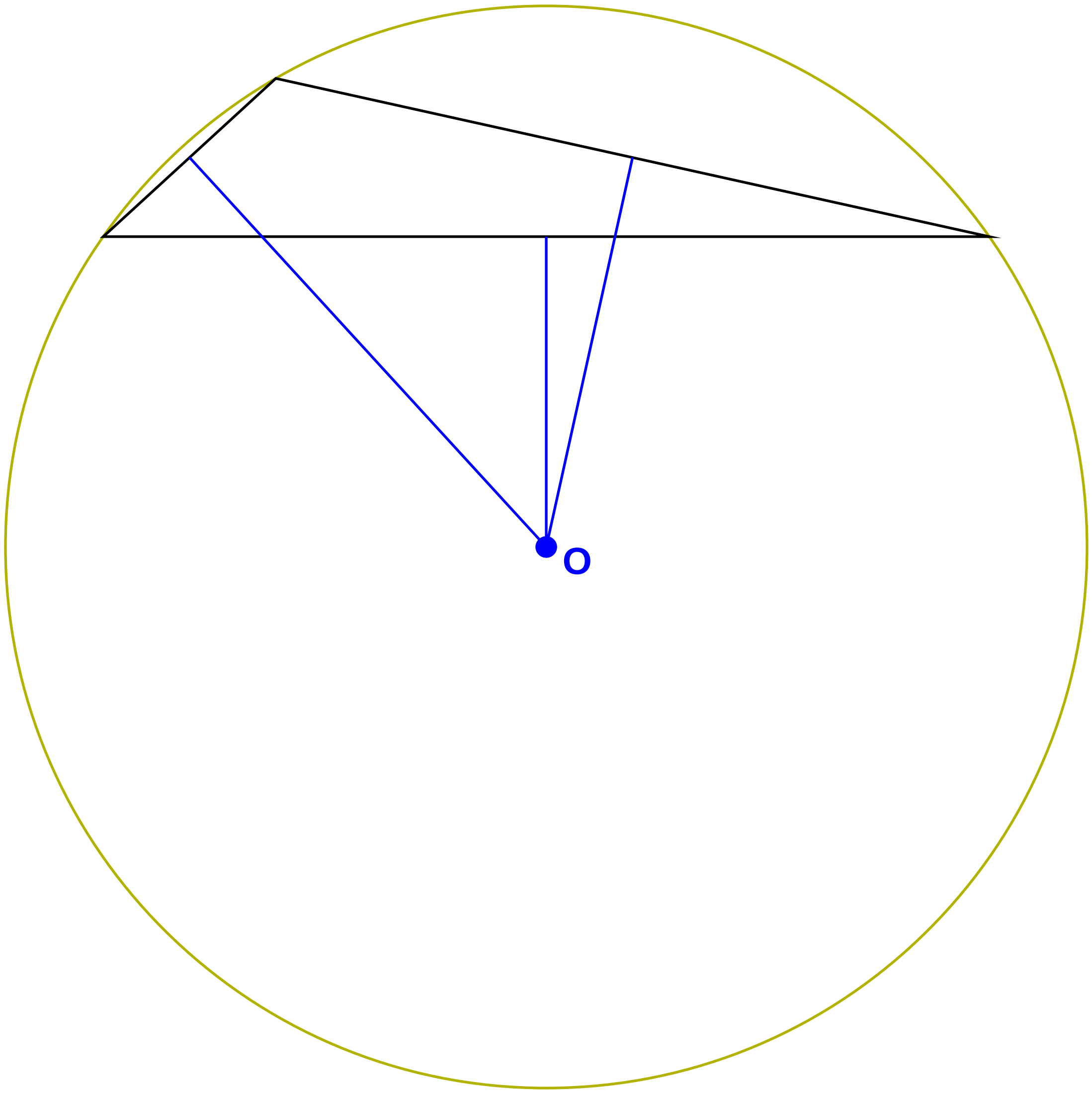
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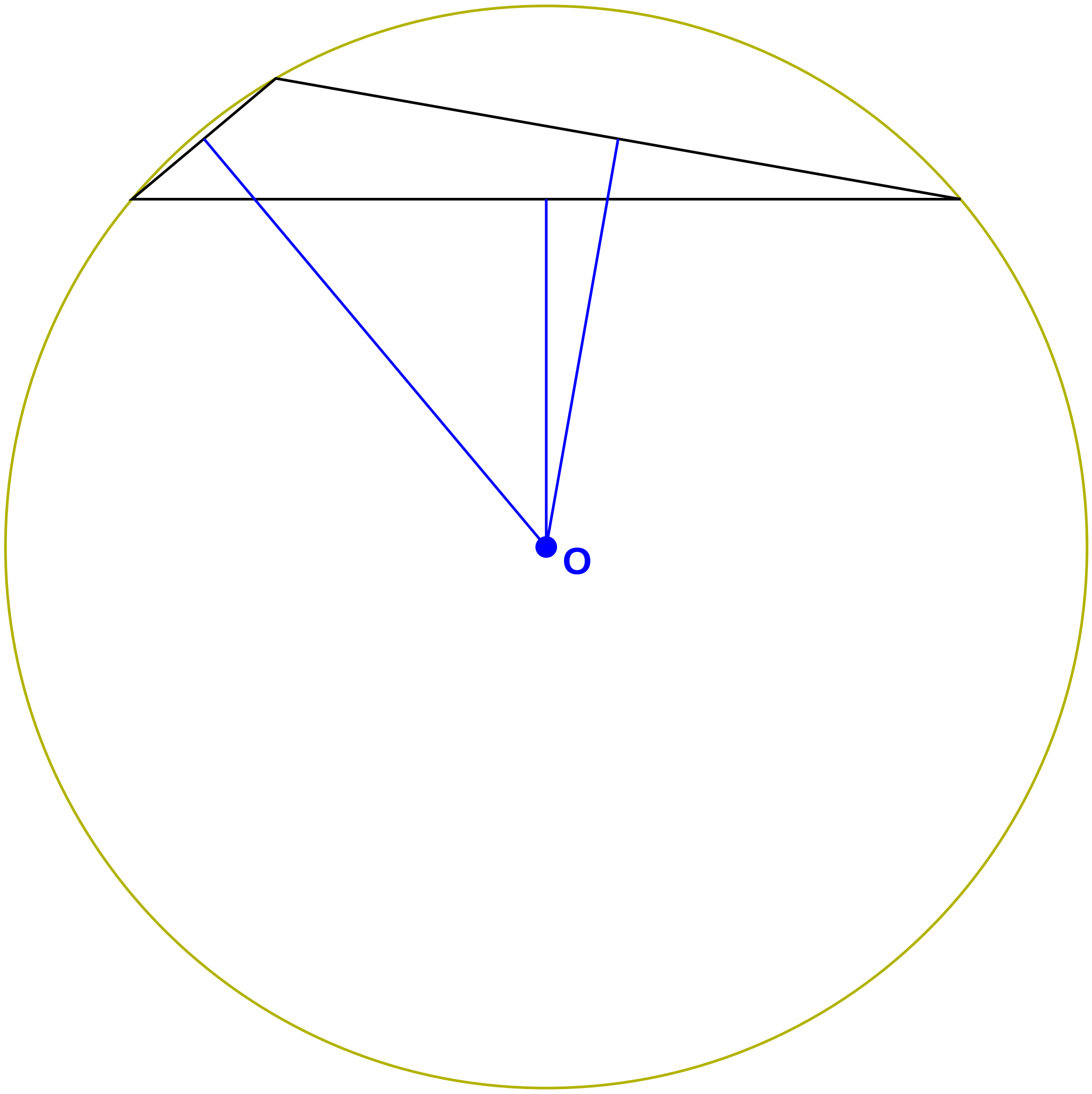
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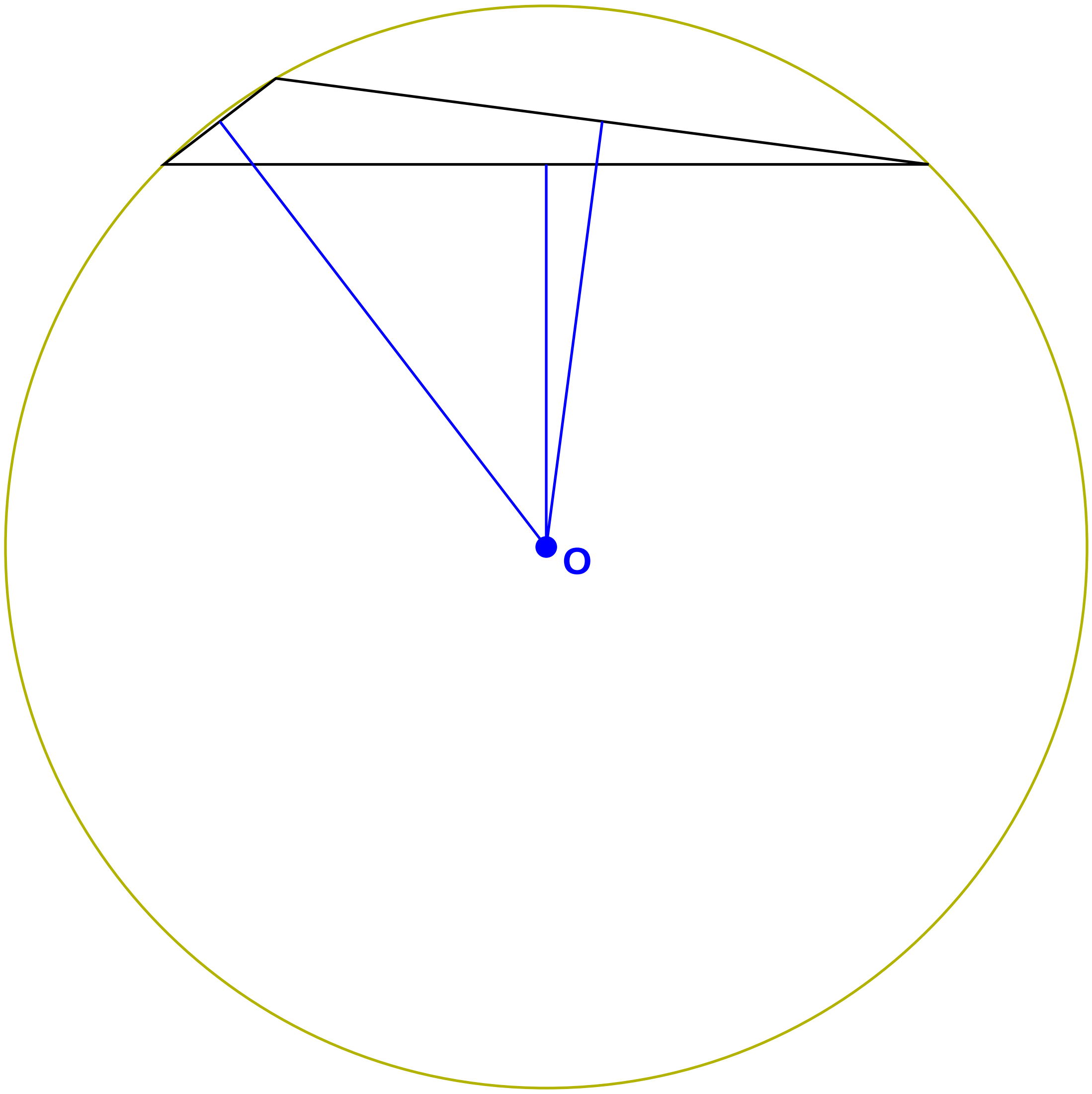
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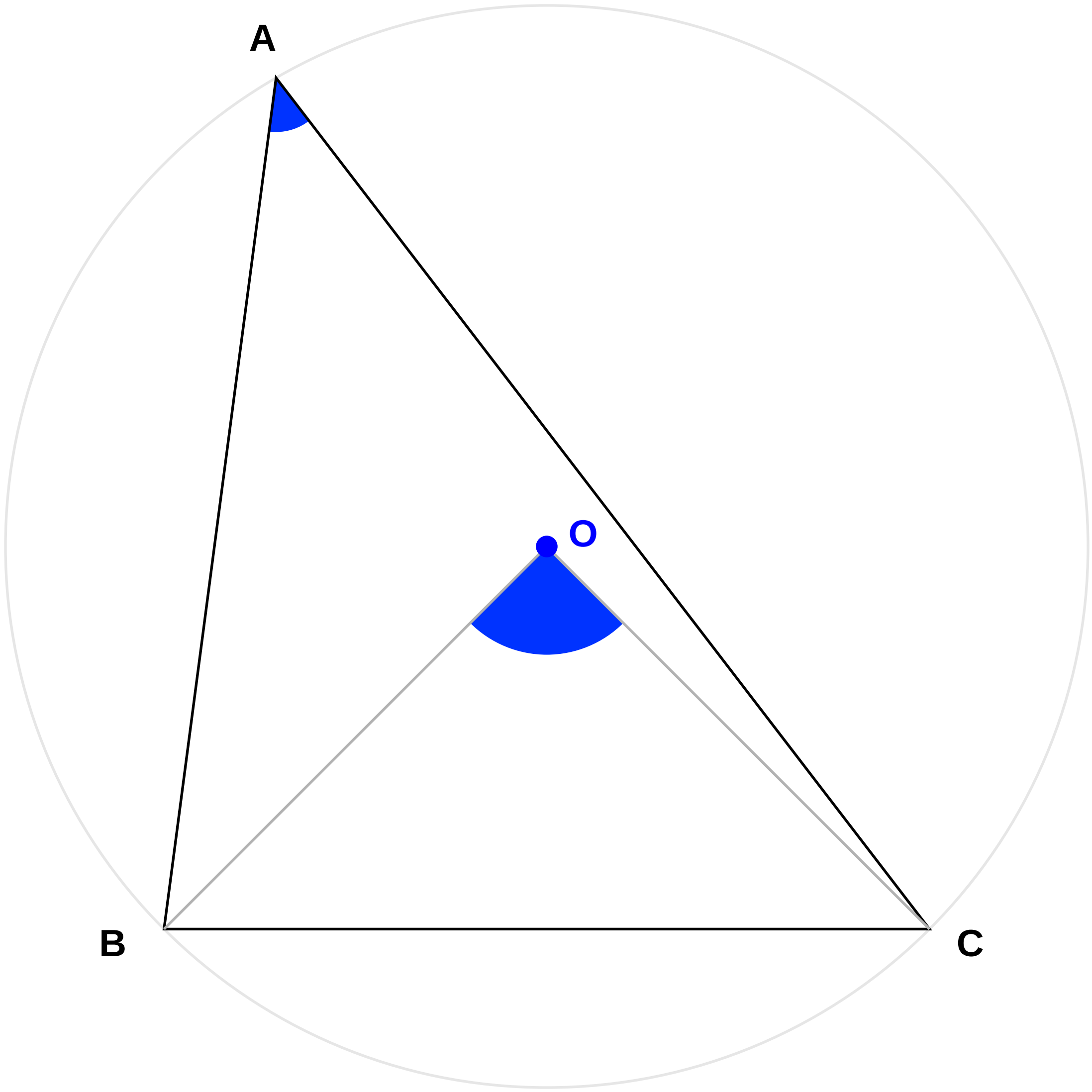
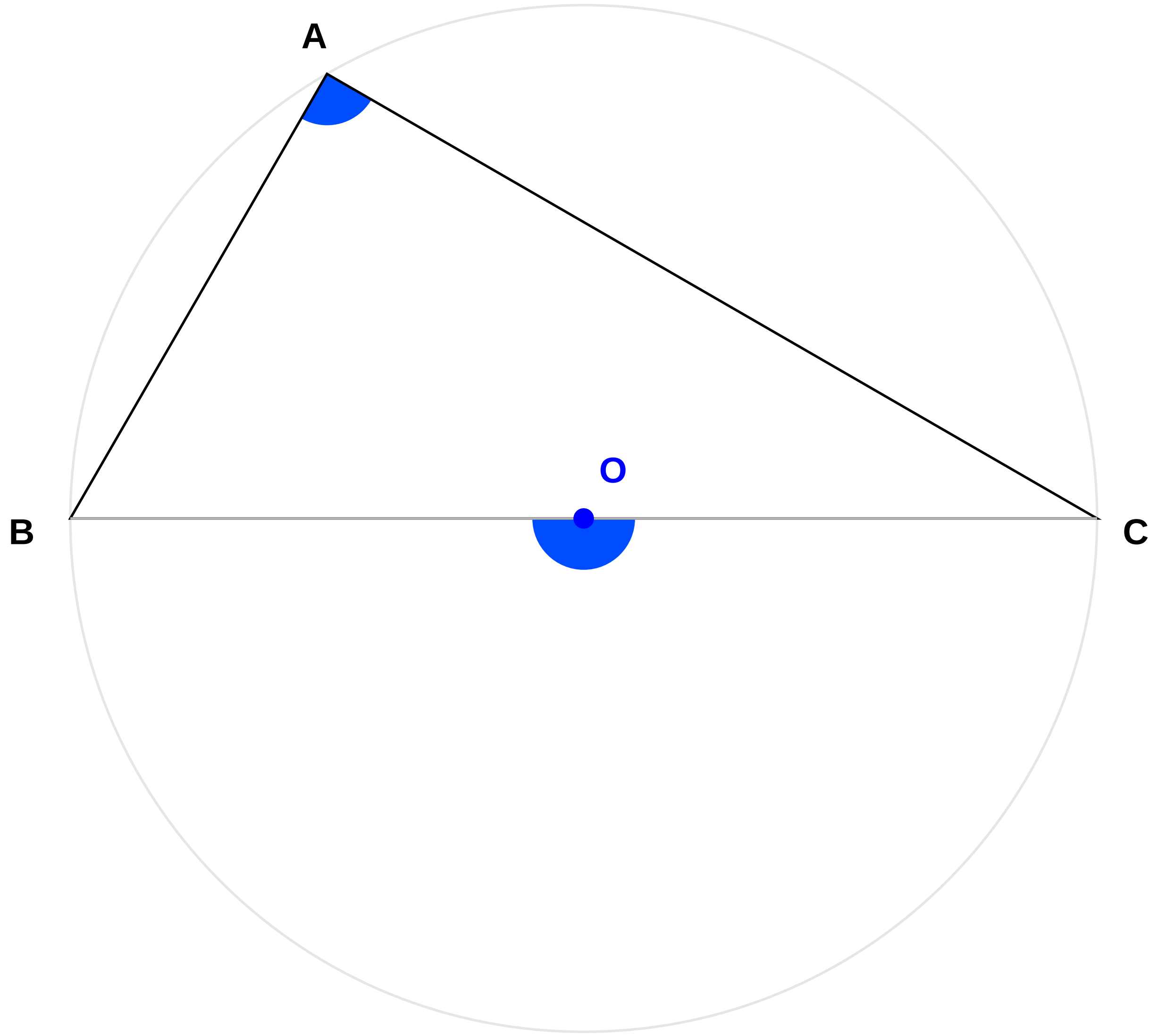


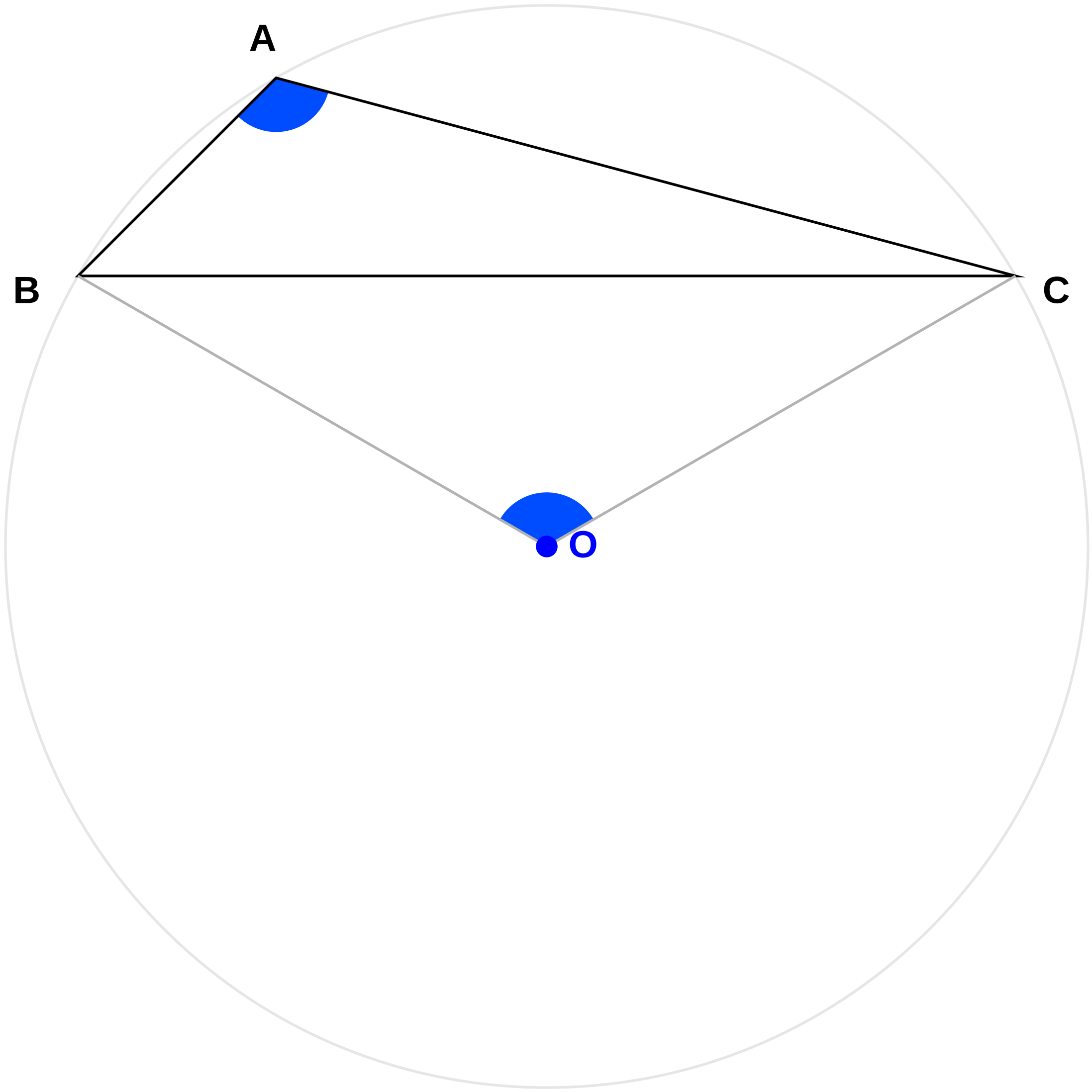
Since the prevous is true, one can apply Euclid's proposition III 20 and see that the angle at the circumcenter subtended by the sides is double the corresponding angle of the triangle.

ie. angleBOC = 2(angleBAC)



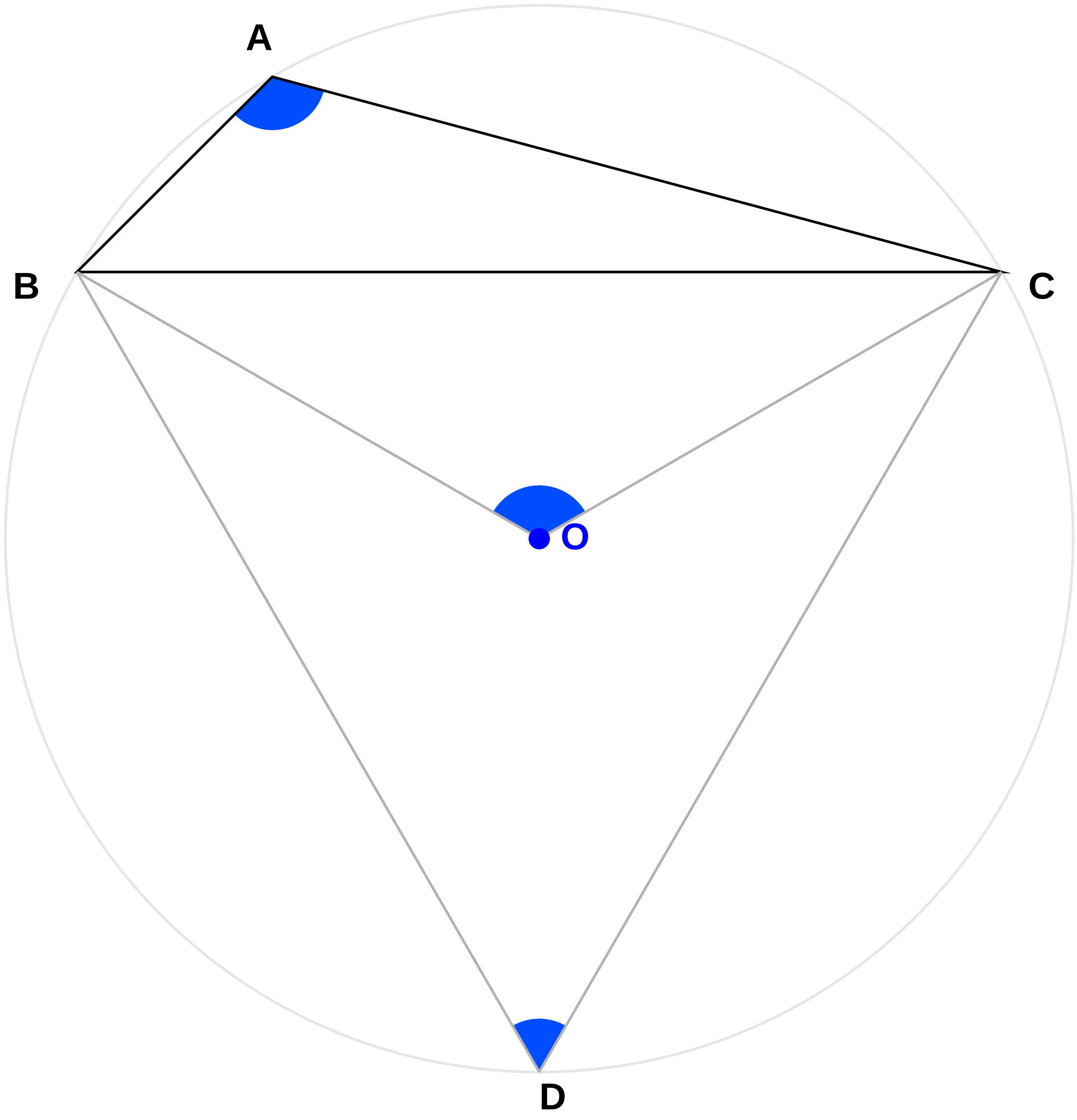
This is also true with right triangles:

ie. angleBOC = 180 = 2(90) = 2(angleBAC)



However, for an obtuse triangle the relationship becomes

angleBOC = 2(180 - angleBAC) ...



... to show that the relationship is true, let D be any point on the circle opposite to A such that ABCD is a cyclic quadrilateral.

angleBOC = 2(angleBDC)

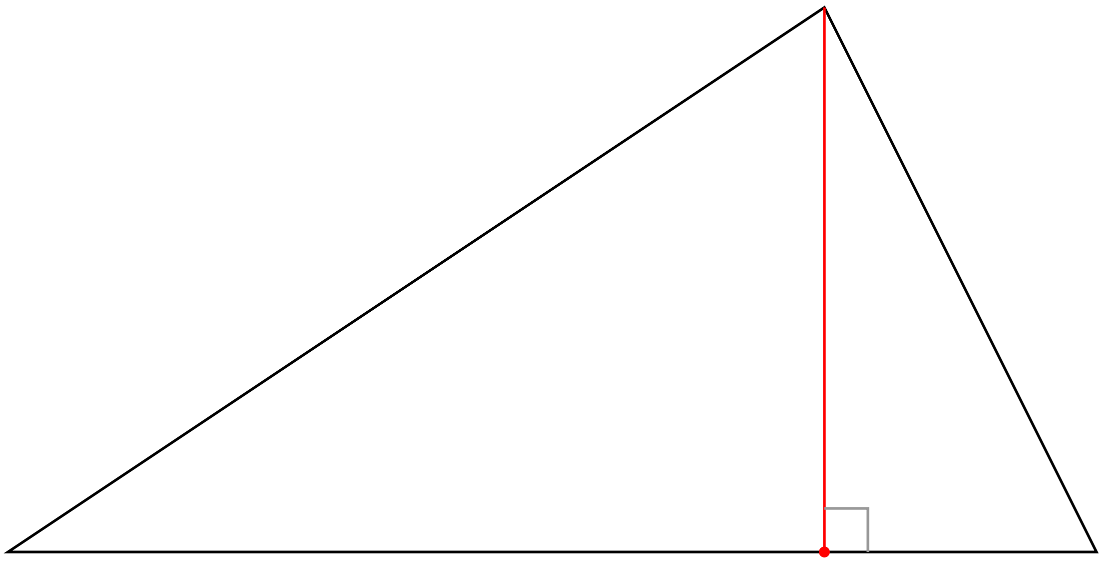
angleBDC = 180 - angleBAC

hence, angleBOC = 2(180 - angleBAC) .

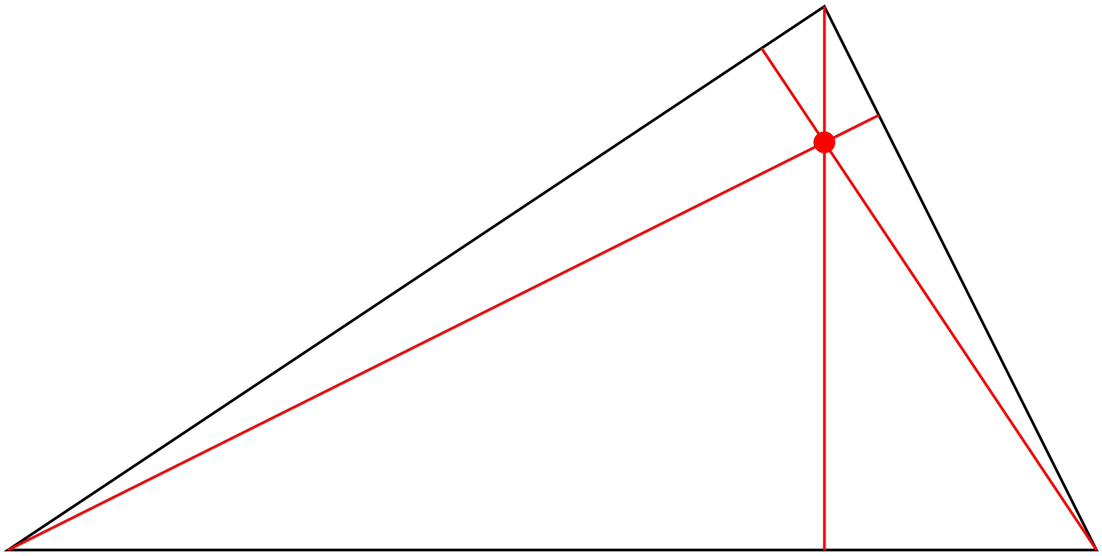
2. The Orthocenter (H)

2.1 The Orthocenter (H)

- the formation of H



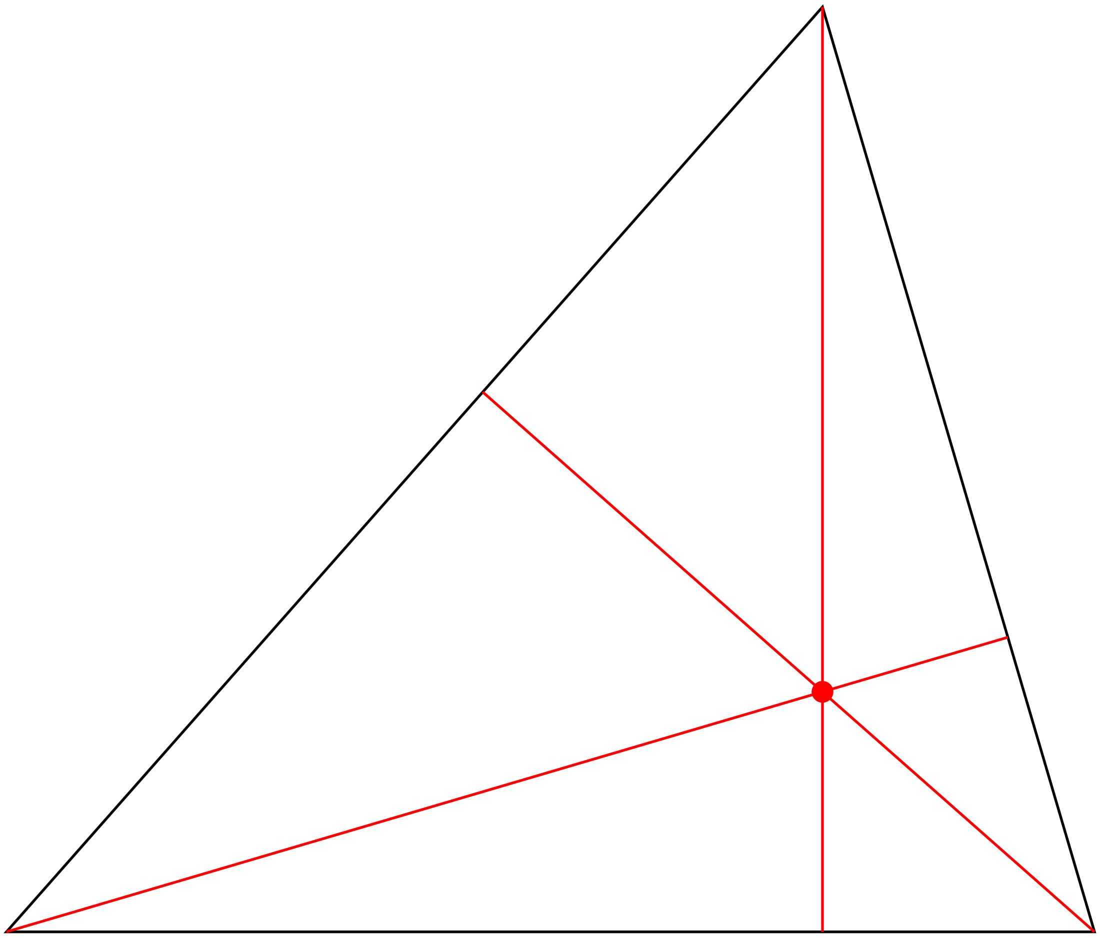
Every triangle has three altitudes; each one is made by "dropping down", from a vertice, a line perpendicular to the opposite side.



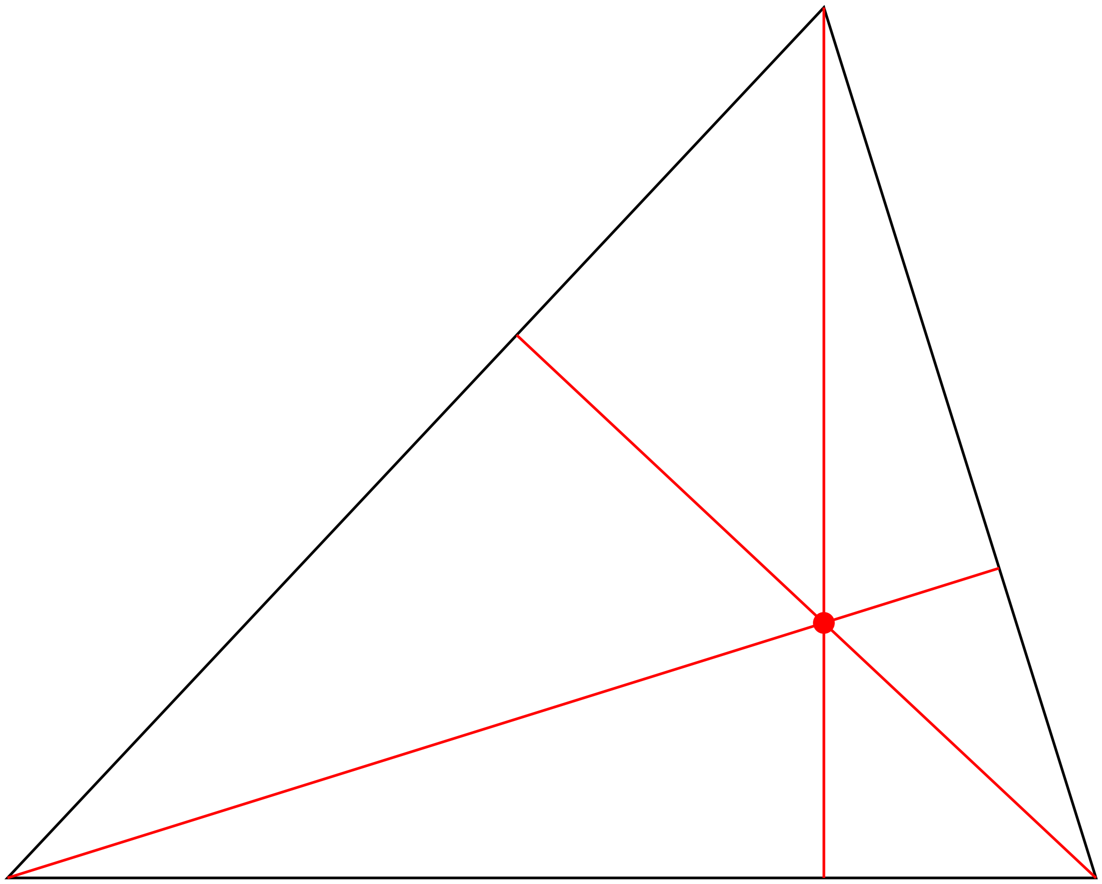
Where all three altitudes meet is where the orthocenter, H, is.

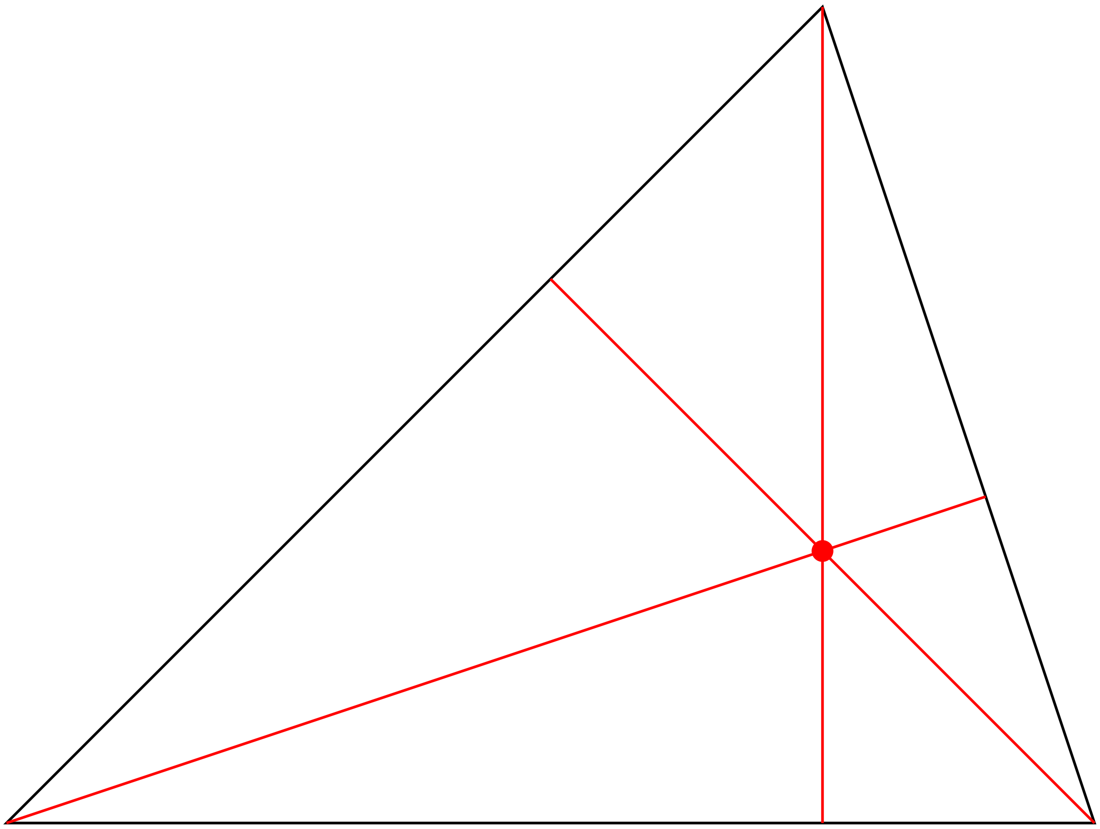
2.2 The Orthocenter (H)

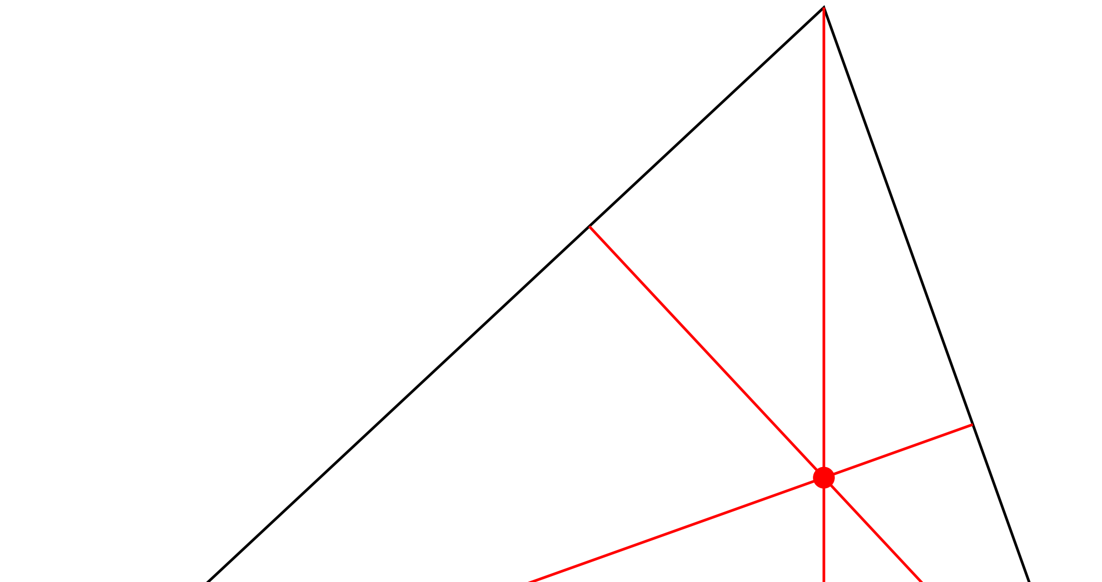
- the position of H

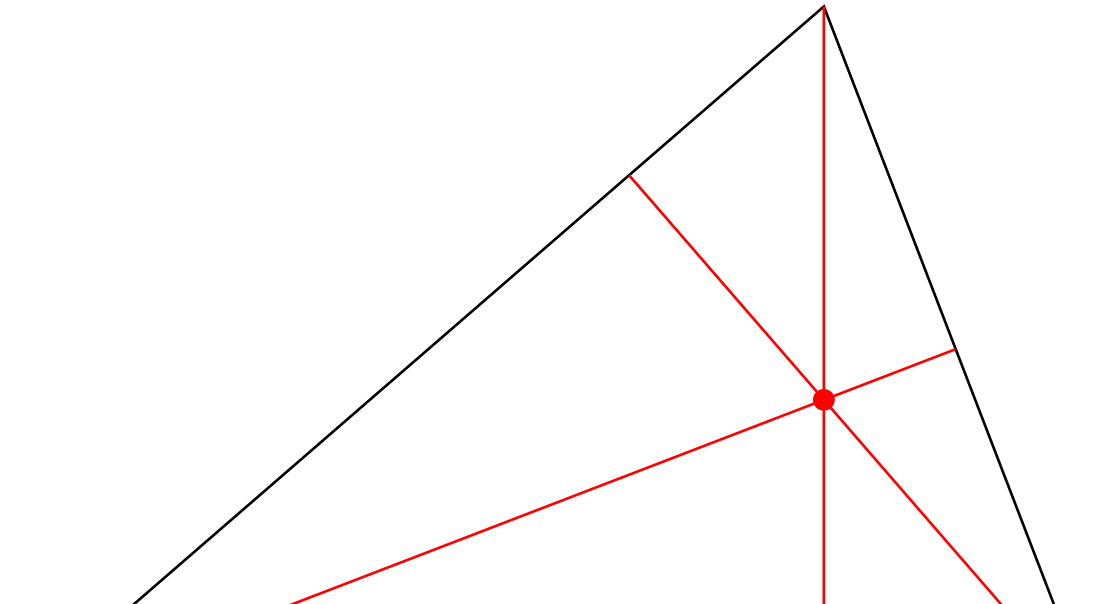


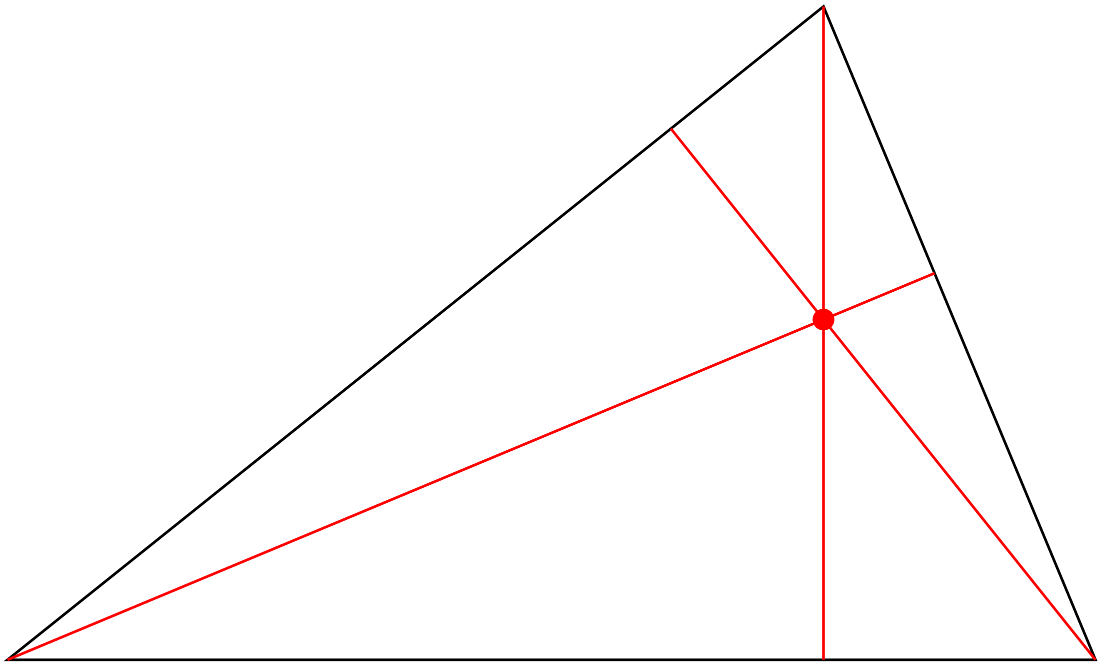
For an acute triangle, its orthocenter is inside the figure.

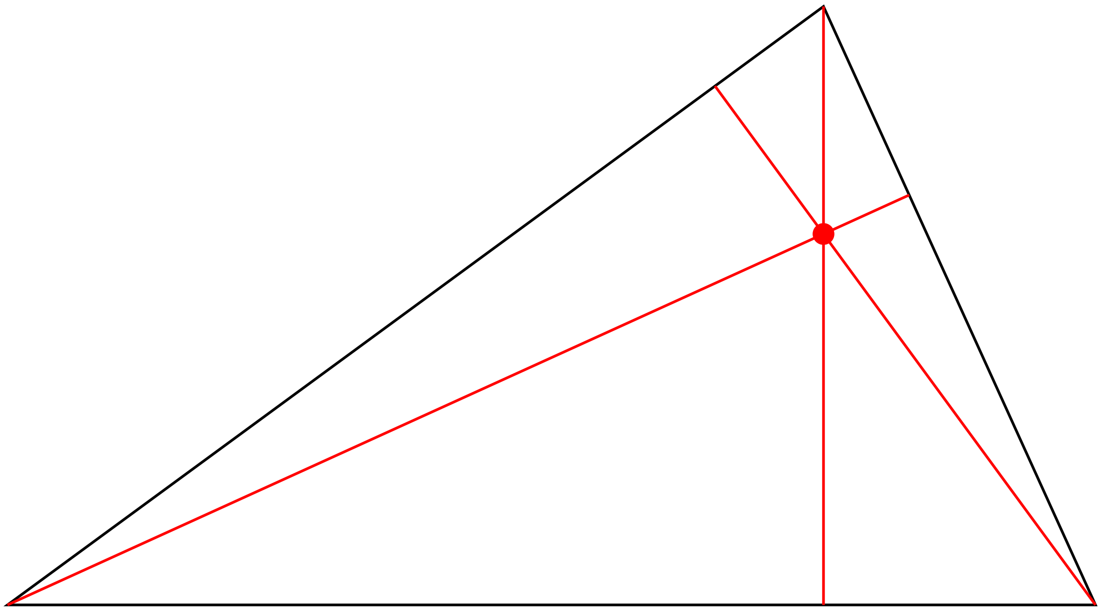


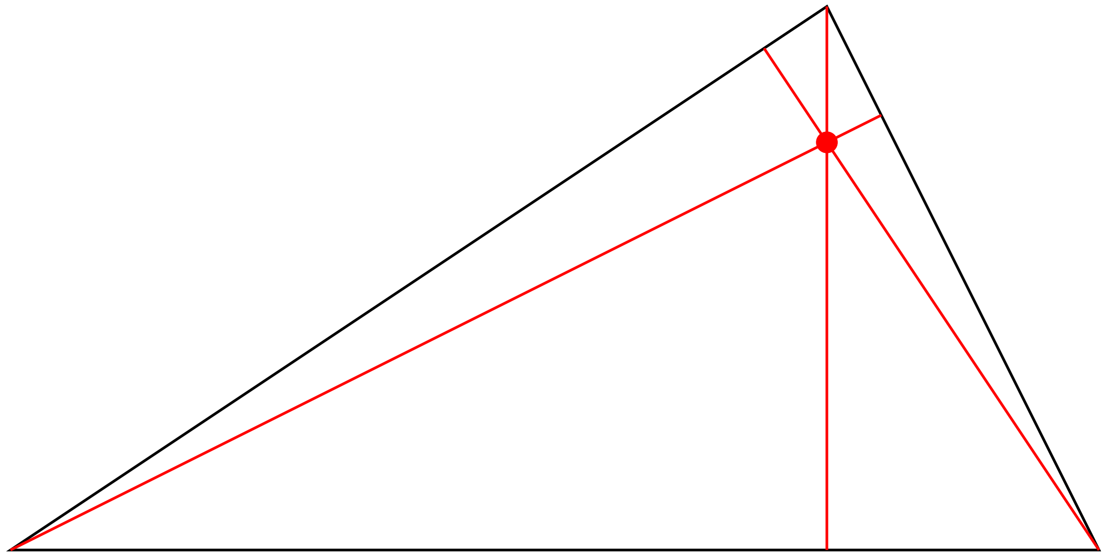


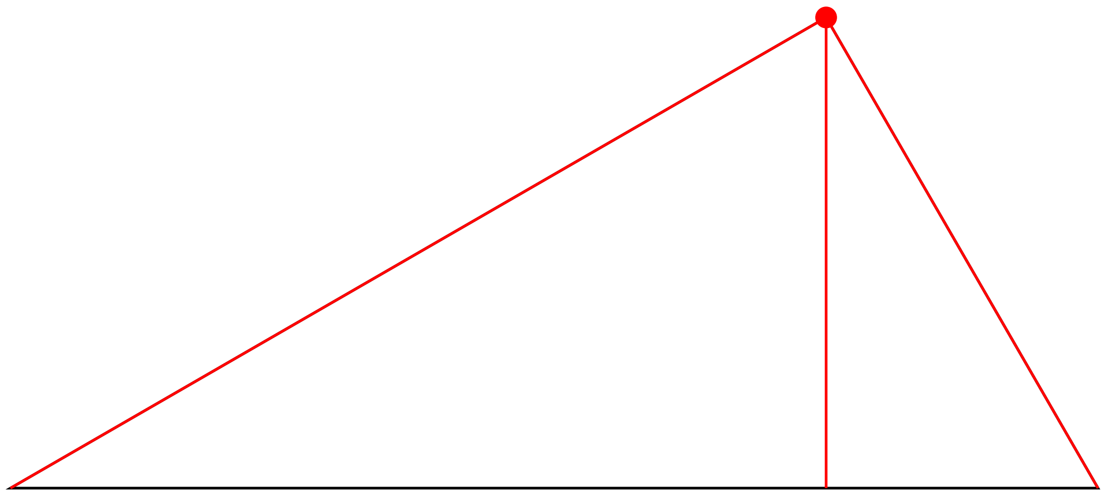




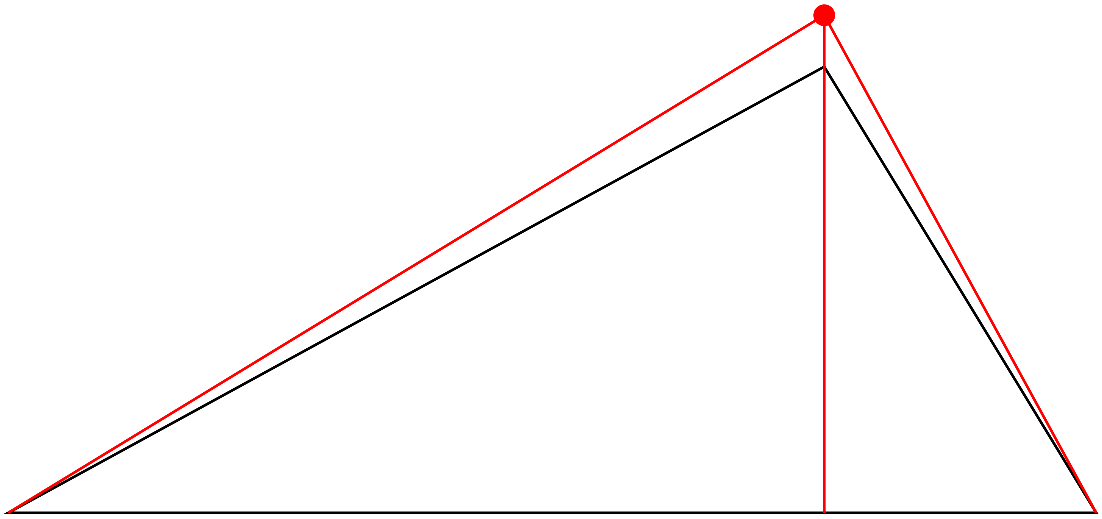




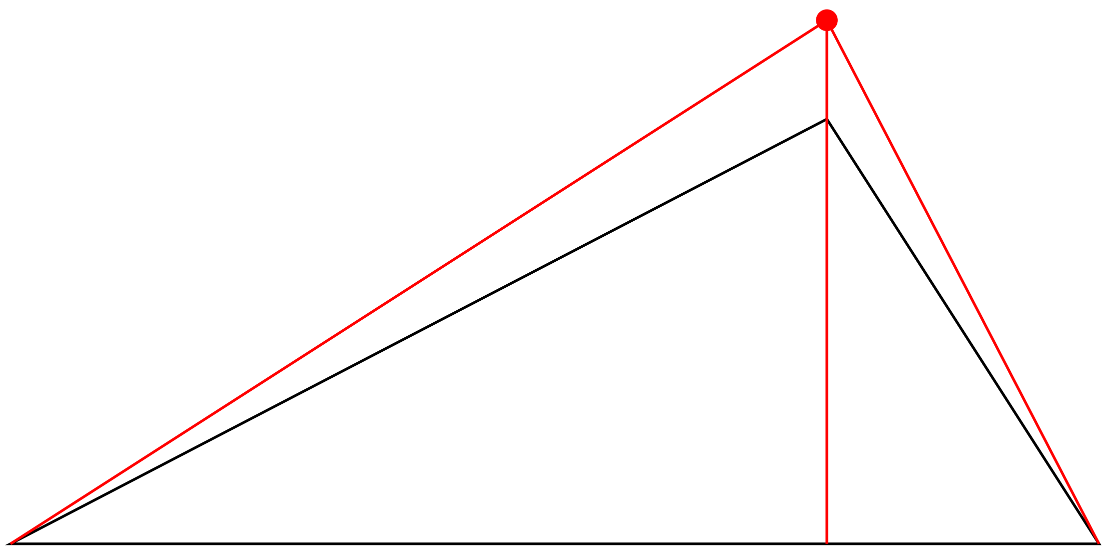


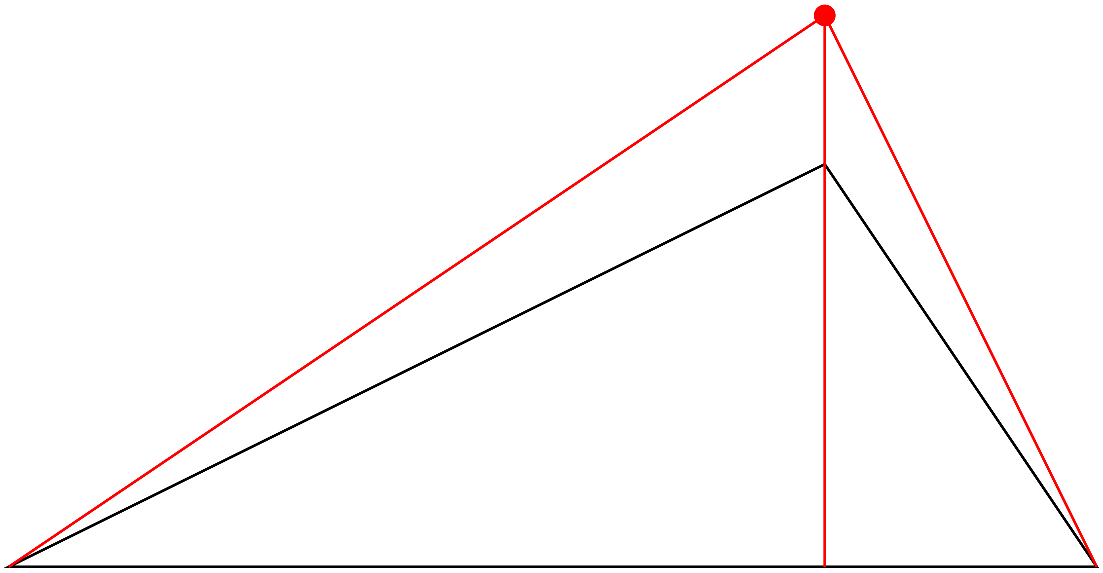


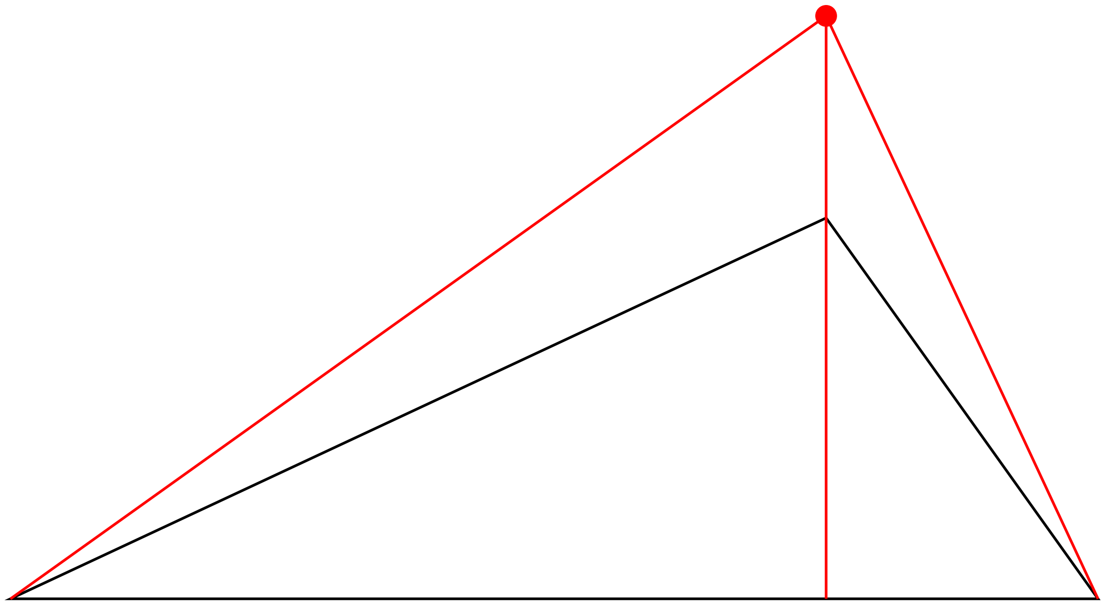
The orthocenter of a right triangle is at its right angle,

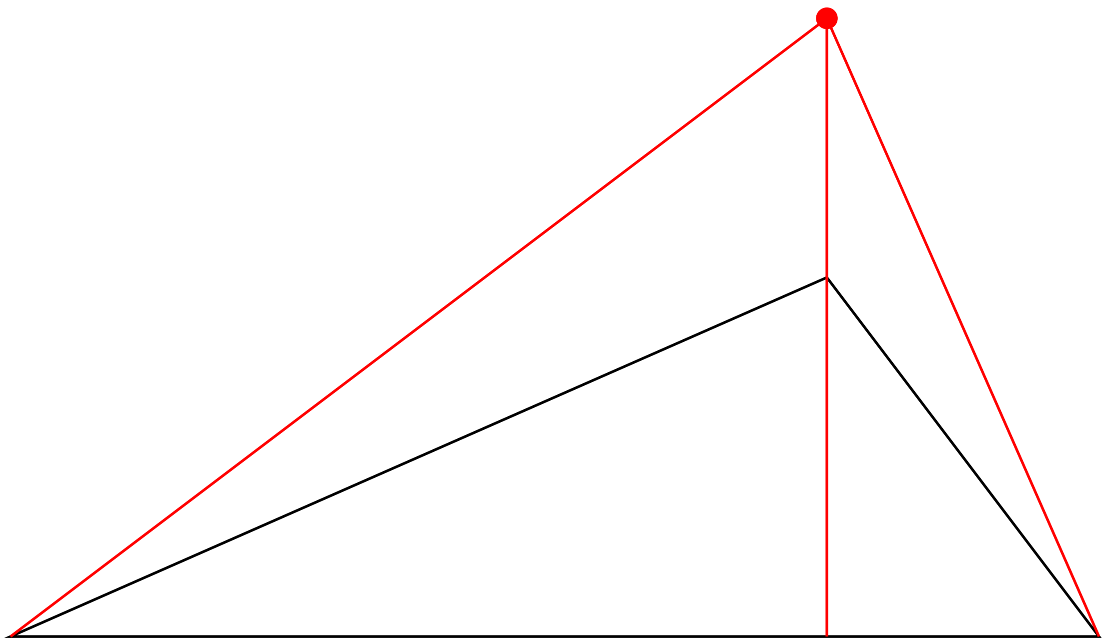


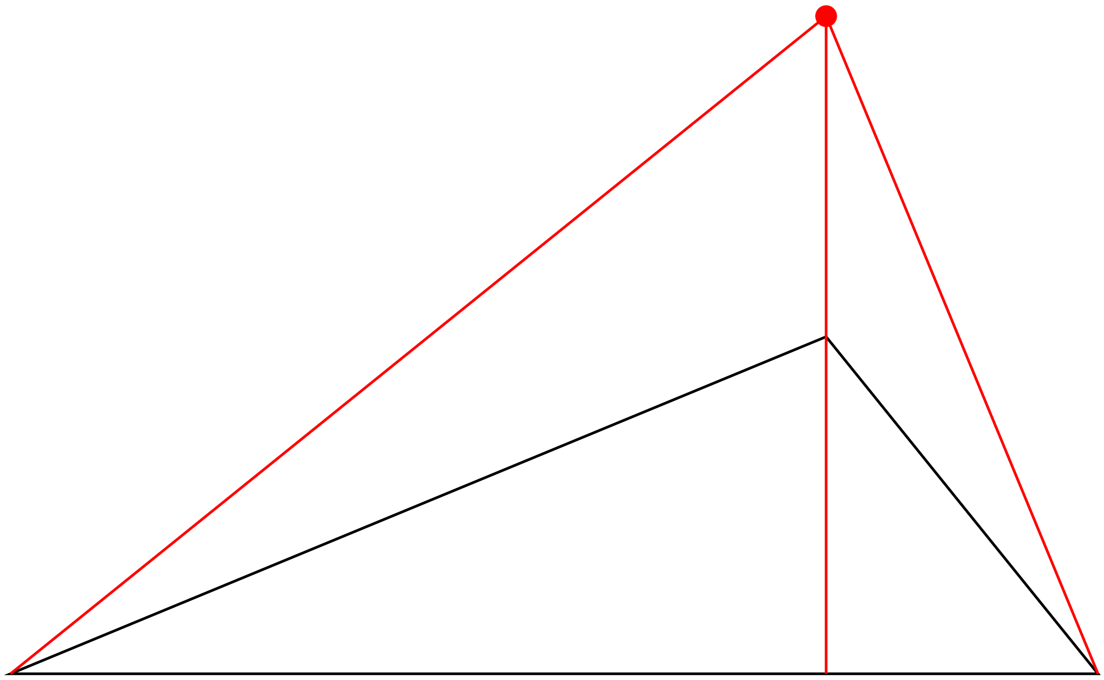
and for an obtuse triangle, its orthocenter is on the extension of the altitudes outside of the obtuse angle.





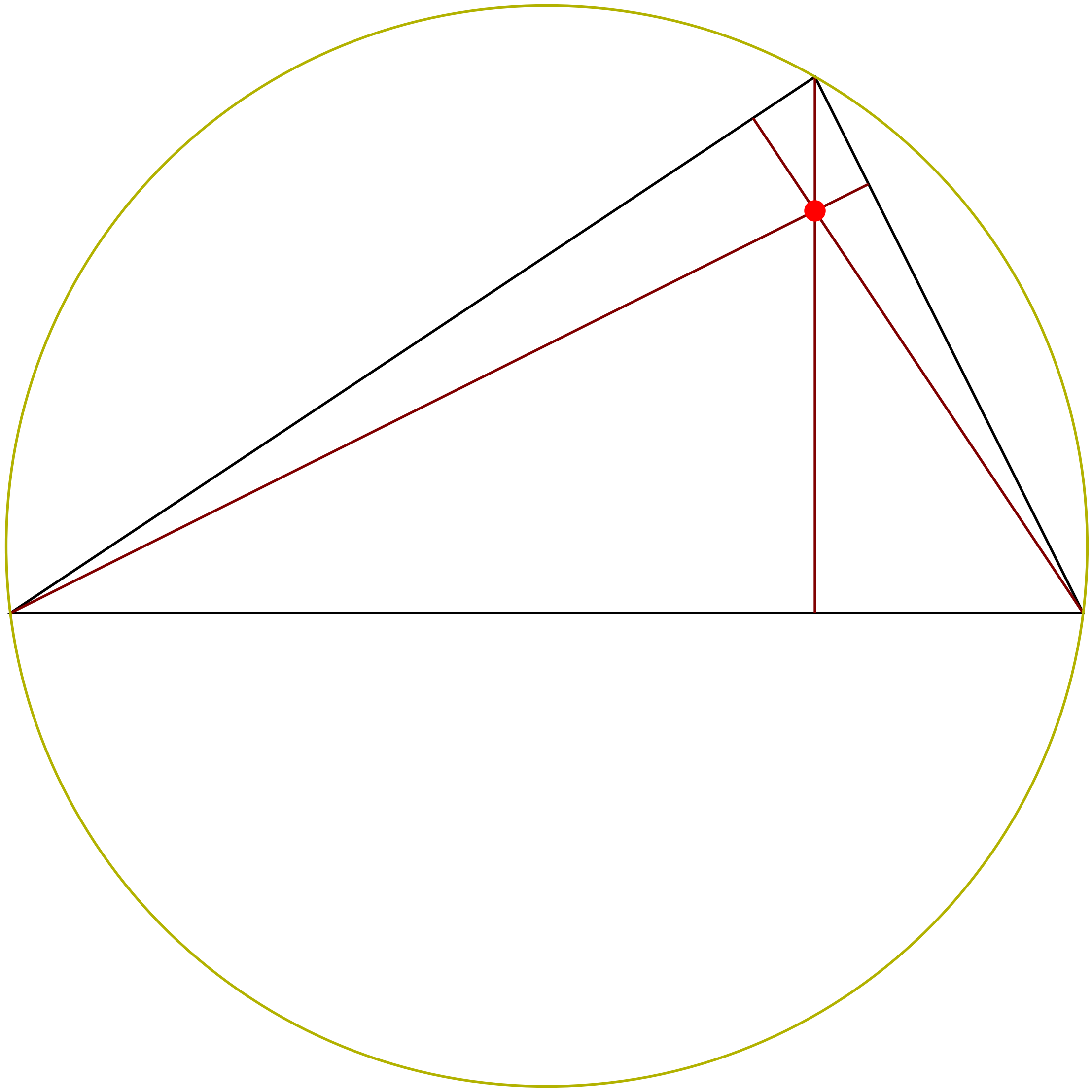




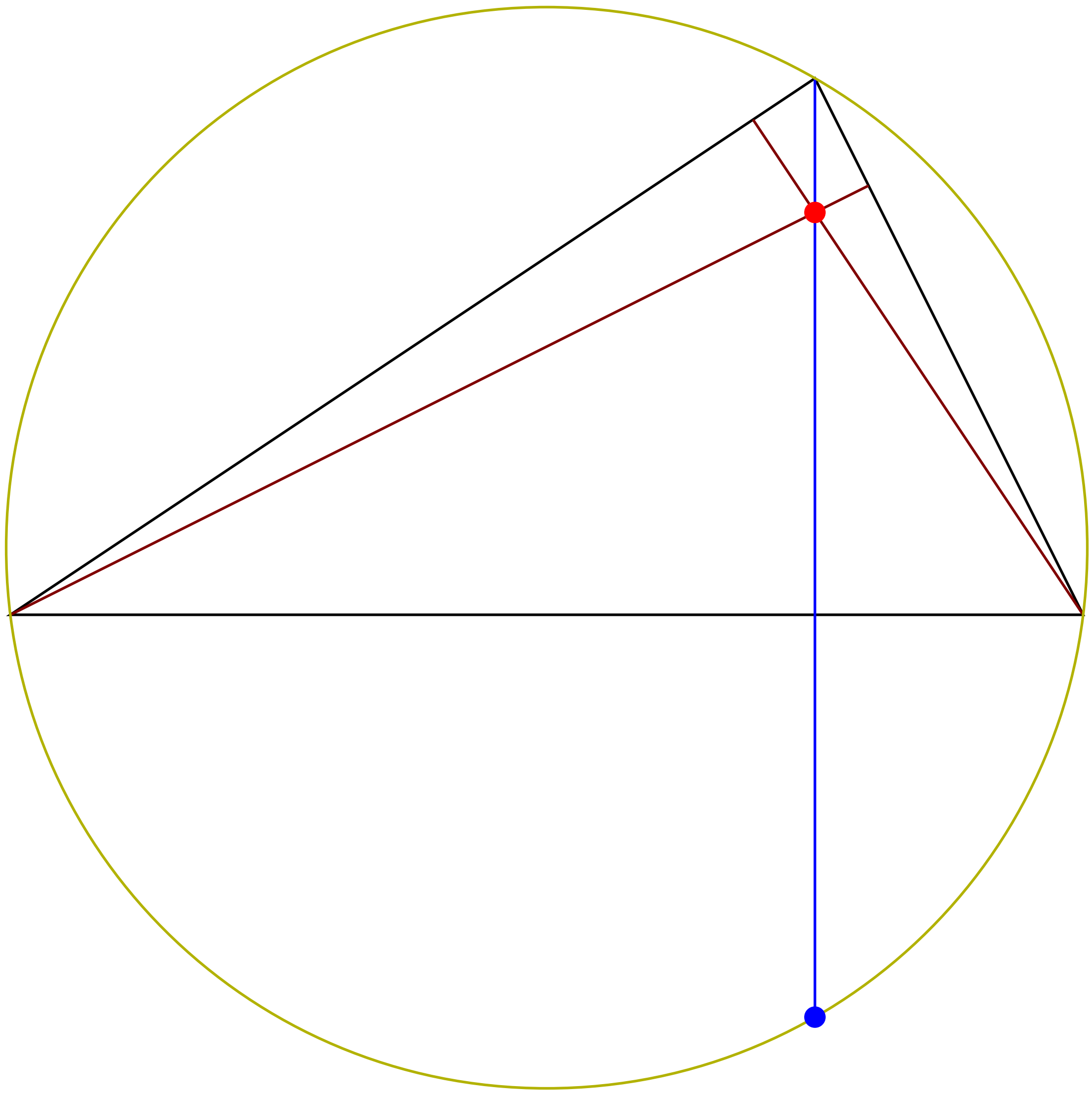


2.3 The Orthocenter (H)

- a special property of H

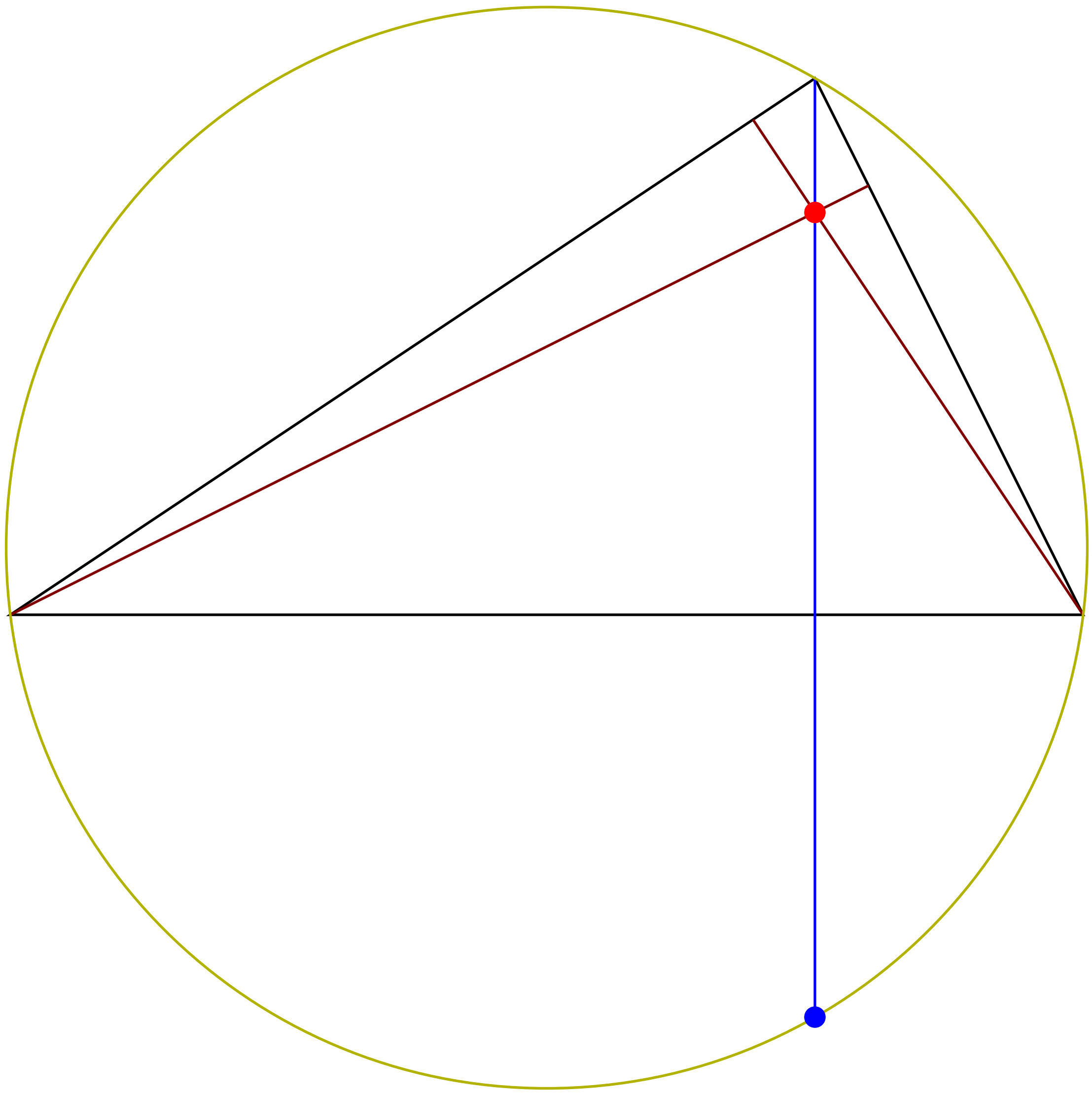


Let a triangle be inscribed in a circle;

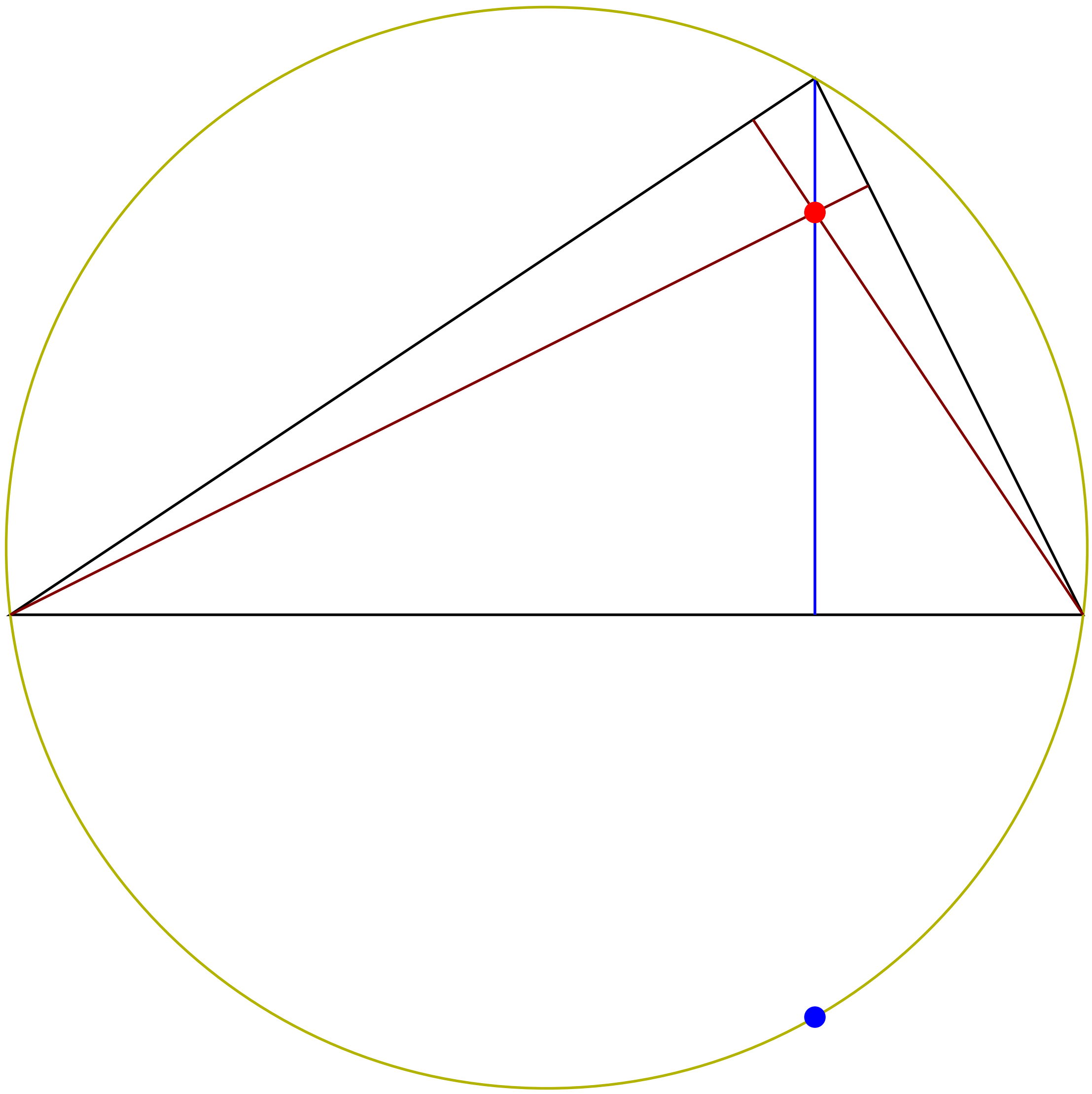


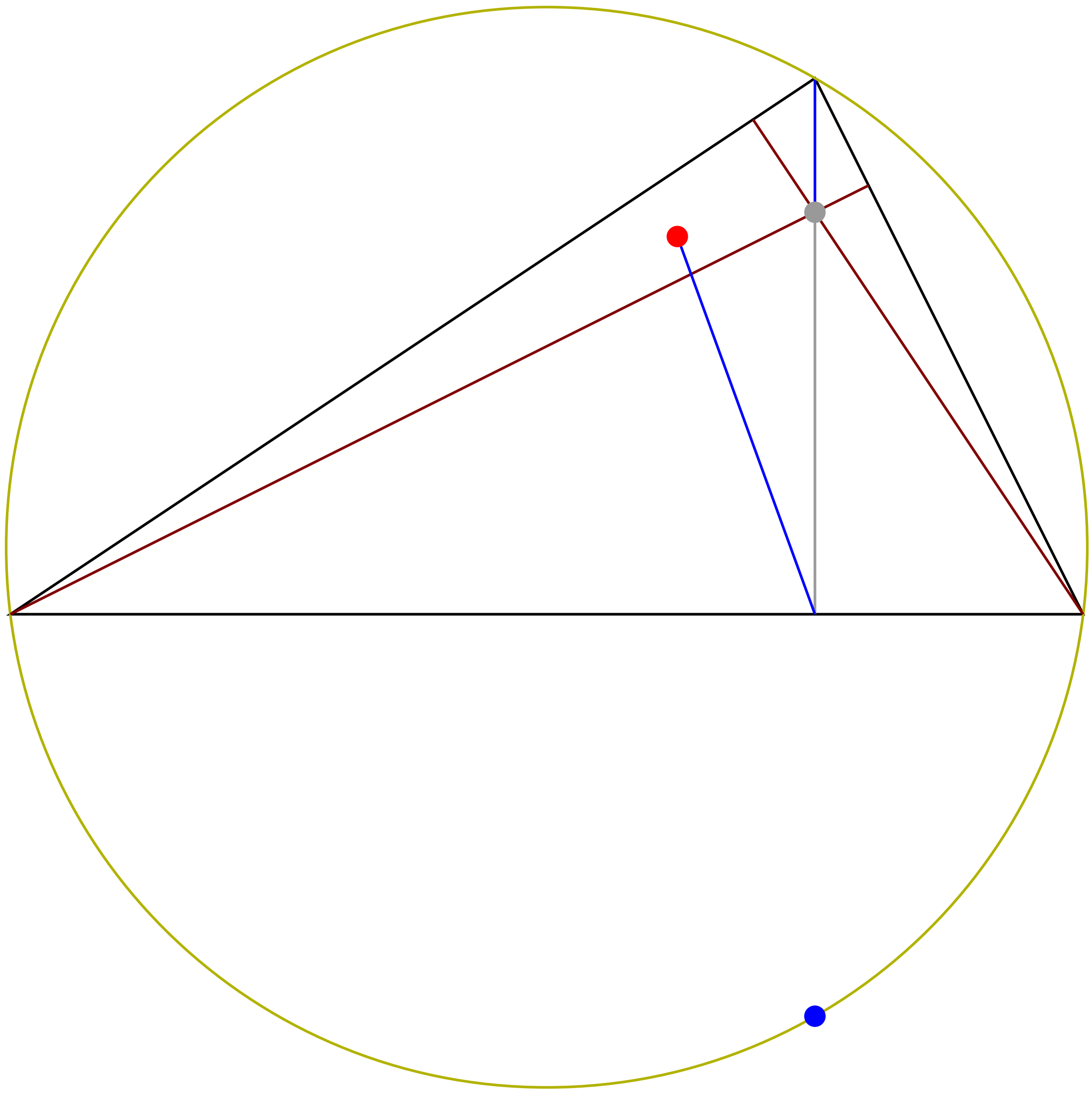
Let a triangle be inscribed in a circle;

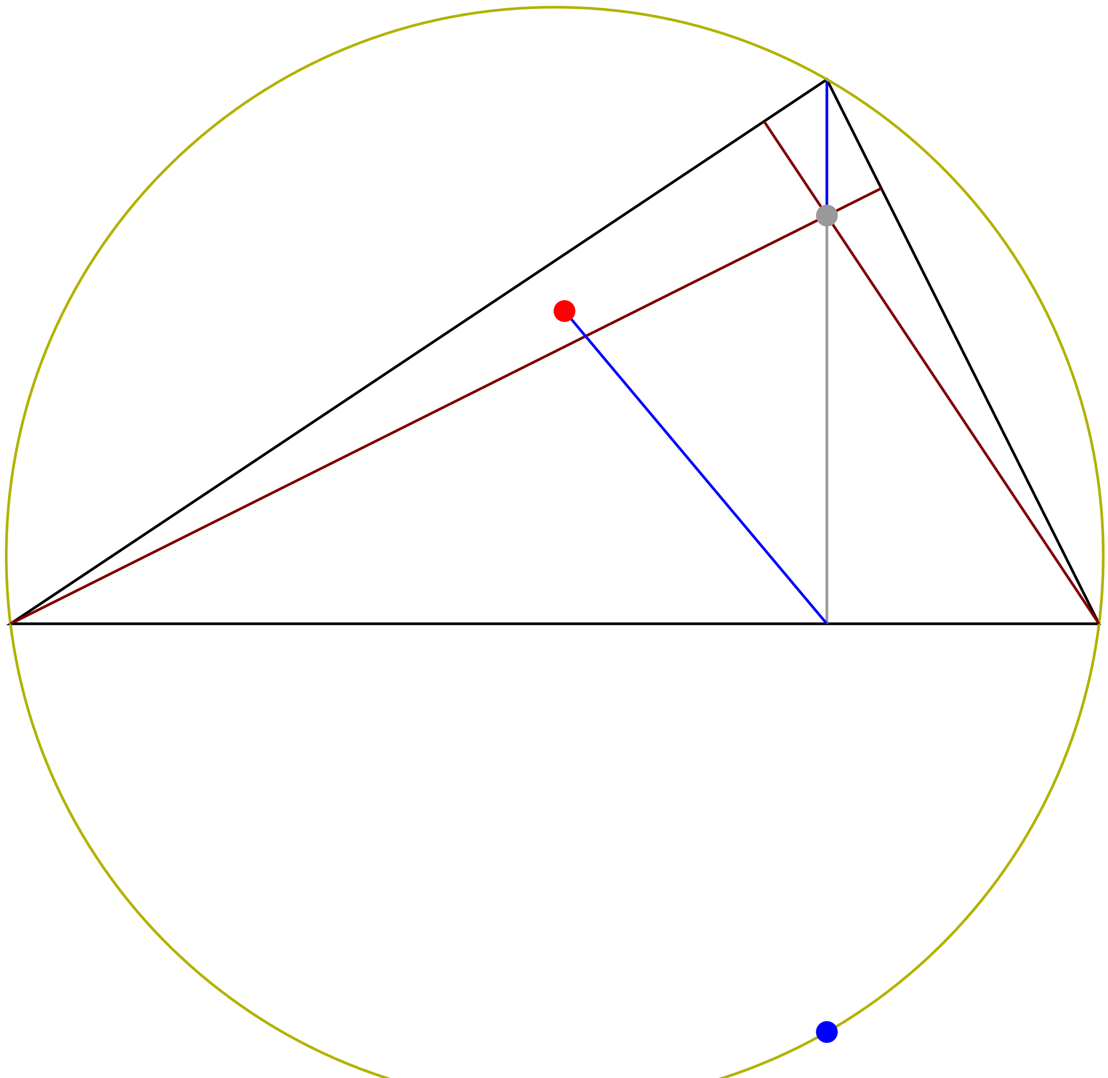
then let one altitude be extended so that it touches the circle at both ends.

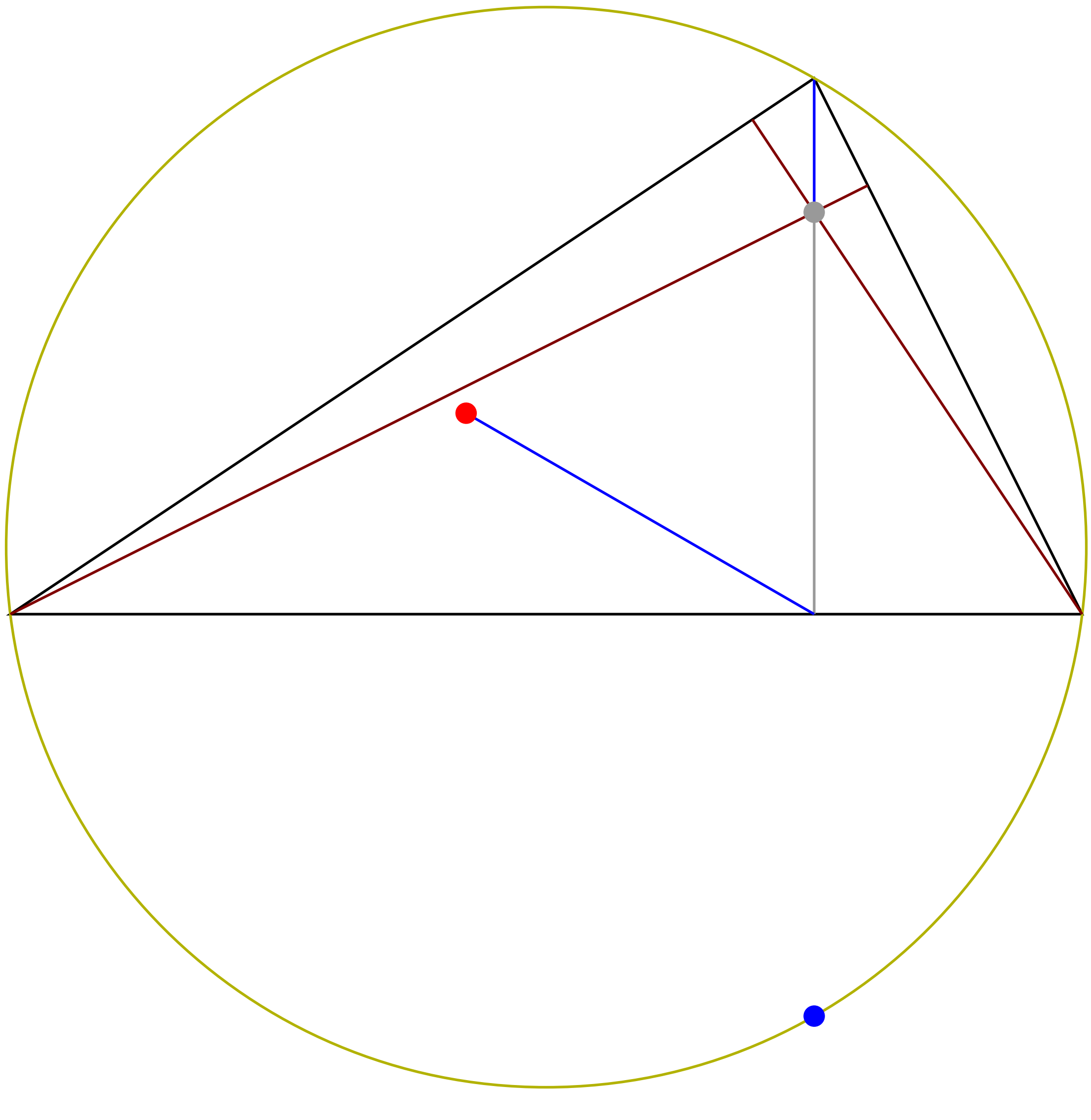


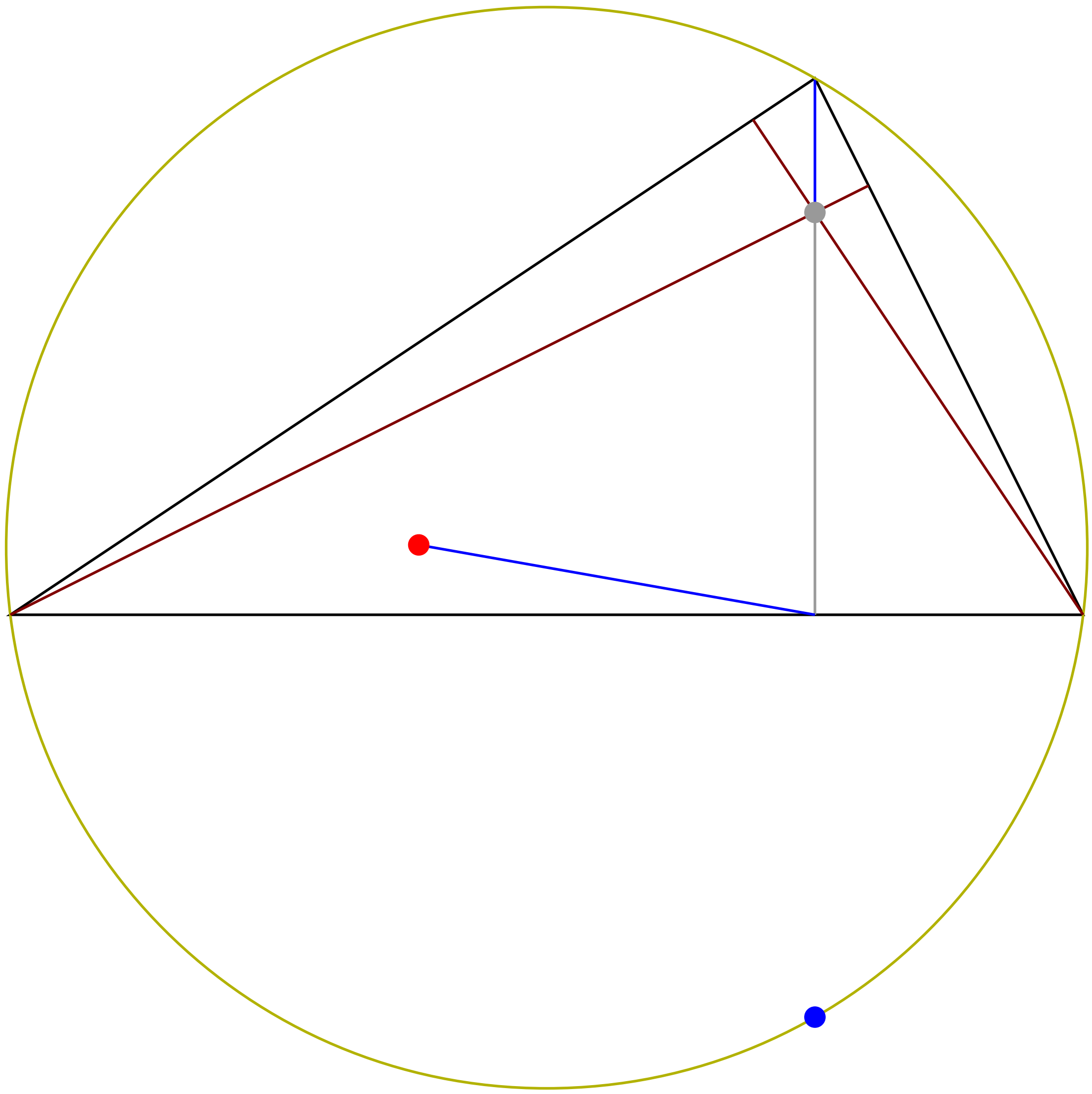
It can be shown that the segment of this altitude from the orthocenter to the side is equal to its extended segment in length.

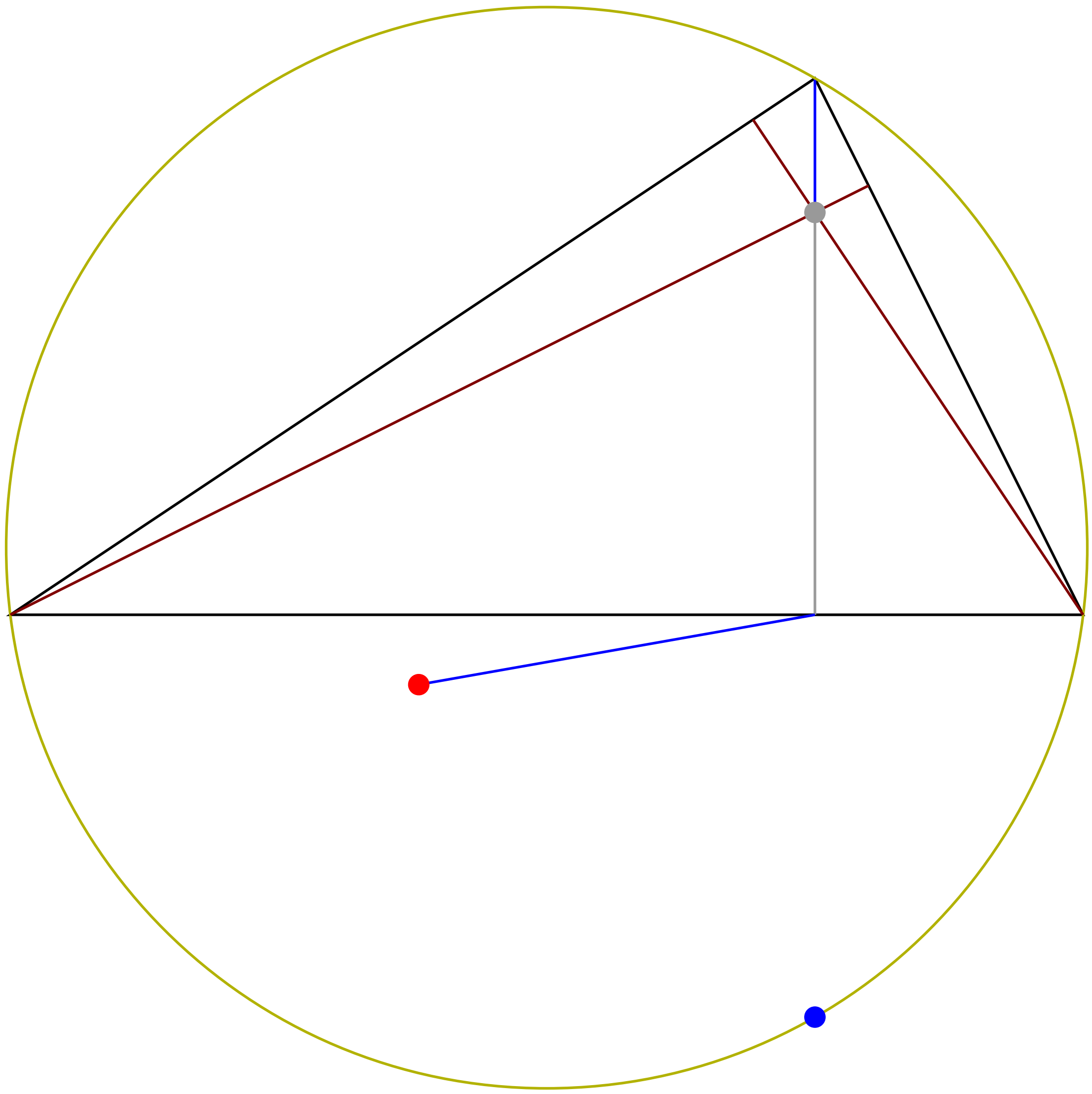


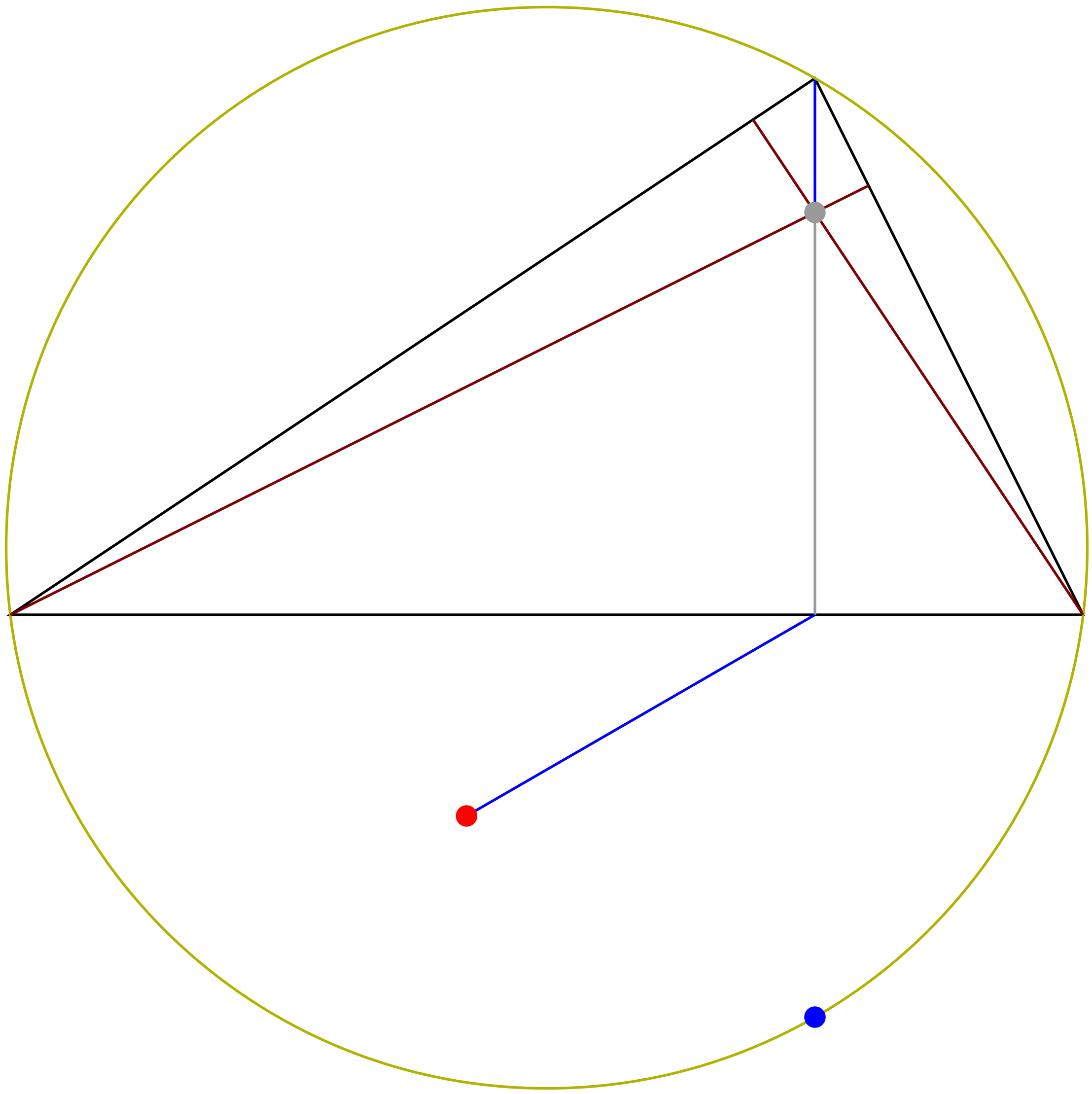


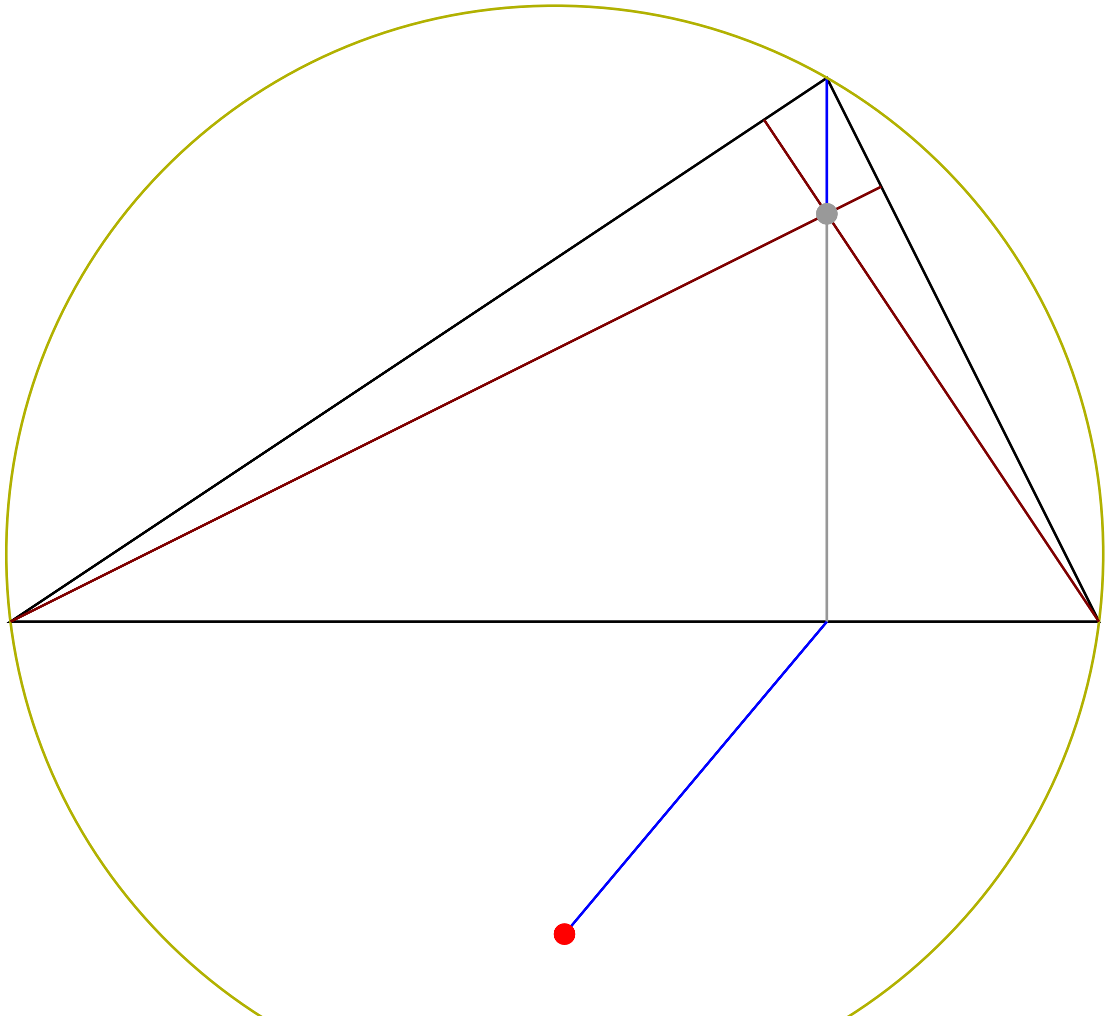


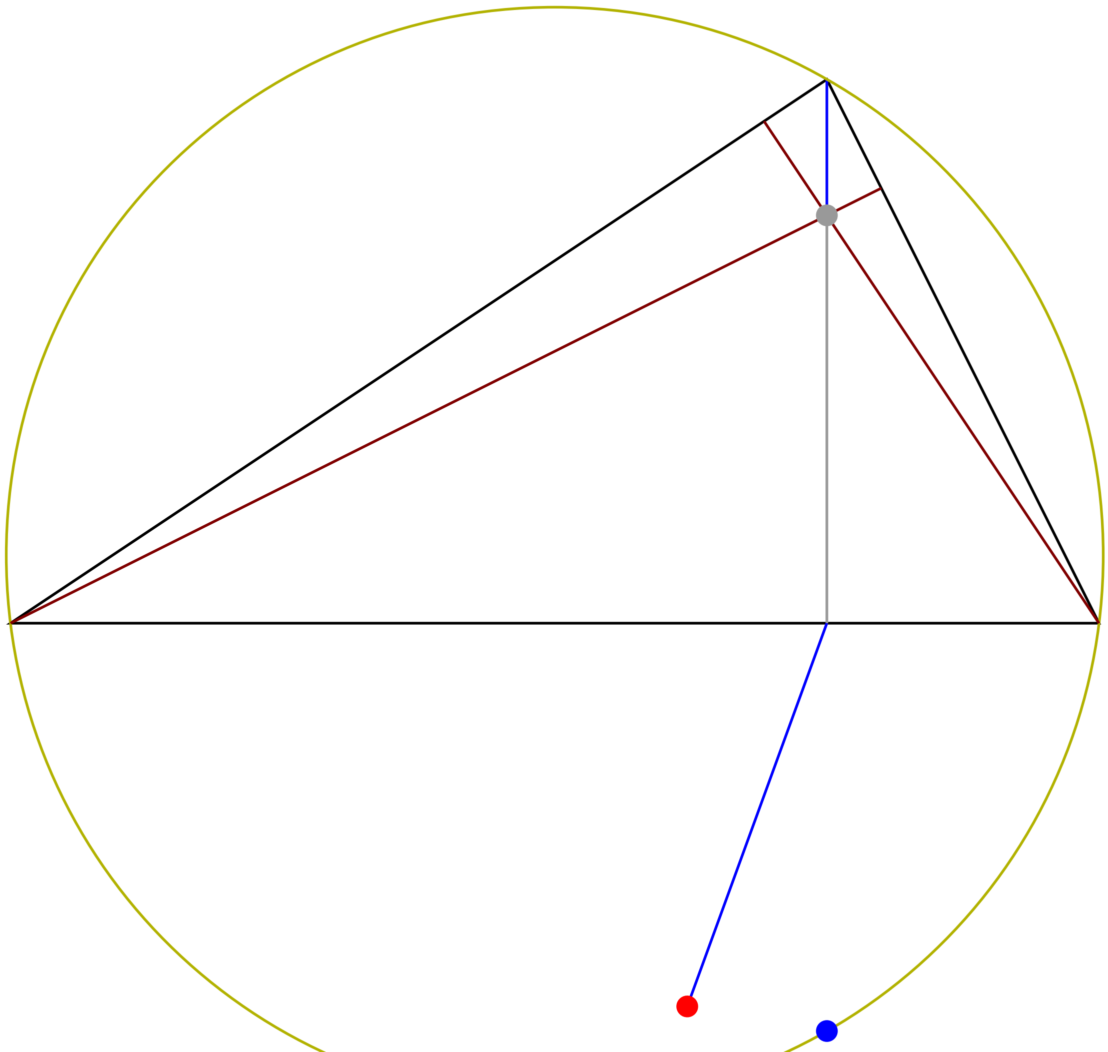


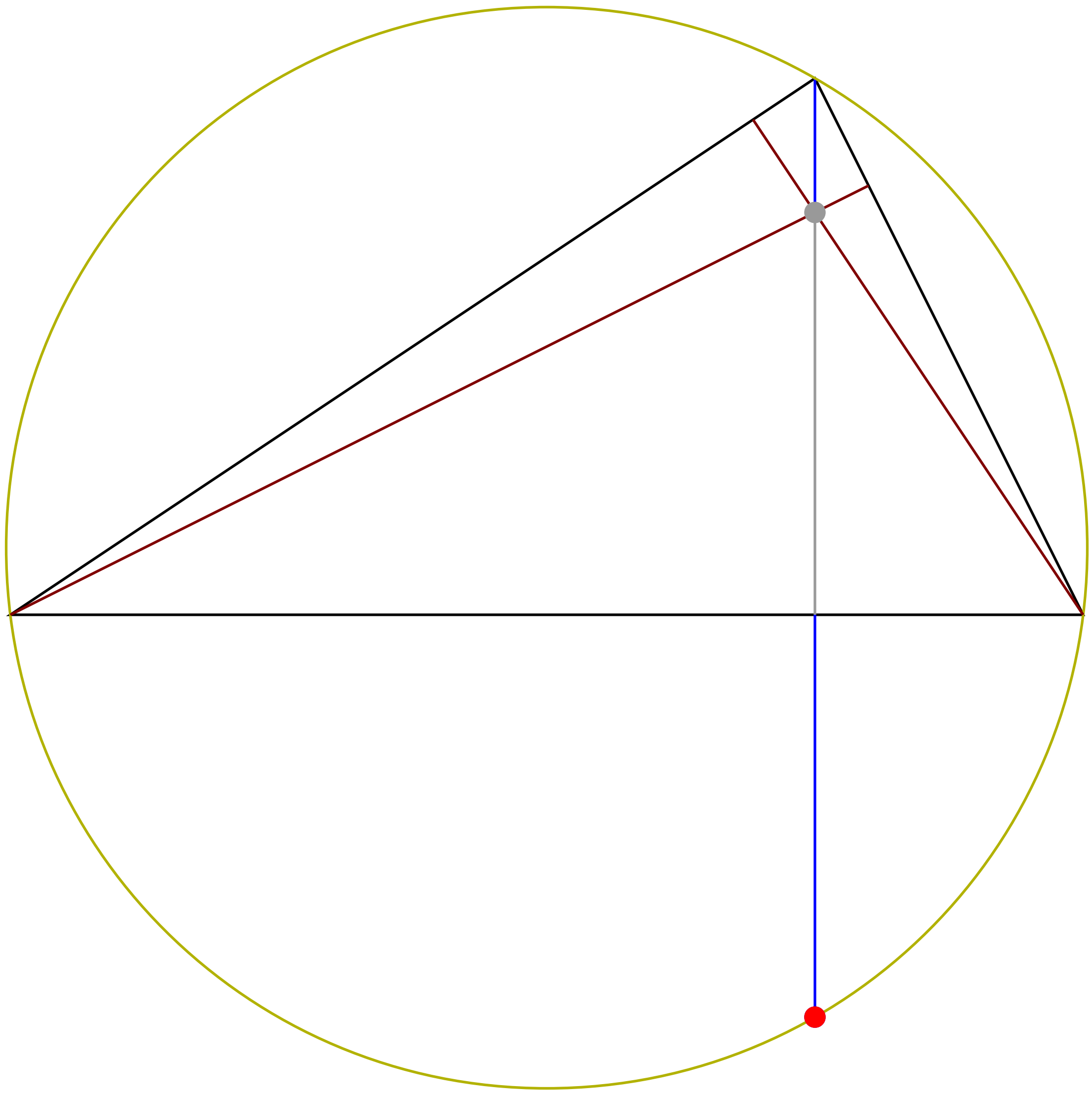


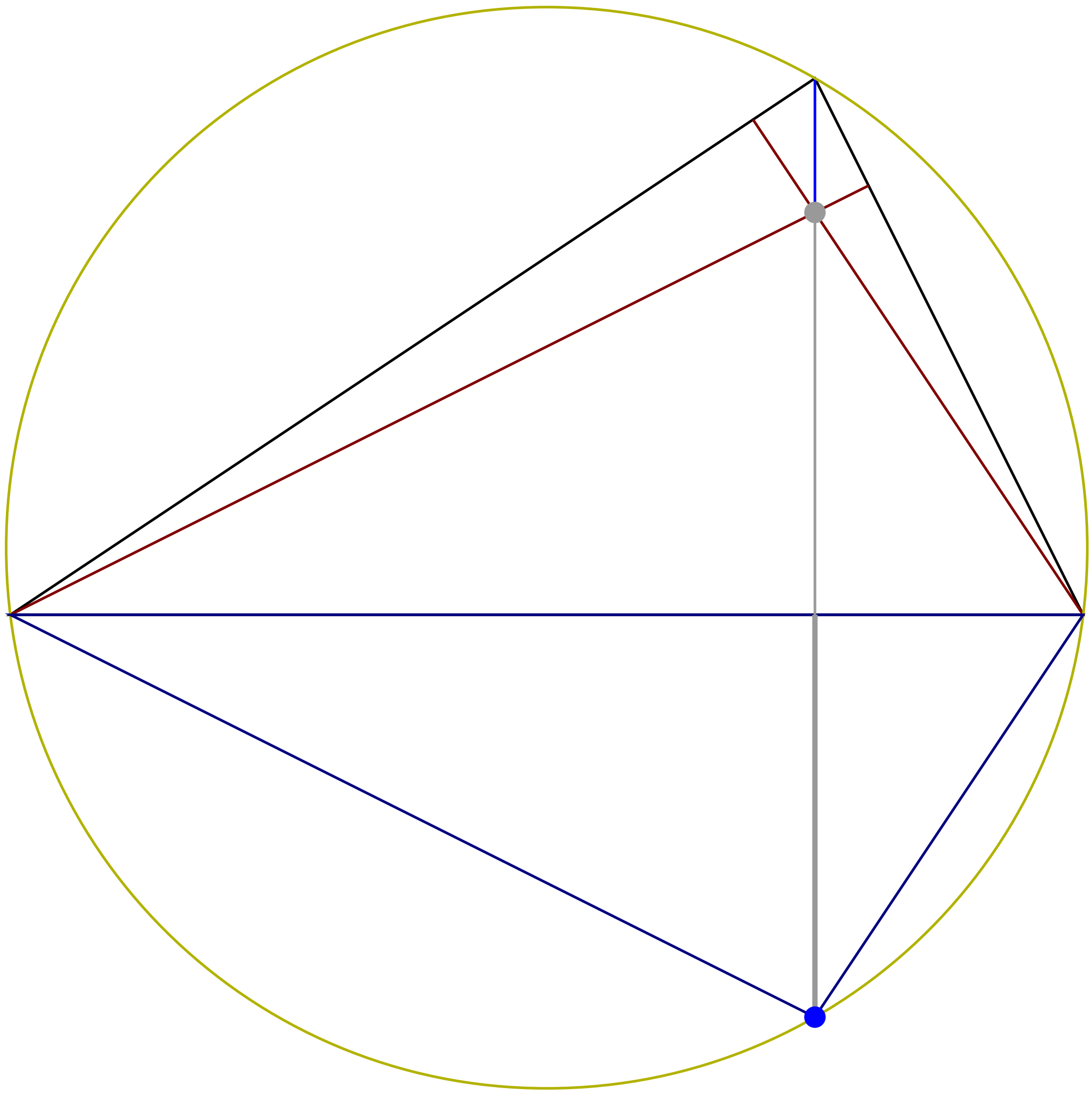




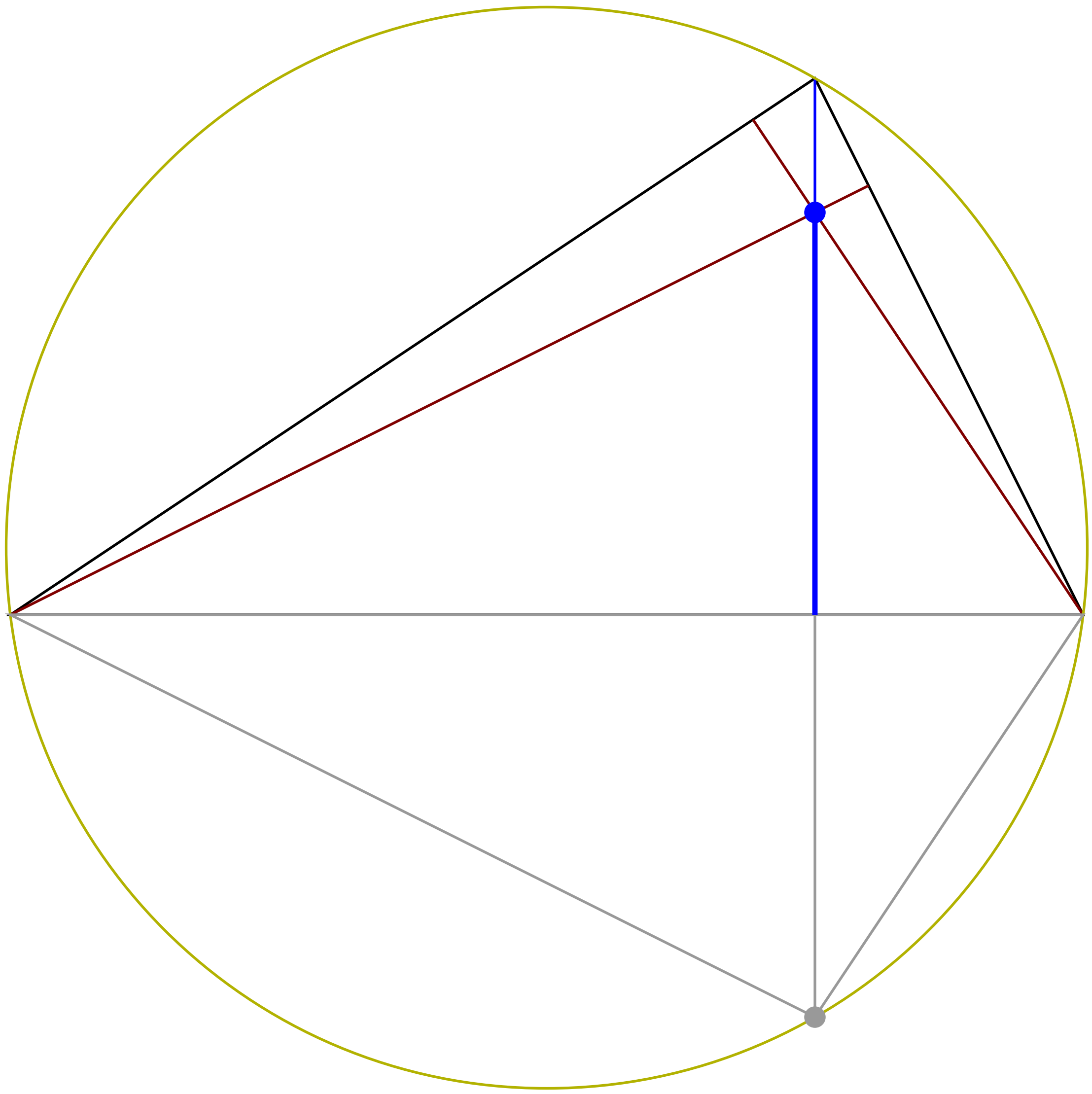




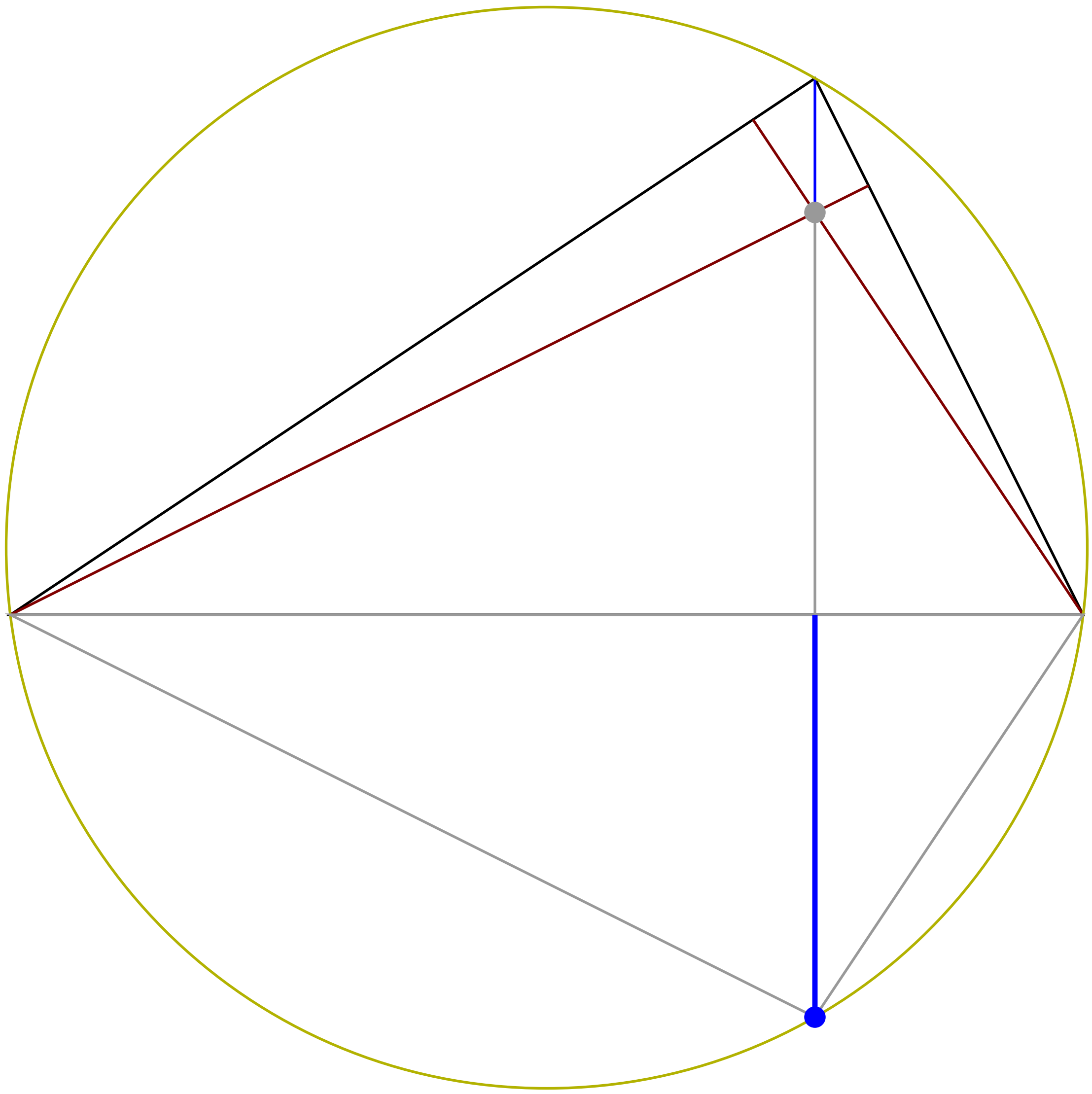




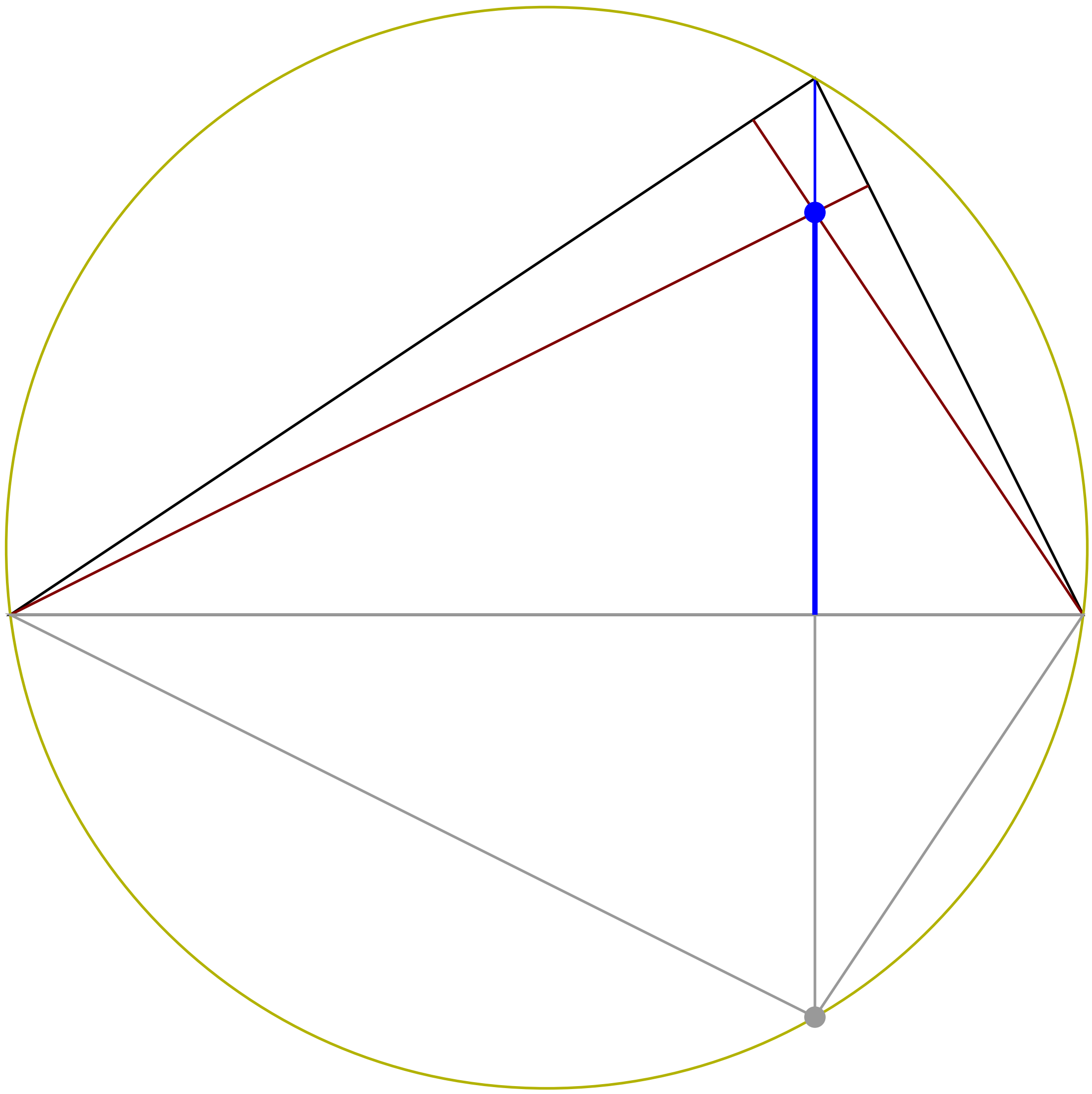
Now let the new point on the circle be joined with the other two vertices and form a new triangle.



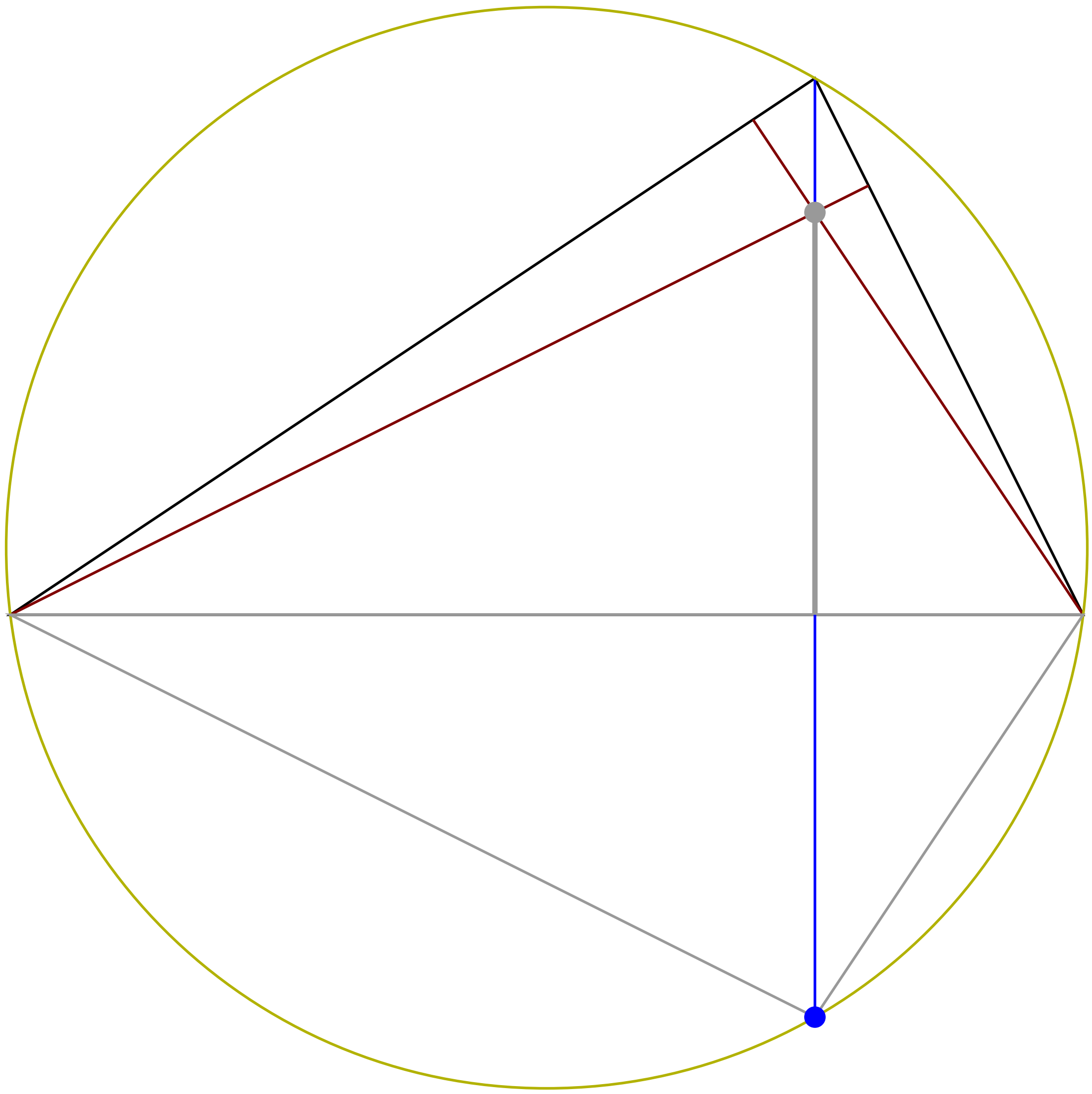
Since the two blue lines are equal in length,



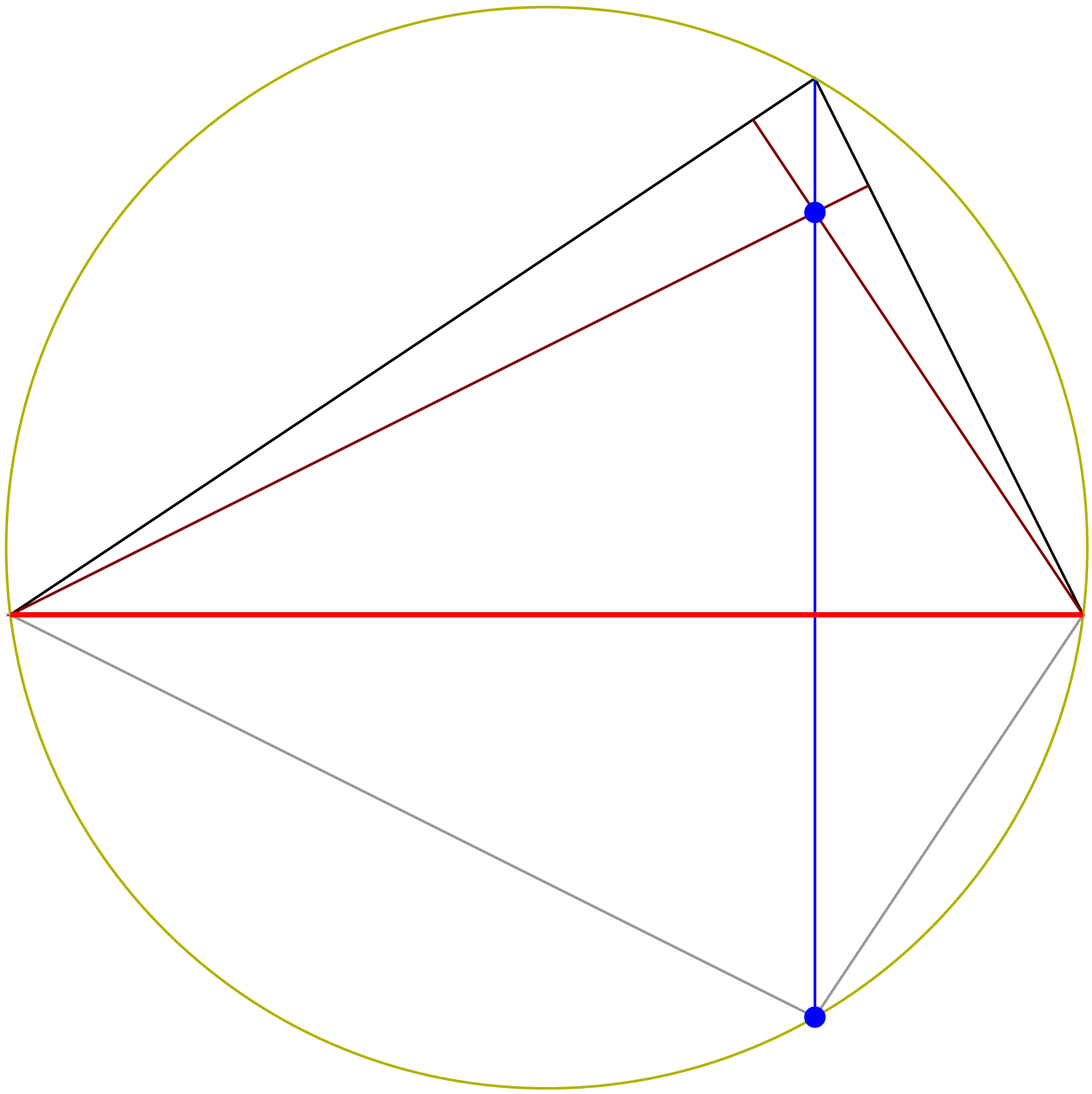
Since the two blue lines are equal in length,



Since the two blue lines are equal in length,

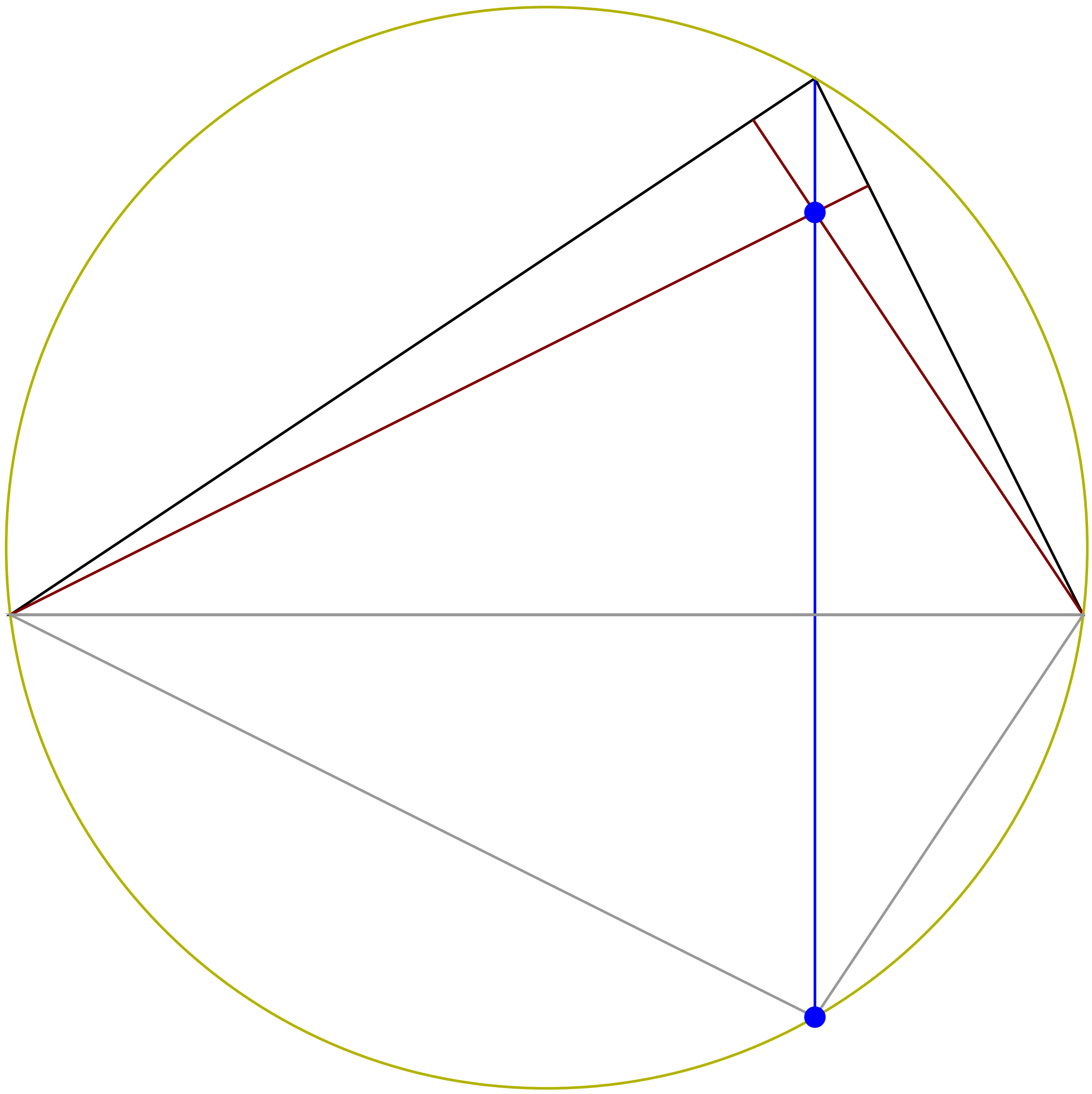


Since the two blue lines are equal in length,



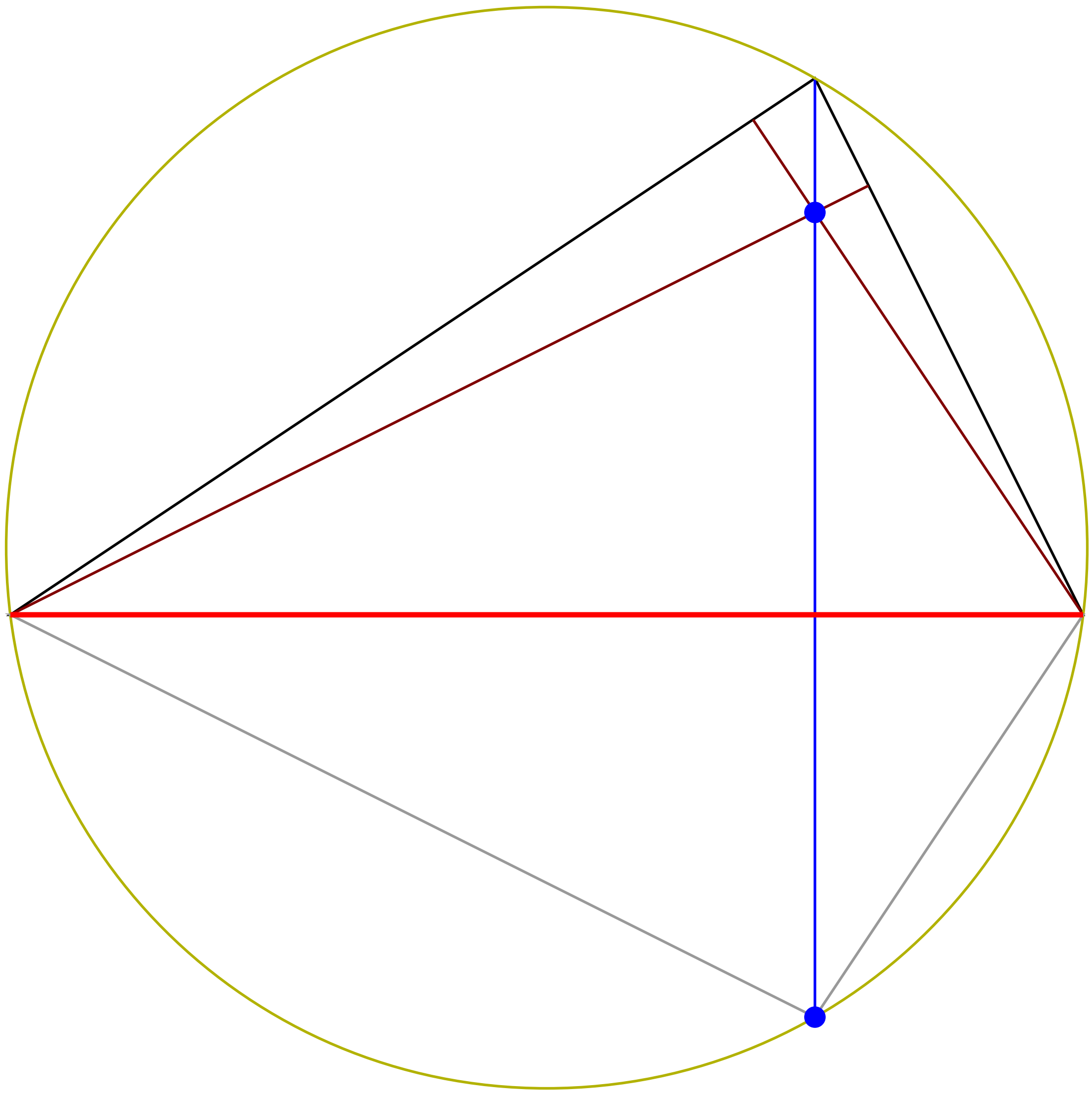
Since the two blue lines are equal in length,

and since the red line is common,



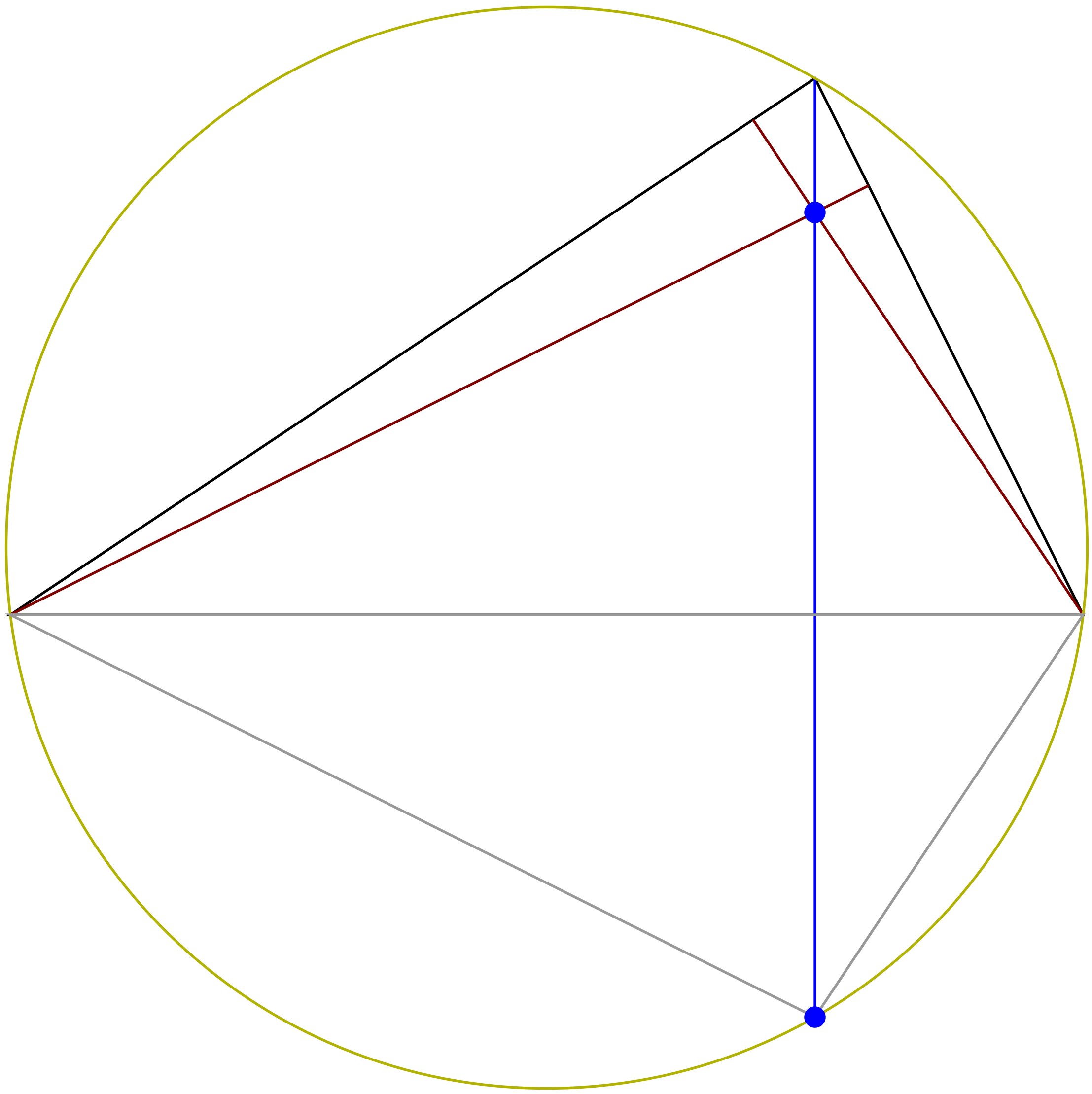
Since the two blue lines are equal in length,

and since the red line is common,



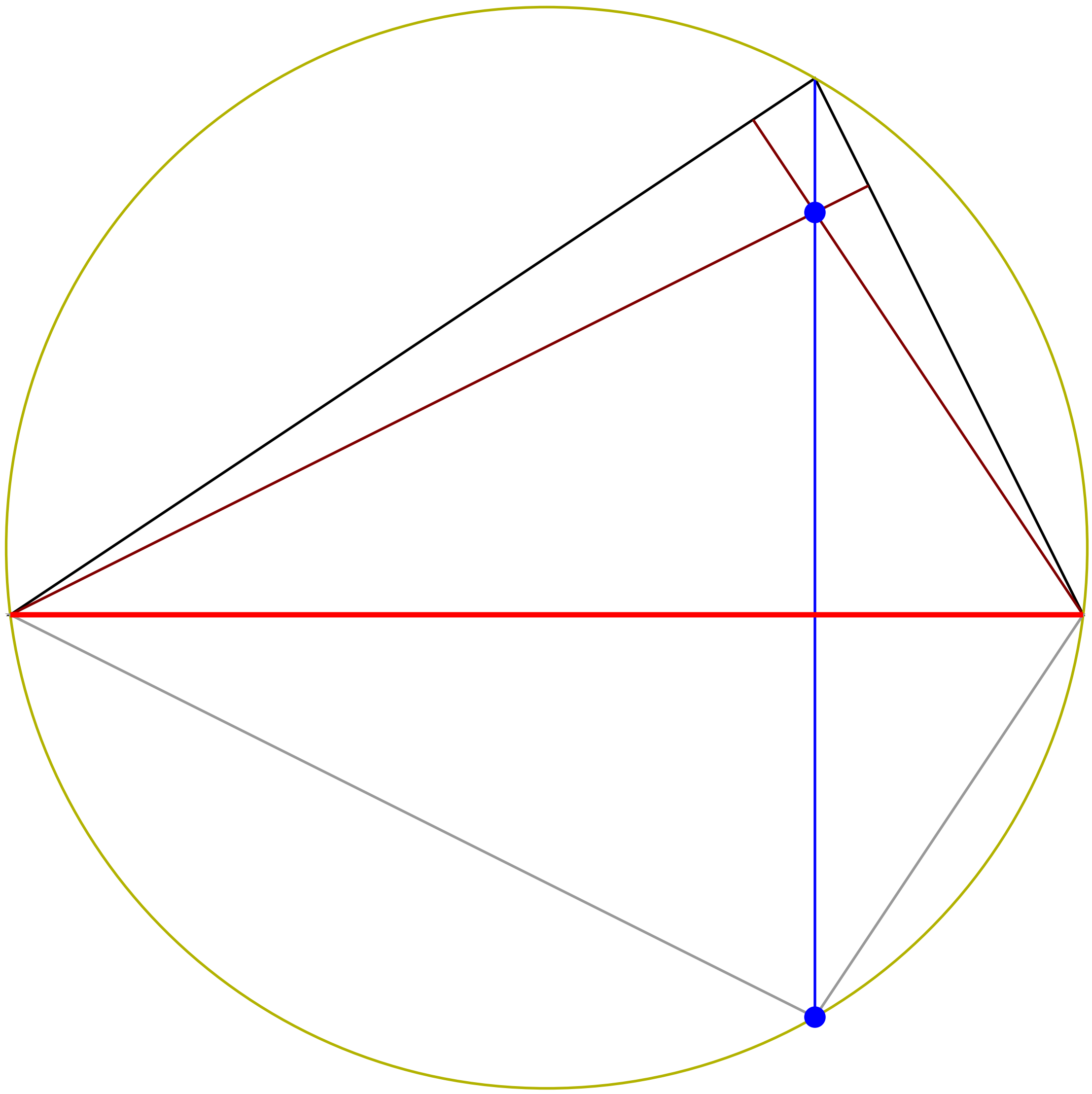
Since the two blue lines are equal in length,

and since the red line is common,



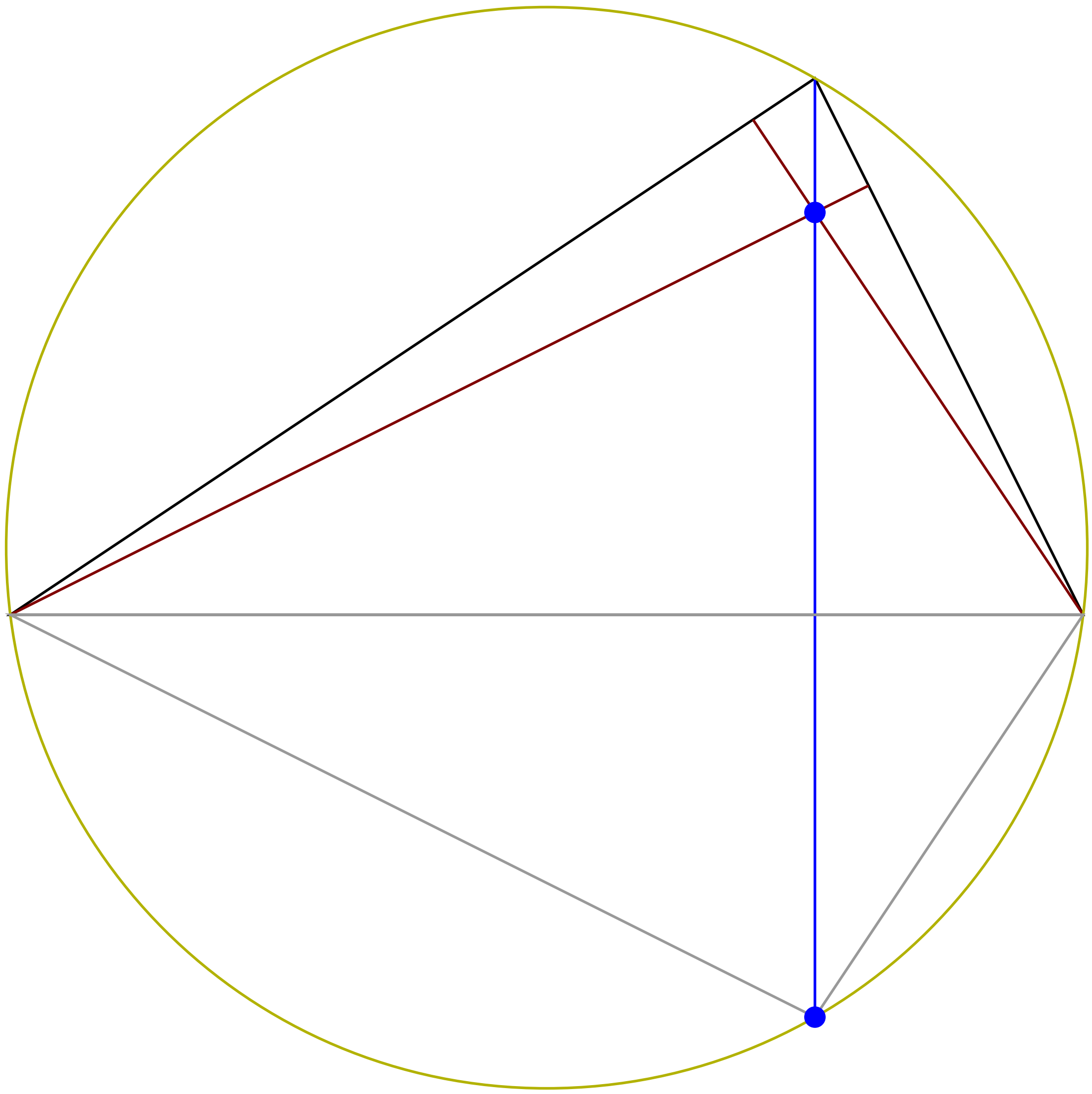
Since the two blue lines are equal in length,

and since the red line is common,



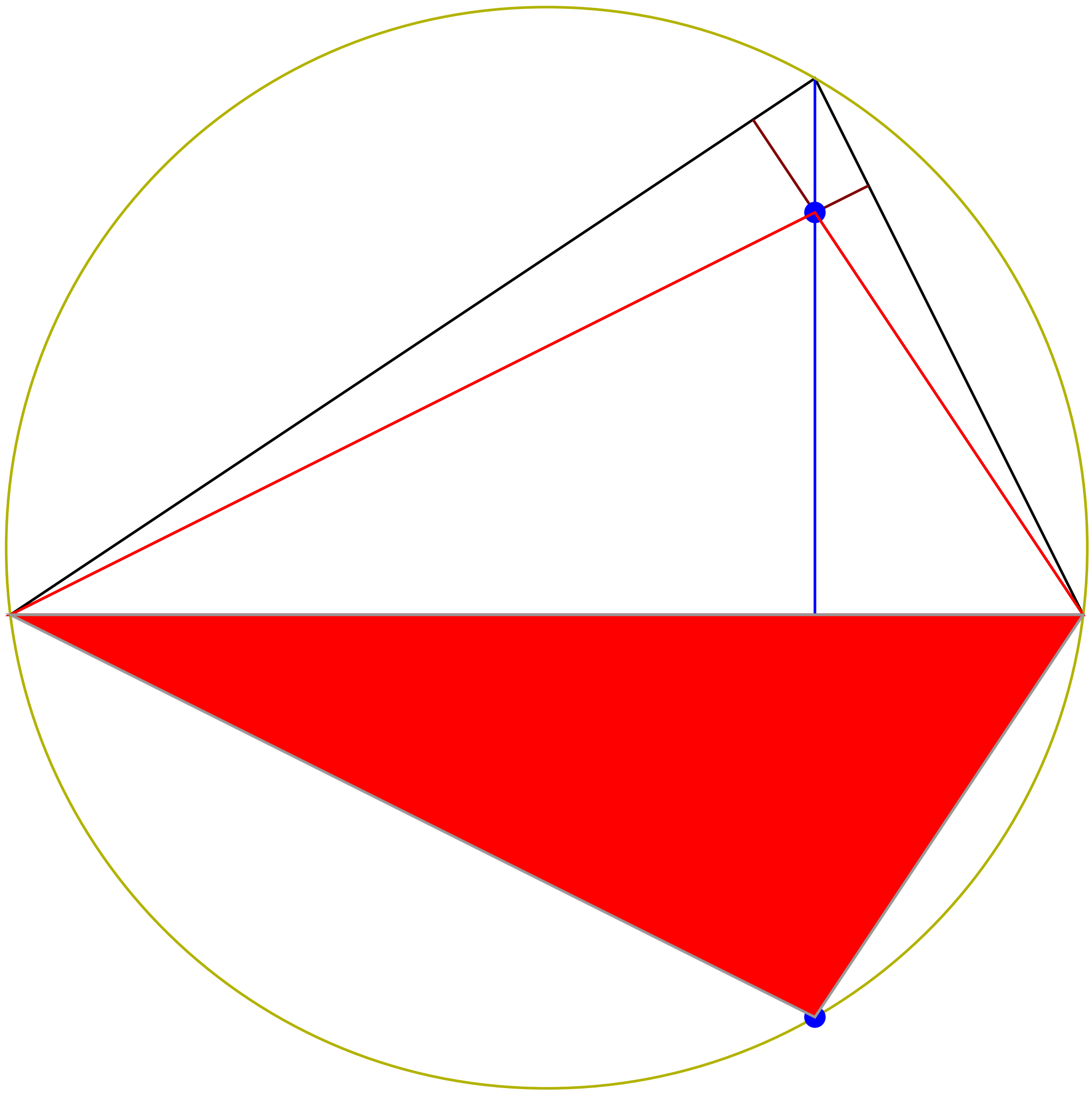
Since the two blue lines are equal in length,

and since the red line is common,



Since the two blue lines are equal in length,

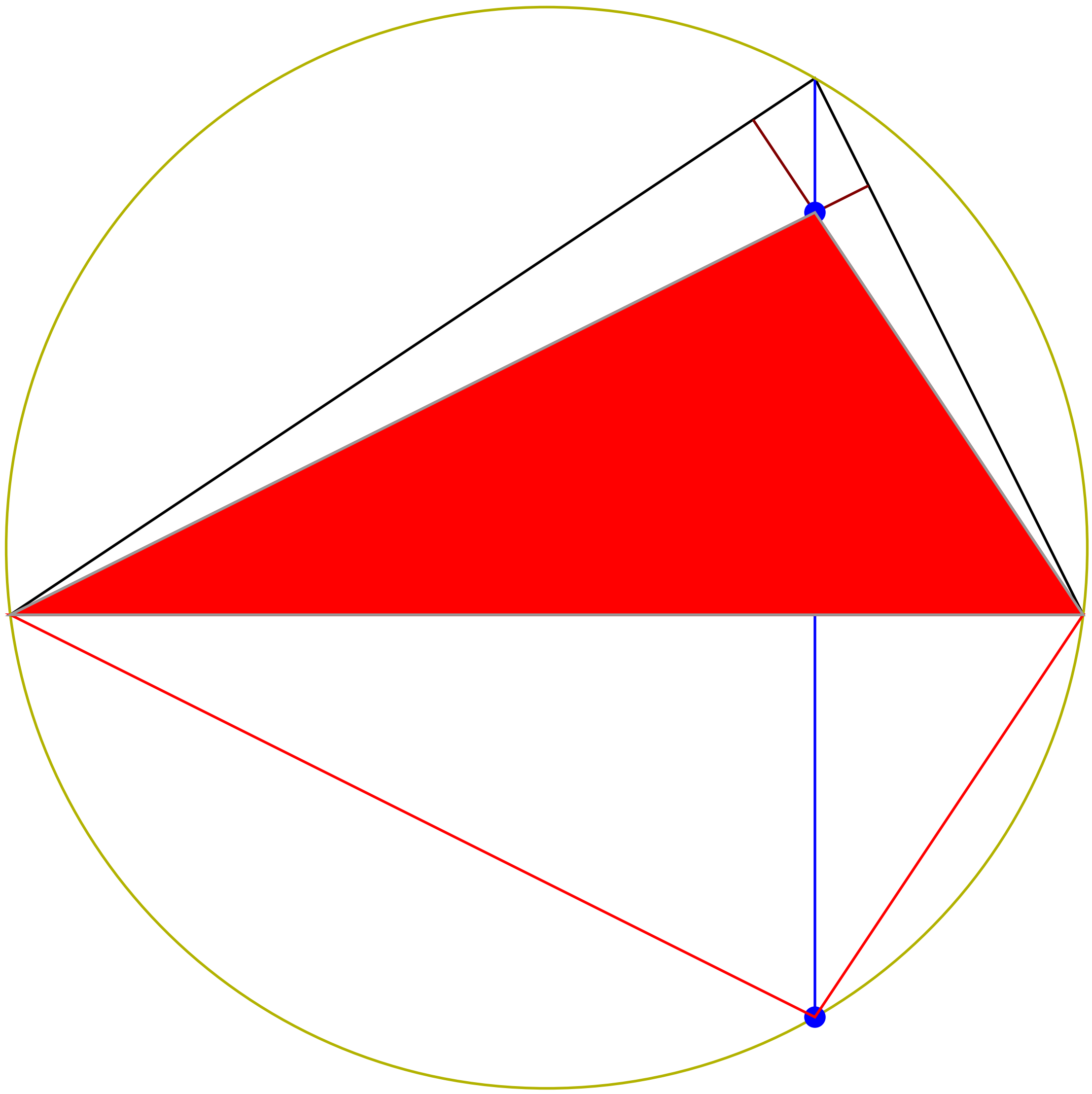
and since the red line is common,



Since the two blue lines are equal in length,

and since the red line is common,

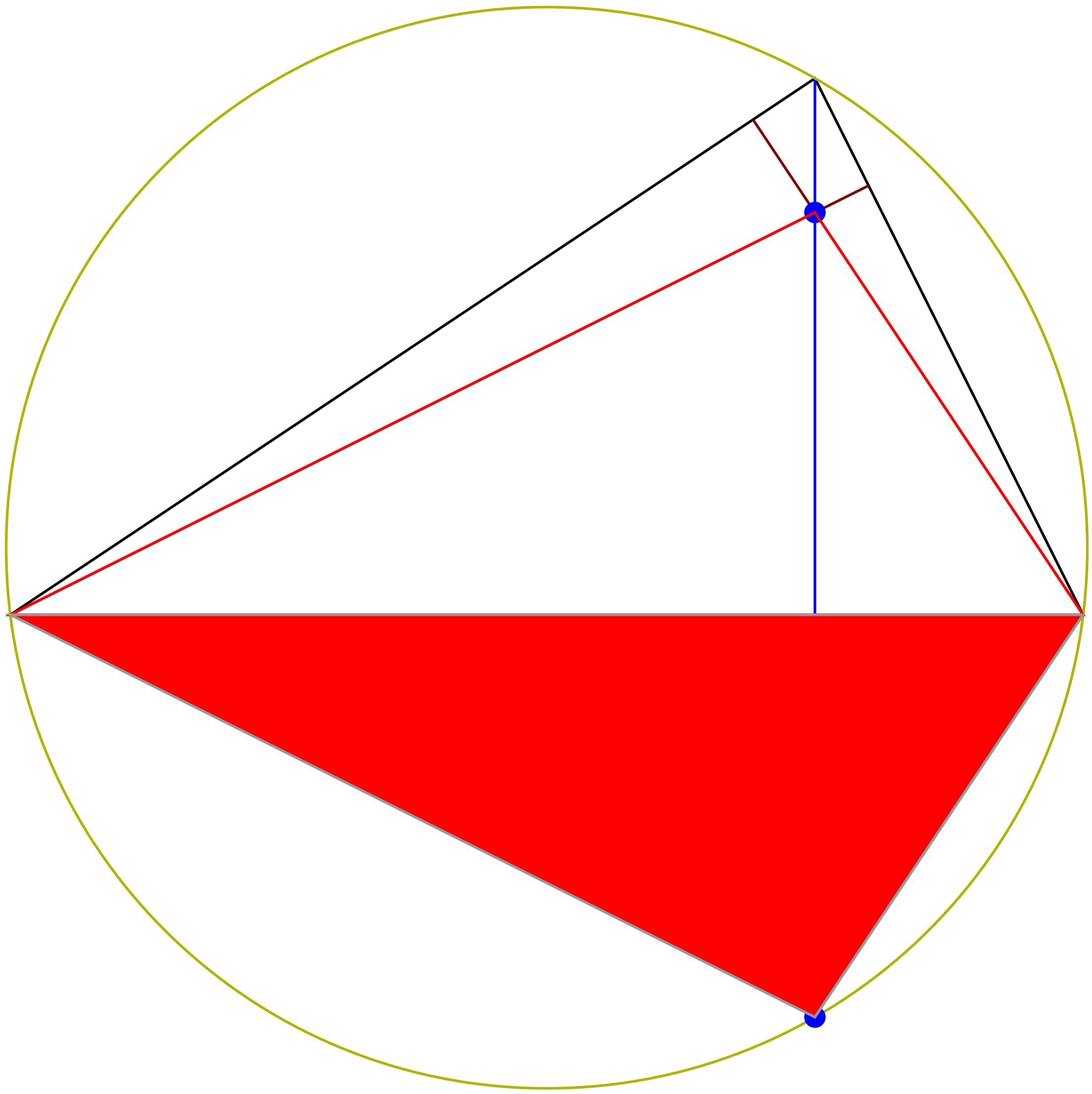
THEREFORE, the two triangles have the same area.



Since the two blue lines are equal in length,

and since the red line is common,

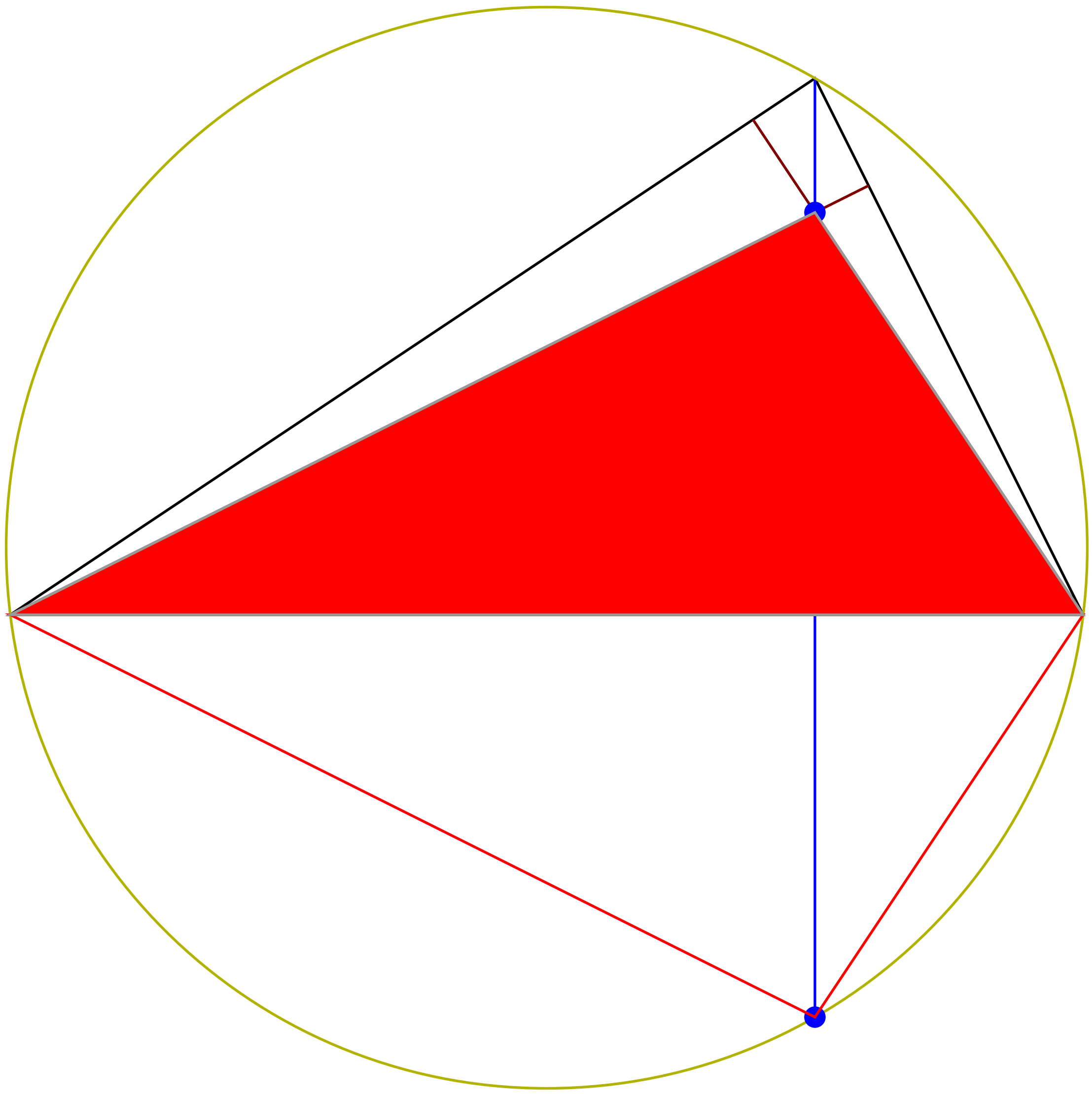
THEREFORE, the two triangles have the same area.



Since the two blue lines are equal in length,

and since the red line is common,

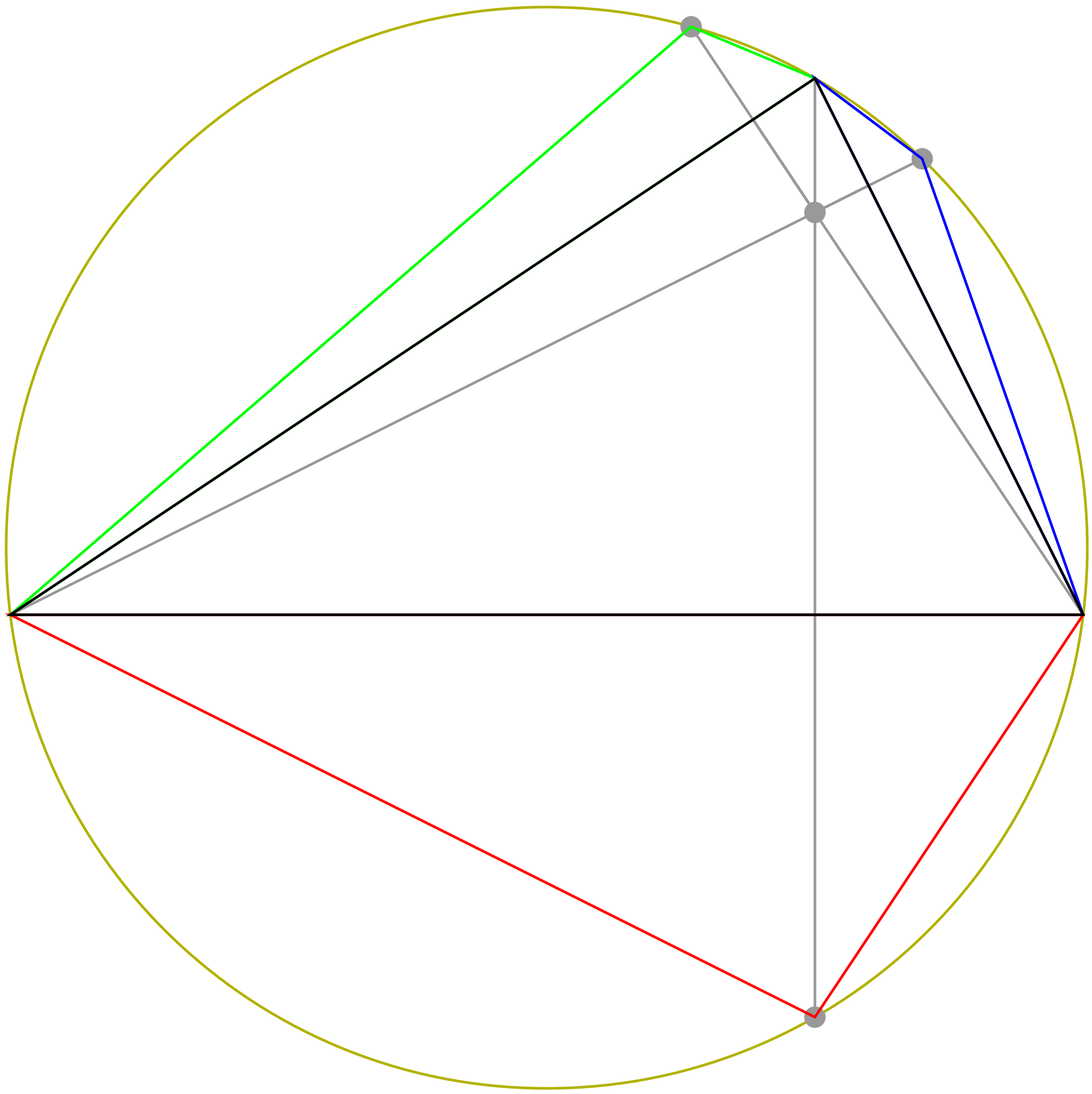
THEREFORE, the two triangles have the same area.



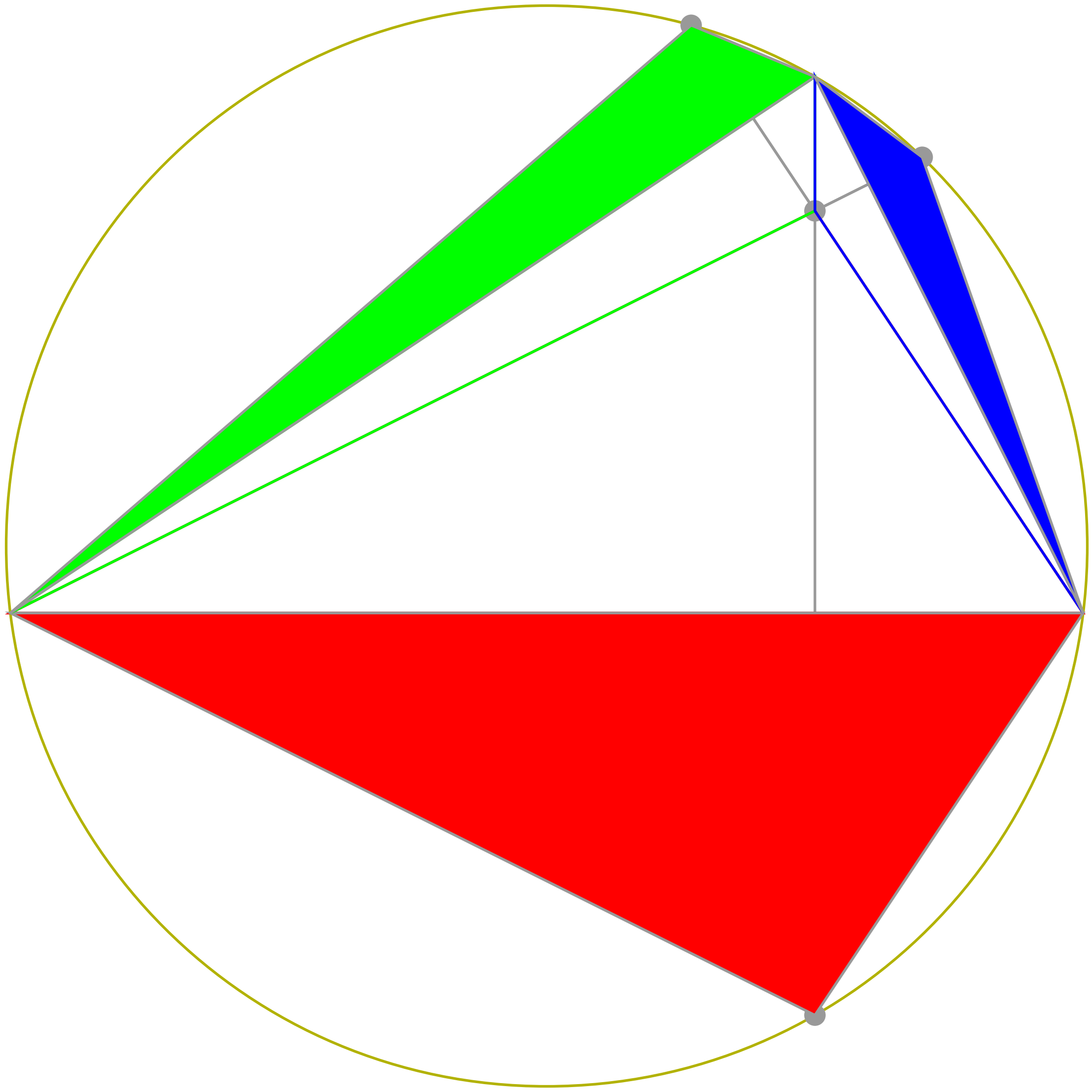
Since the two blue lines are equal in length,

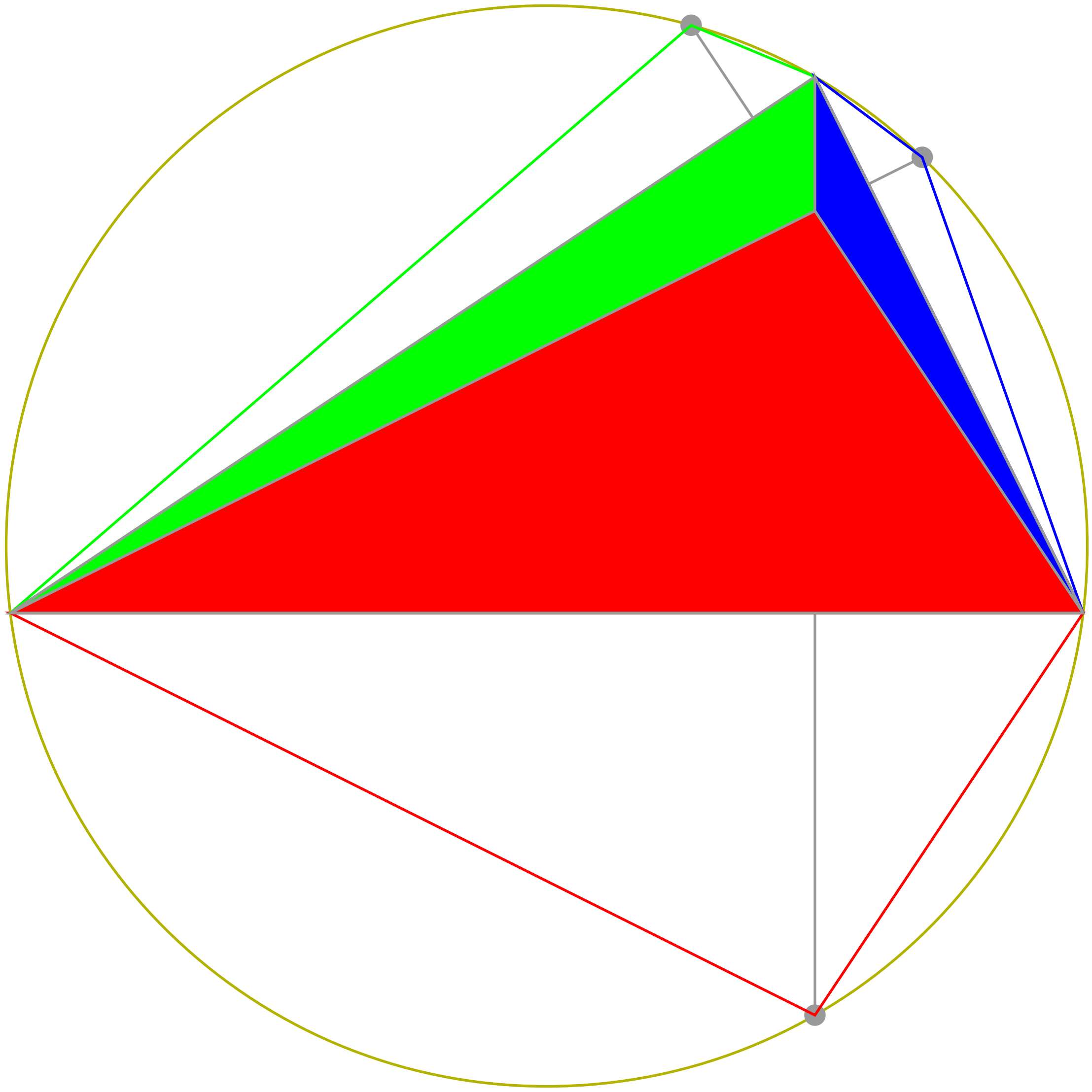
and since the red line is common,

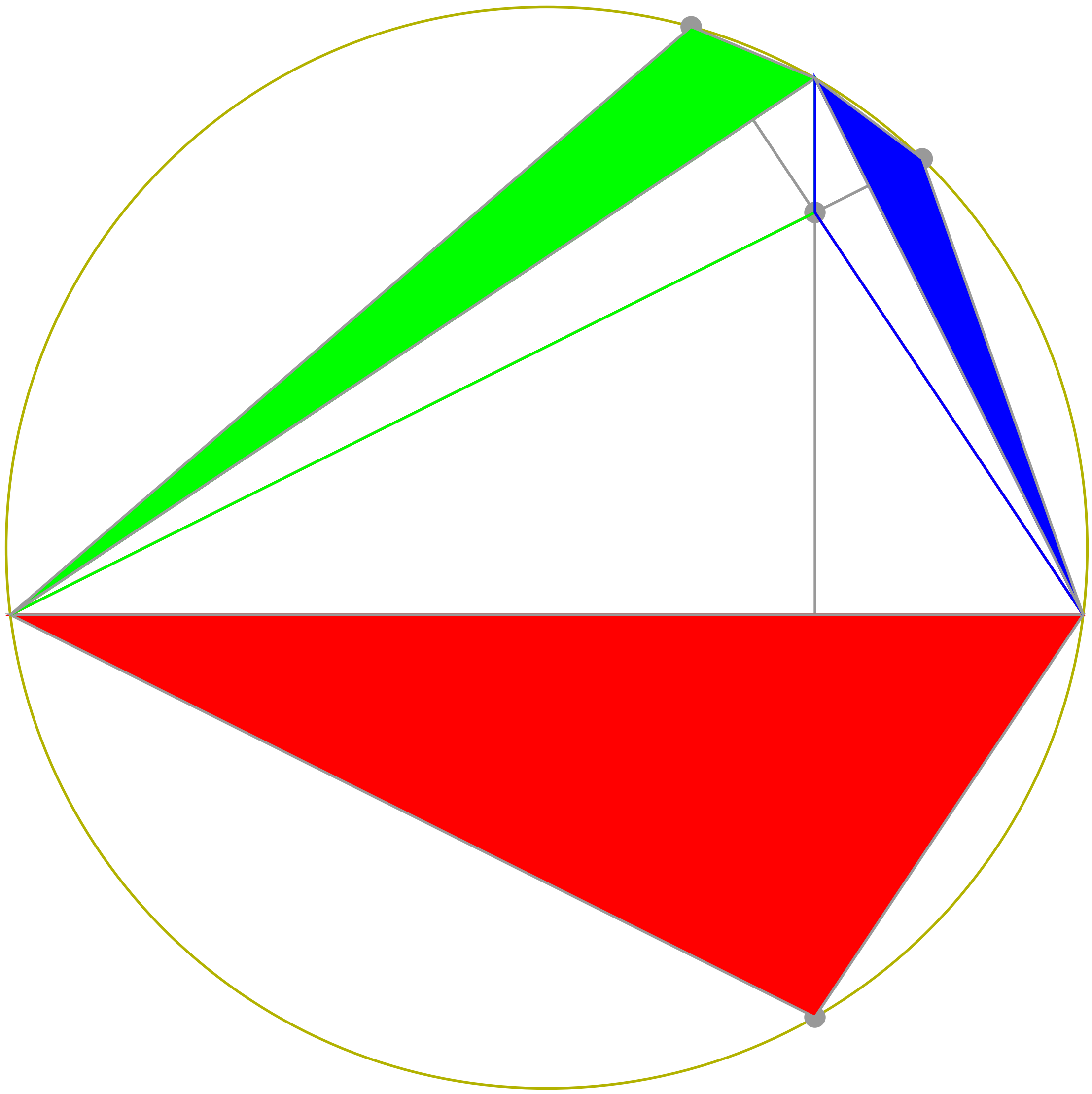
THEREFORE, the two triangles have the same area.

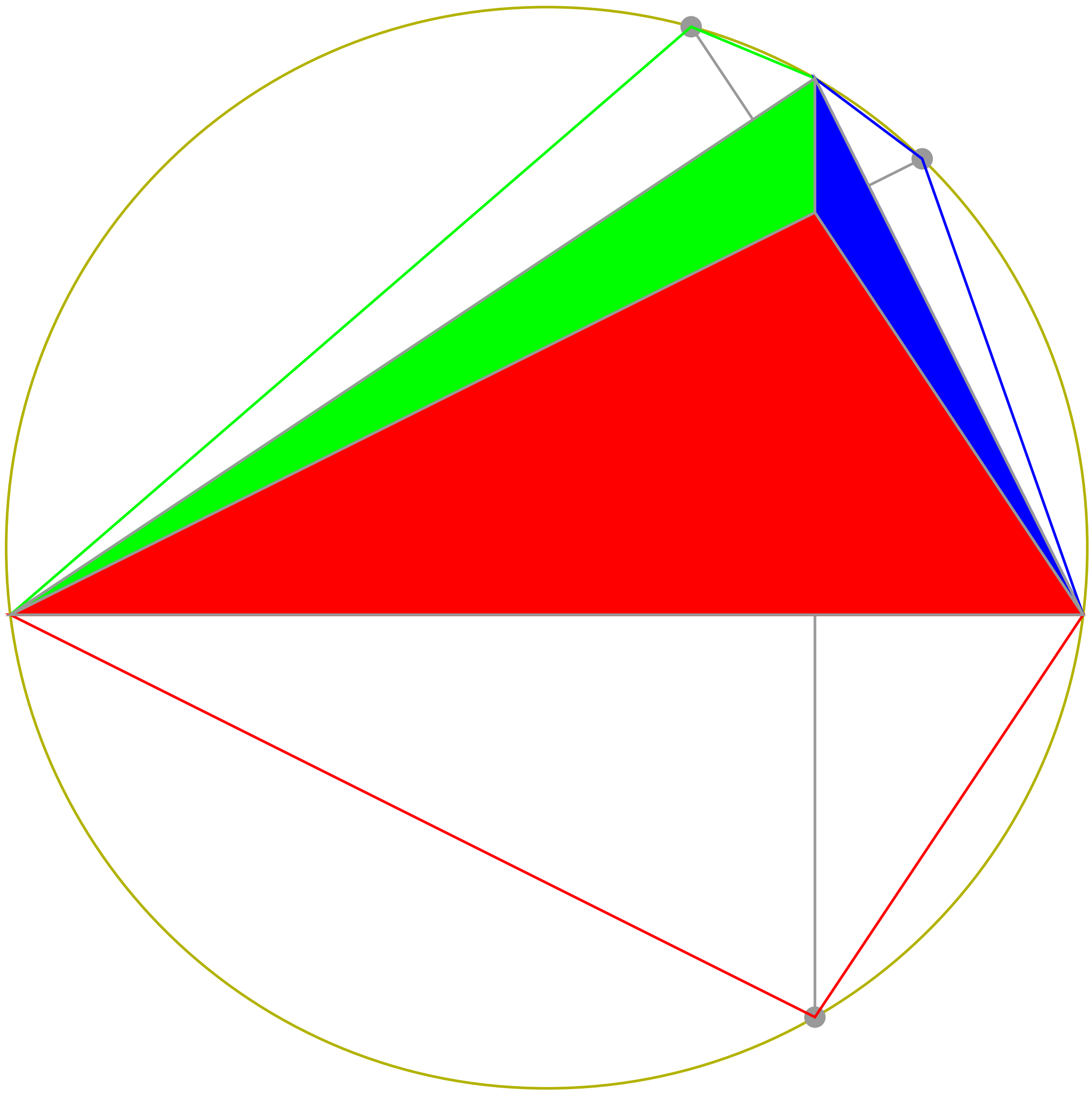


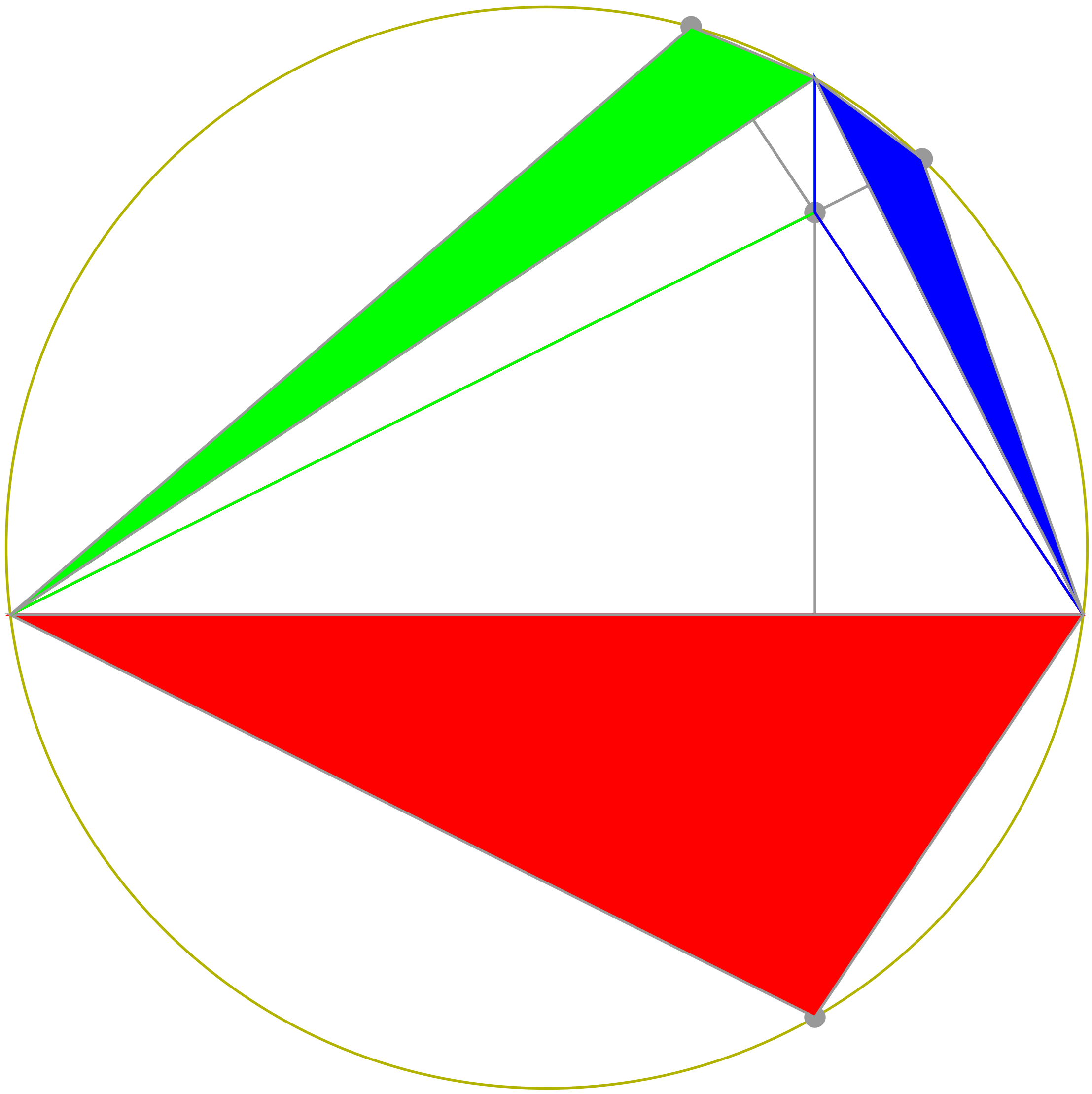
The same is true for the other two altitudes and their extensions as well.

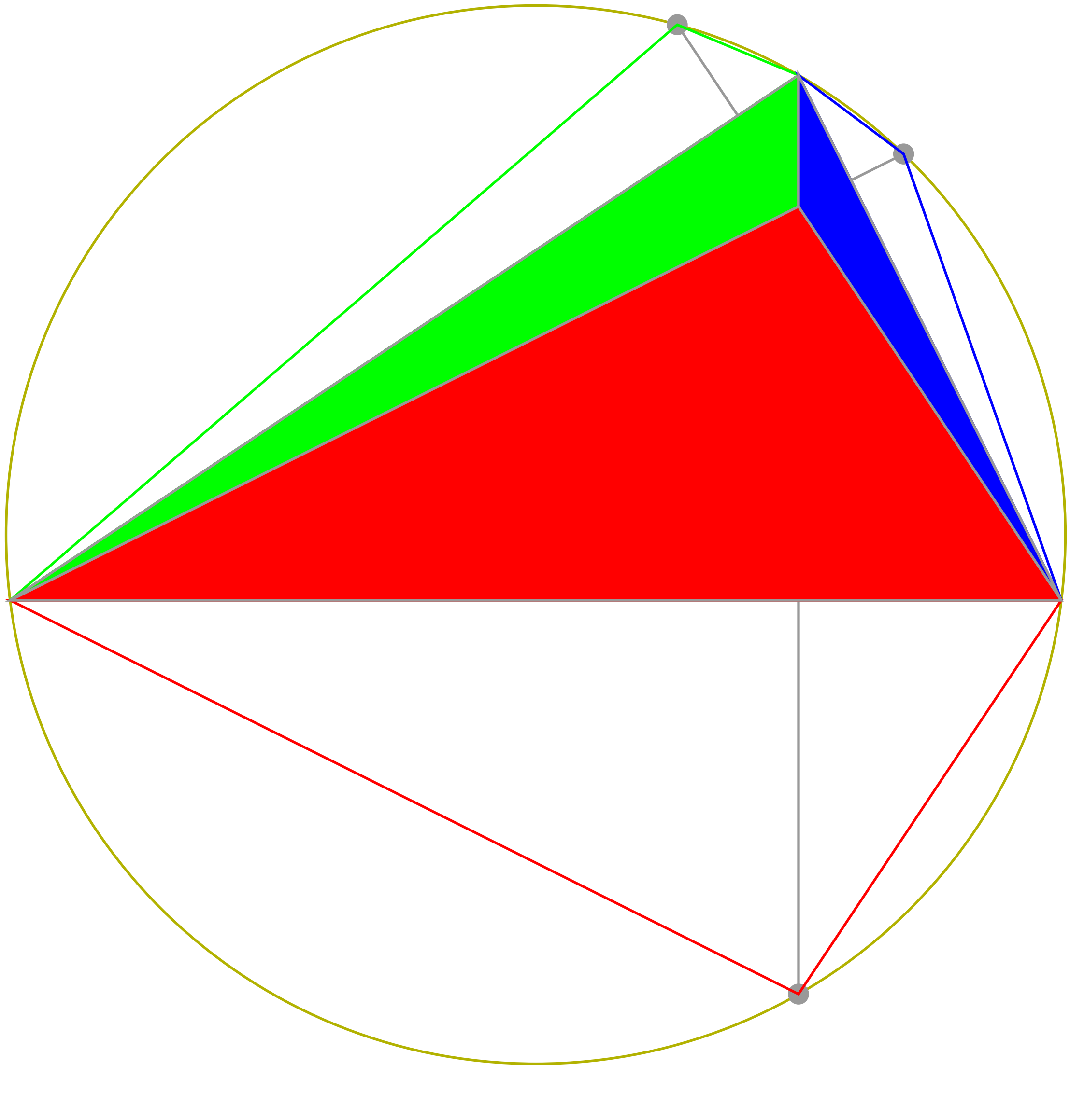








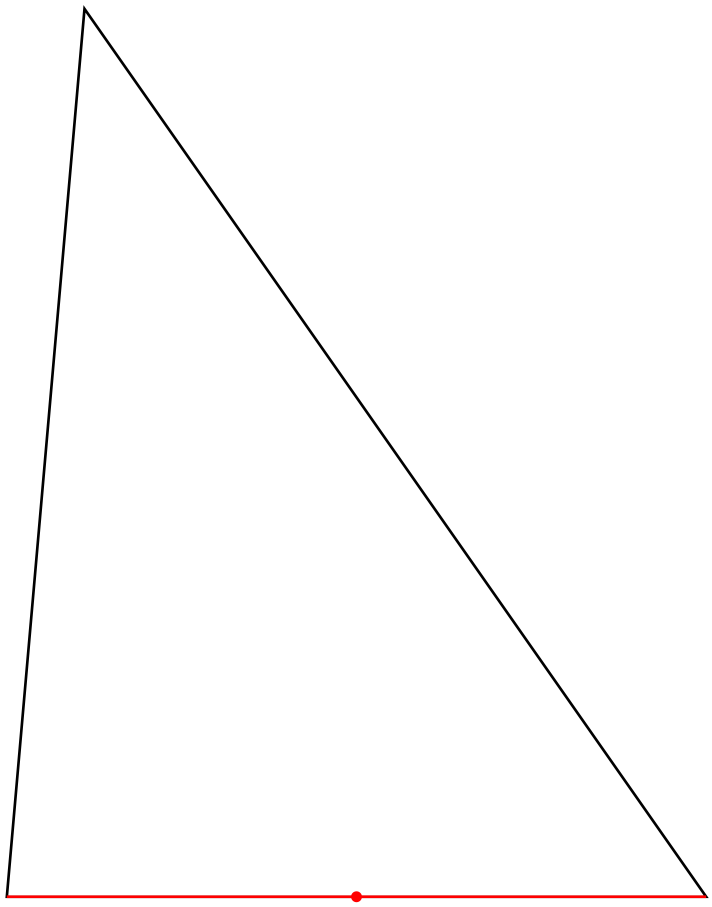




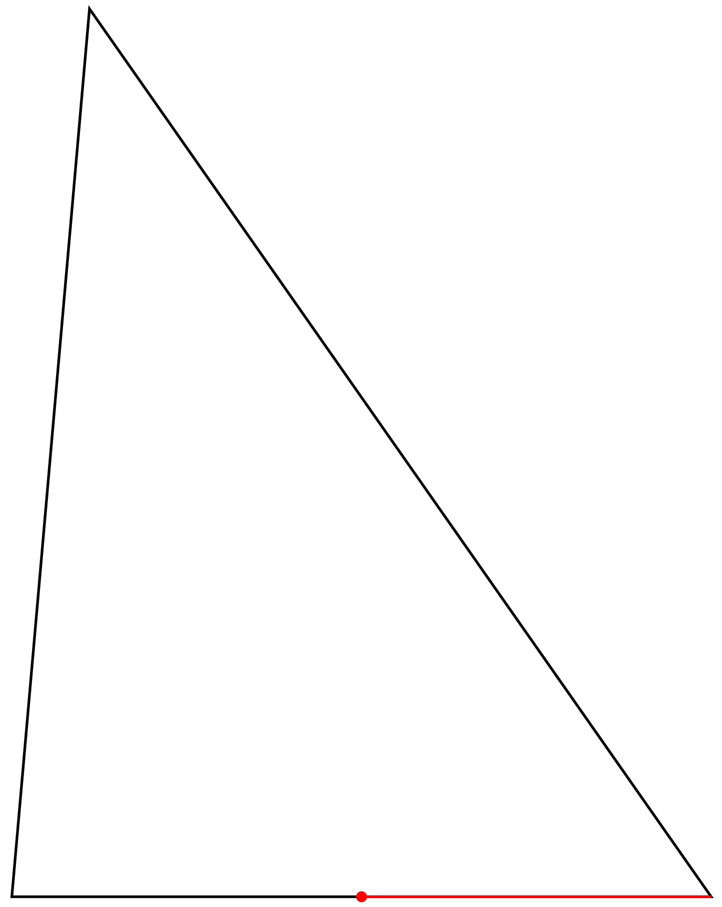
3. The Centroid (M)

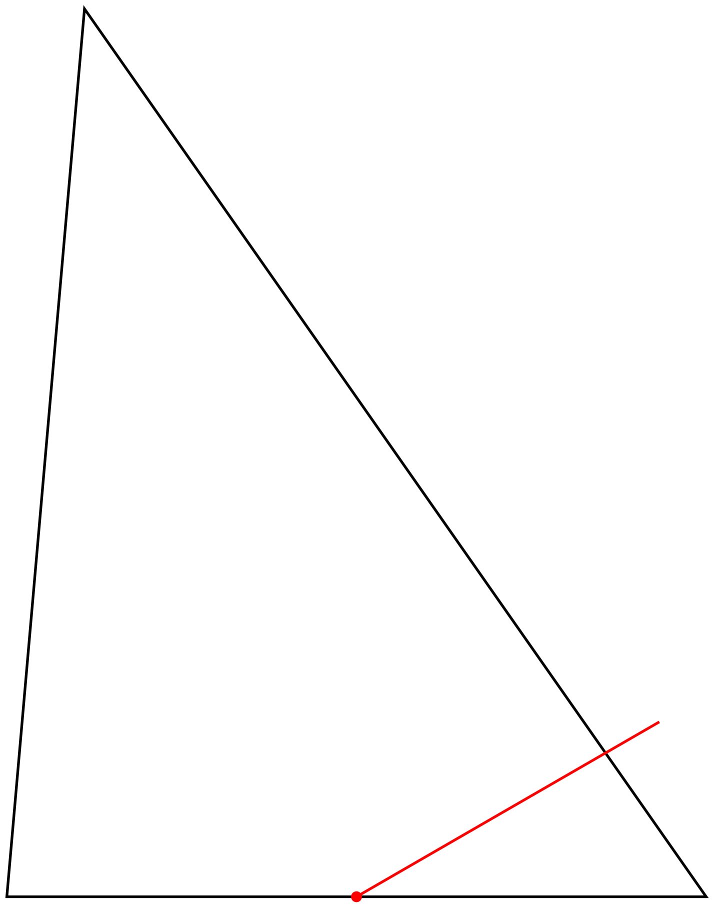
3.1 The Centroid (M)

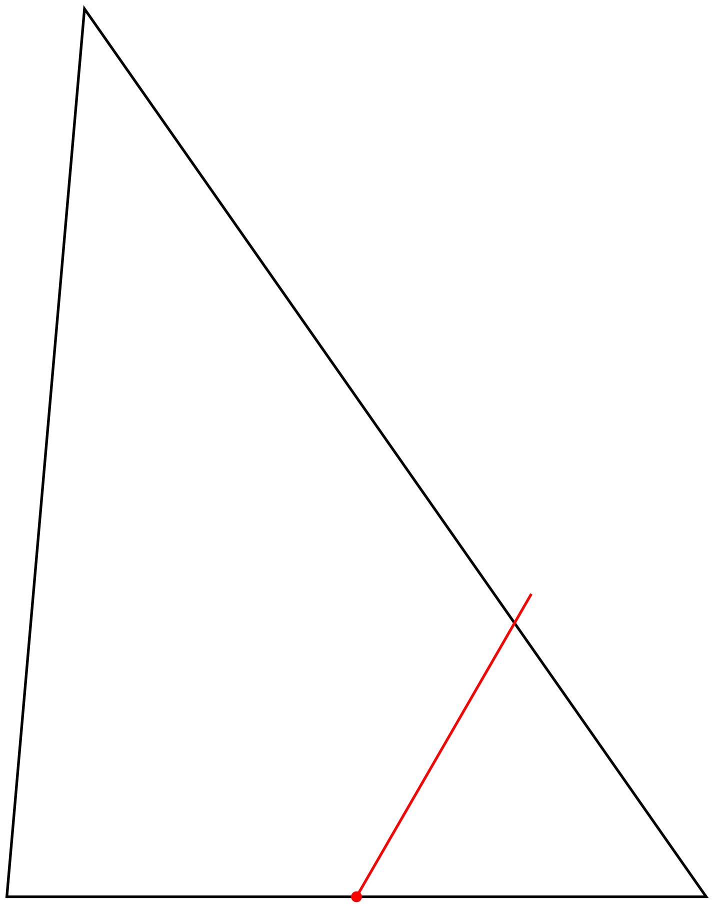
- the formation of M

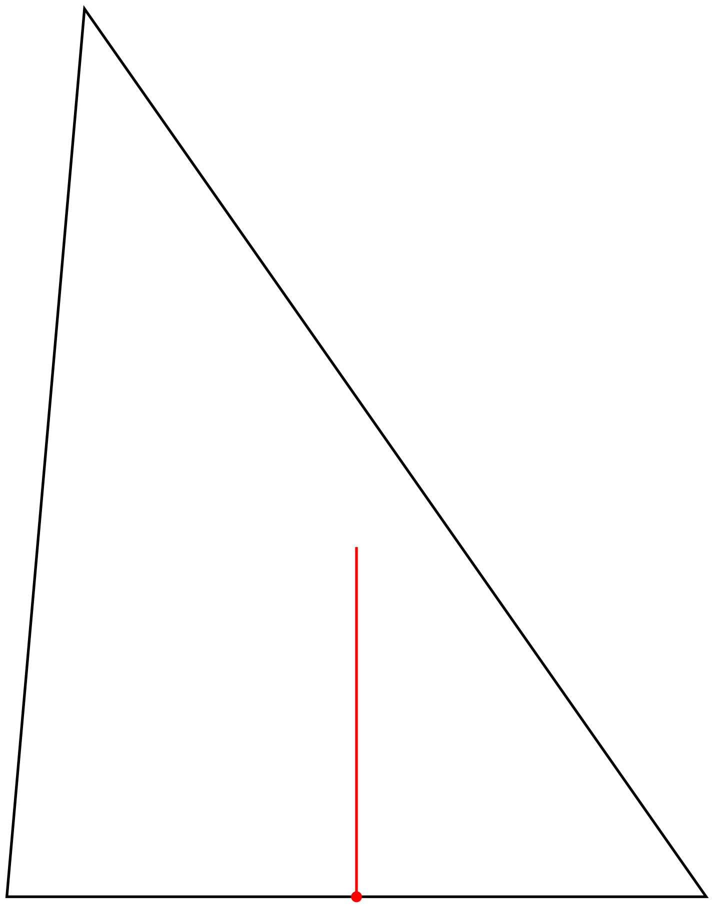


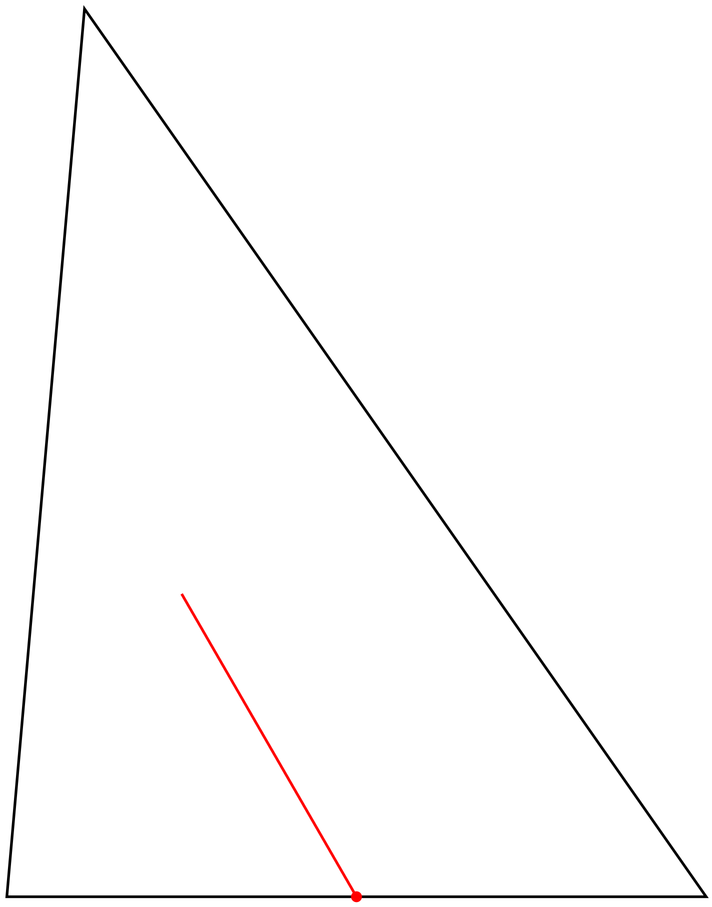
A median of a triangle is constructed by first finding the mid-point of one side,

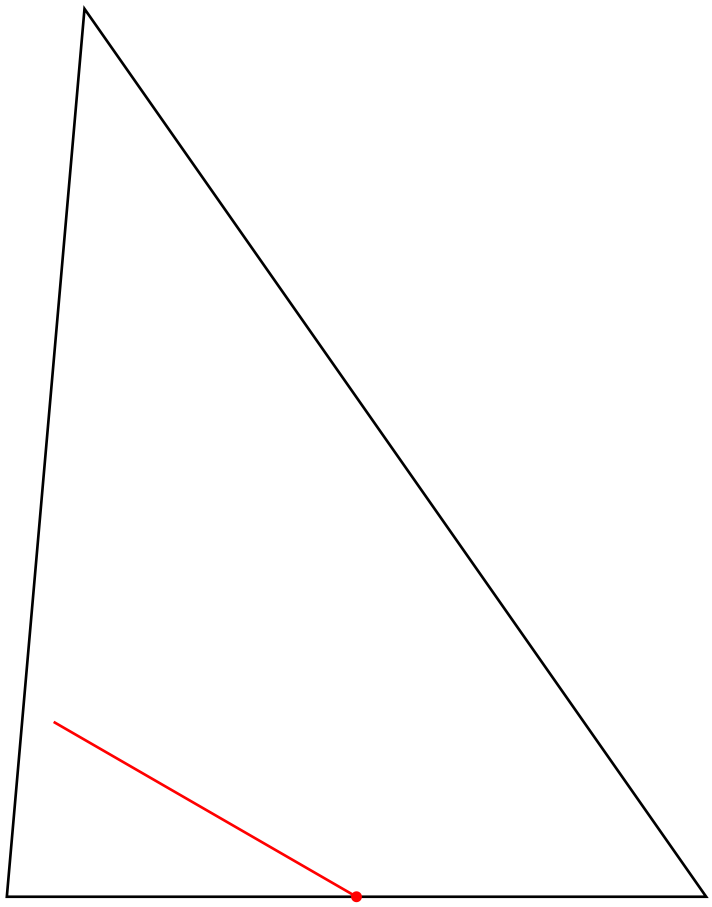


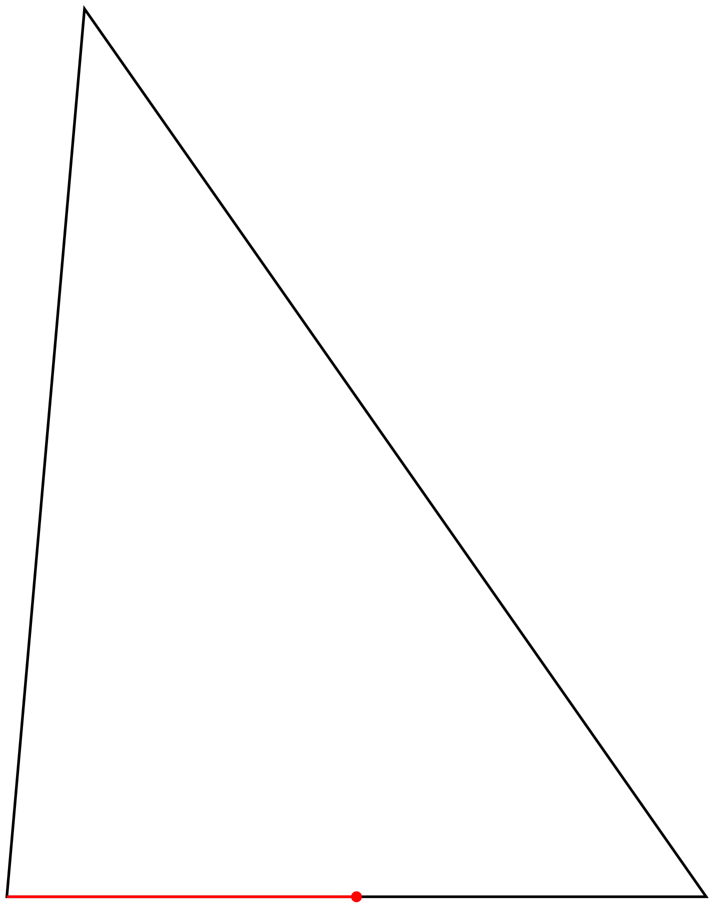


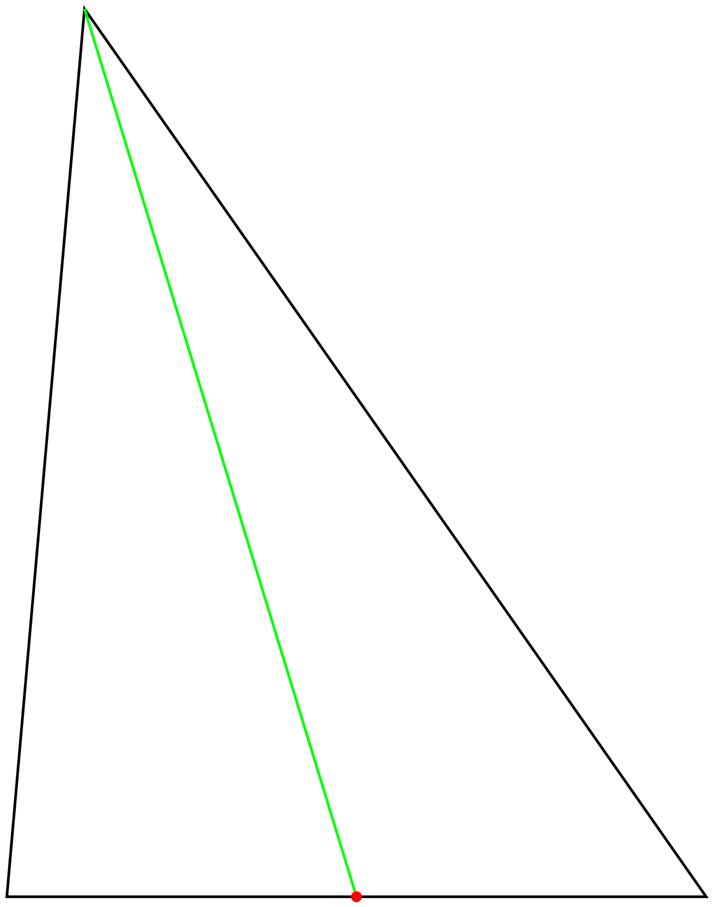




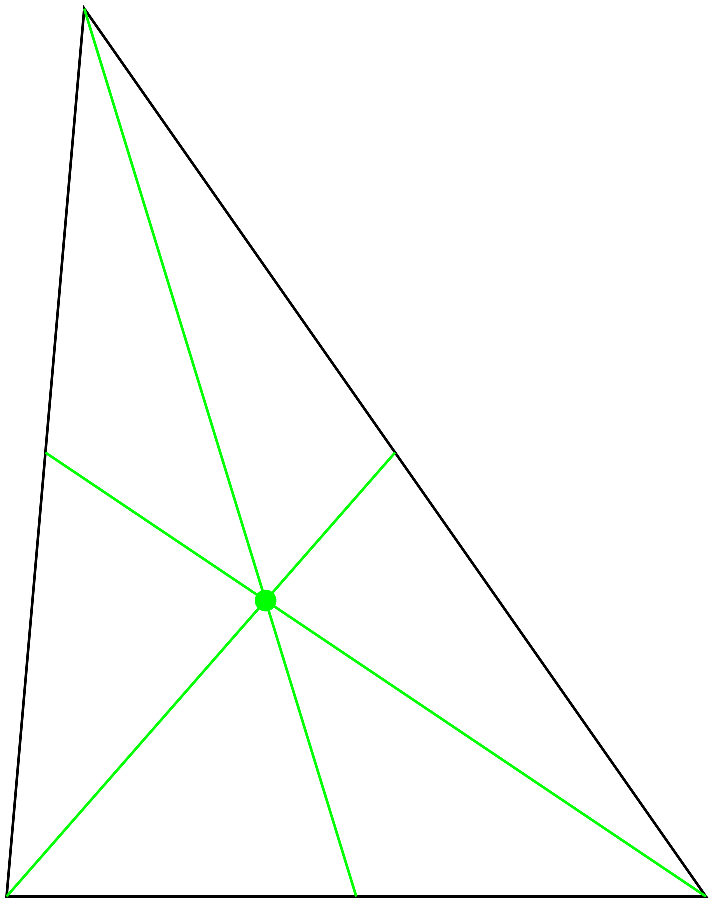








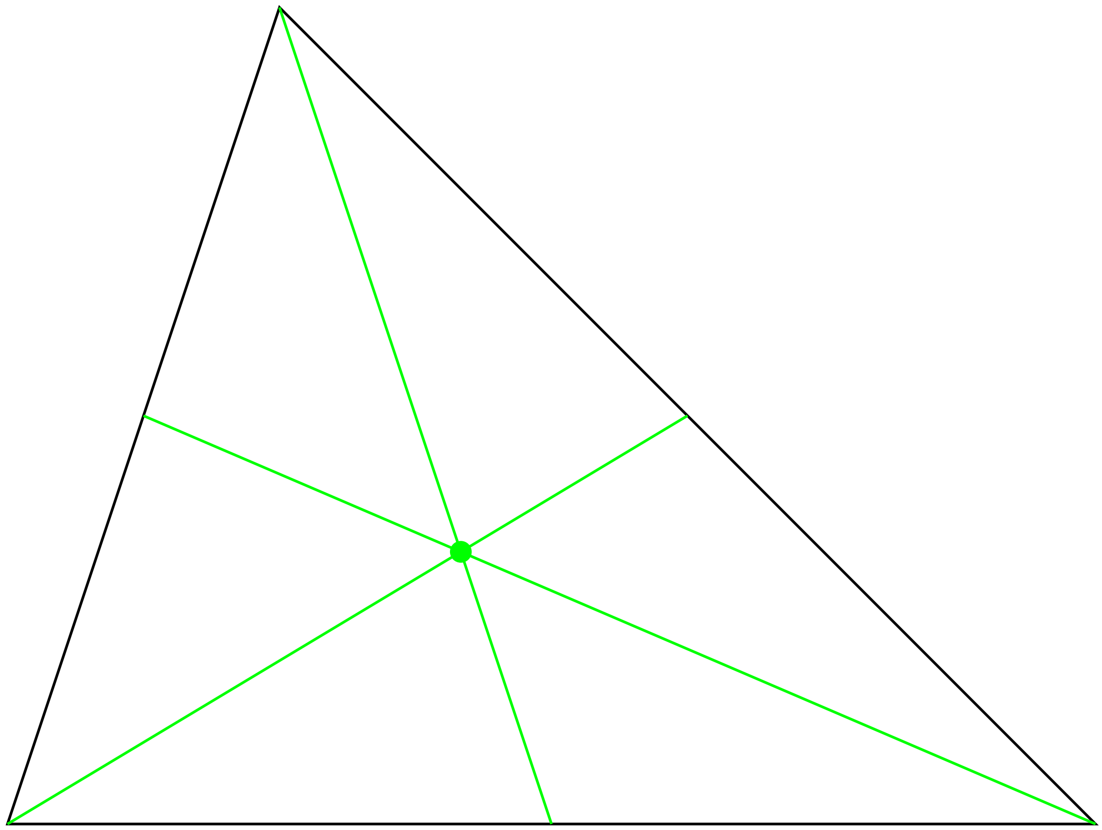
and then joining this mid-point to the vertice opposite to this side.



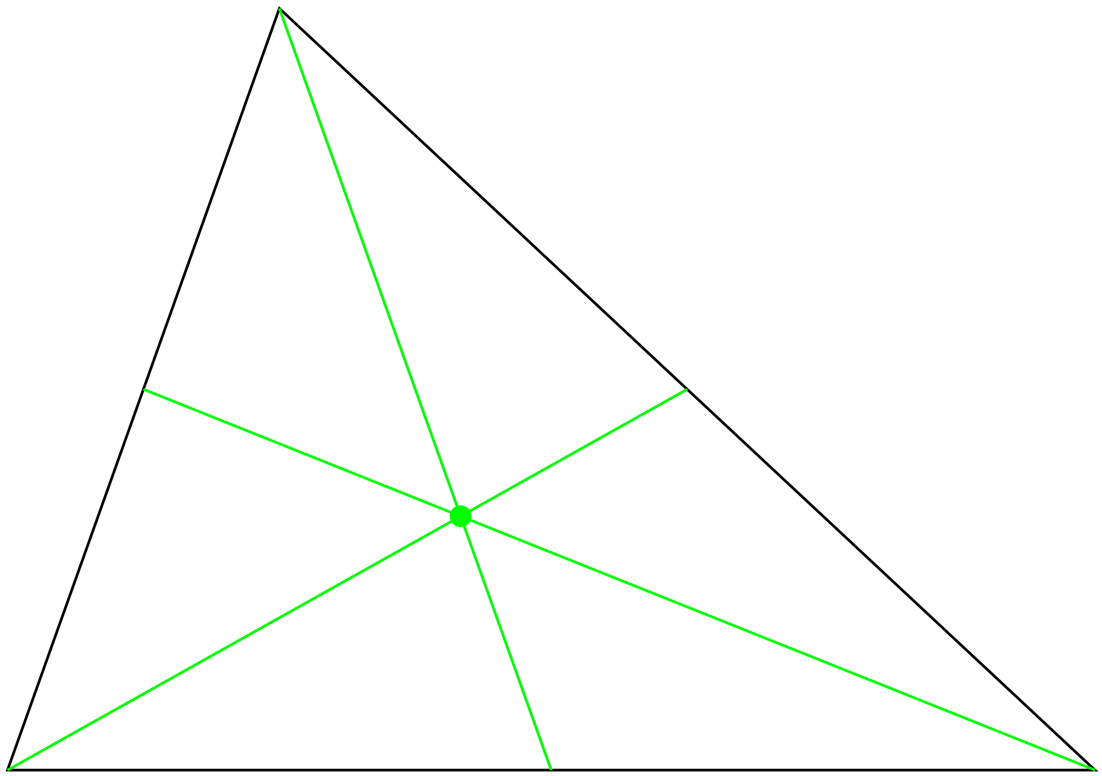
The intersection of the three medians of a triangle is called the centroid.

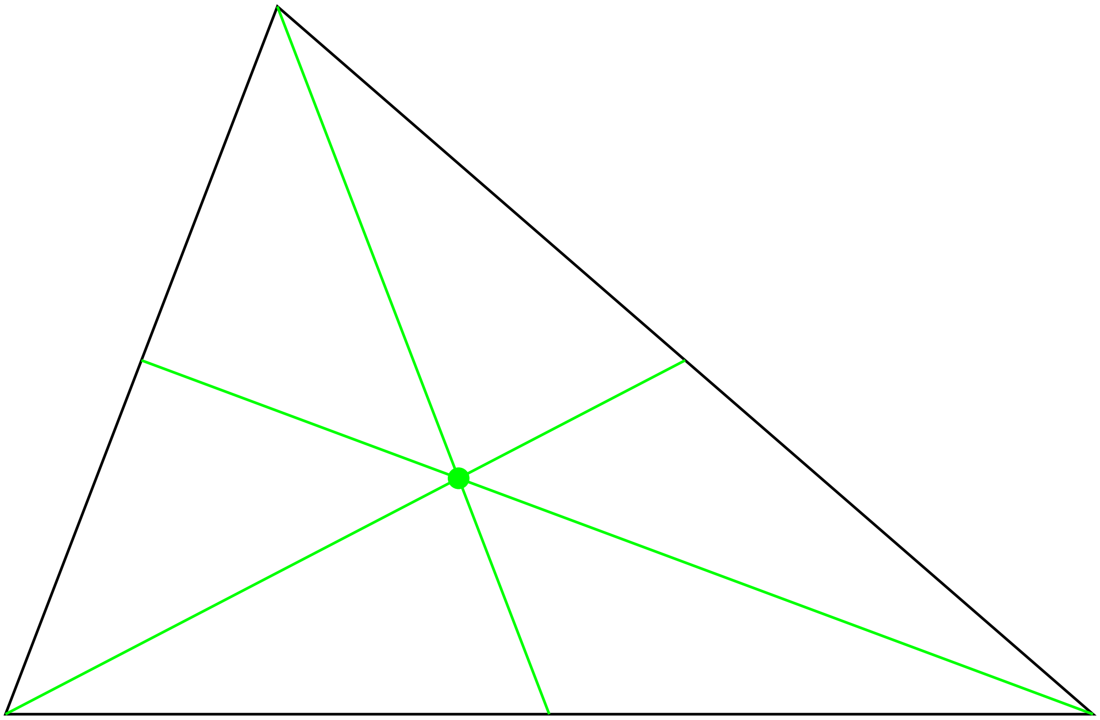
3.2 The Centroid (M)

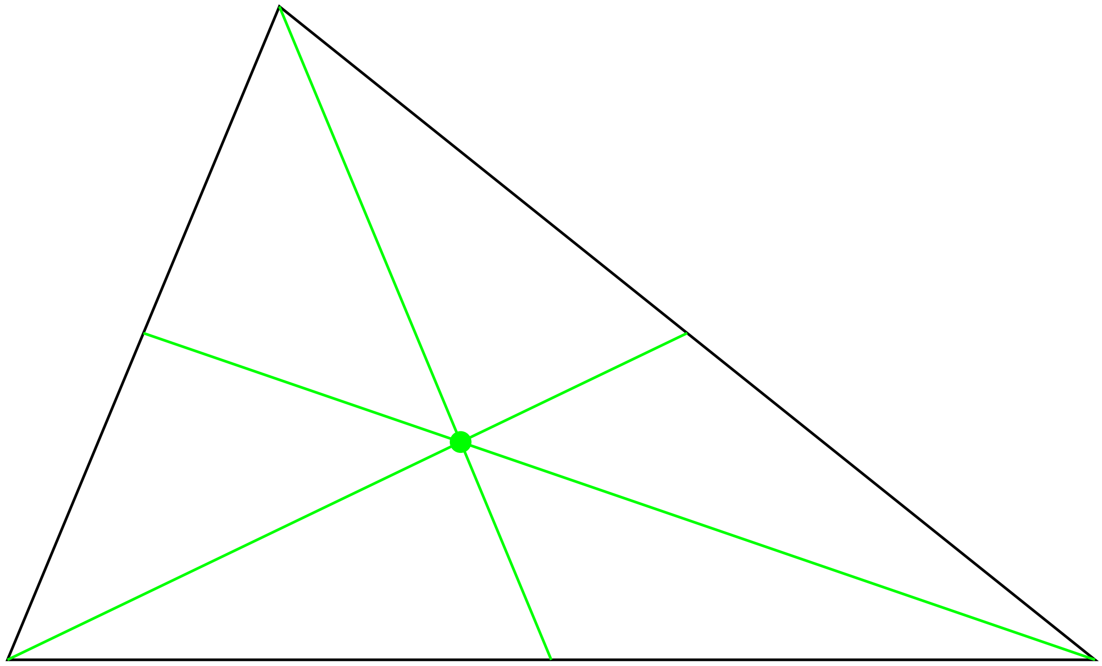
- the position of M

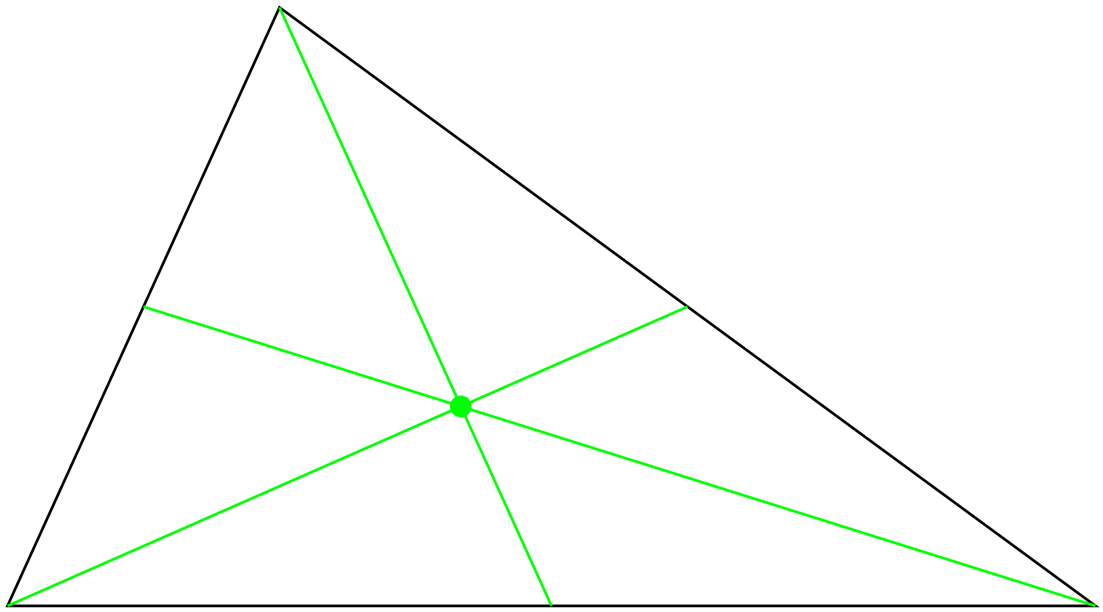


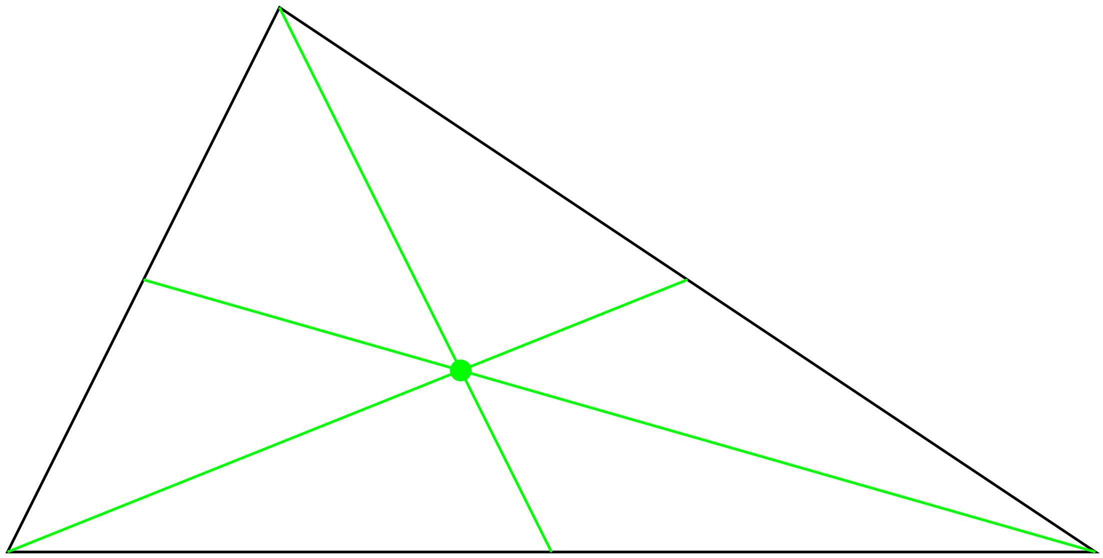
Unlike the previous centers, the centroids of all triangles are within the triangle itself no matter what.

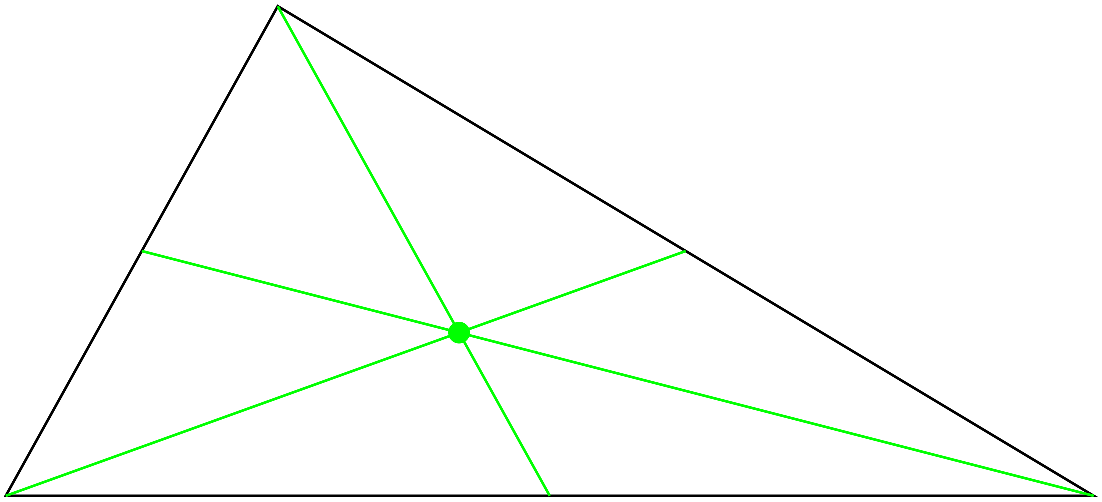


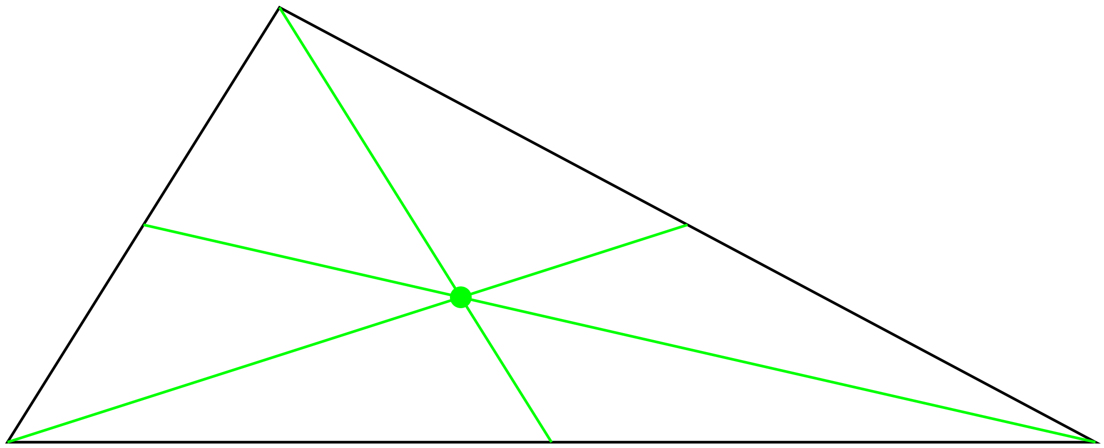


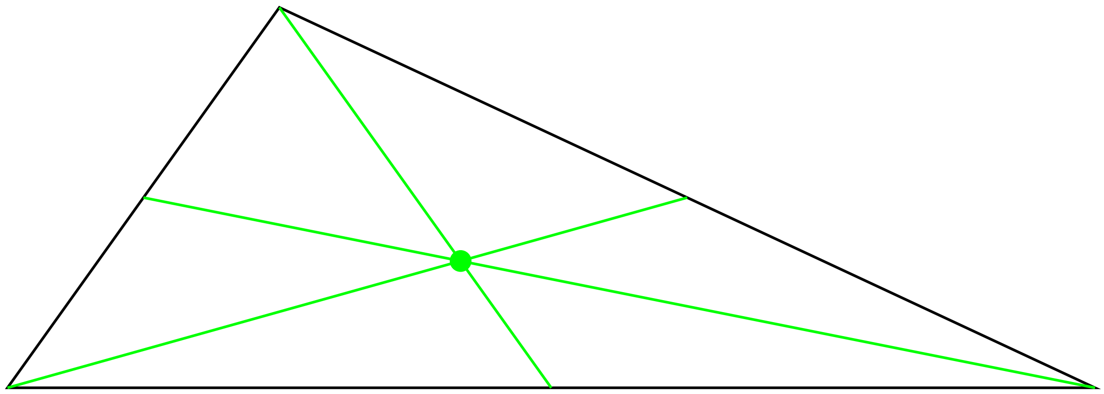


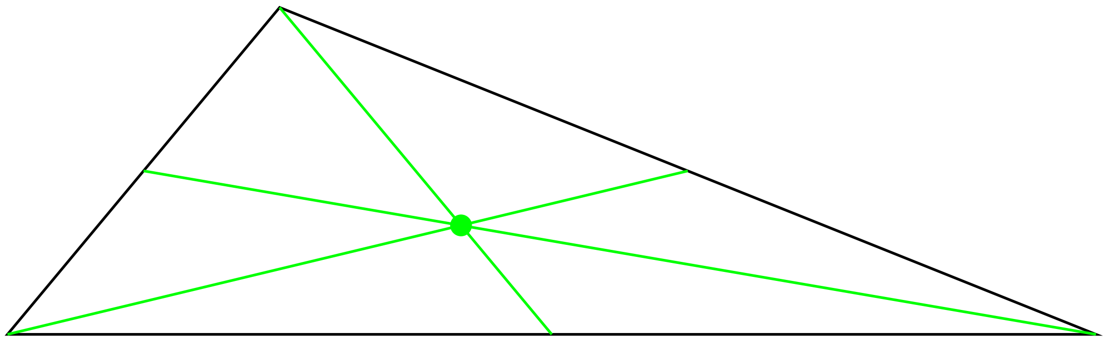


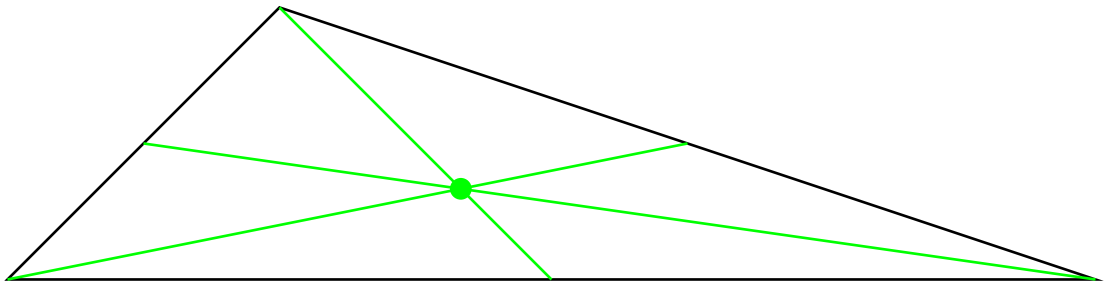


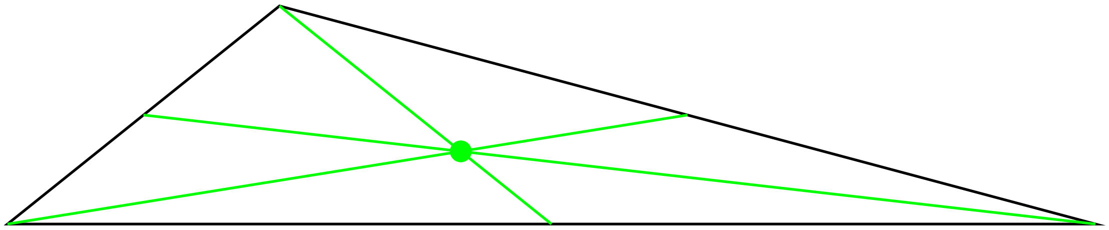






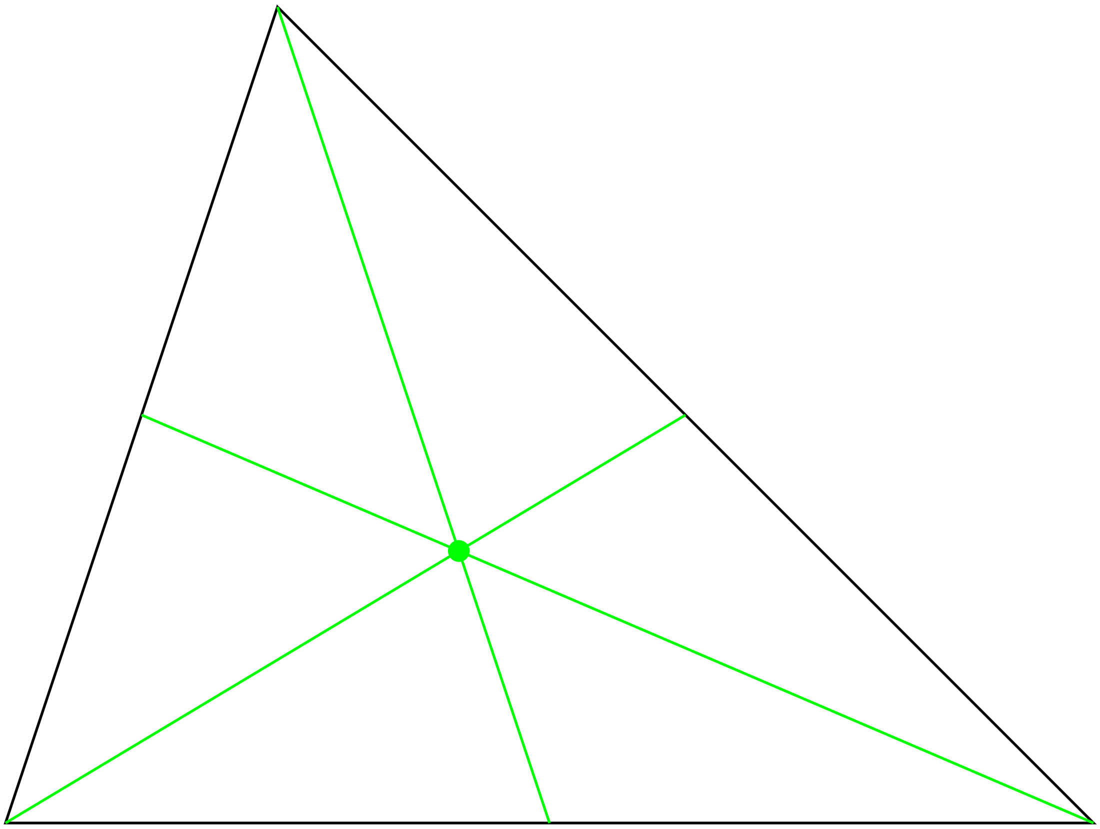




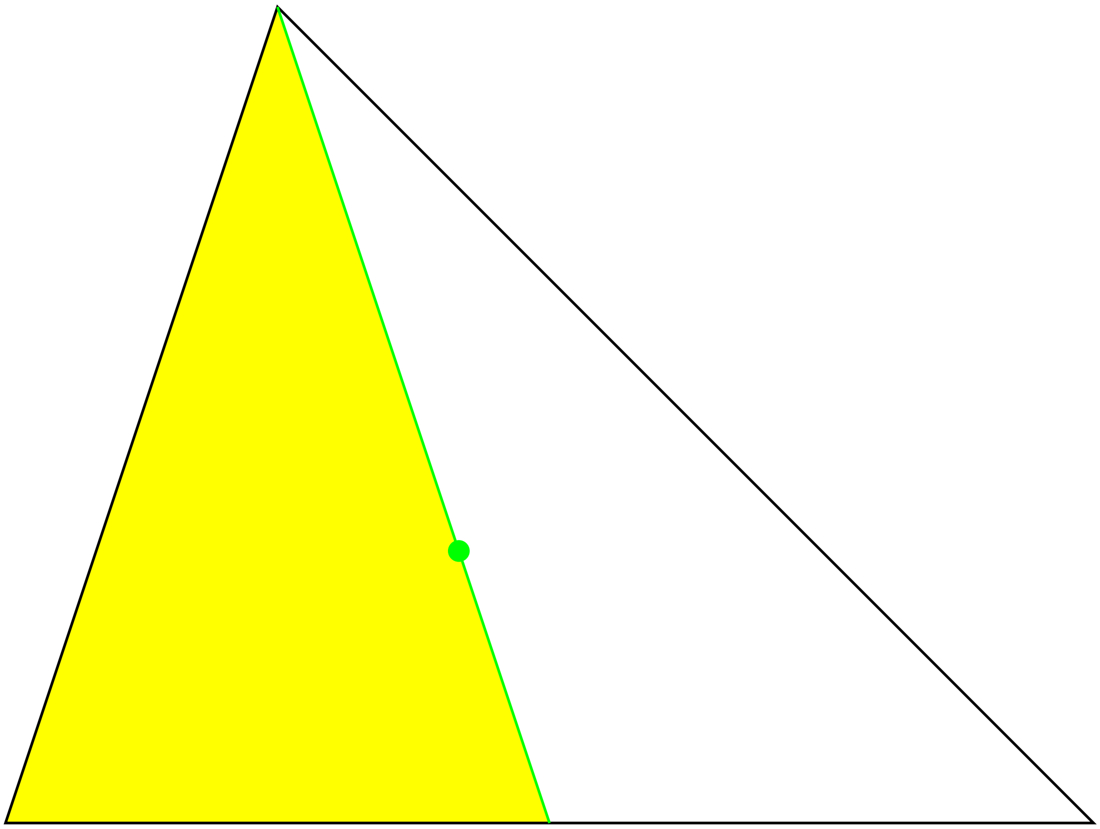


3.3 The Centroid (M)

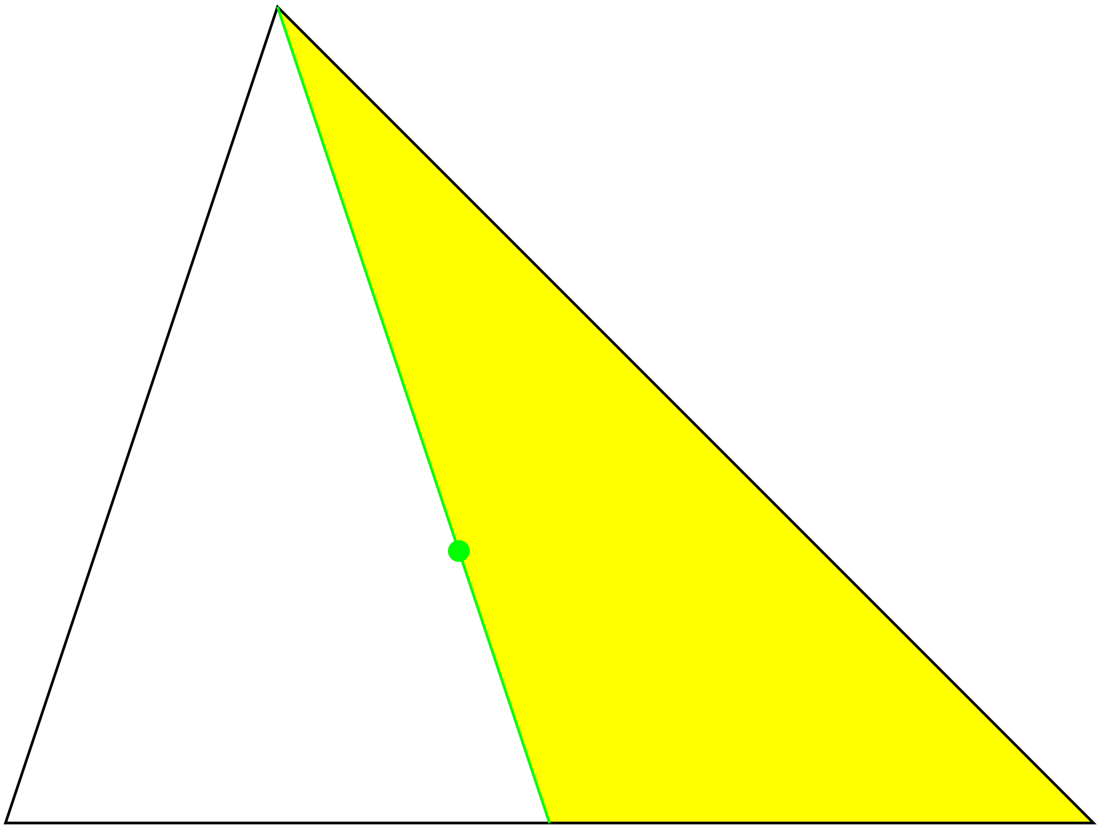
- center of mass



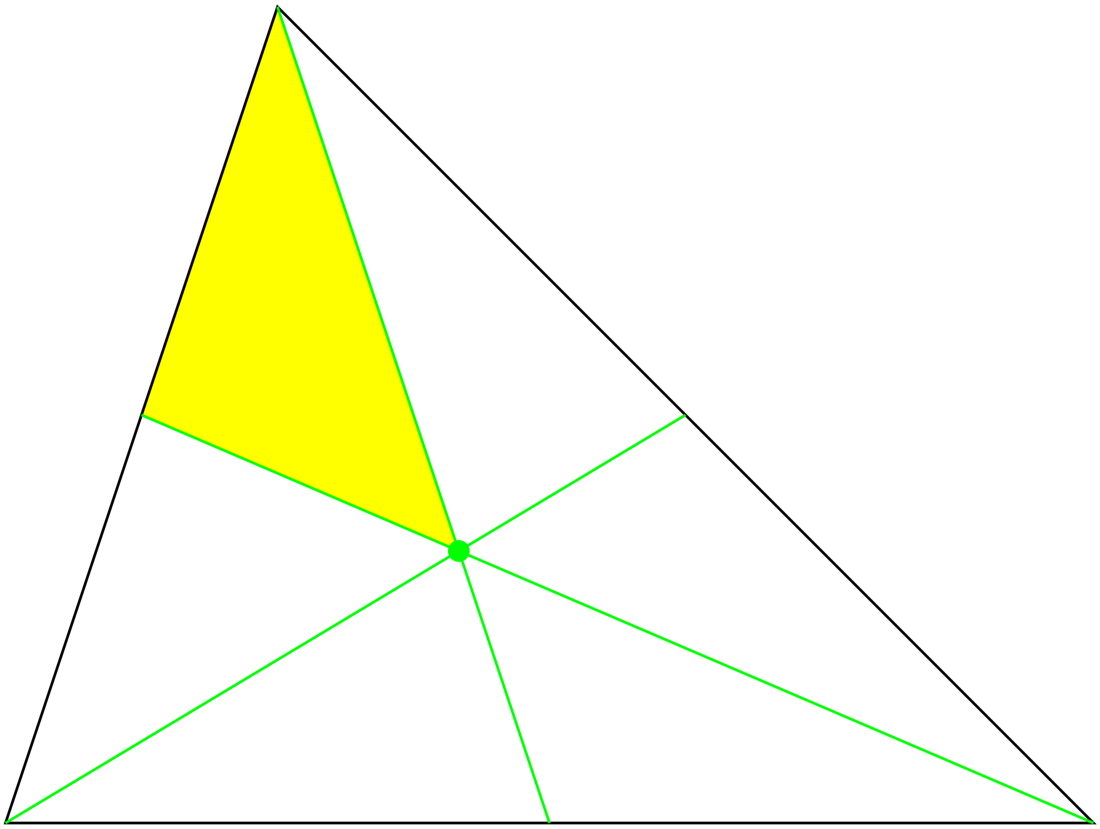
The centroid of a triangle is also its center of mass. In terms of two-dimensional geometry, that is to say that ...



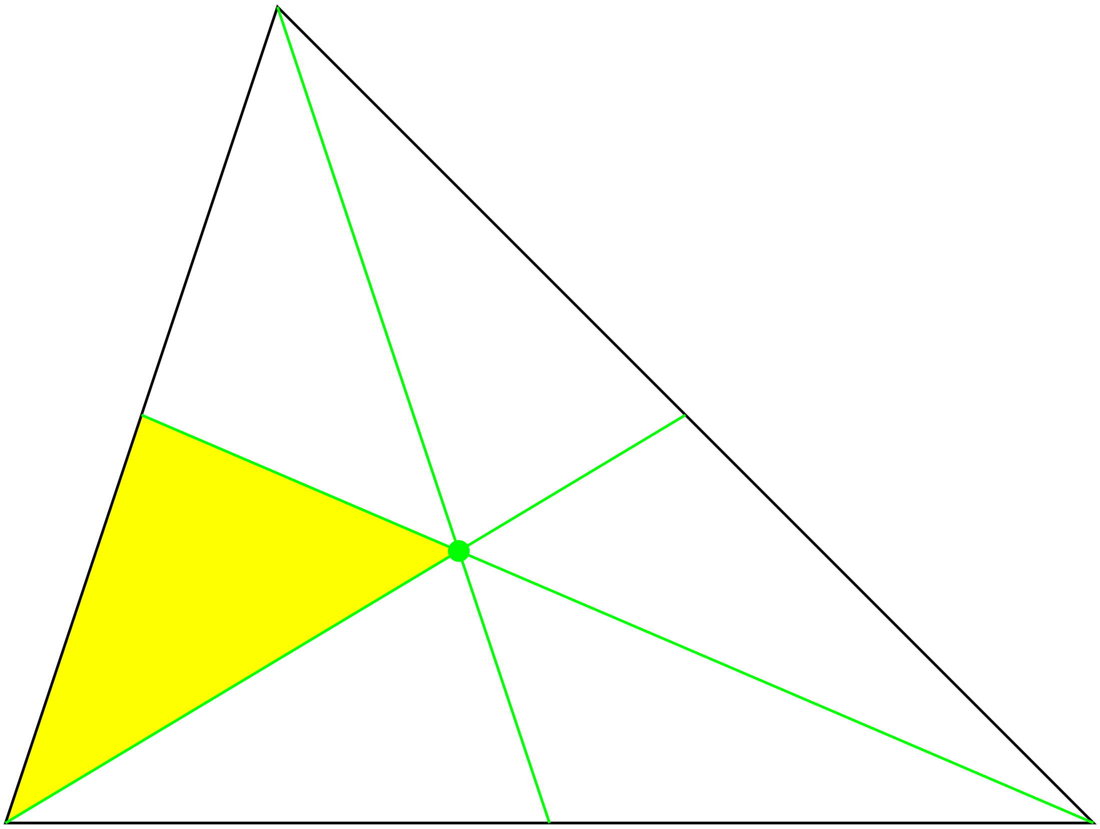
... each median divides the triangle into two triangles of equal areas, and ...



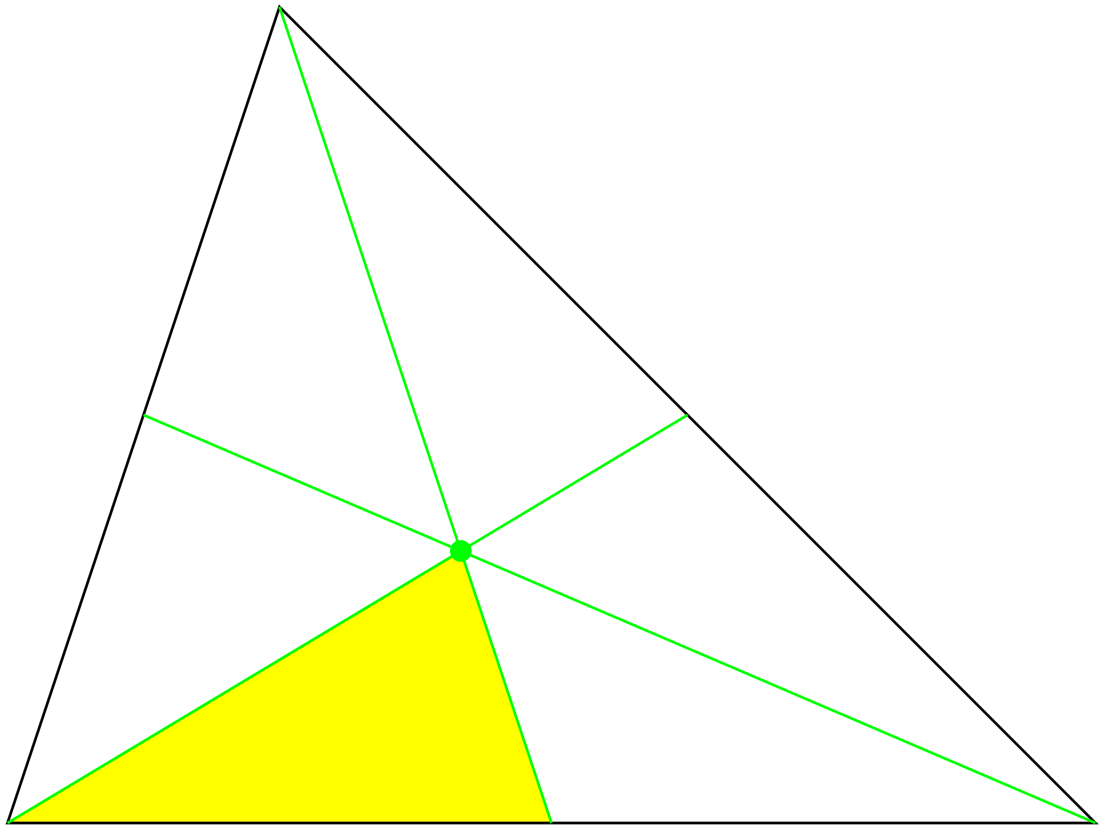
... each median divides the triangle into two triangles of equal areas, and ...



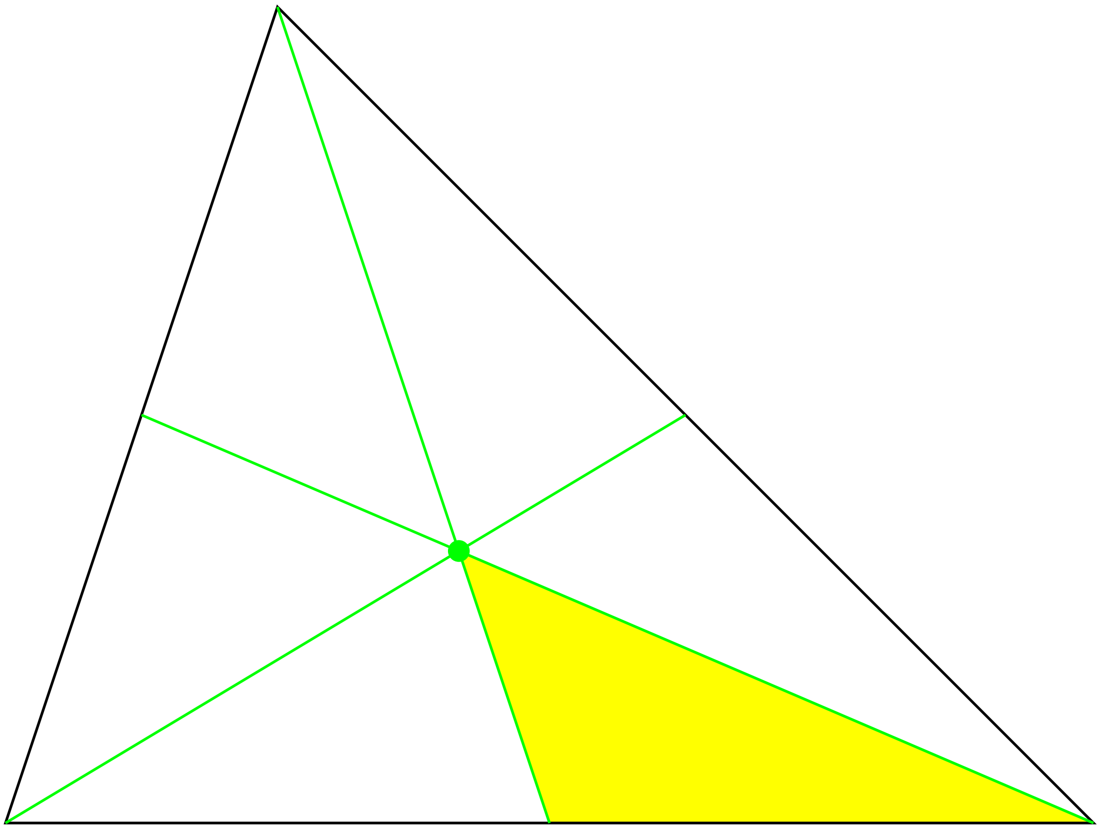
... all three medians together would divide the triangle into six equal portions.



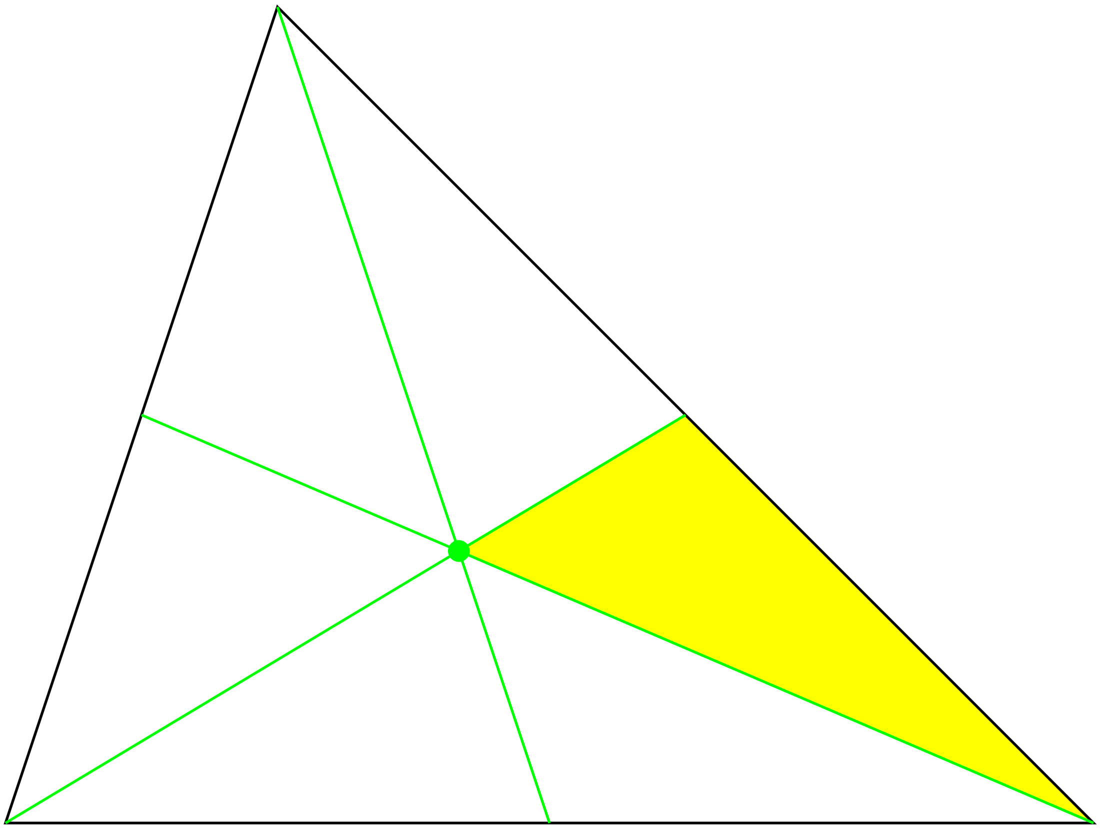
... all three medians together would divide the triangle into six equal portions.



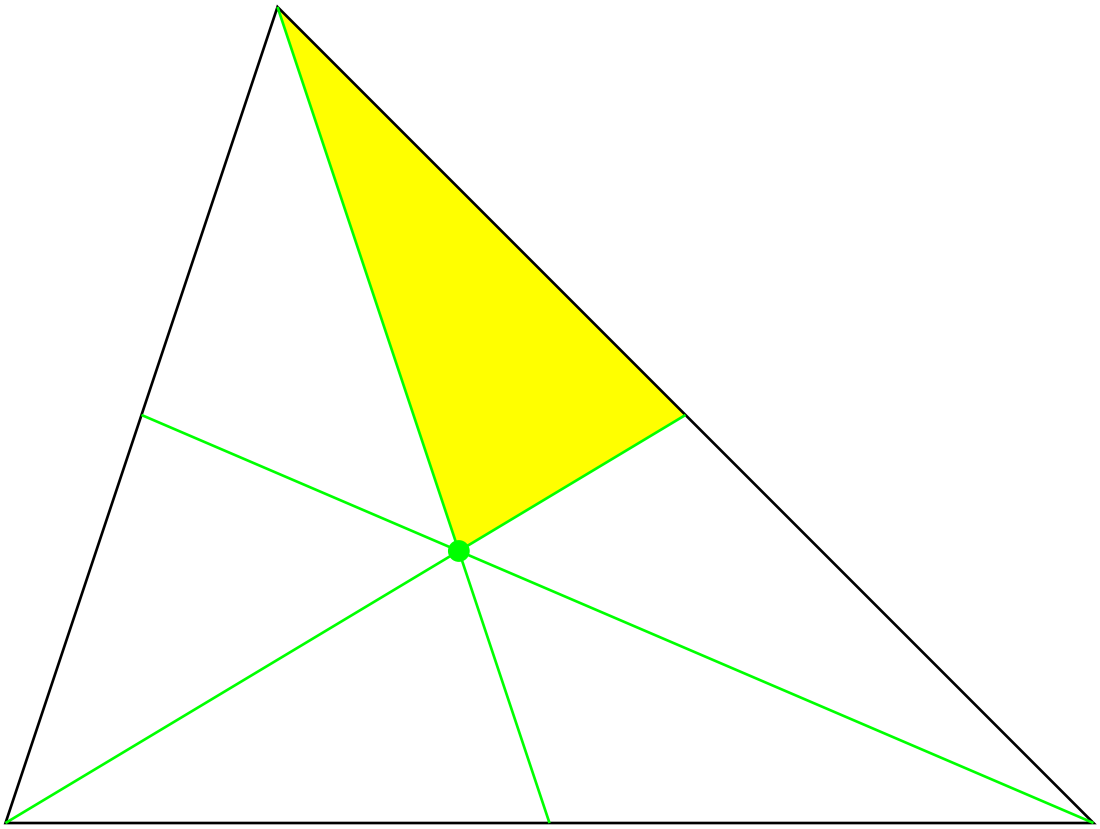
... all three medians together would divide the triangle into six equal portions.



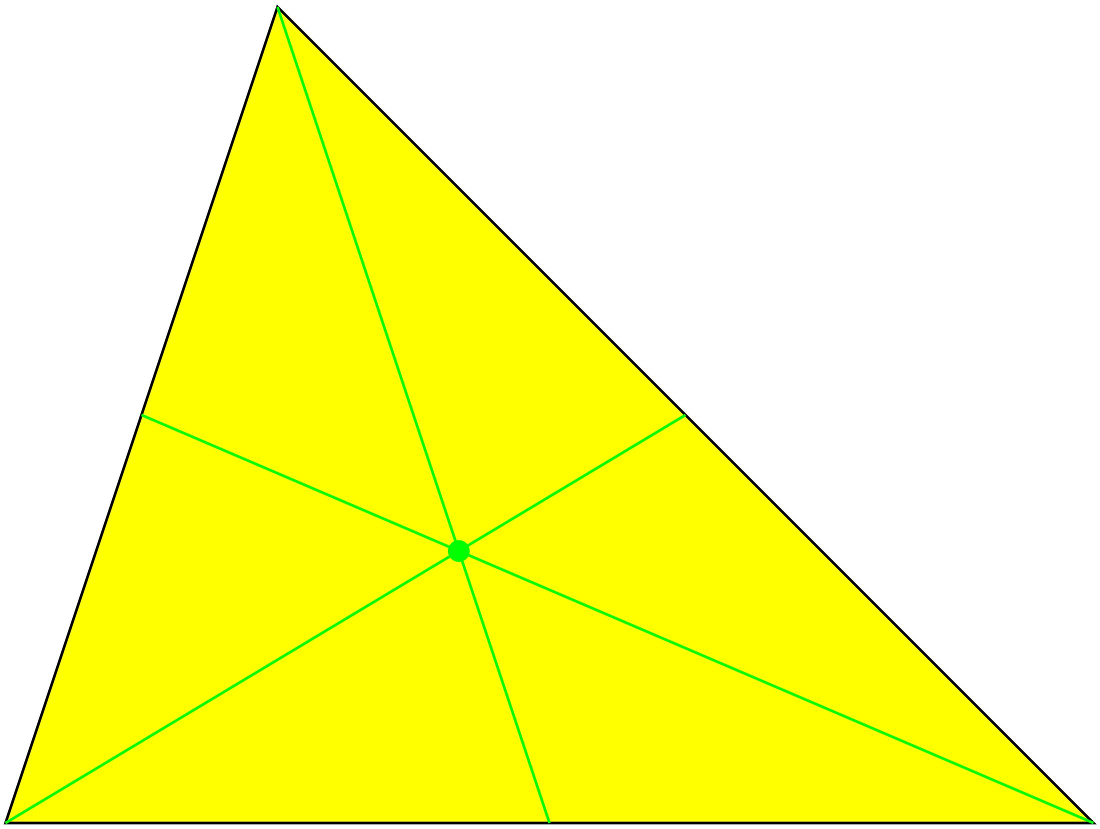
... all three medians together would divide the triangle into six equal portions.



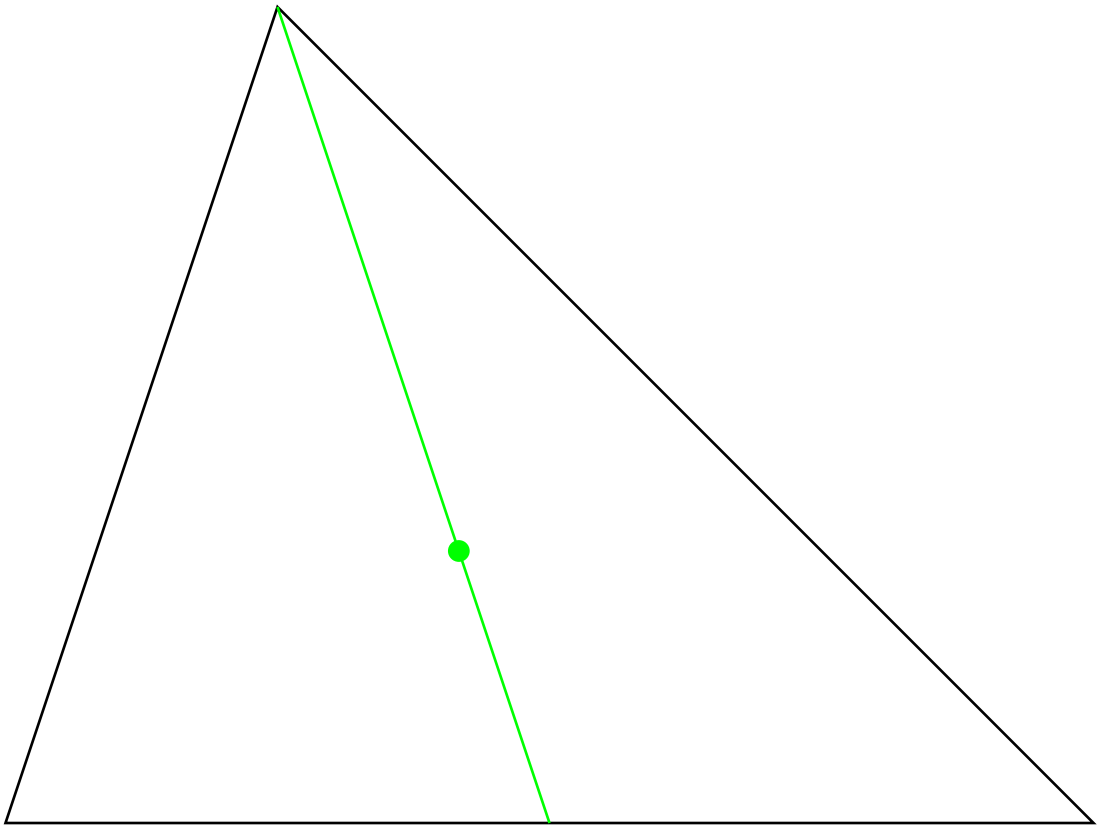
... all three medians together would divide the triangle into six equal portions.



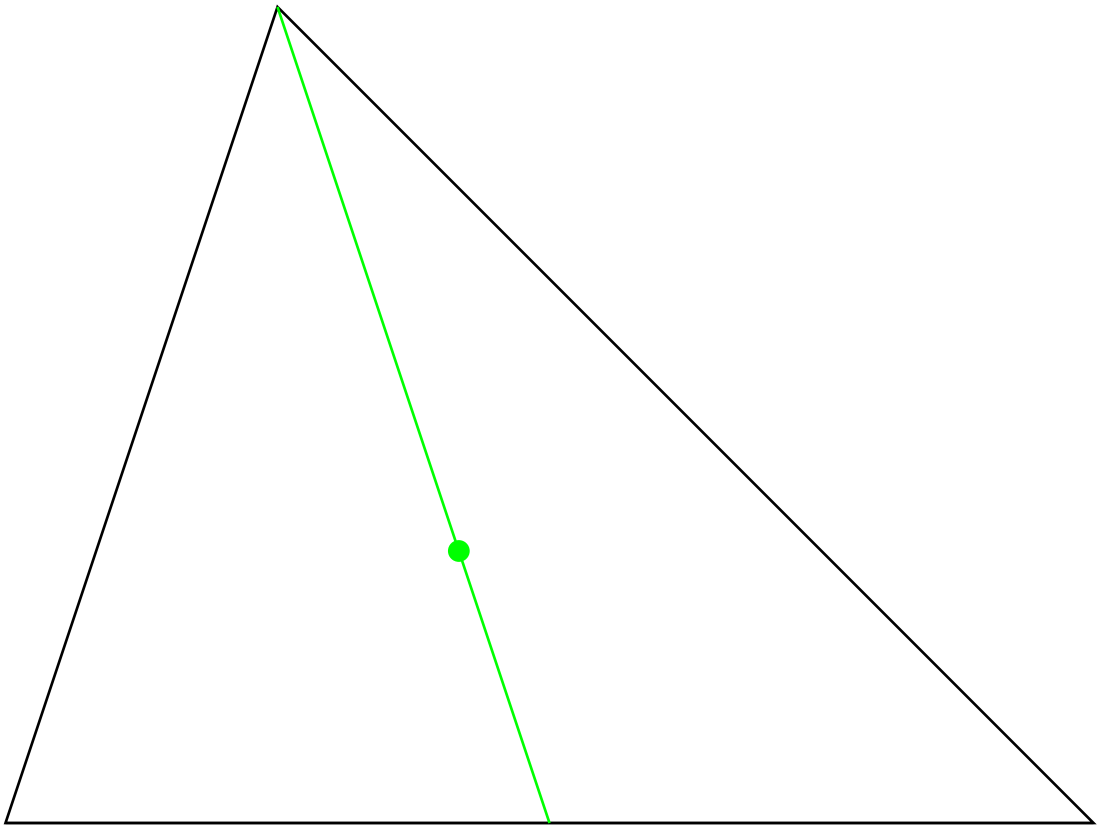
... all three medians together would divide the triangle into six equal portions.

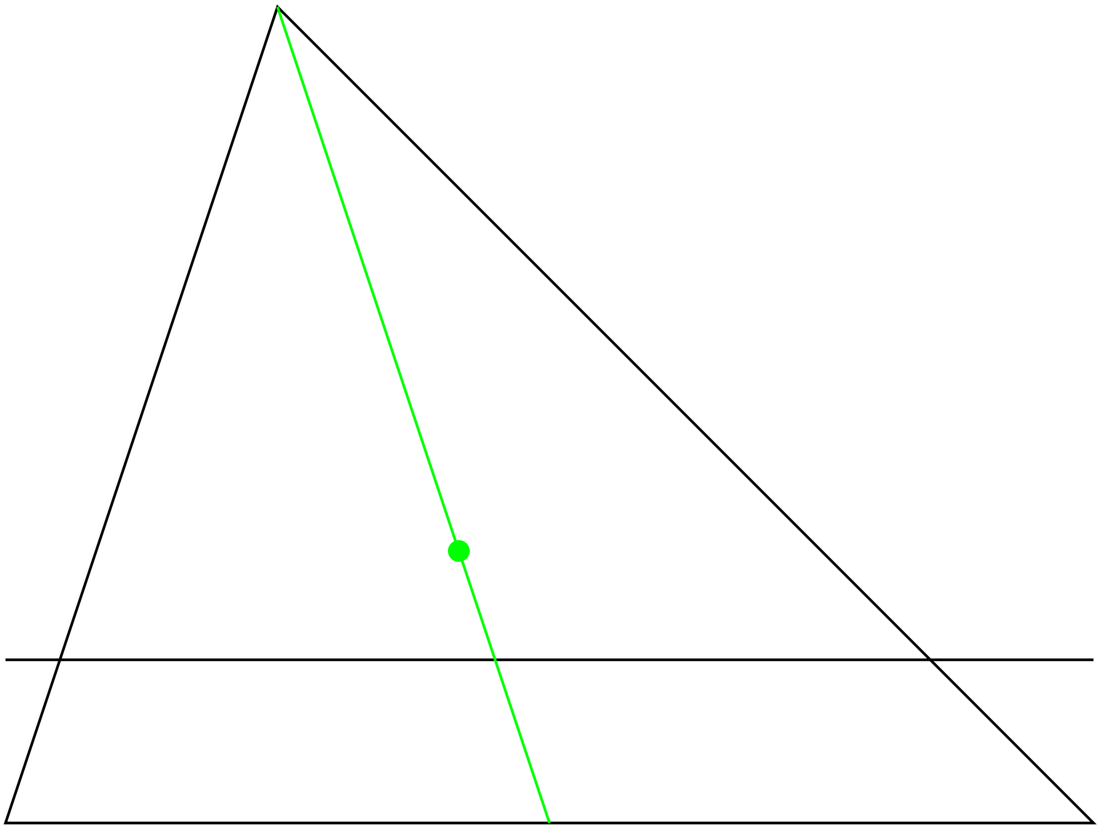


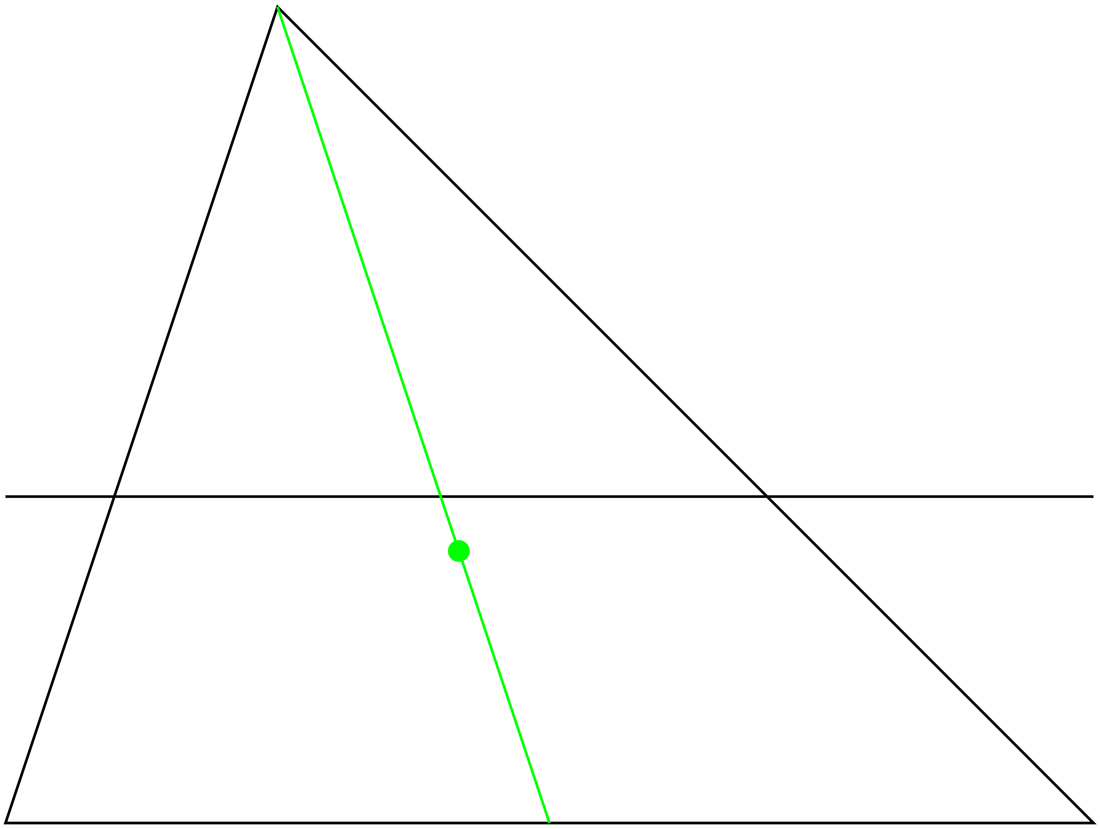
The following is a proof of this:

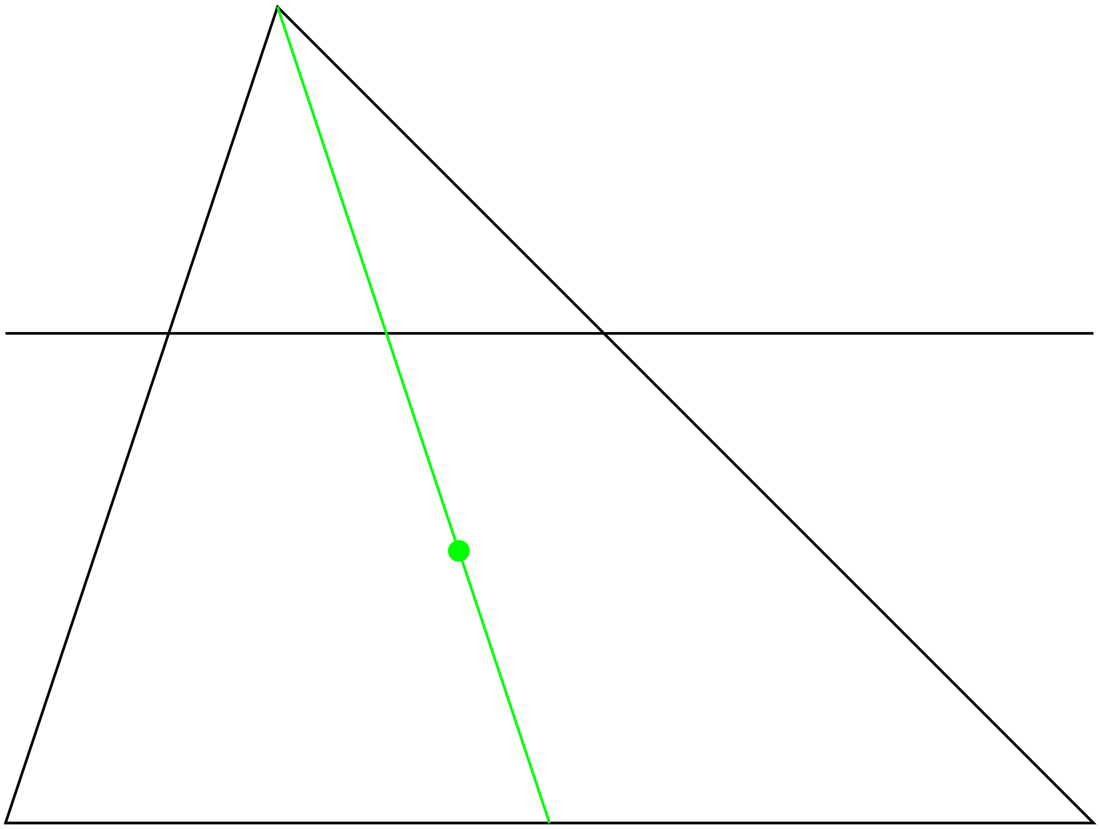


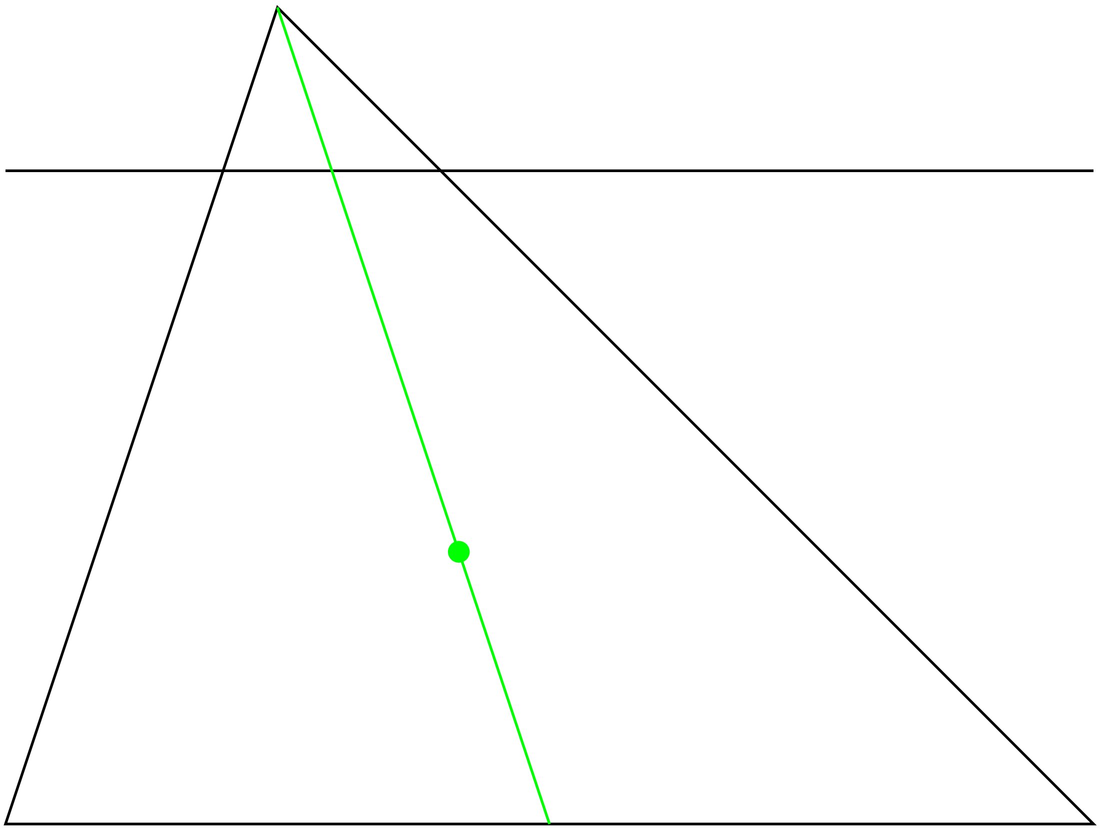
First proof that the two triangles have equal areas.

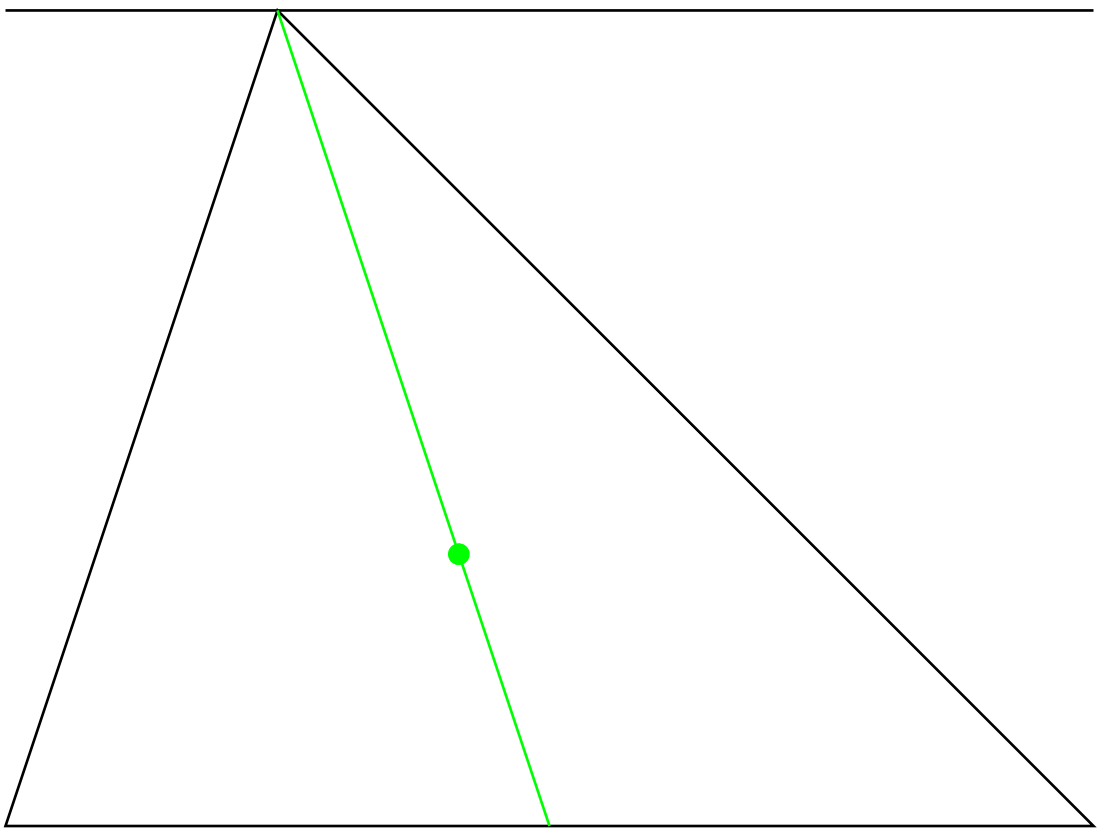


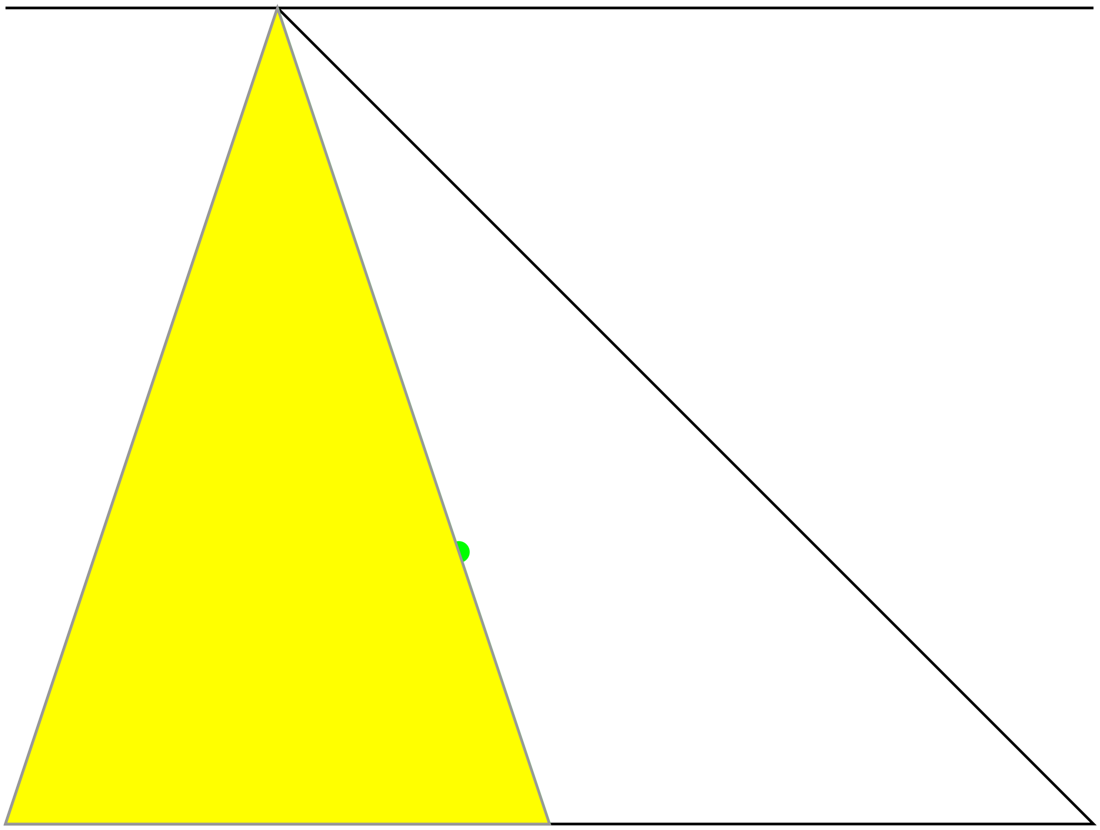


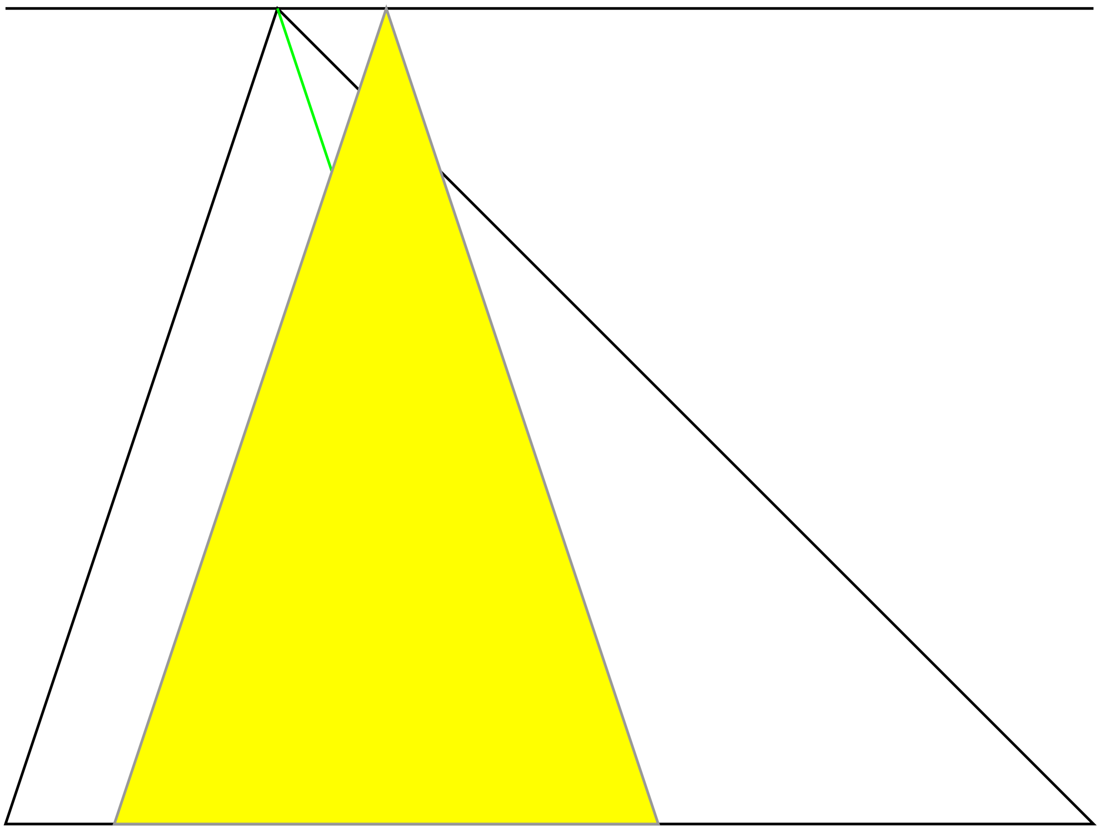


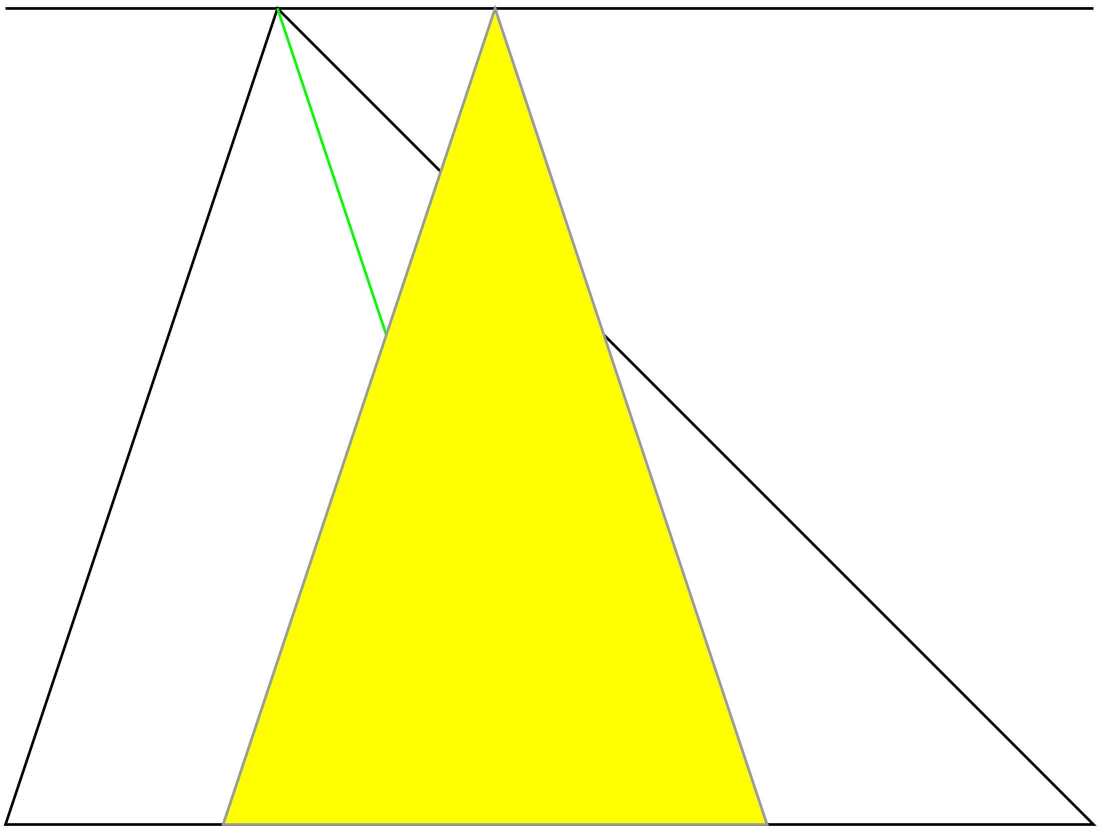


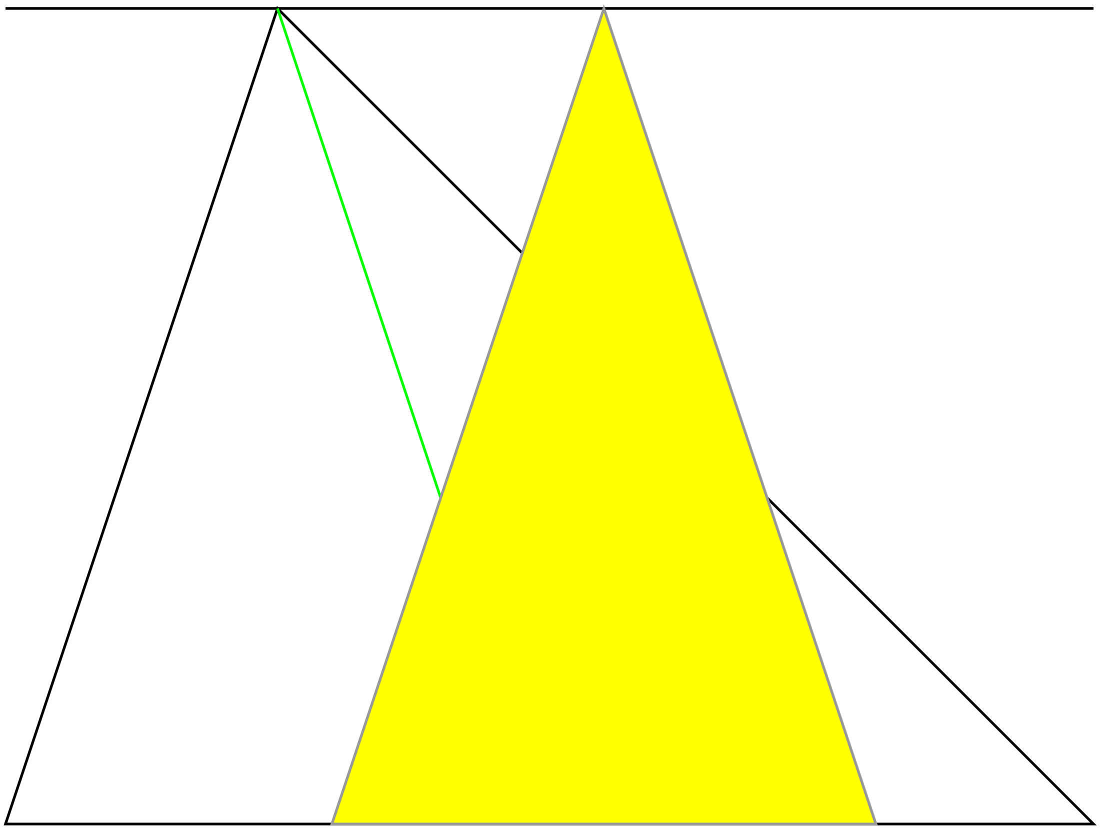


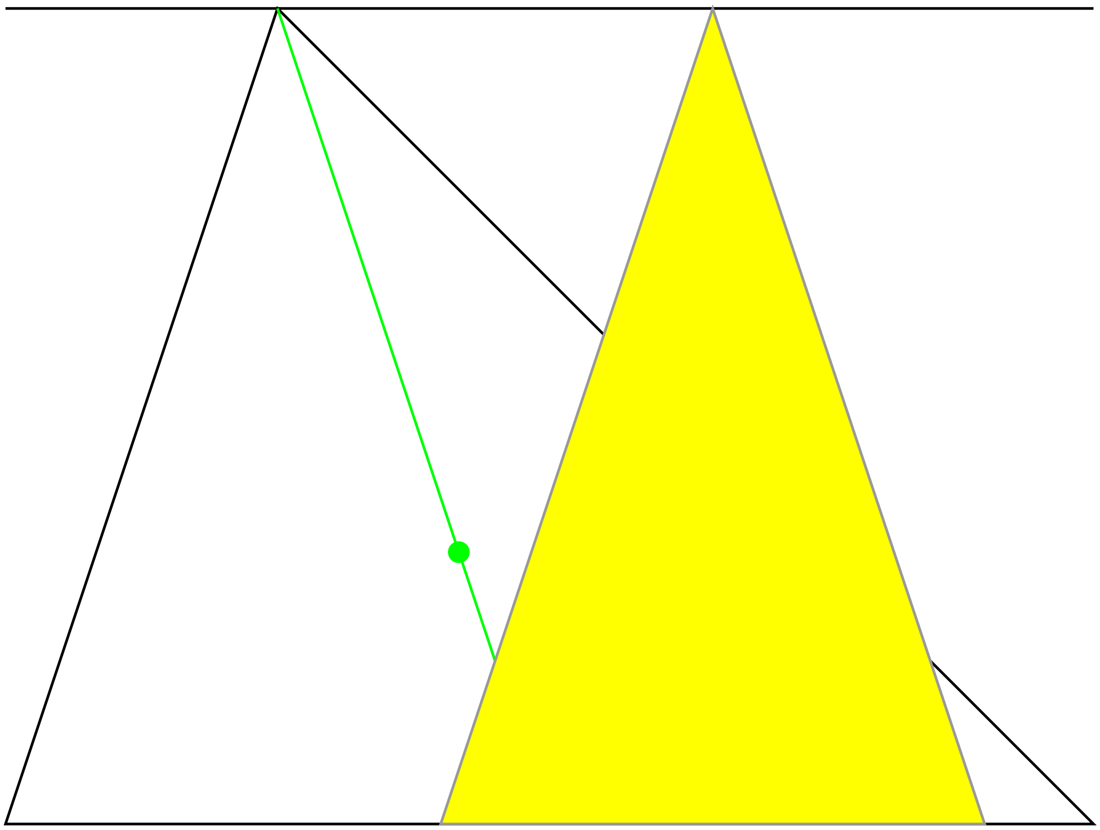


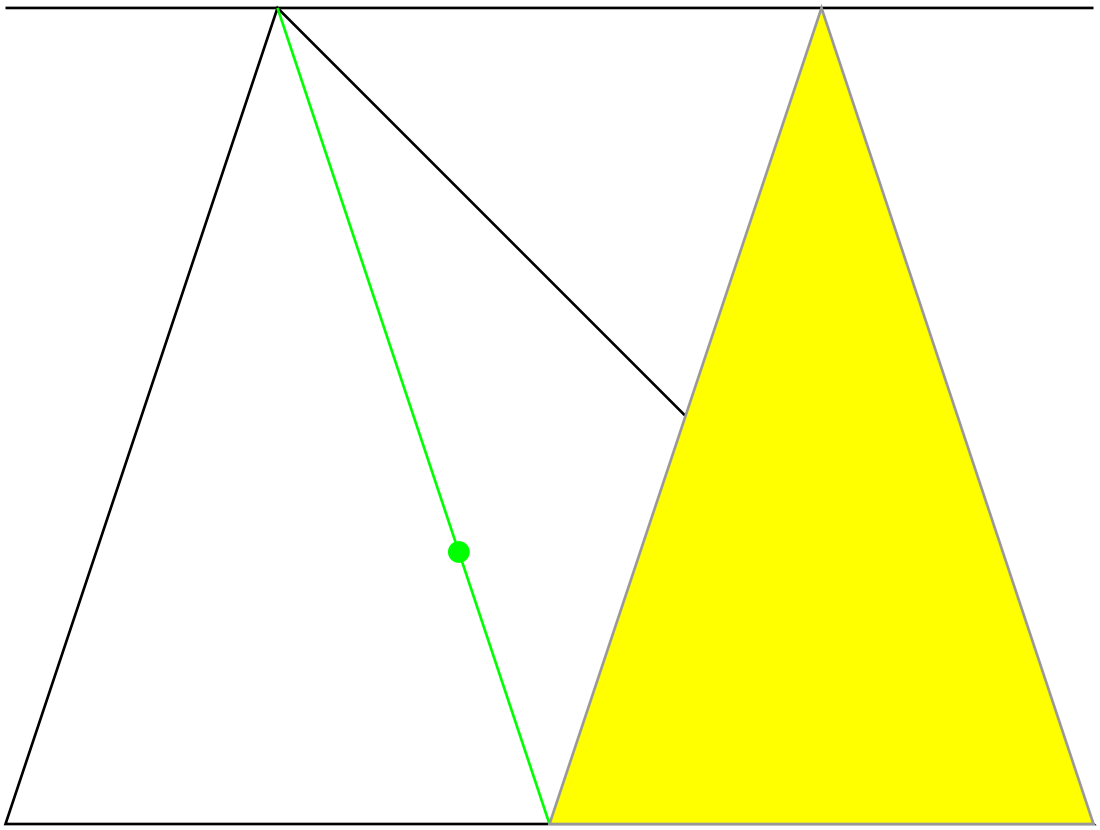


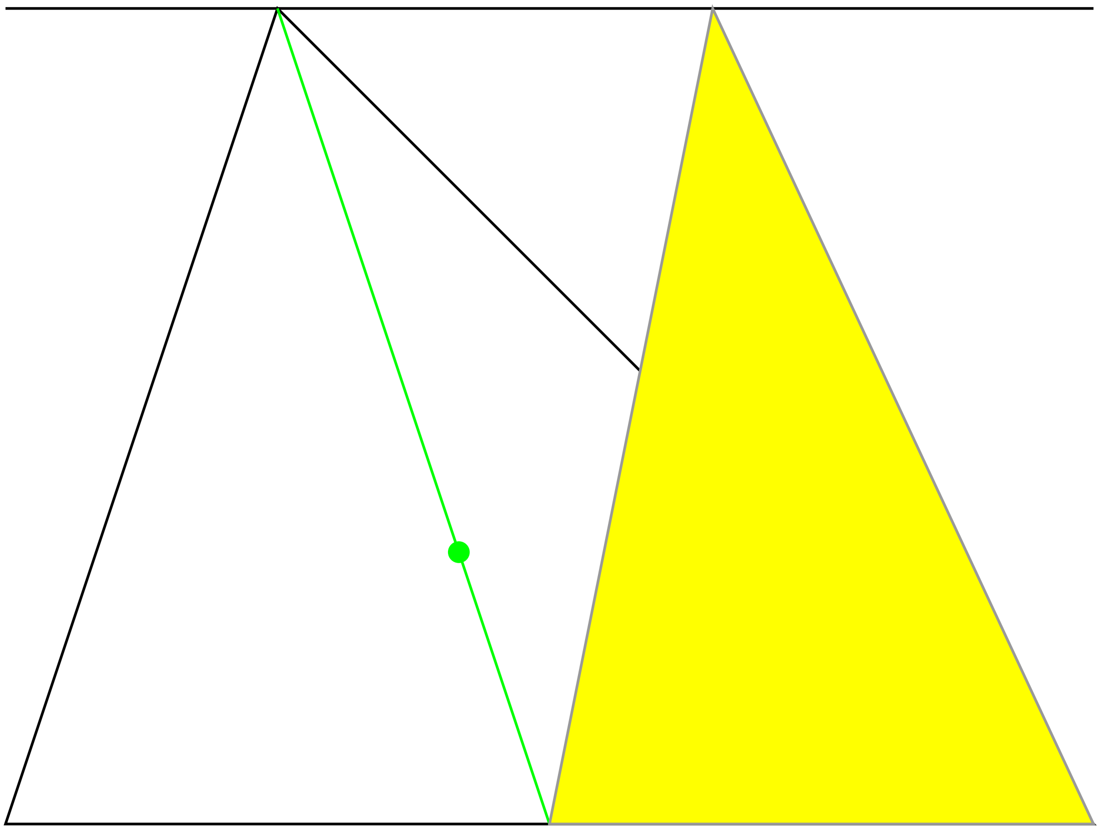


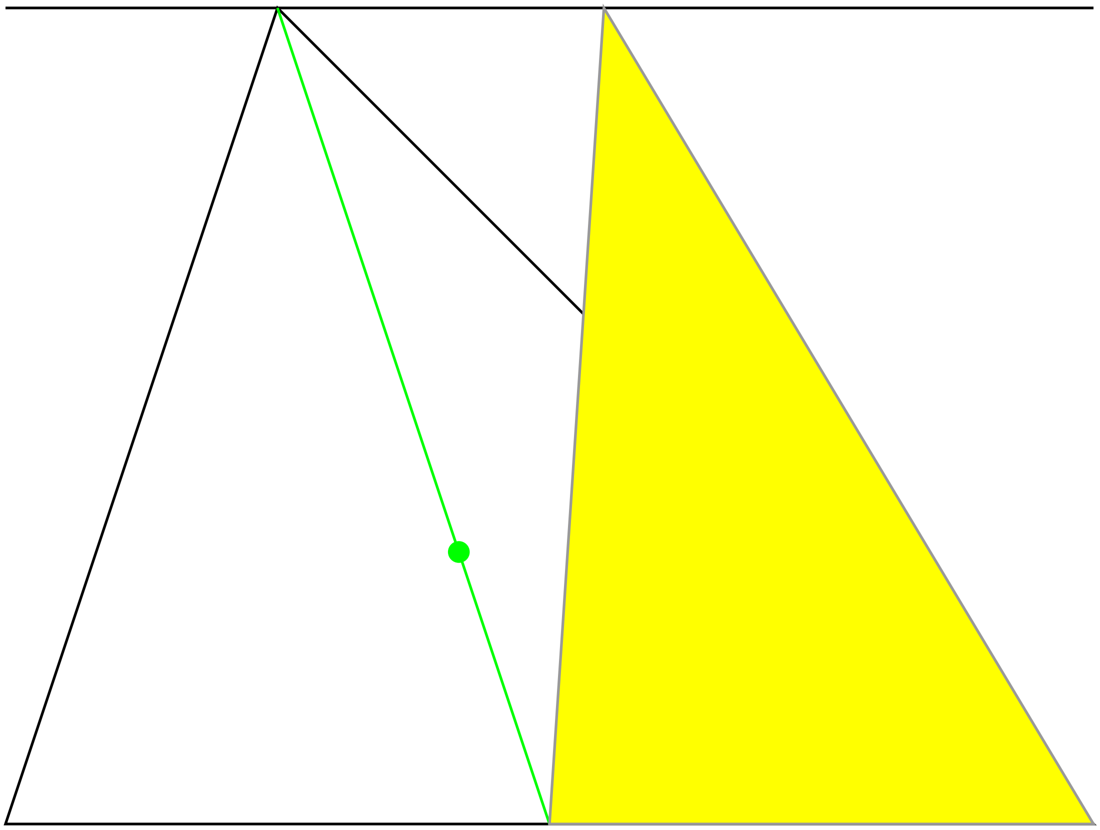


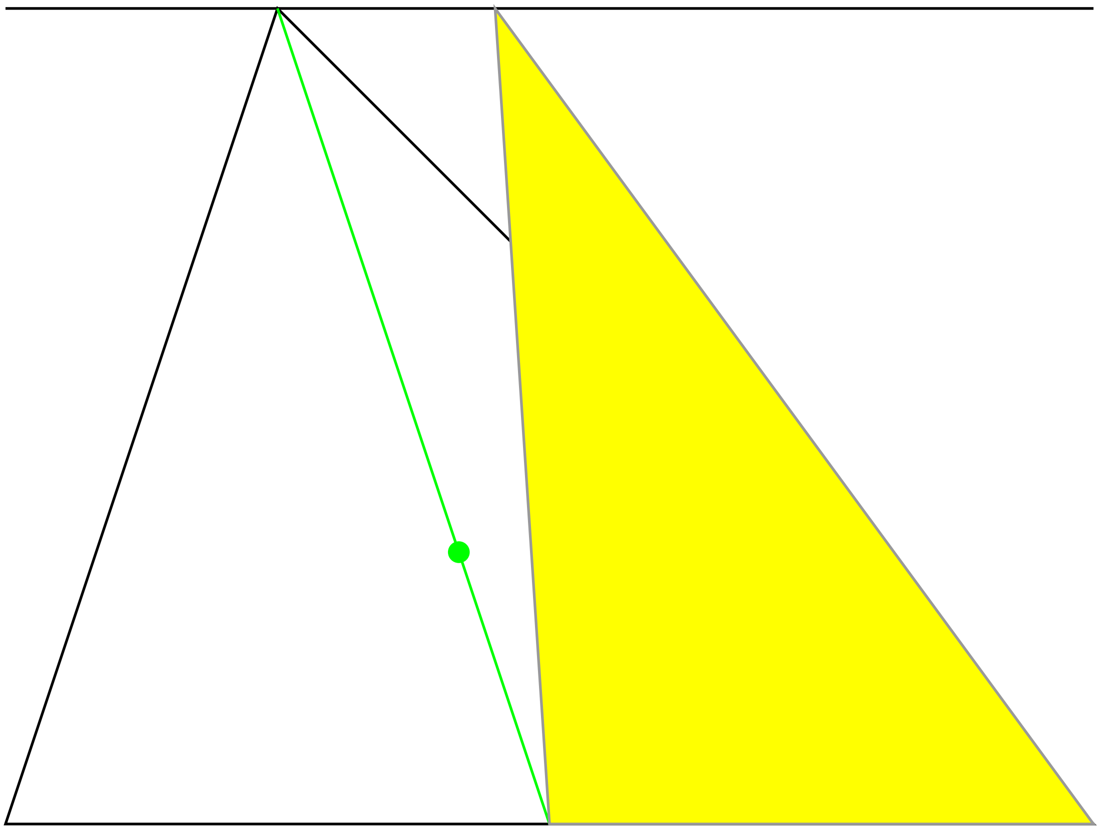


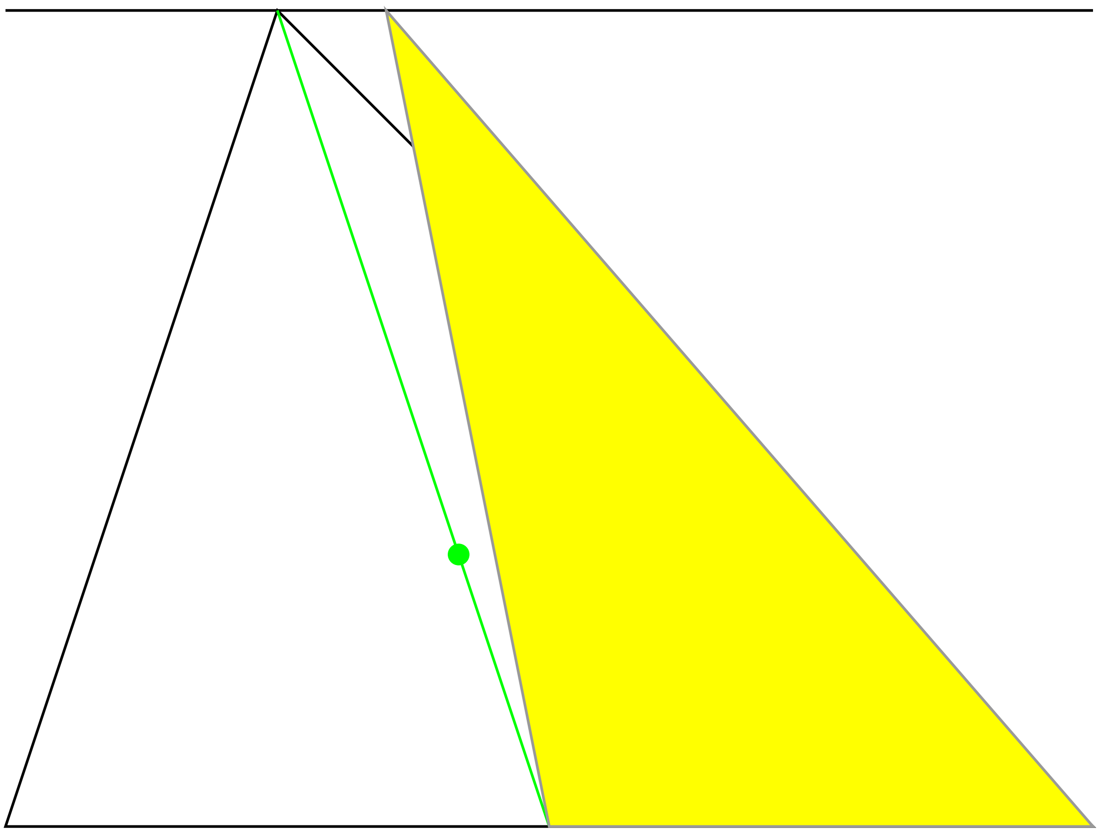


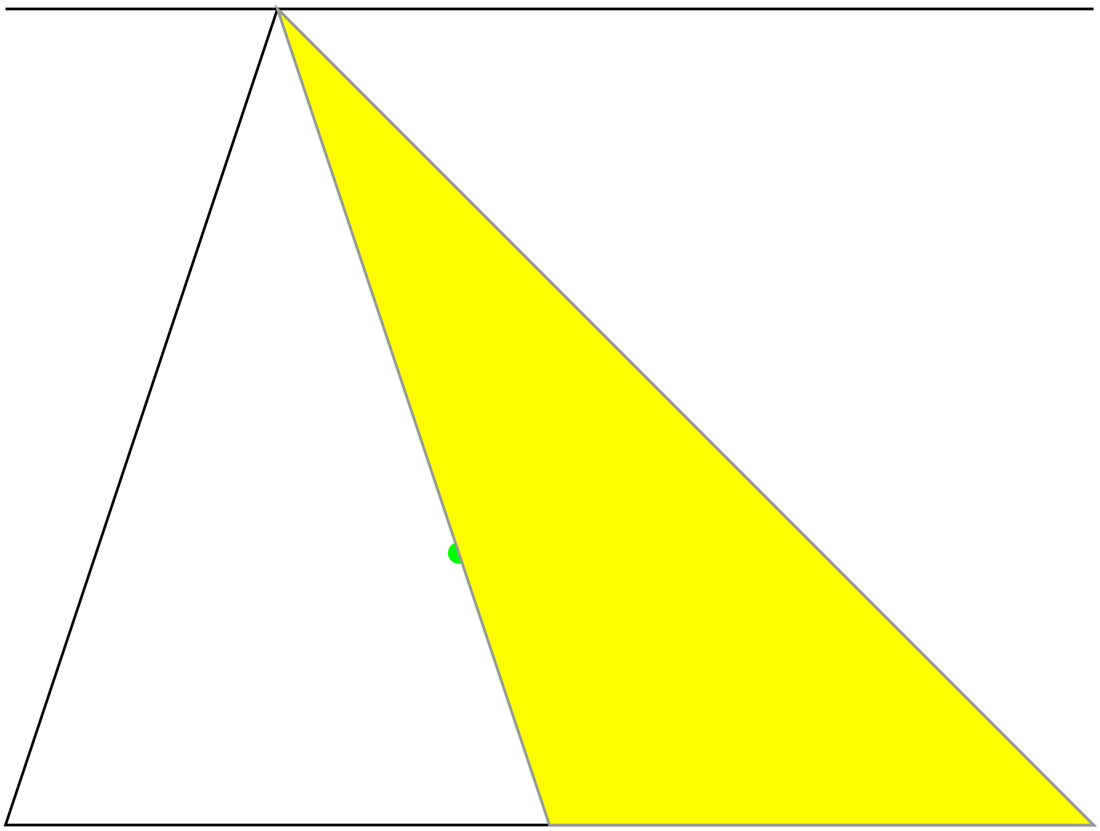


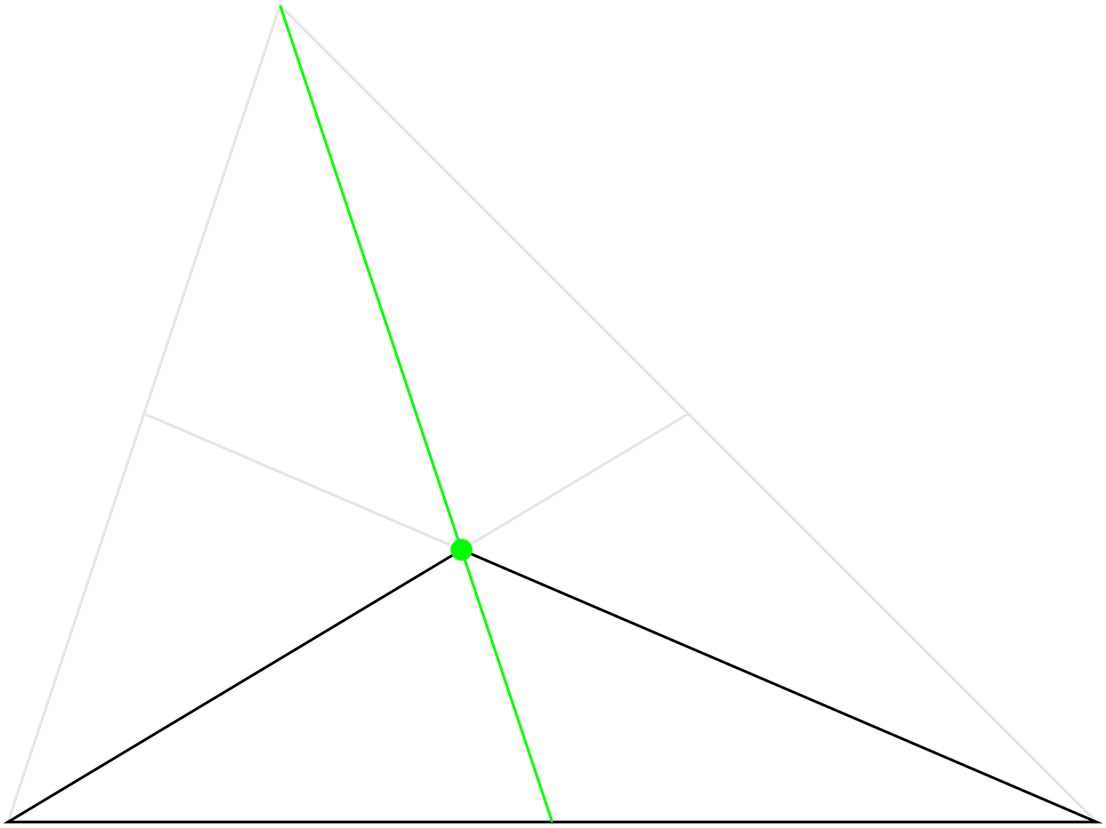




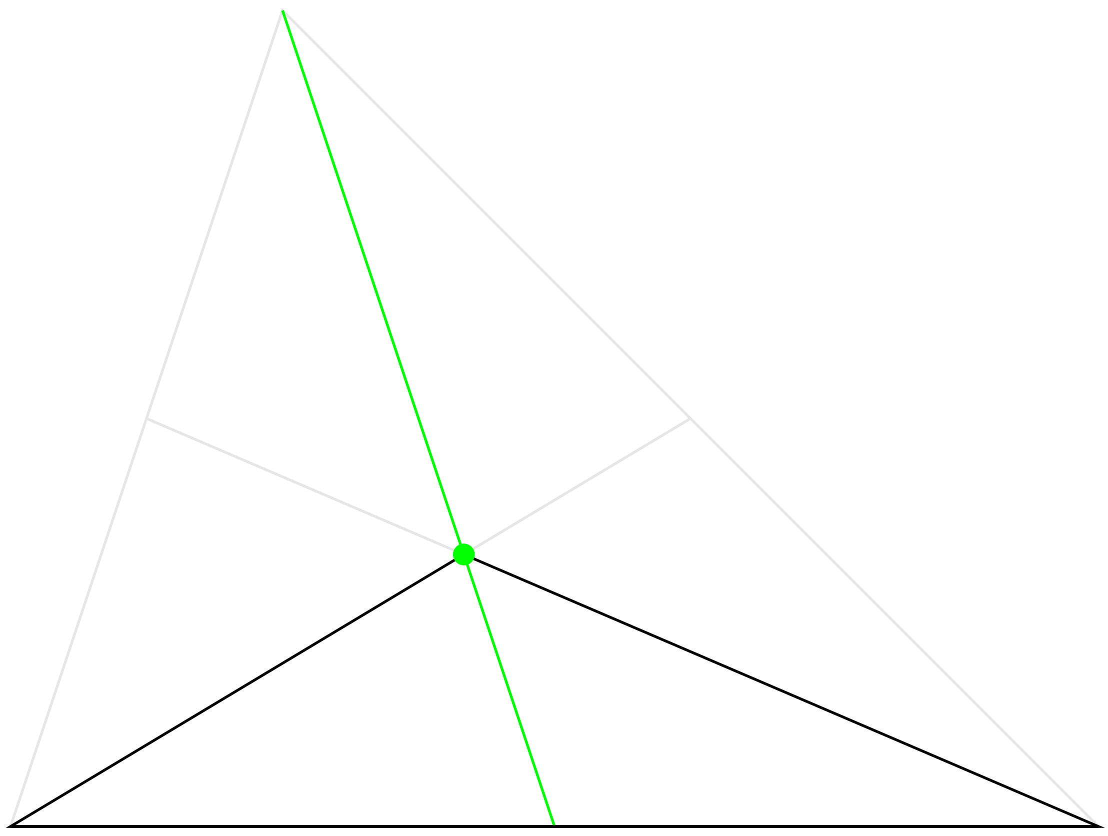


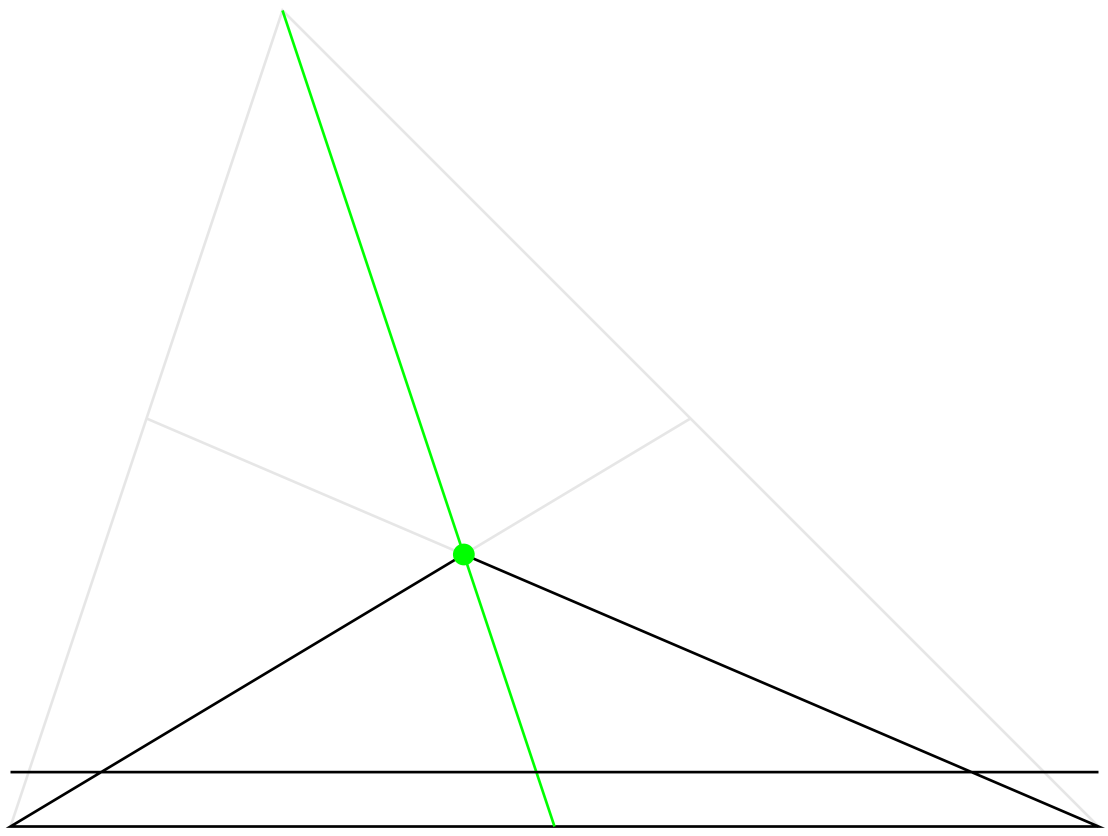


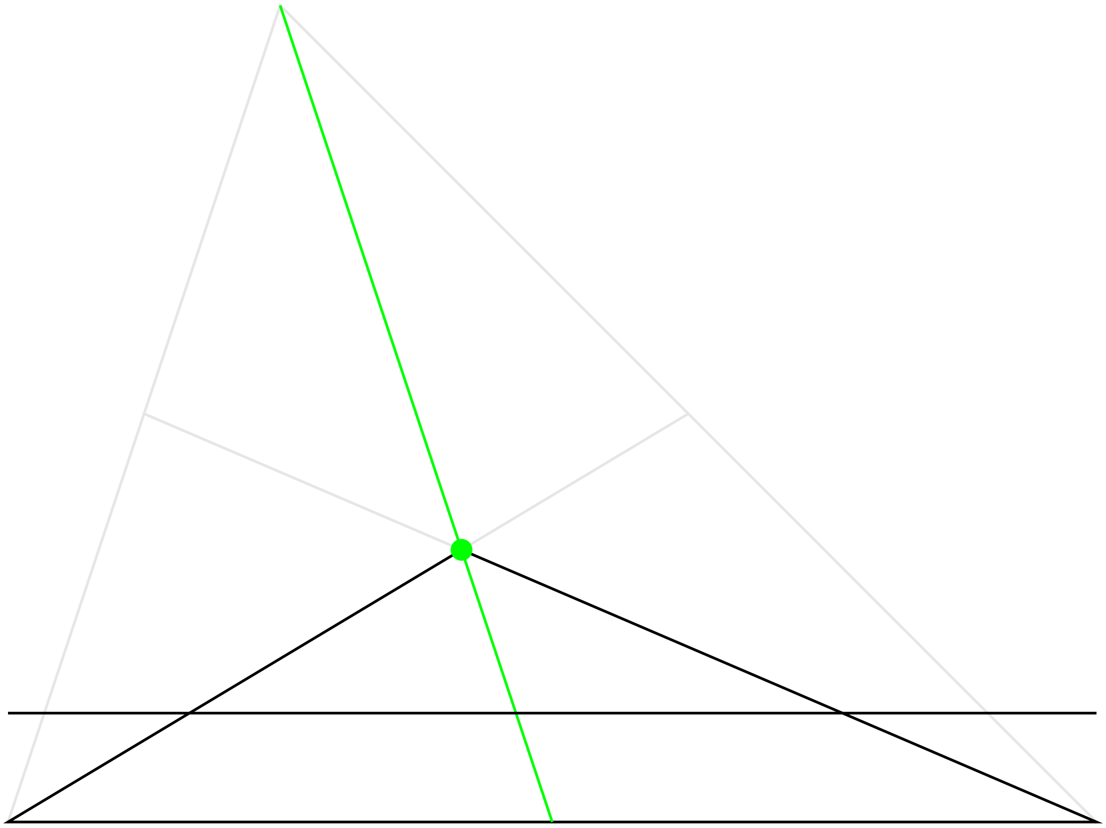


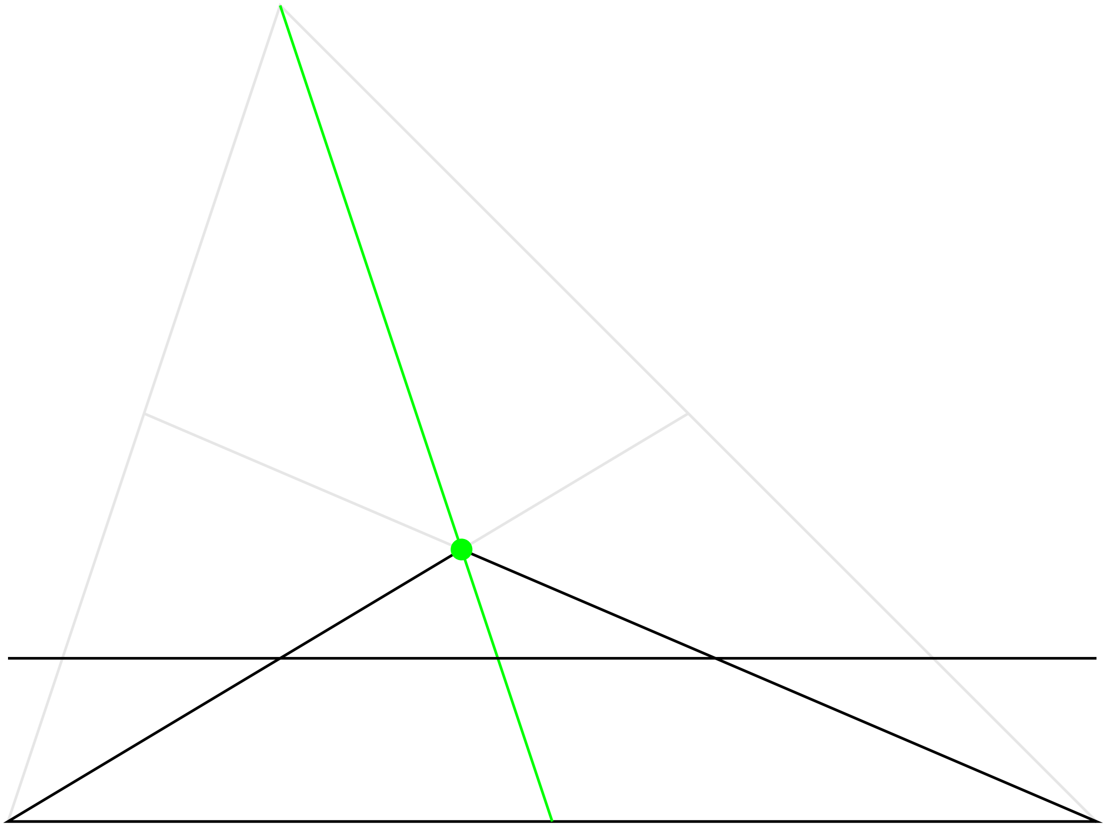


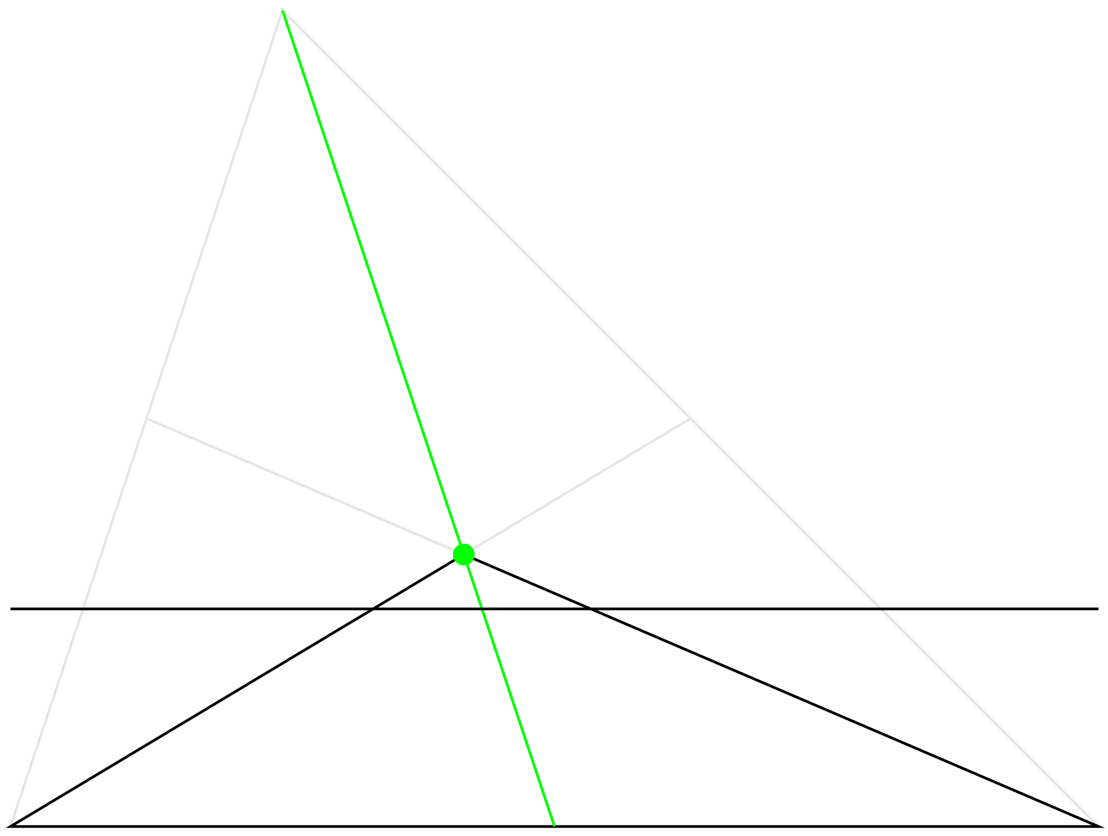
Then proof that two triangles with bases on the same side of the original triangle are equal.

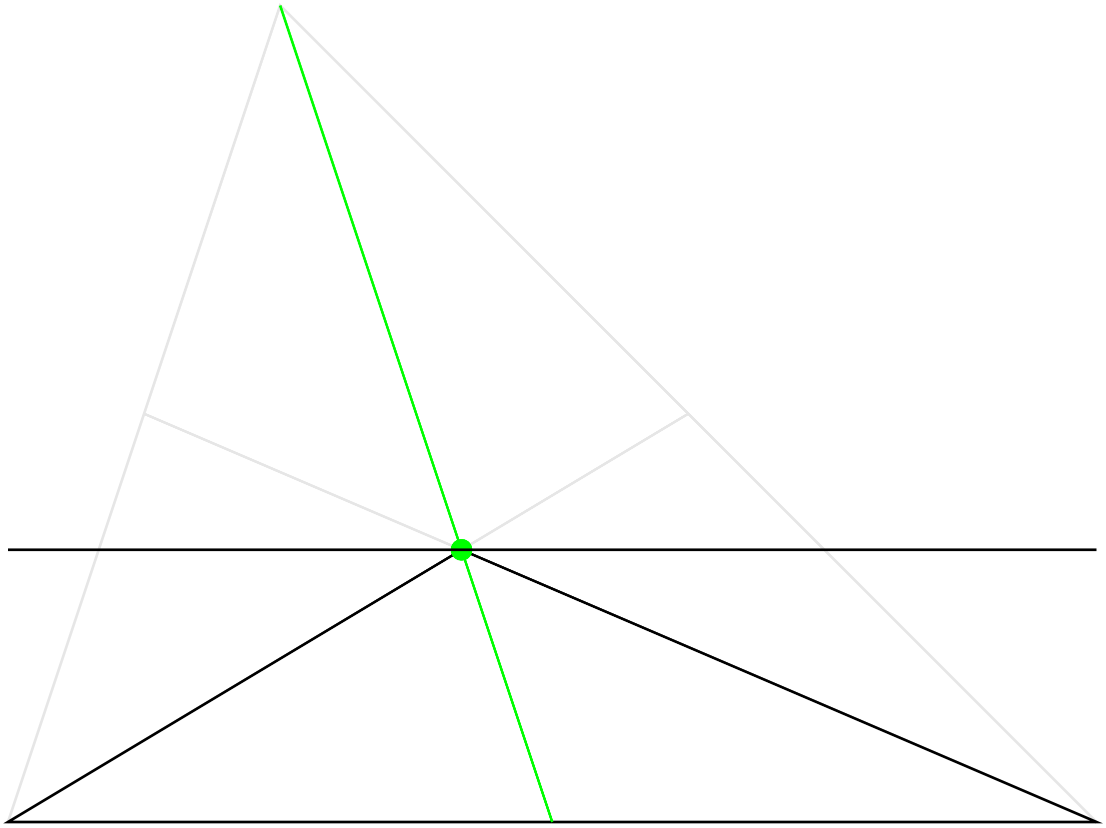


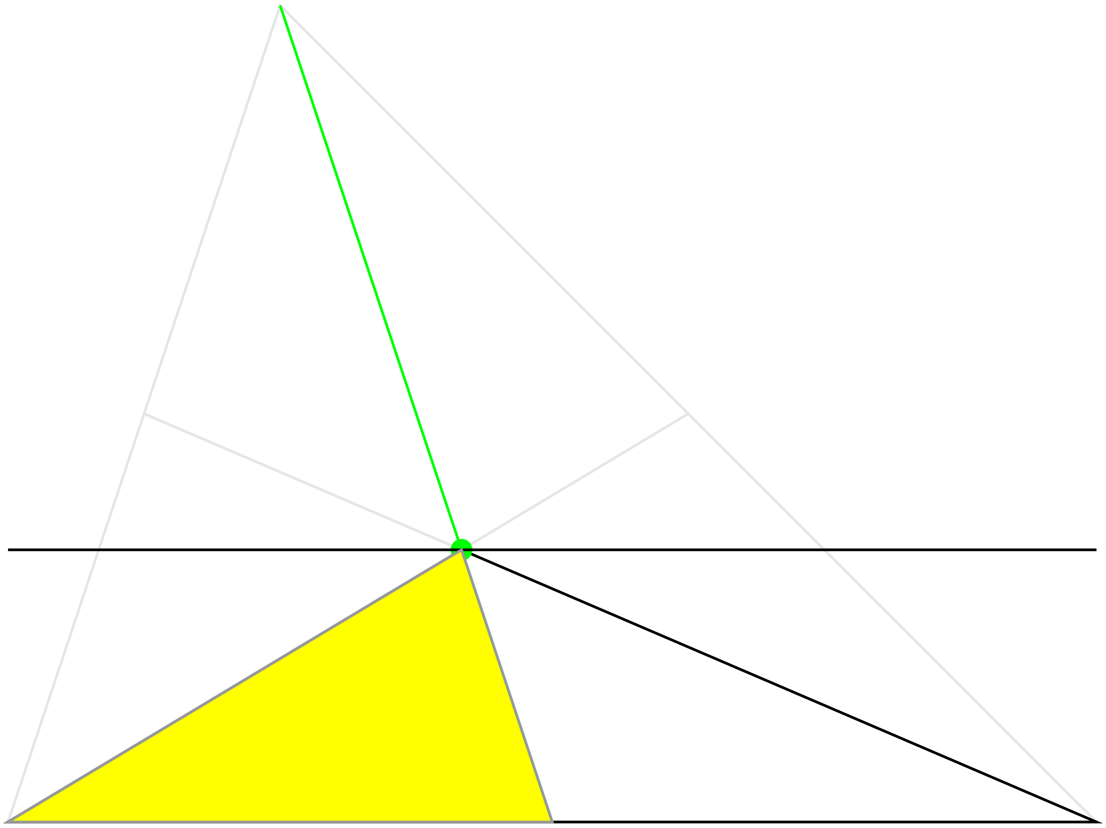


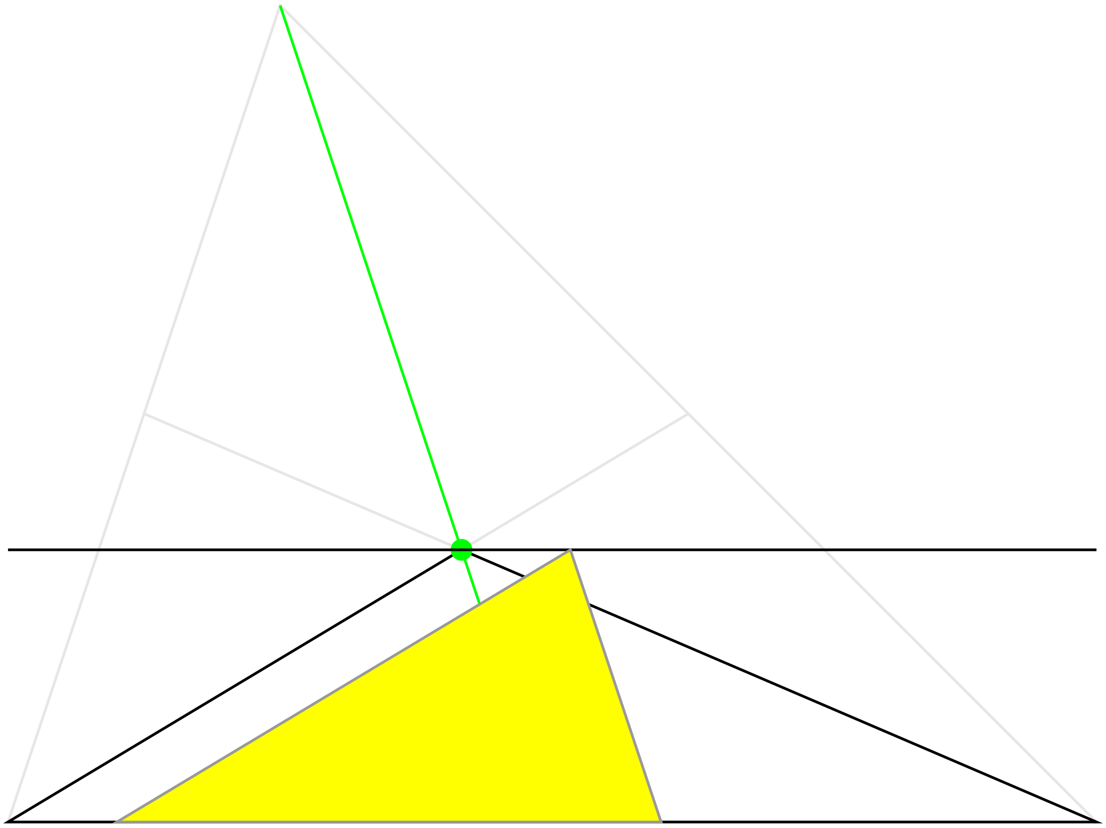


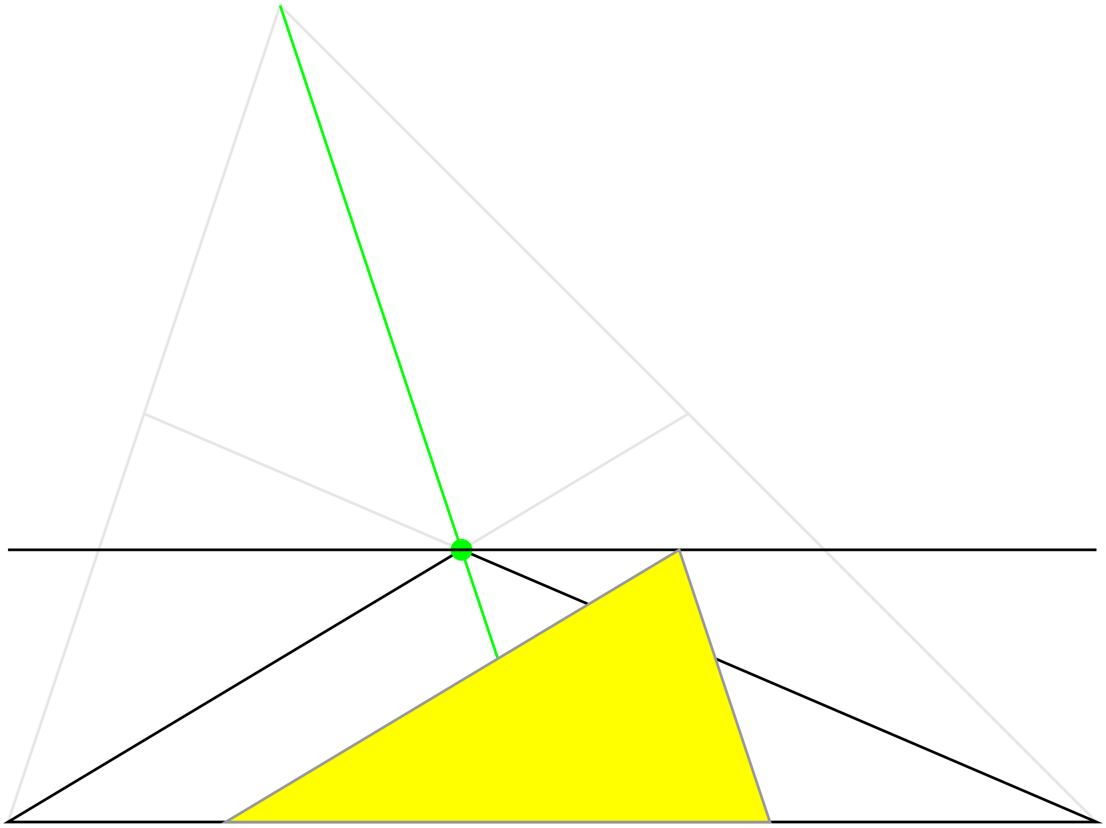


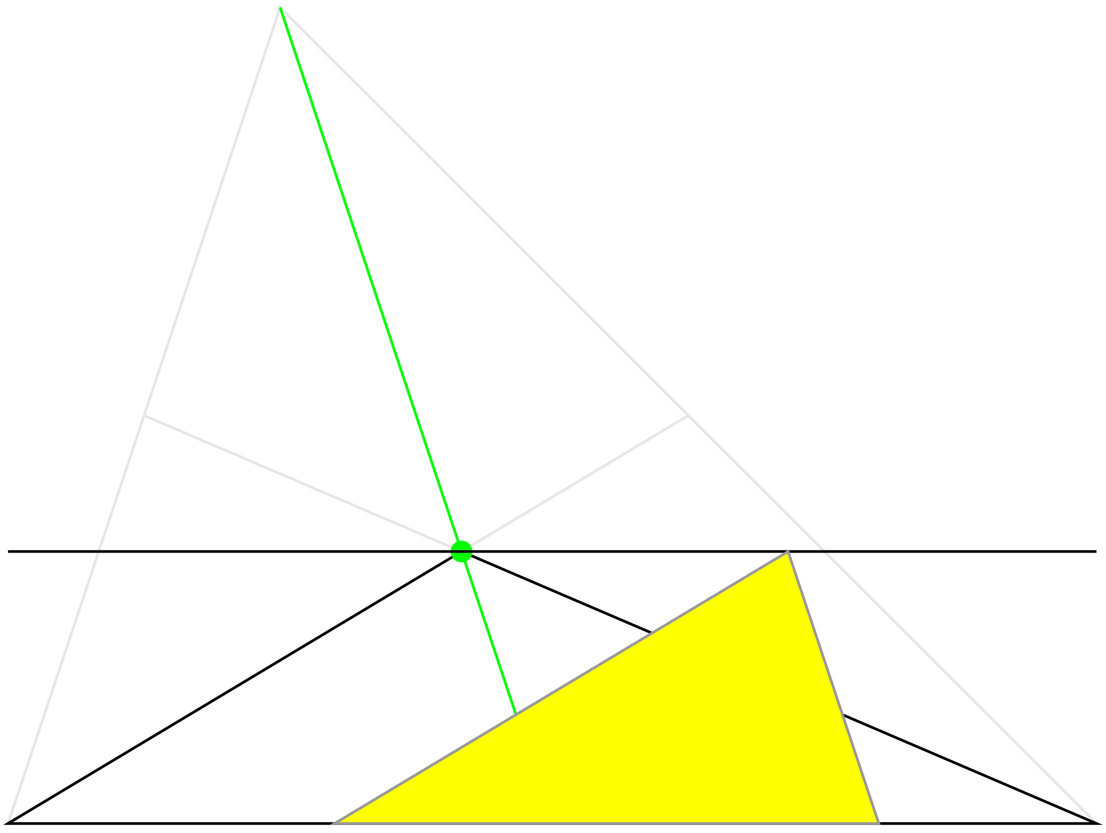


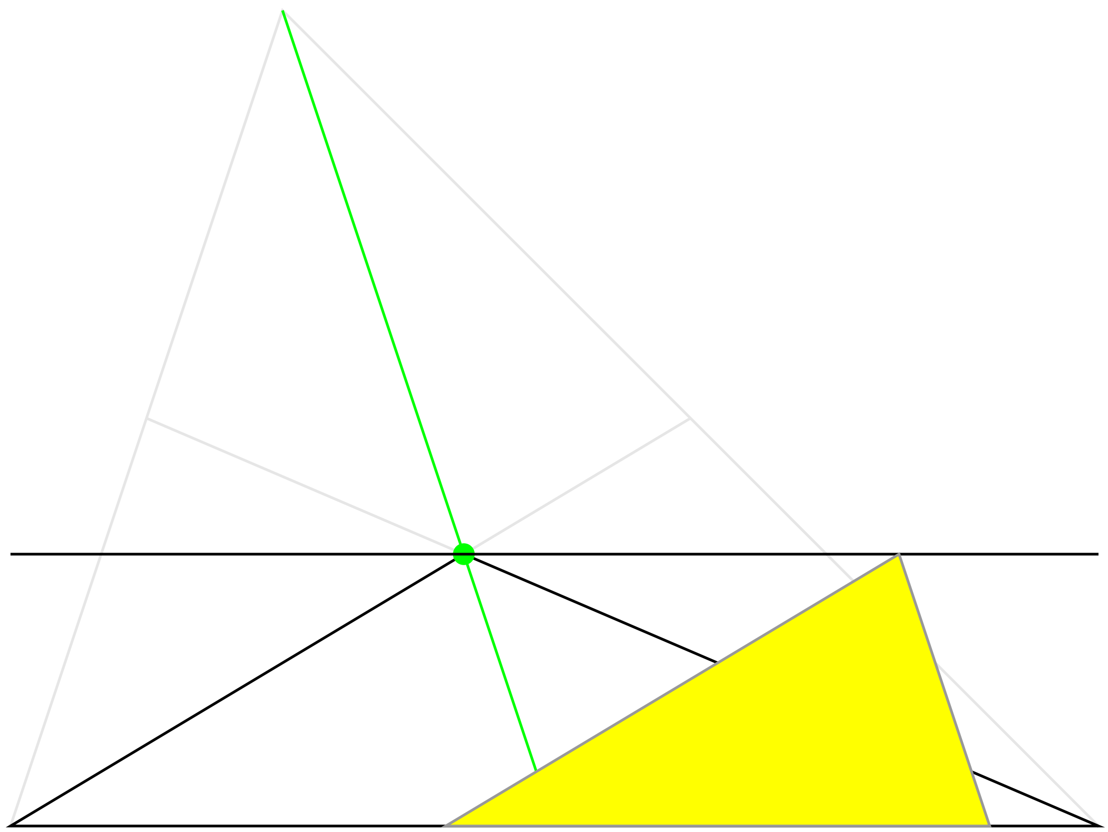


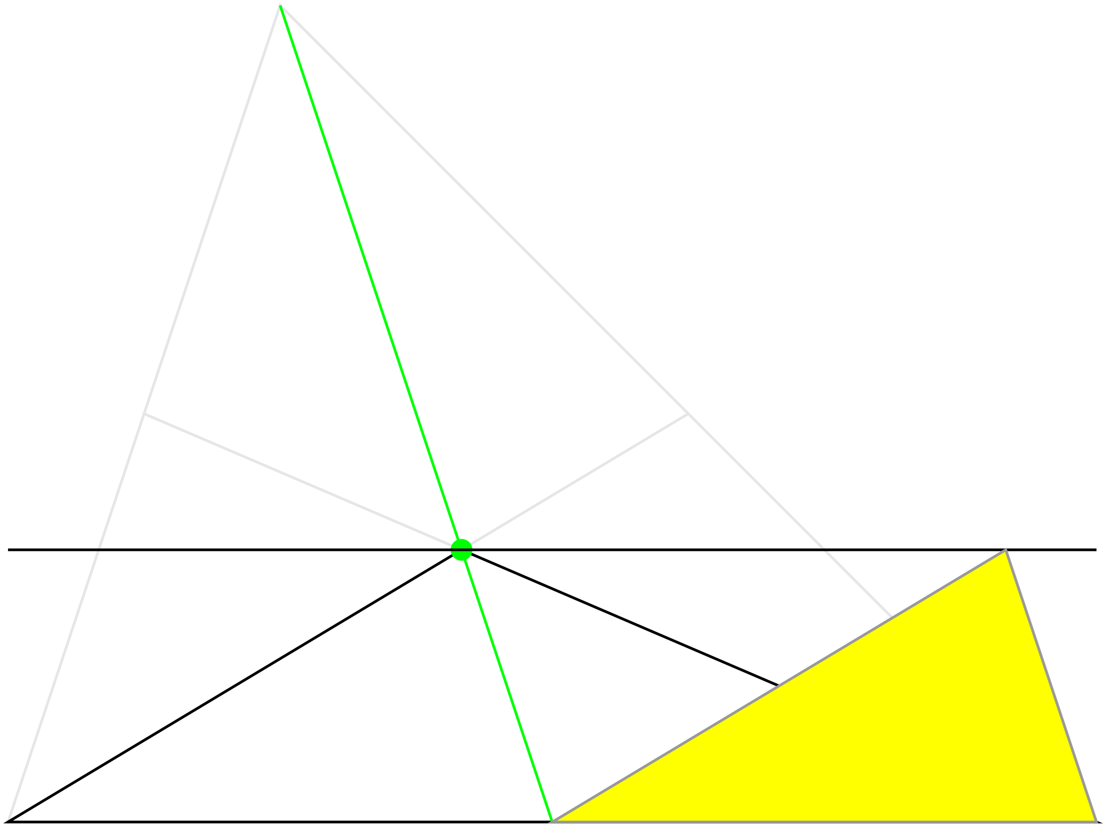


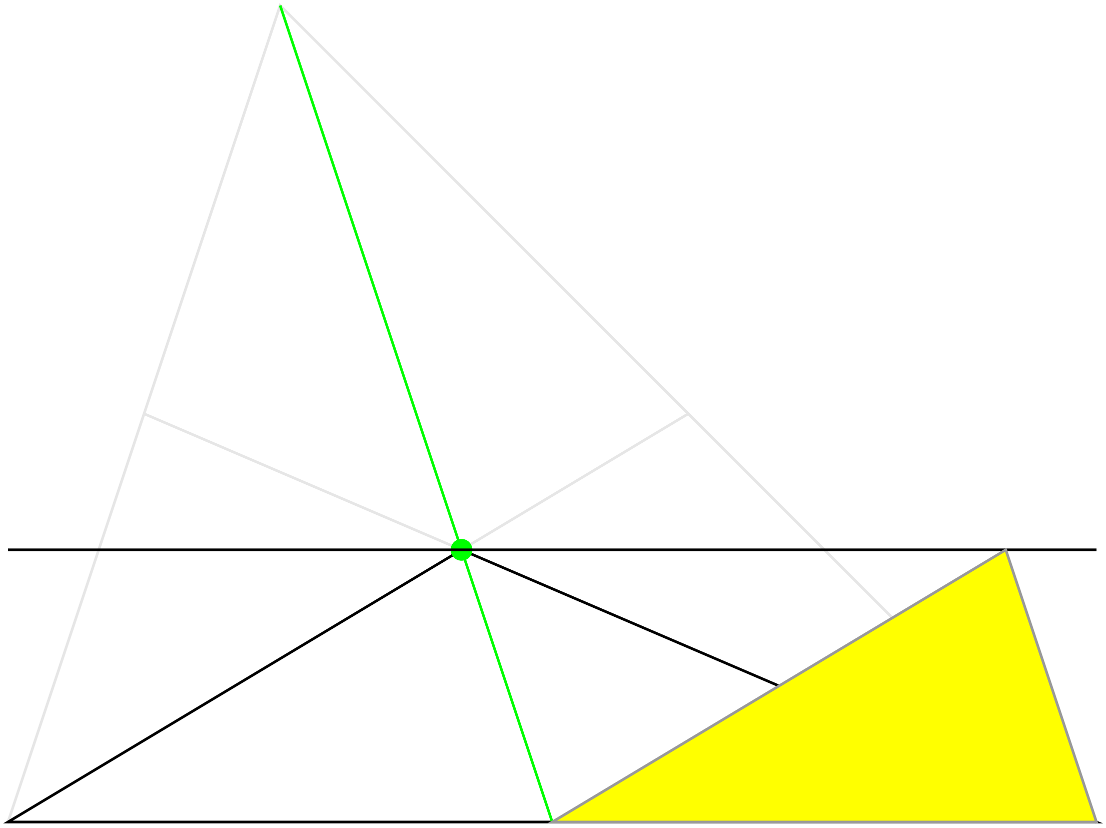


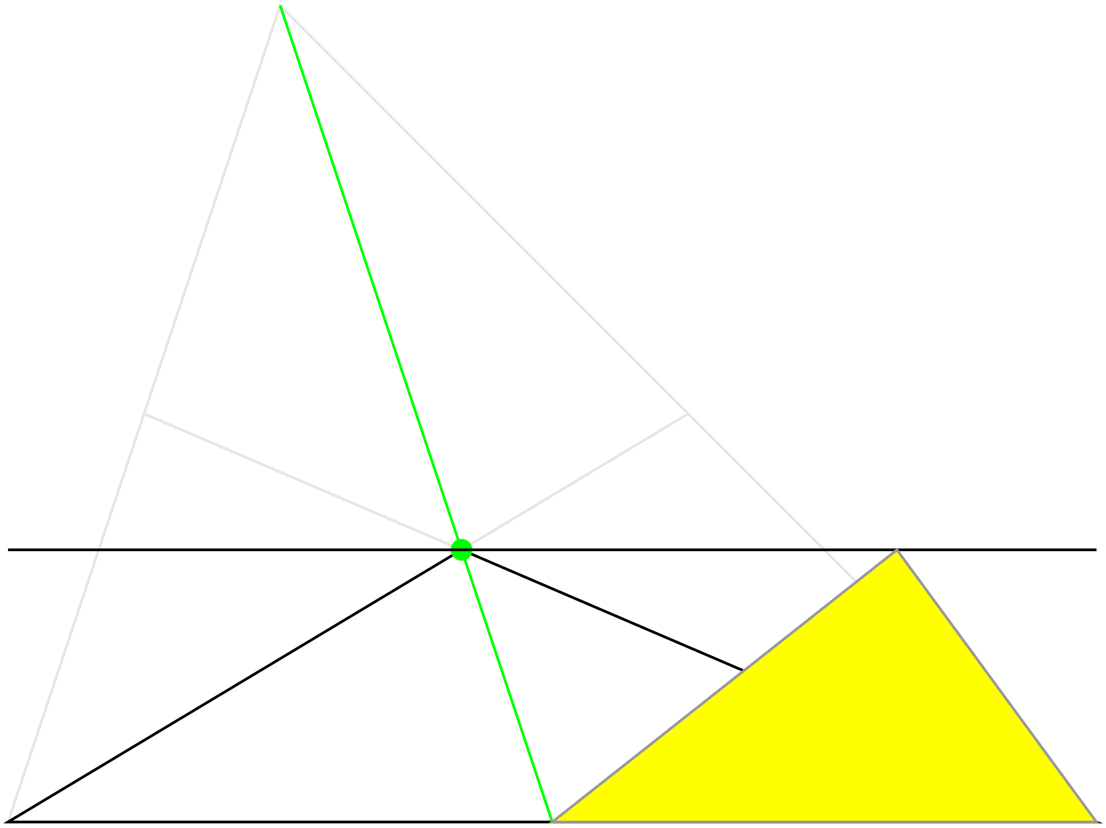


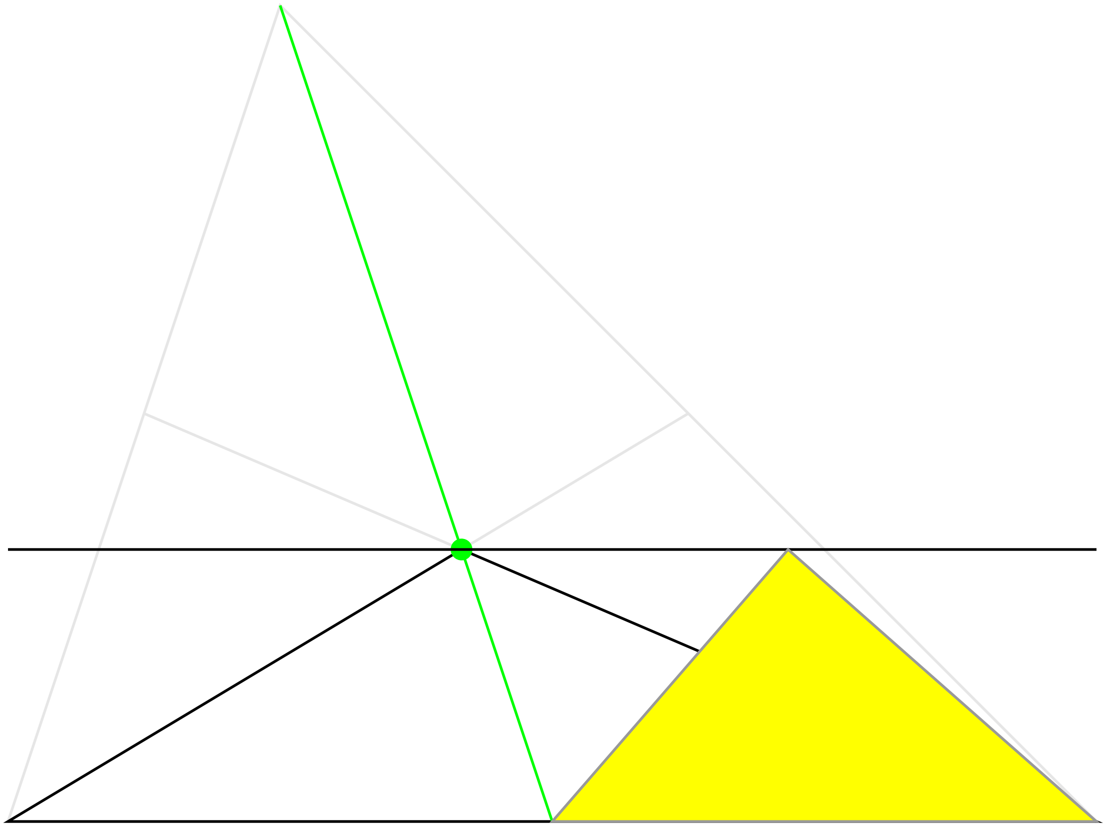


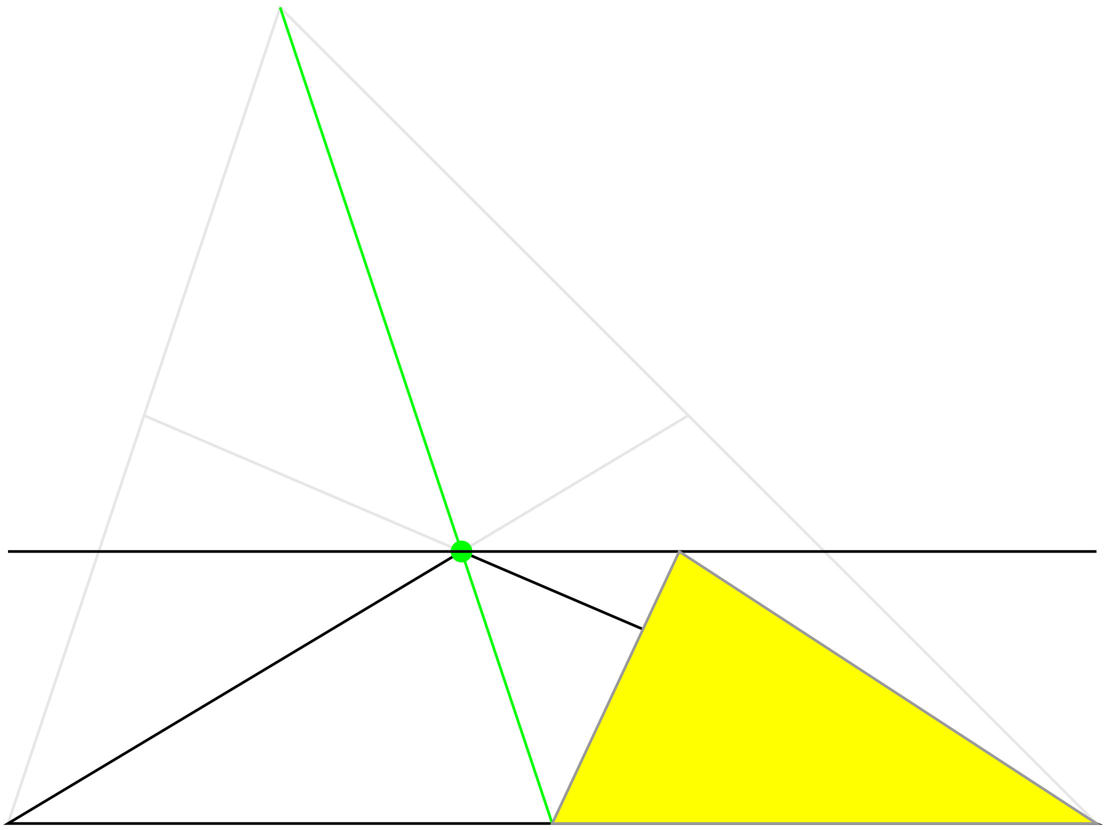


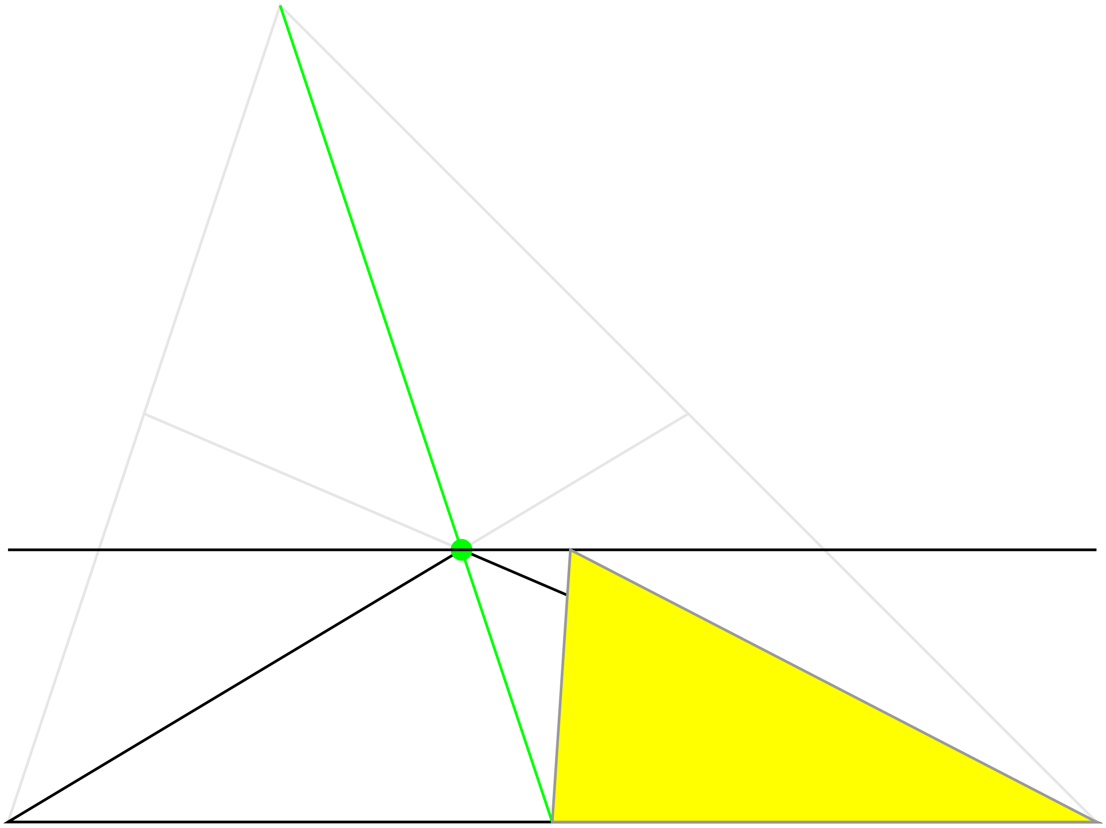


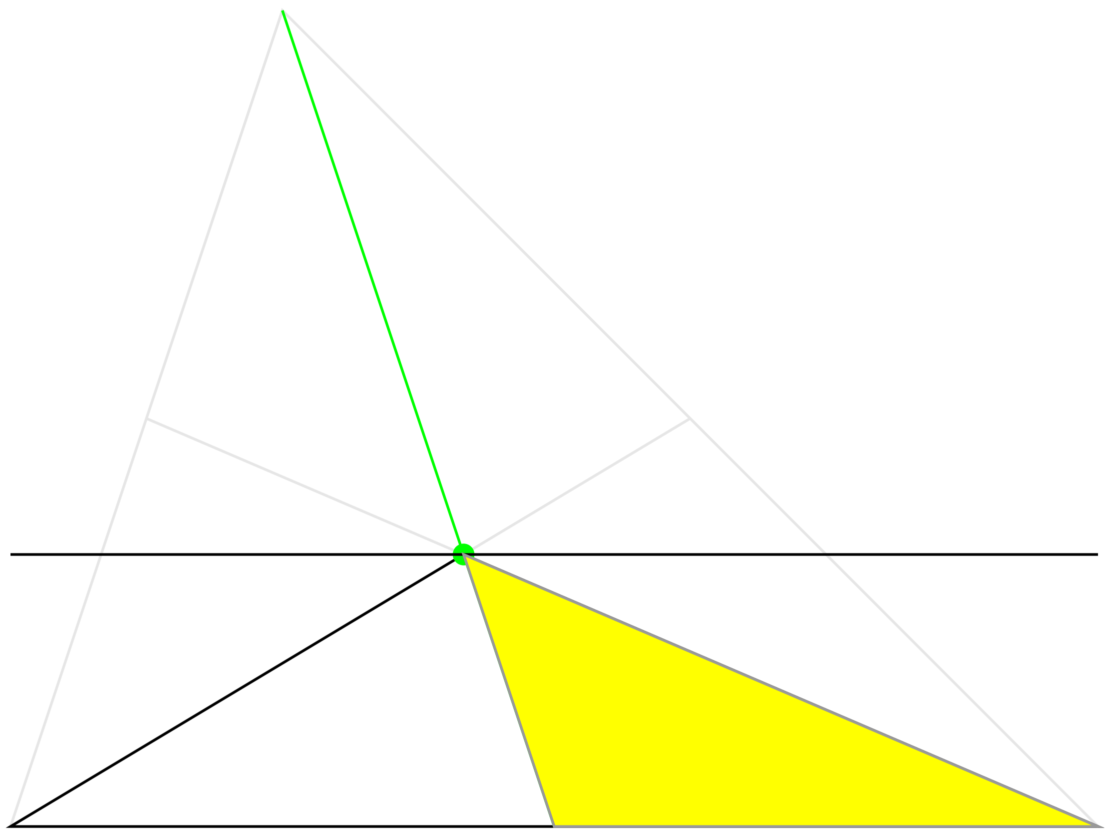


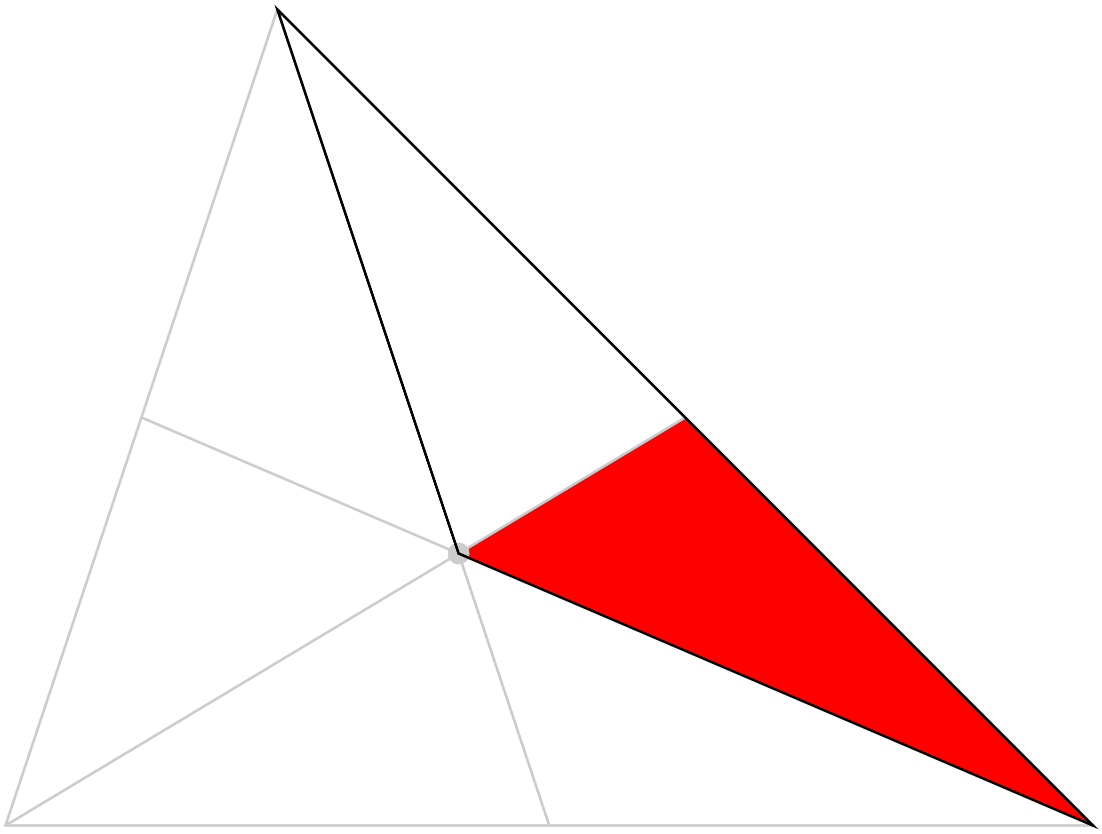




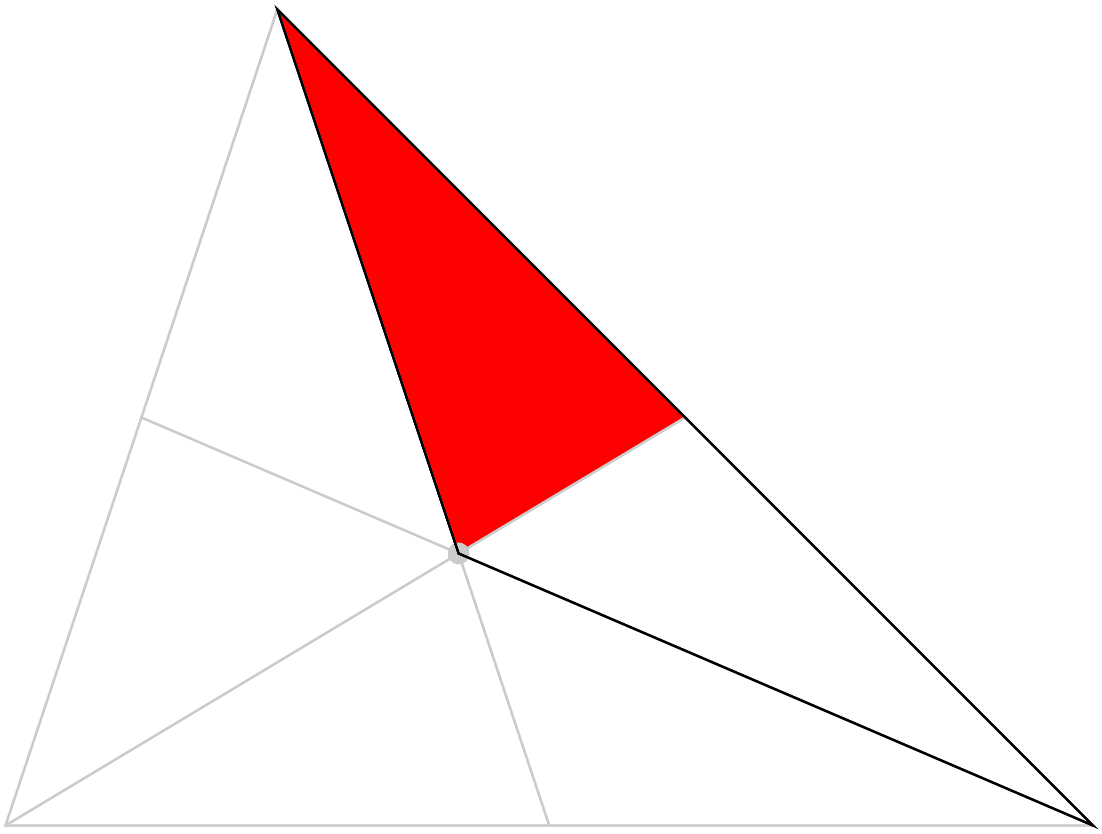








Similar proof can be done to show that each pair of triangles lying on the same side have equal areas.



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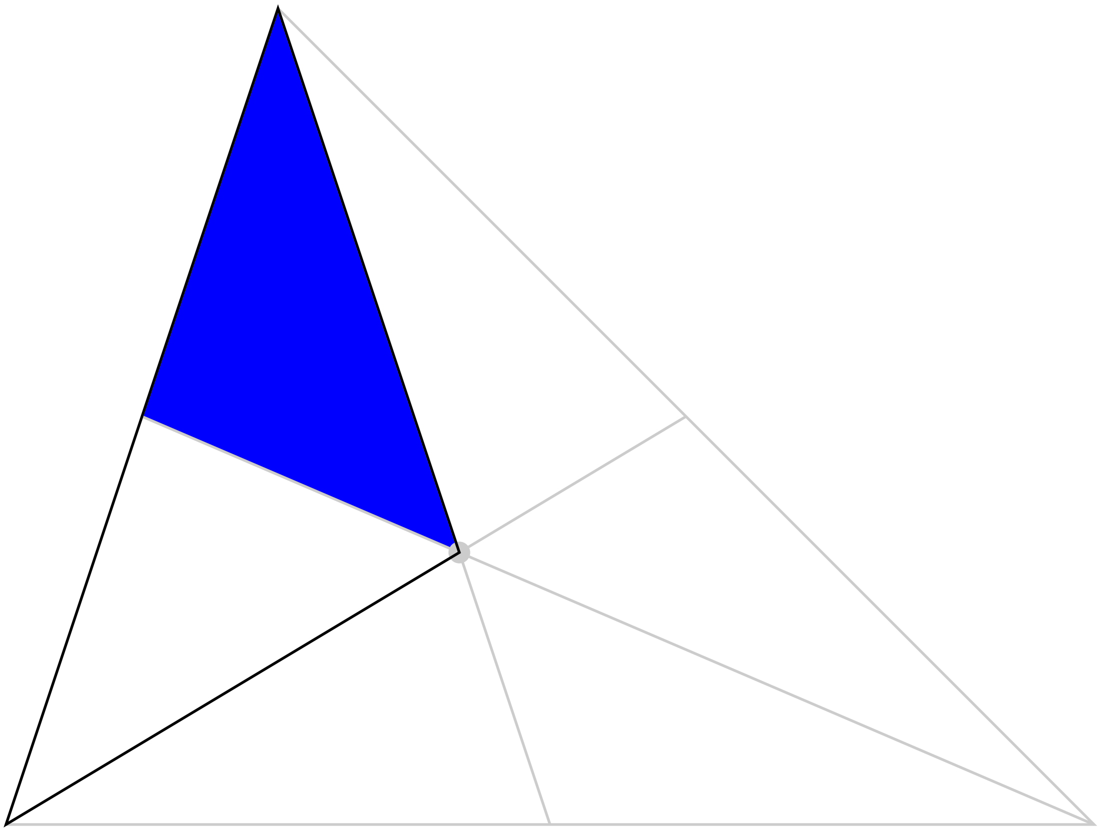

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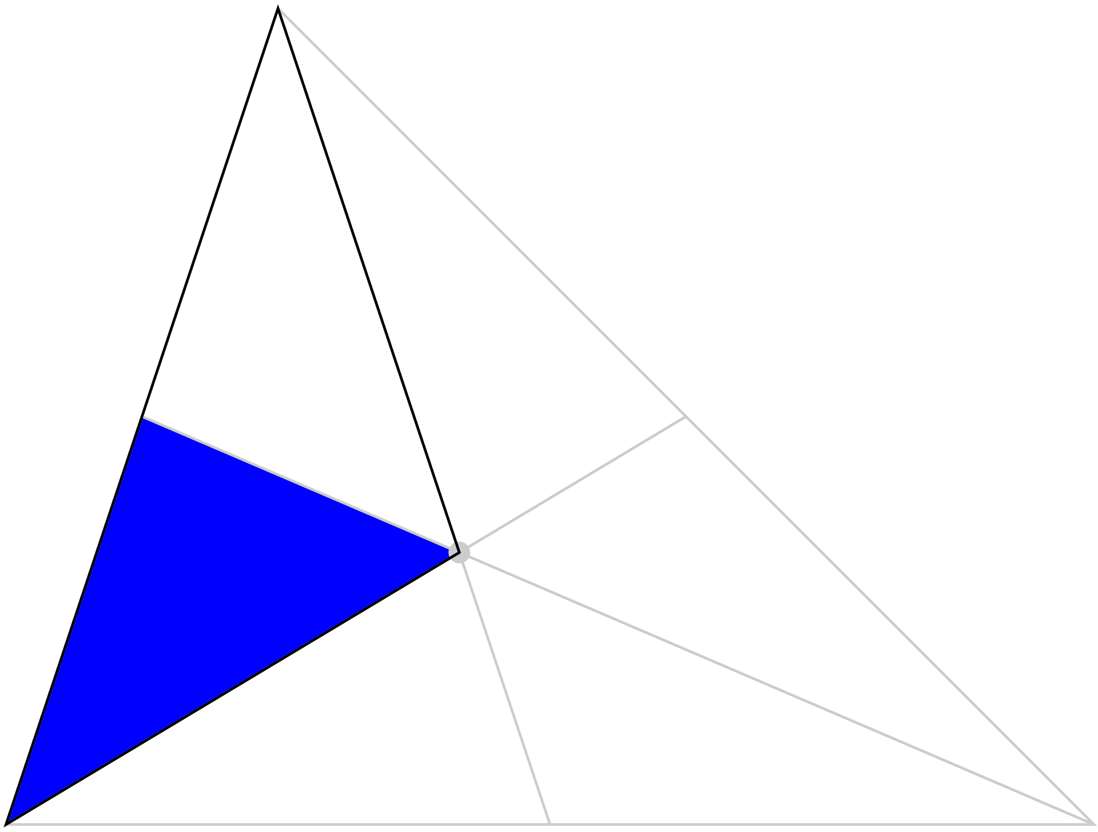

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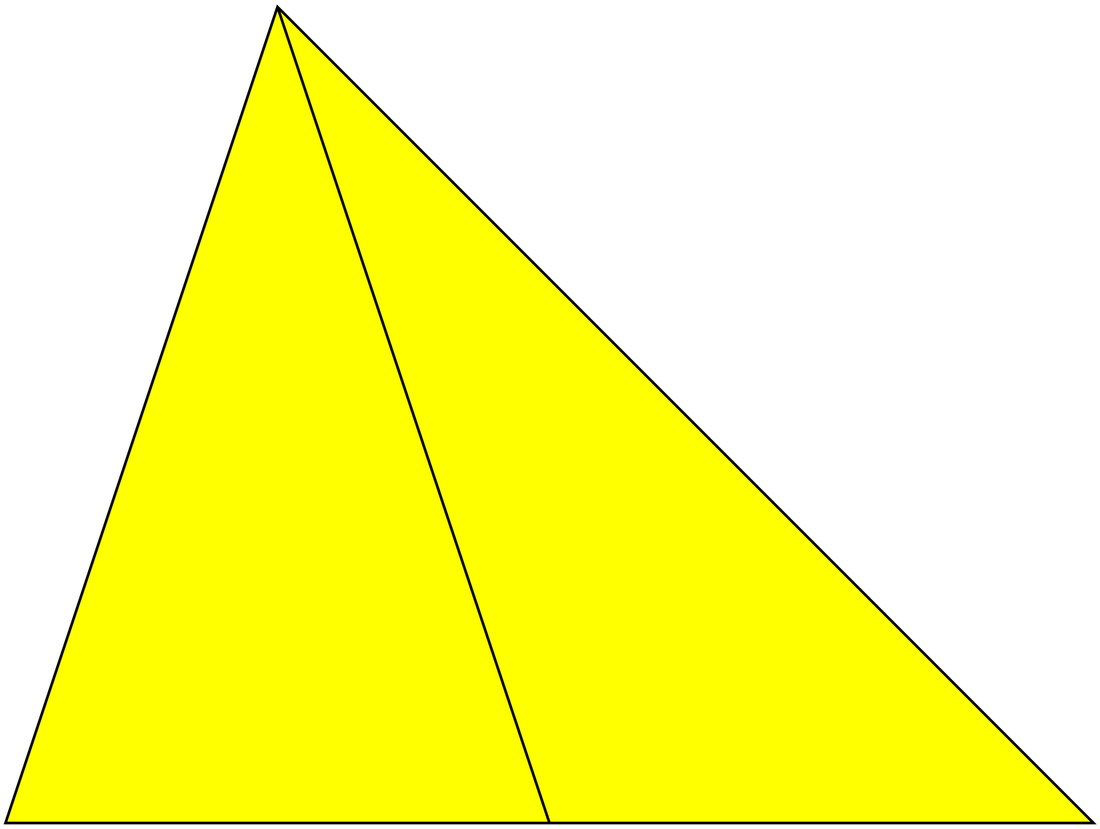
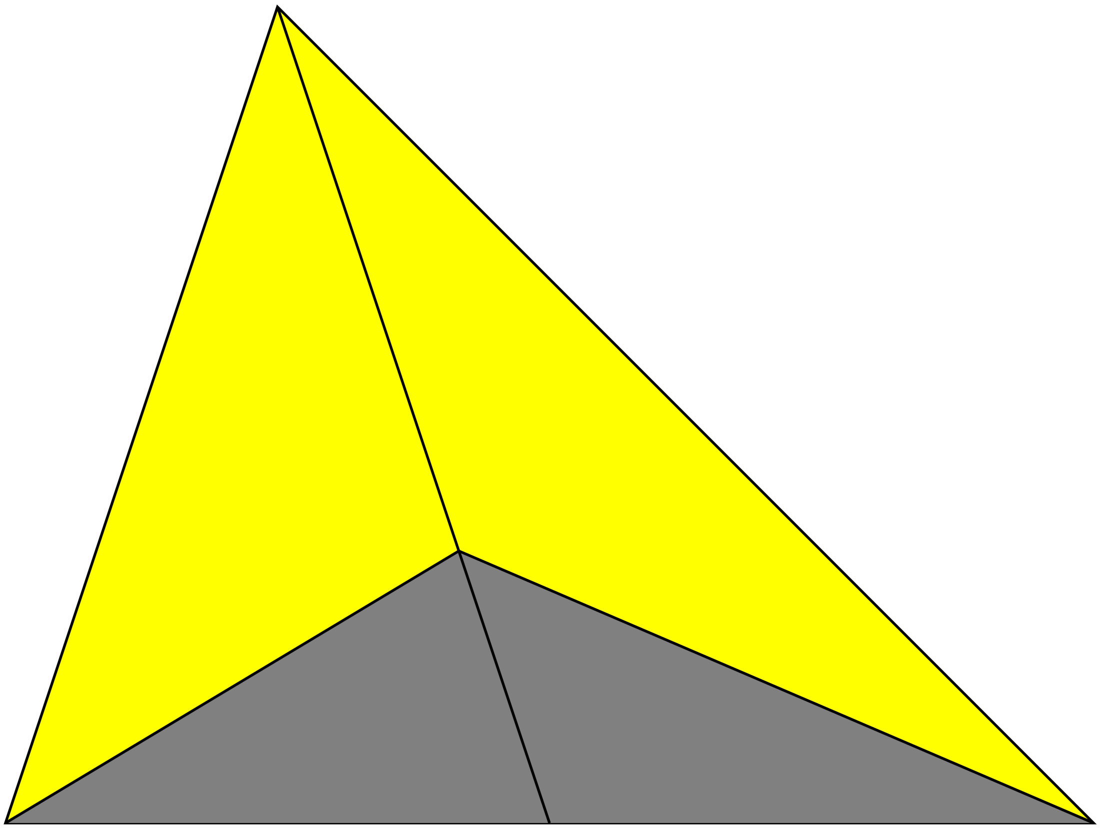
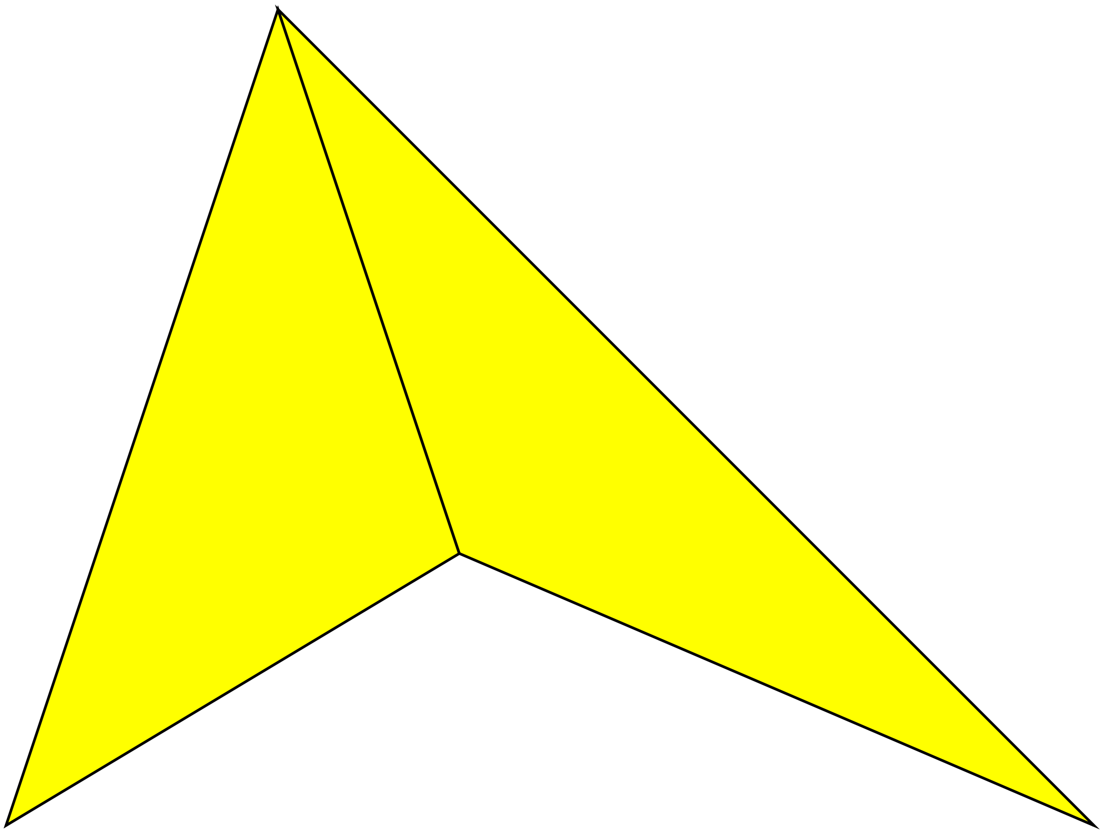


Now, since the two large triangles have same areas,



Since the two large triangles have same areas,

and since the two smaller triangles are also equal,

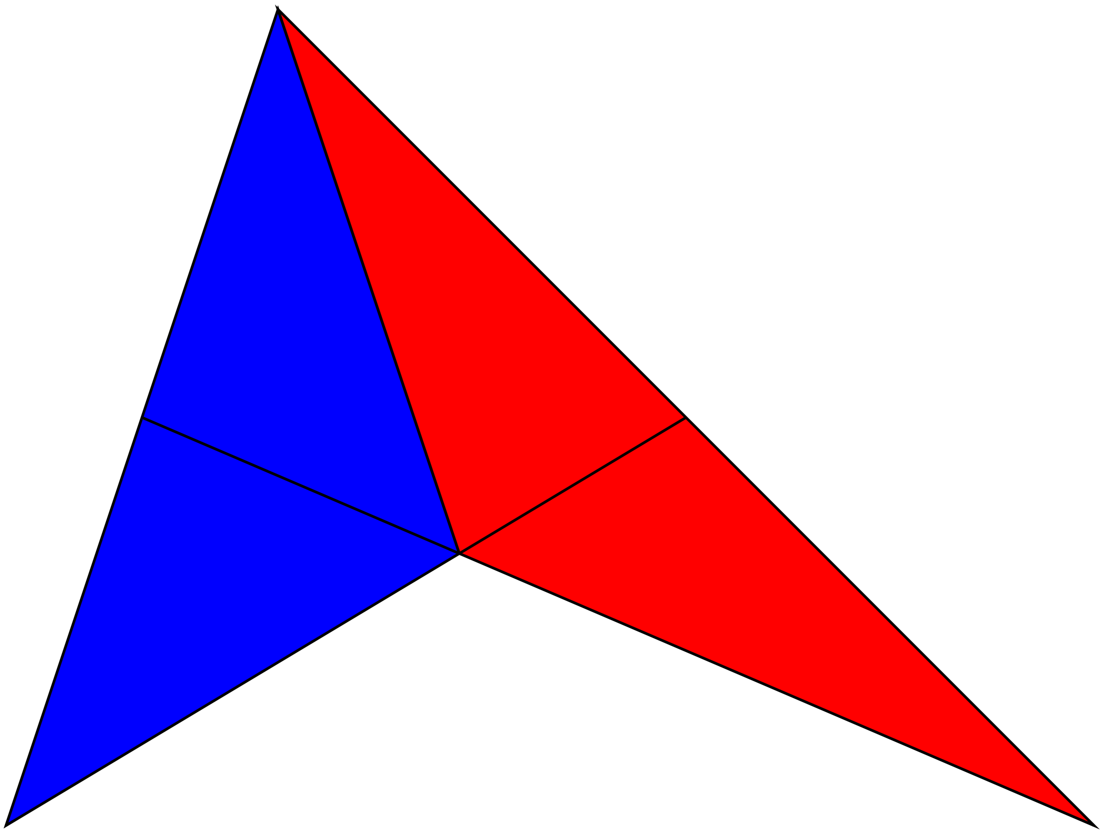


Since the two large triangles have same areas,

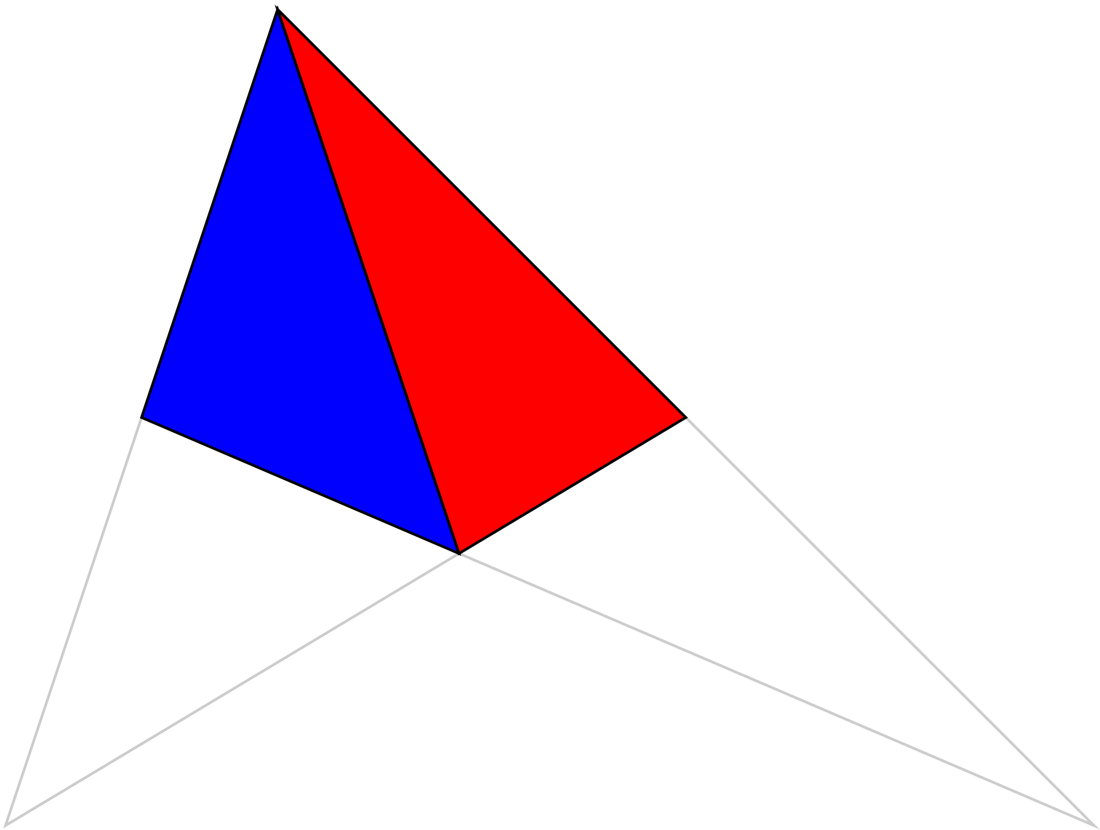
and since the two gray triangles are also equal,

a gray triangle can be subtracted from each yellow

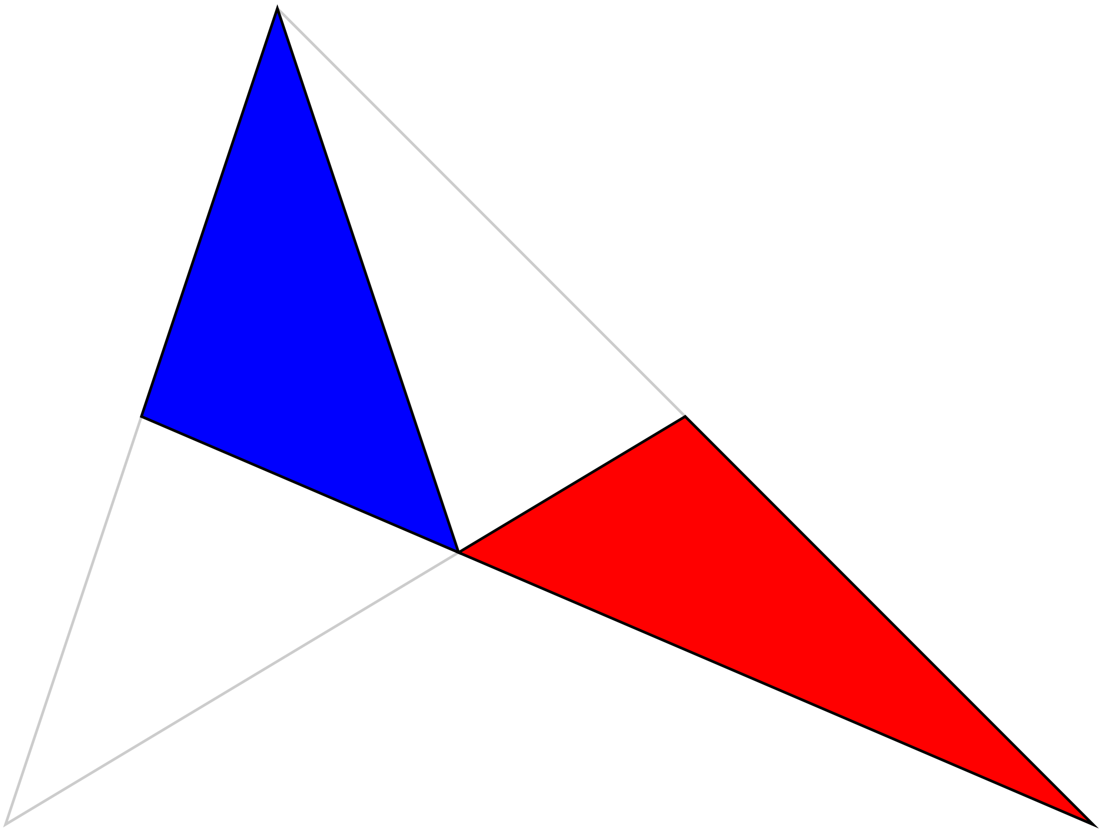
and the remaining parts will be equal.



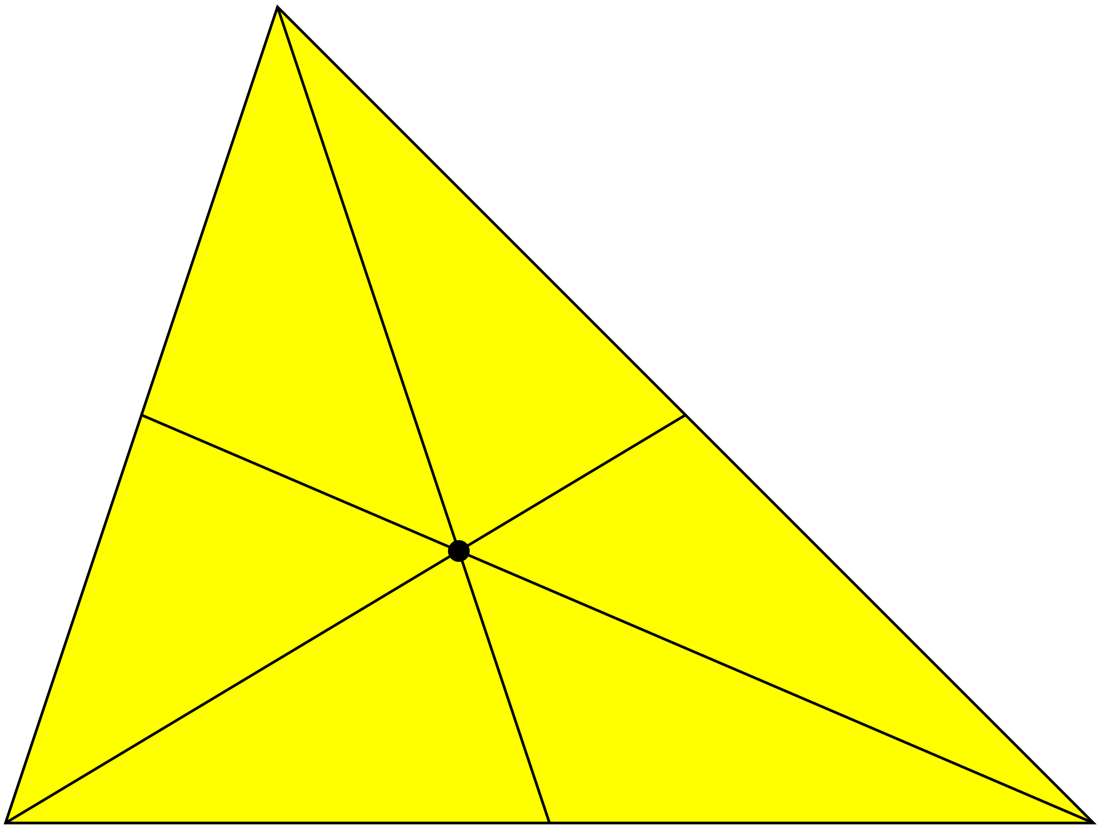
However, each pair of these two are equal...



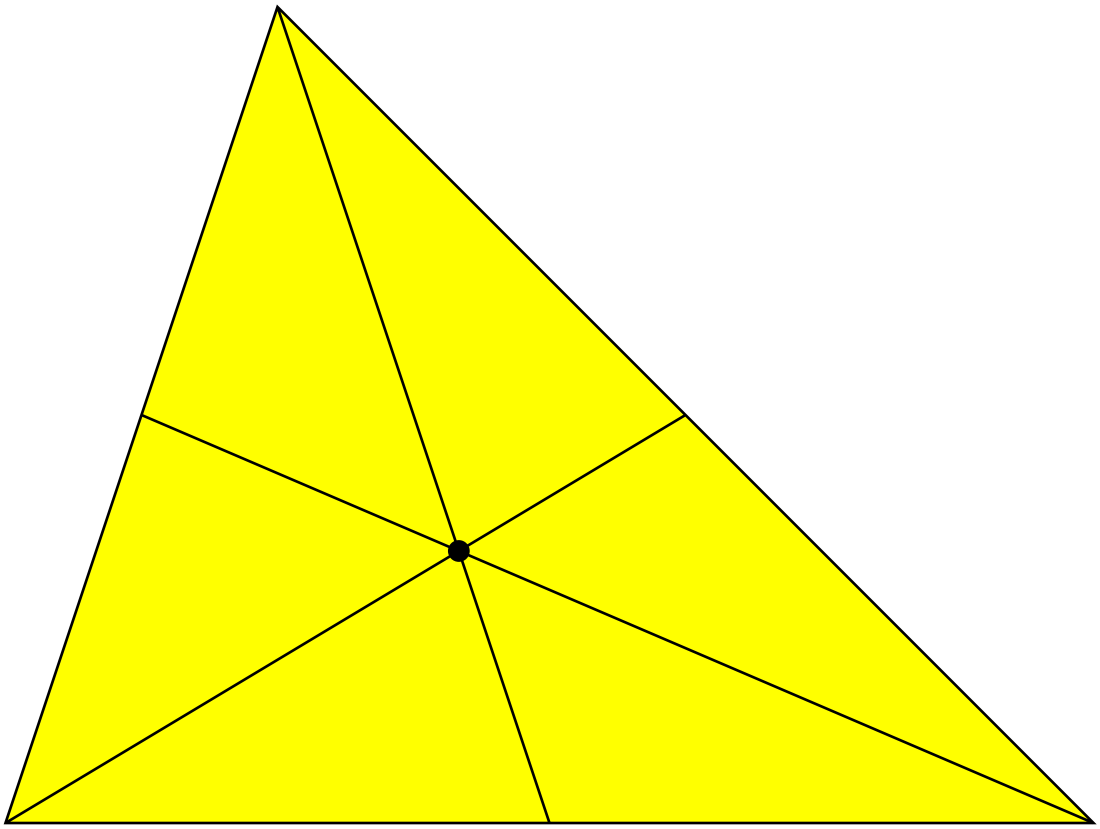
... so it follows that these two are equal,



... and so are these two.

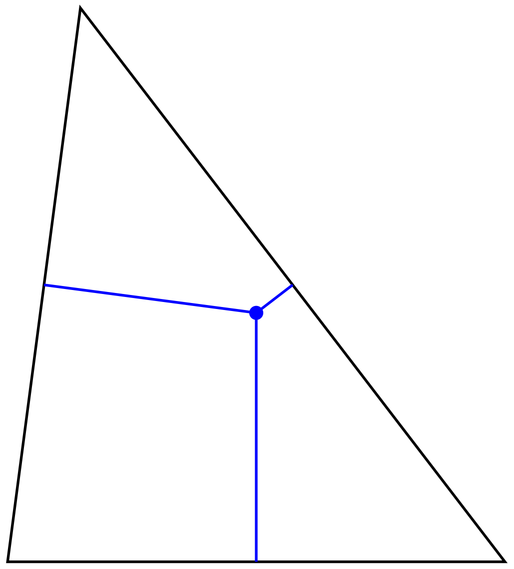


Similar proofs can be applied to all the other groups and consequently show that each of these six triangles have equal areas...

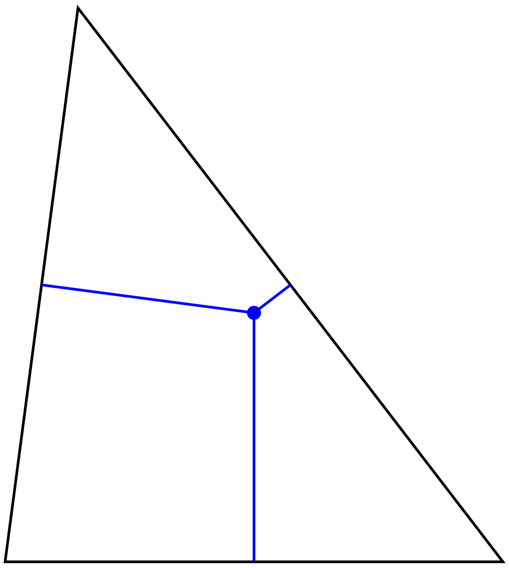


Therefore, the centroid is the center of mass, and all three medians together divide the triangle into six equal portions.

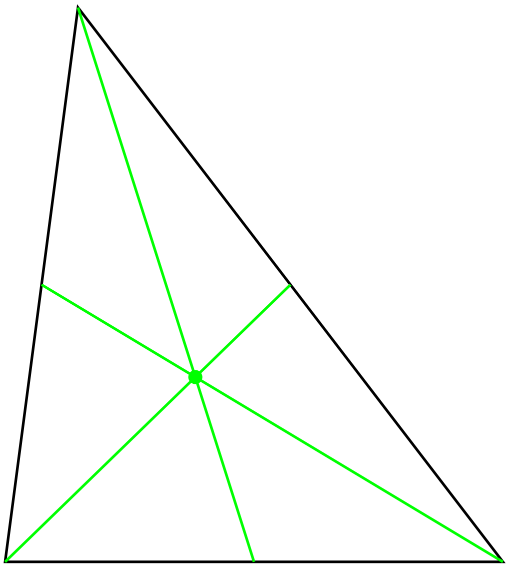
4. The Euler line



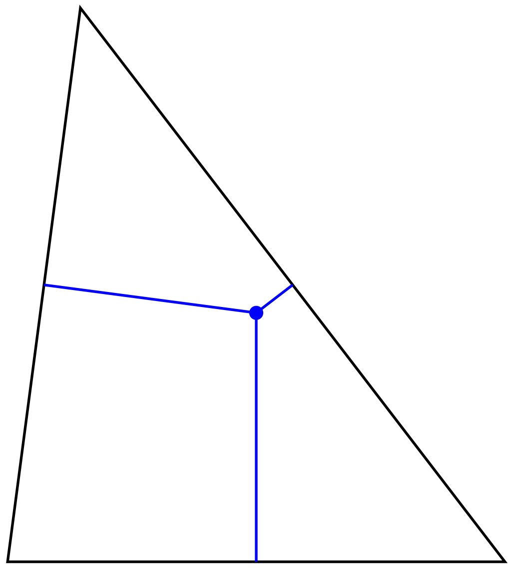
circumcenter(O)



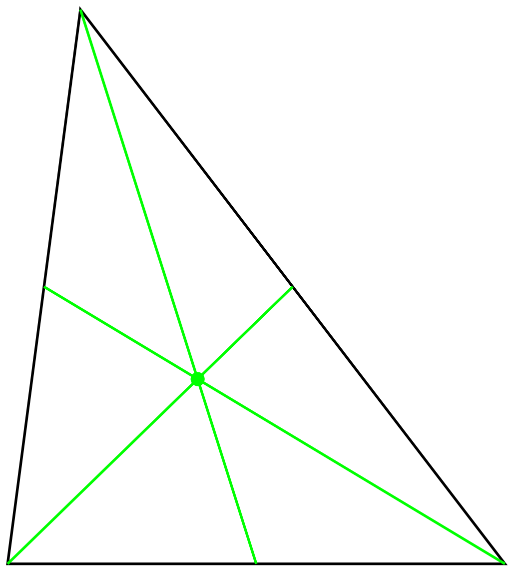
Circumcenter(O)



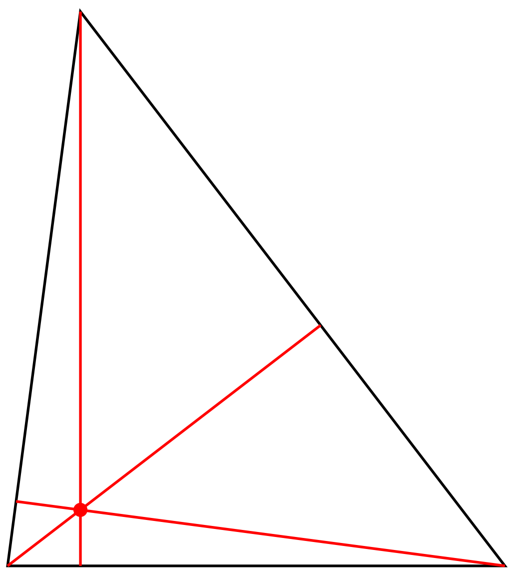
Centroid(M)



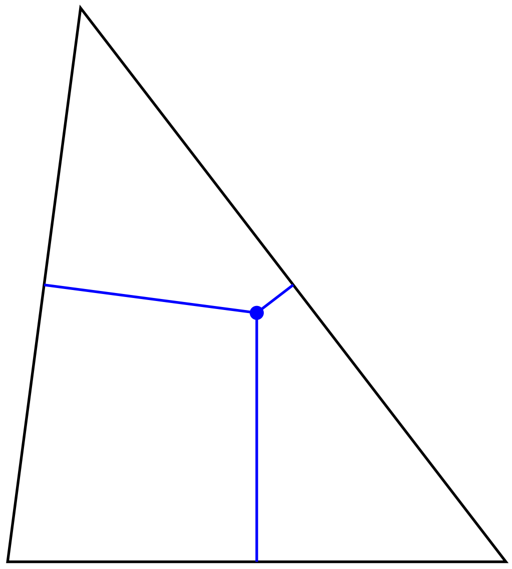
Circumcenter(O)

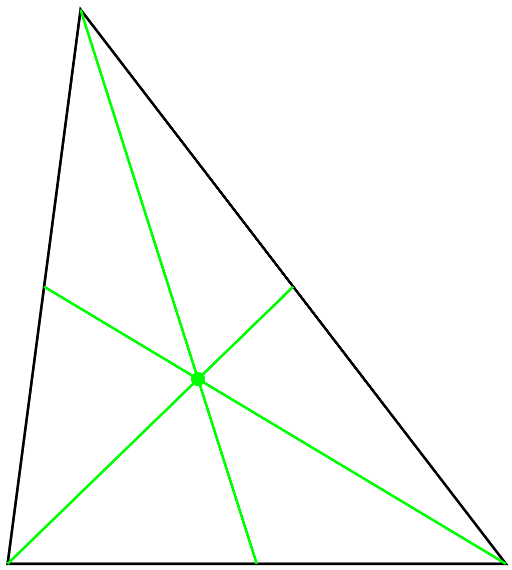


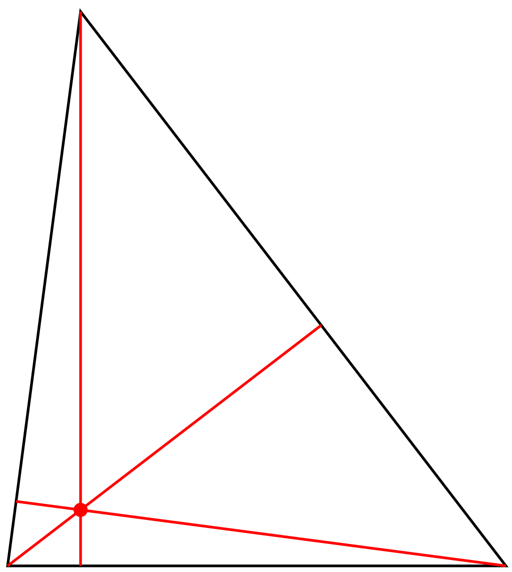
Centroid(M)

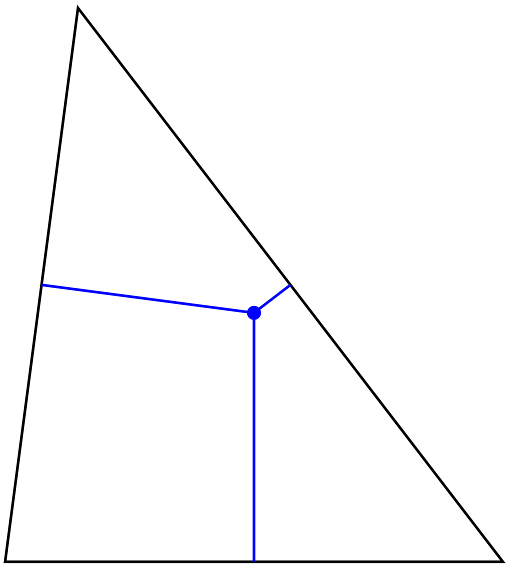


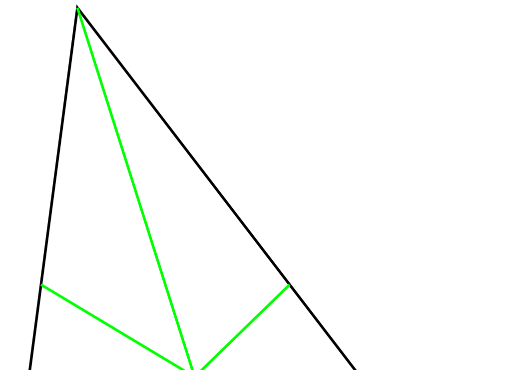
Orthocenter(H)

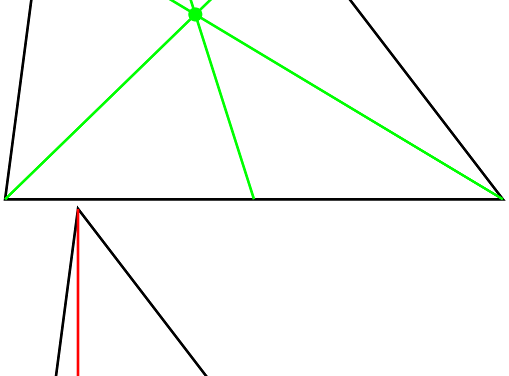


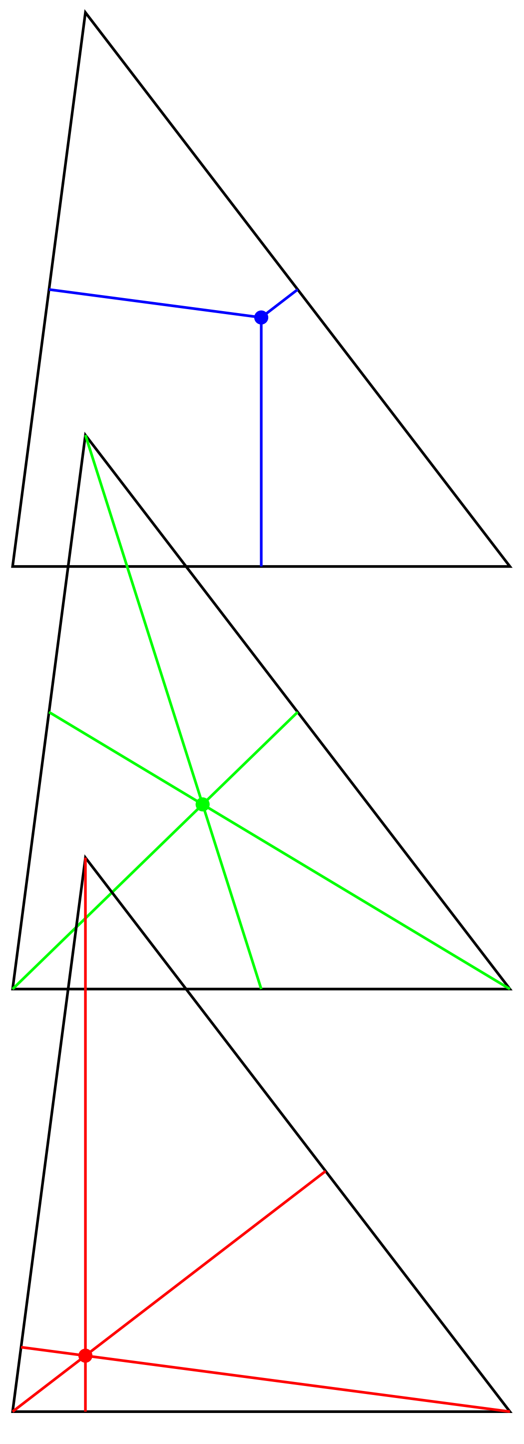


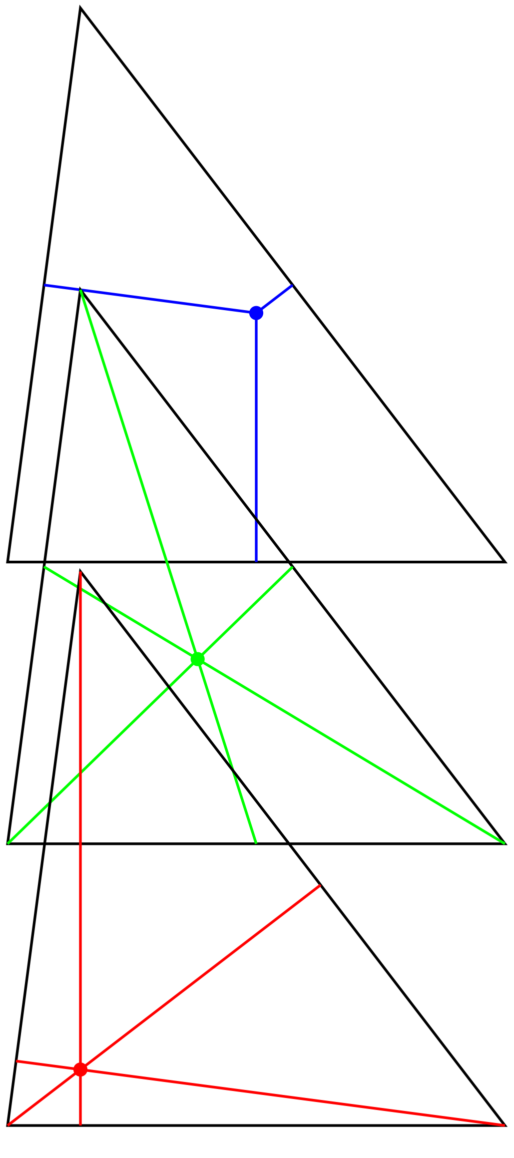


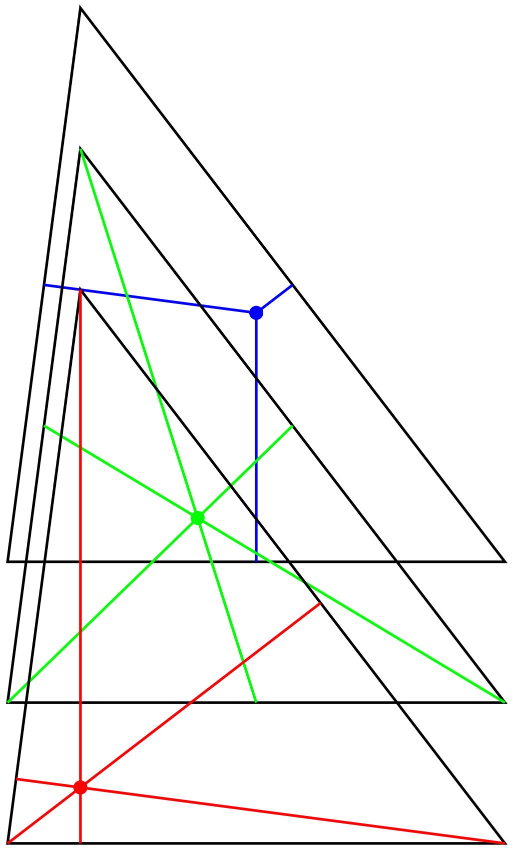


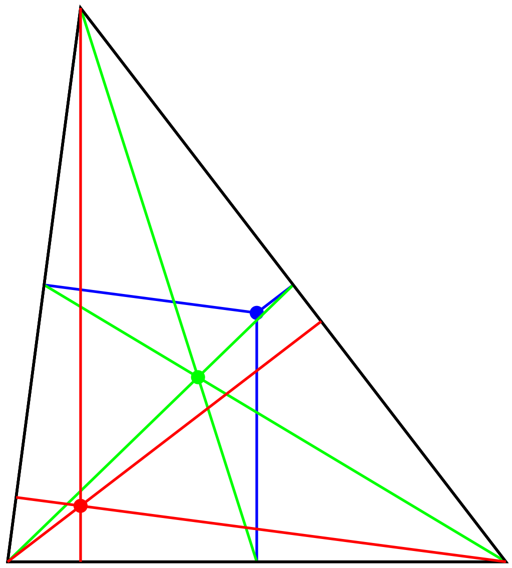


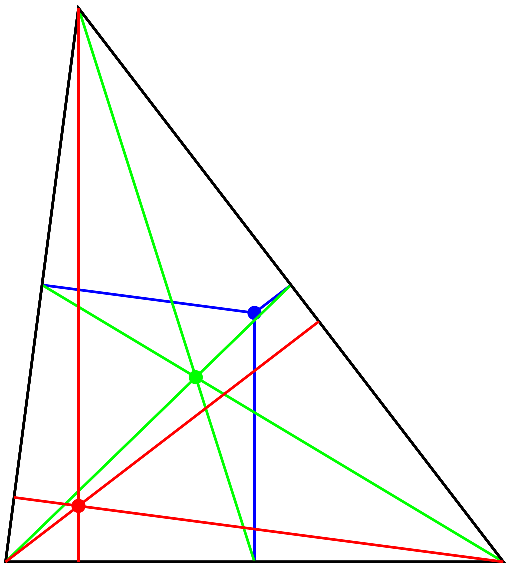


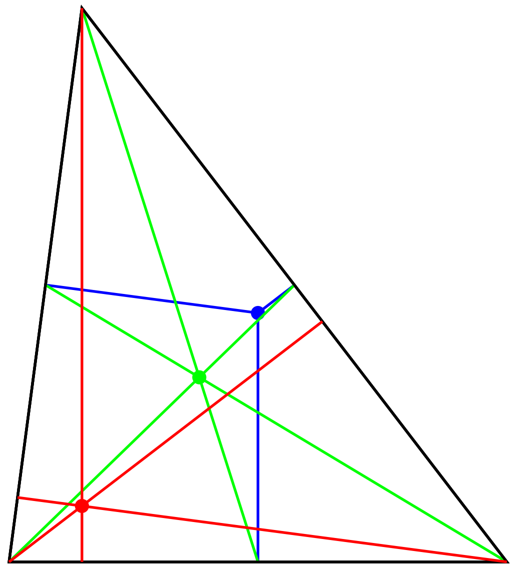


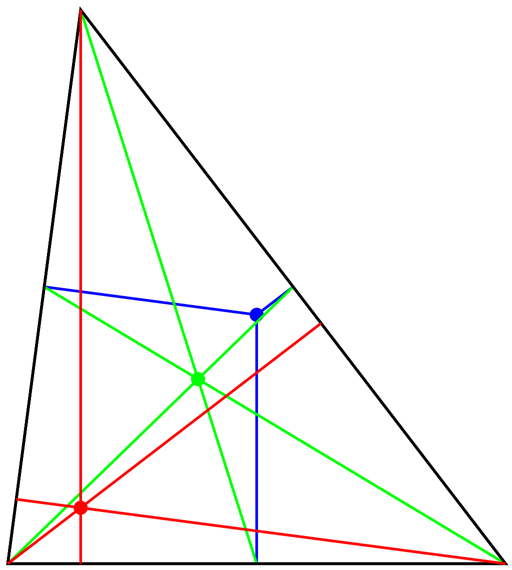


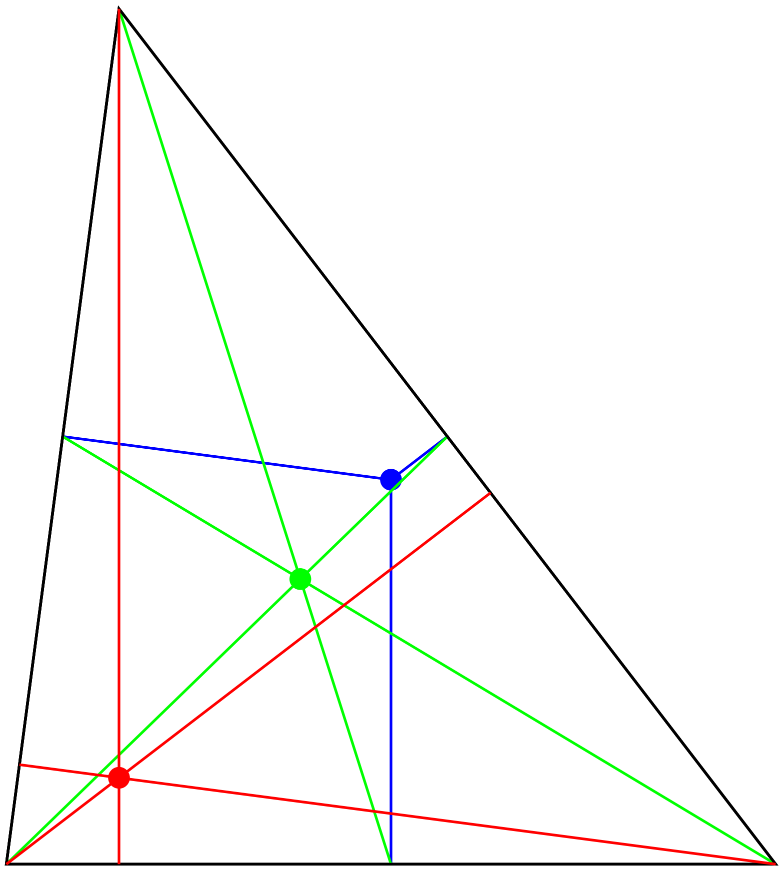




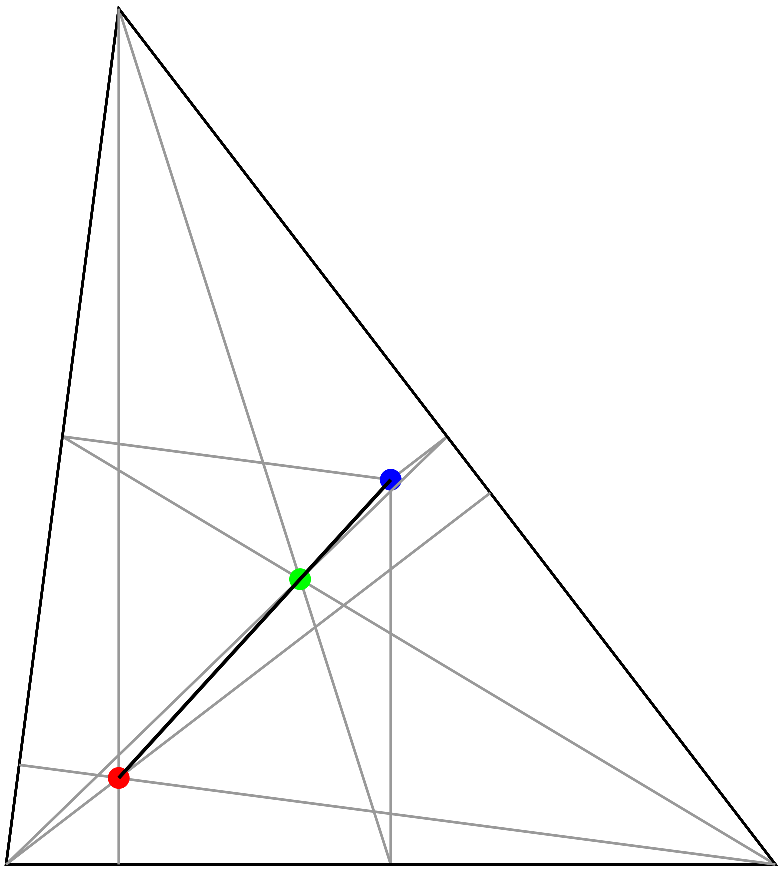








It is most interesting to observe that the three centers, the circumcenter (blue), the centroid (green), and the orthocenter (red) lie on a line!



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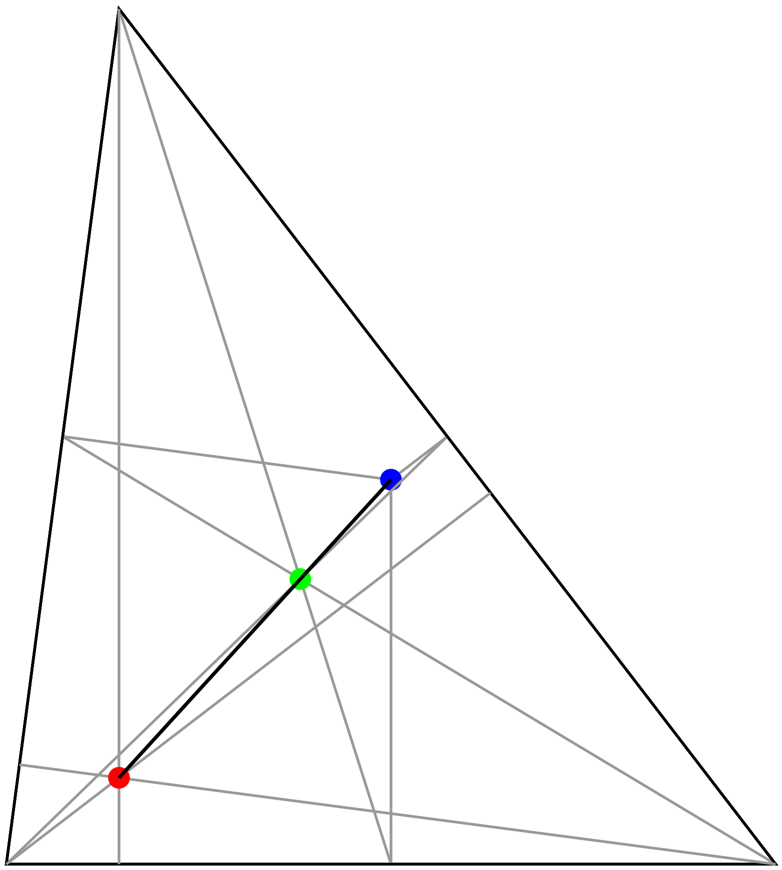
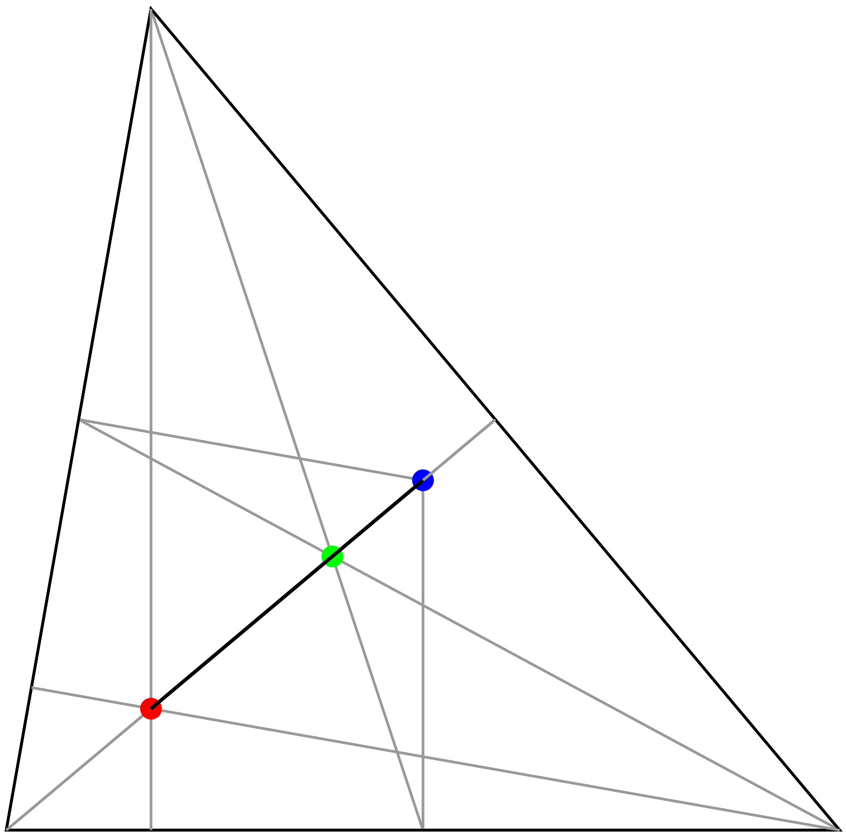
This is the infamous Euler line.



In fact, the Euler line exists in every kind of triangles.



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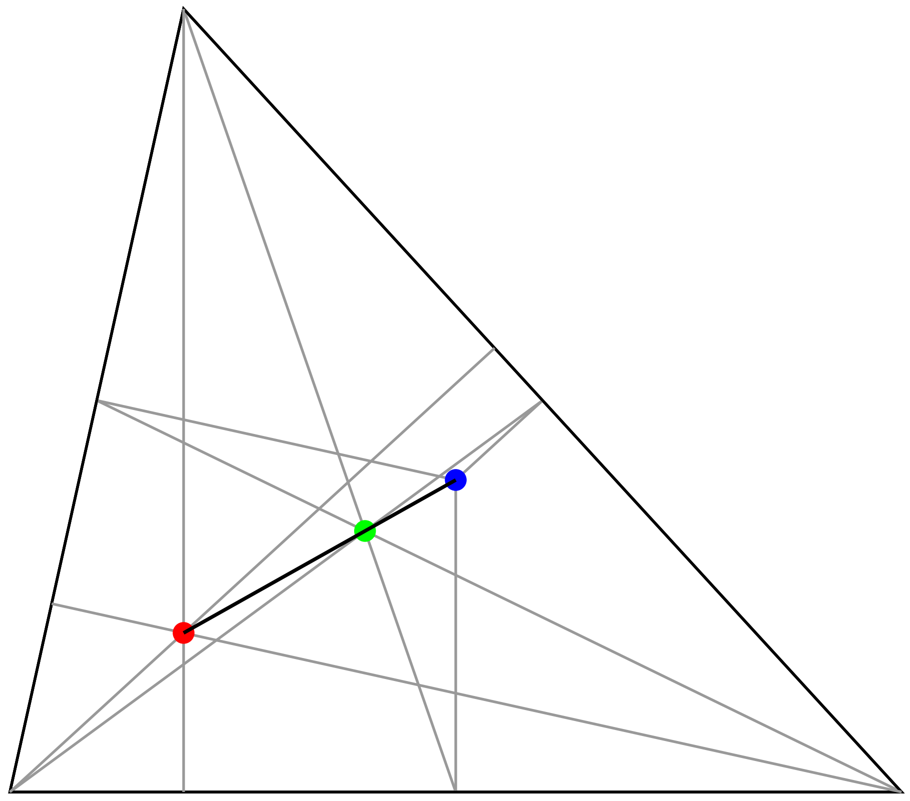

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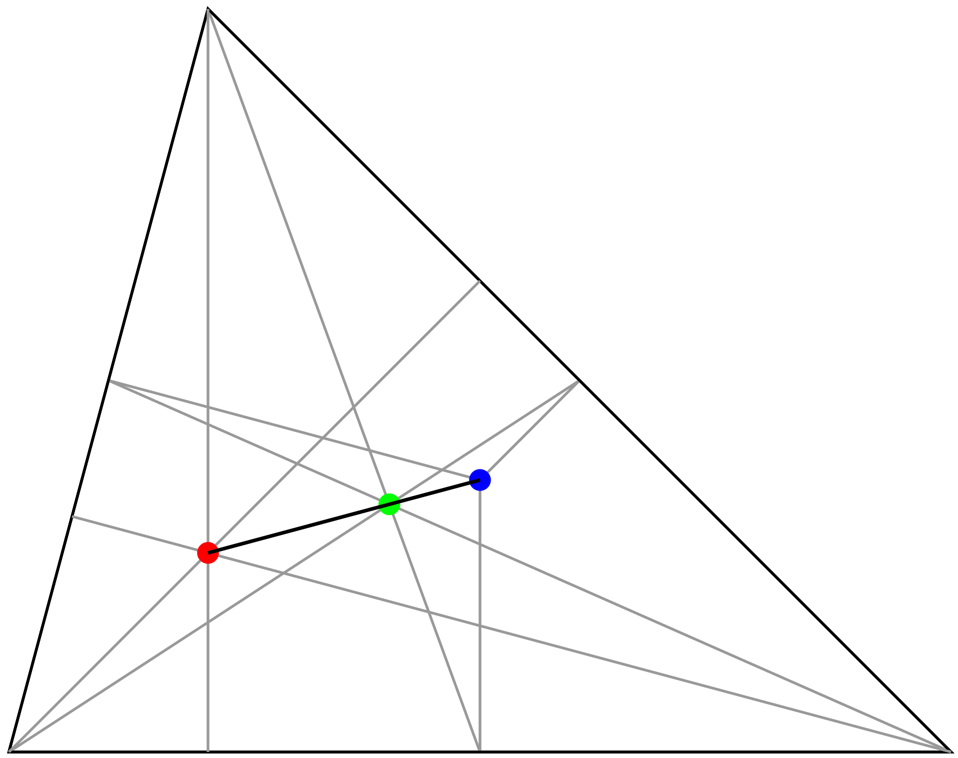

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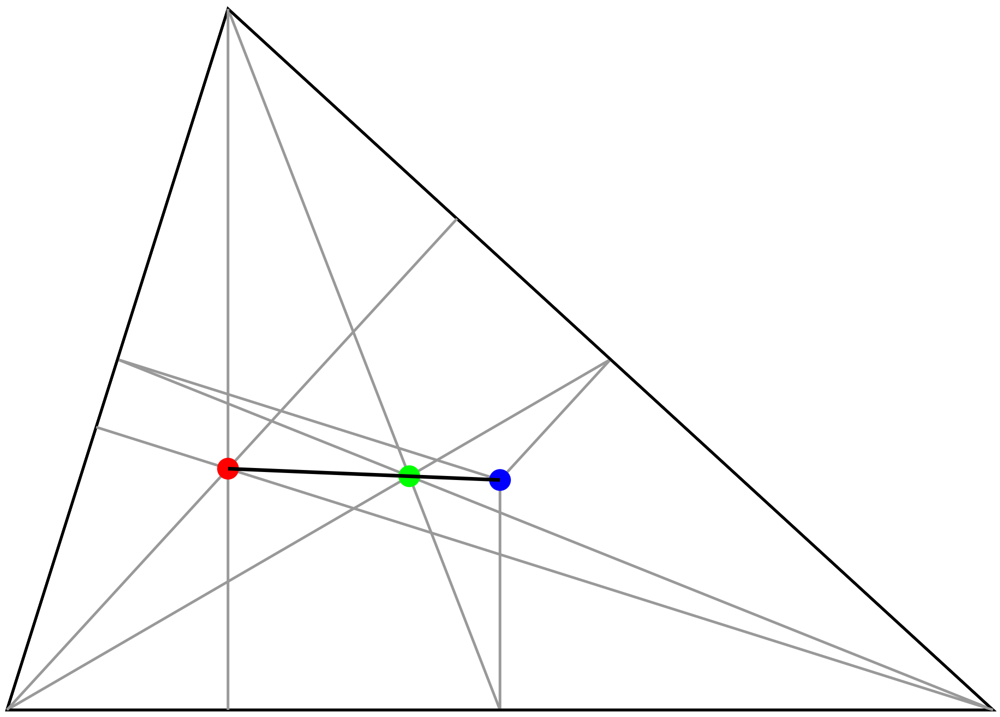

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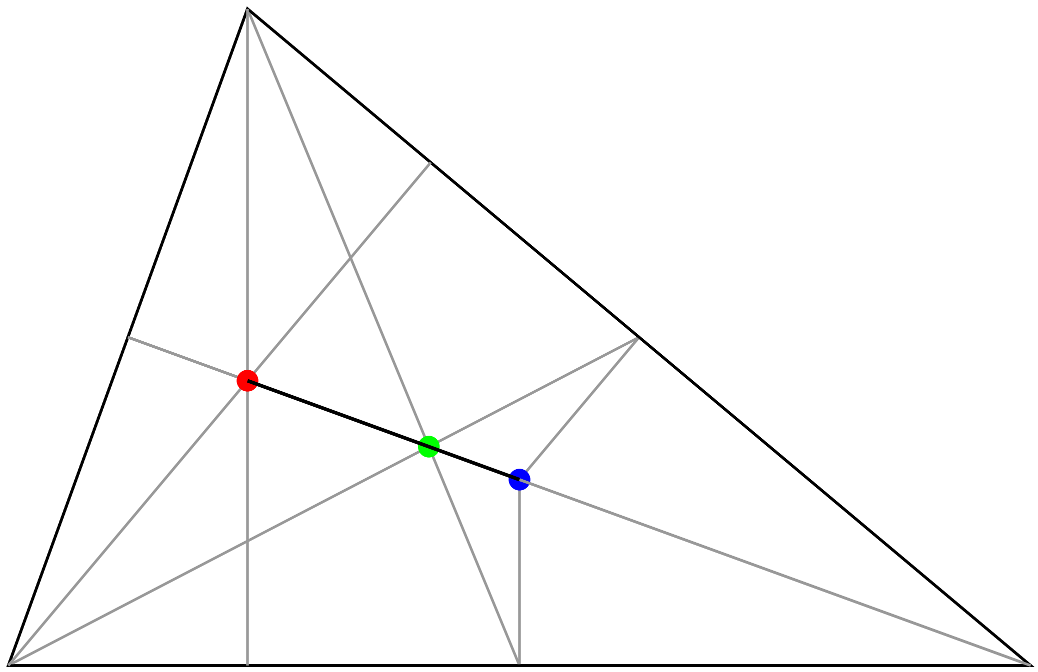
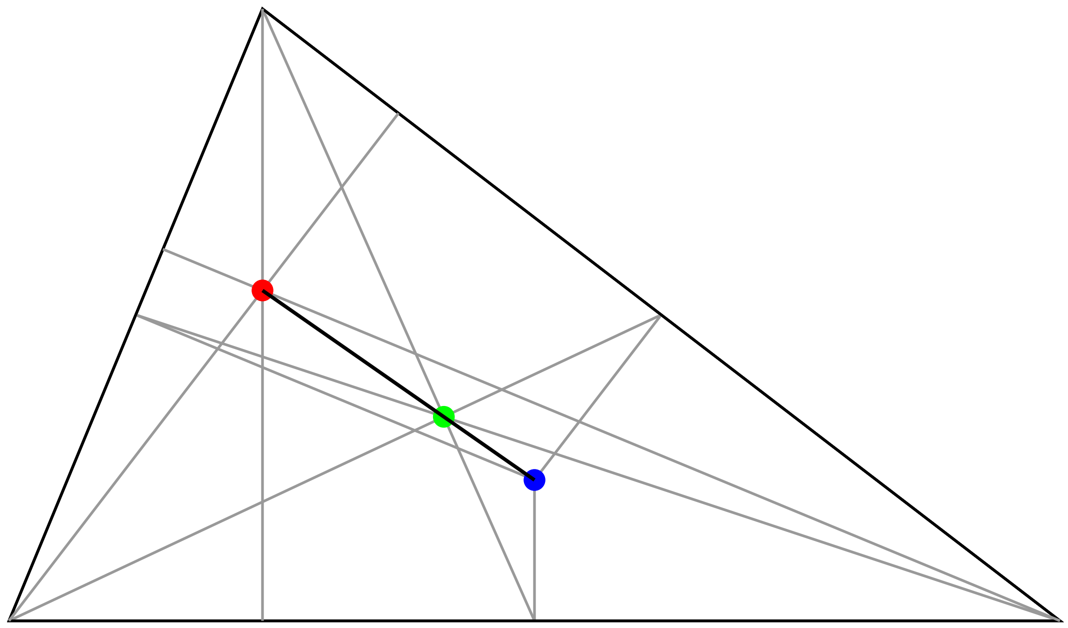


In fact, the Euler line exists in every kind of triangles.



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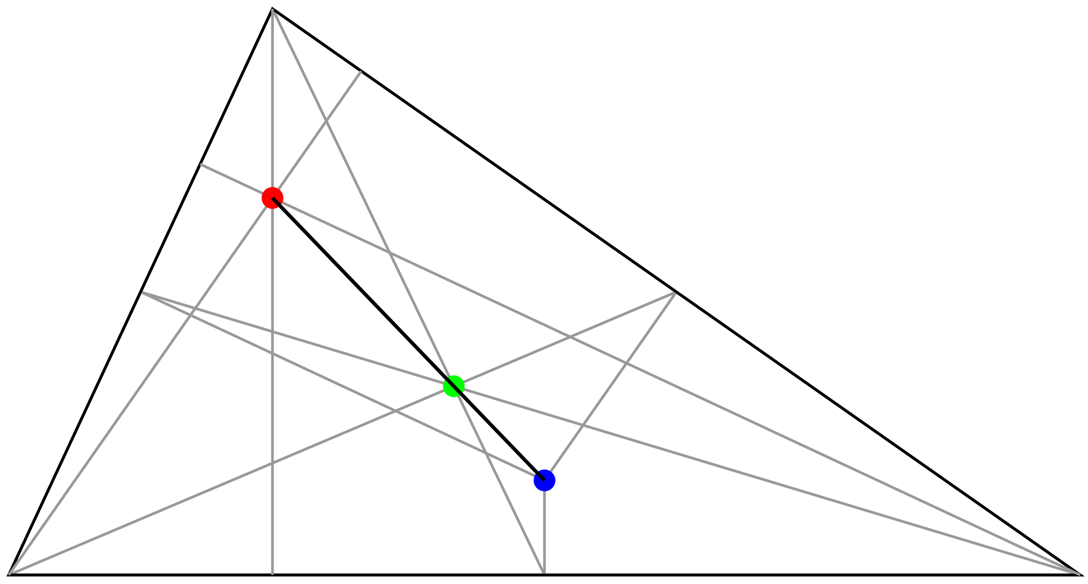

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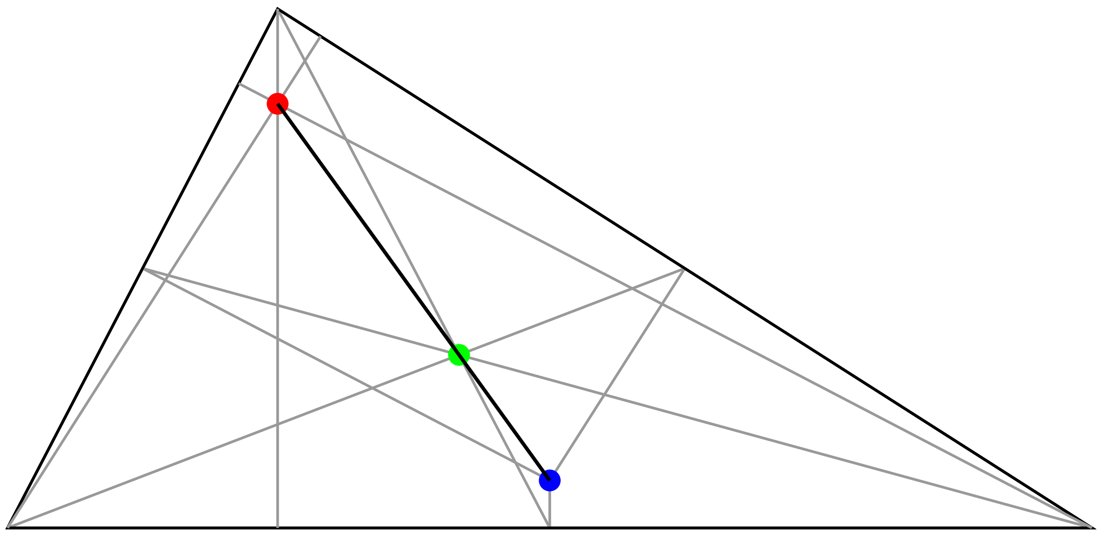

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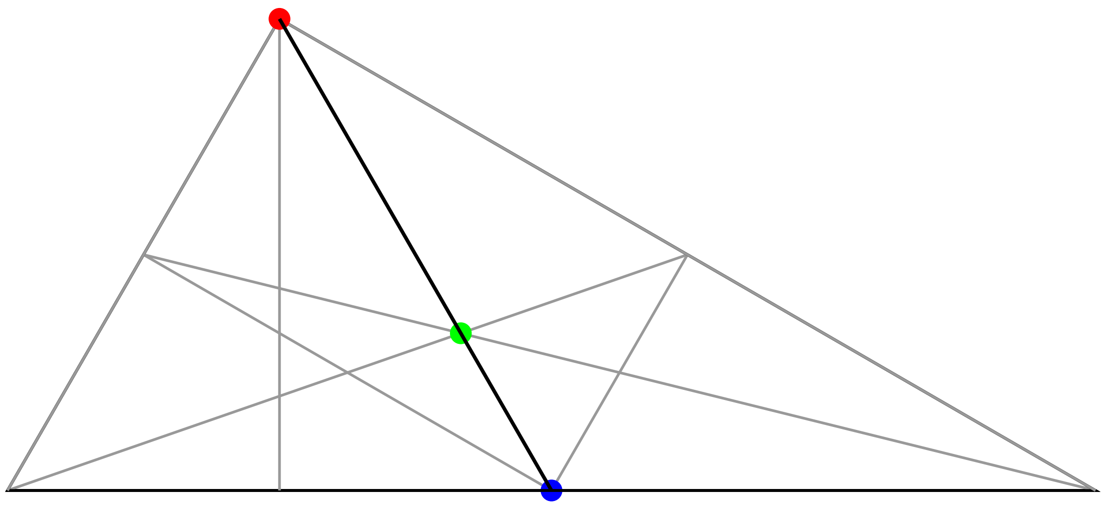

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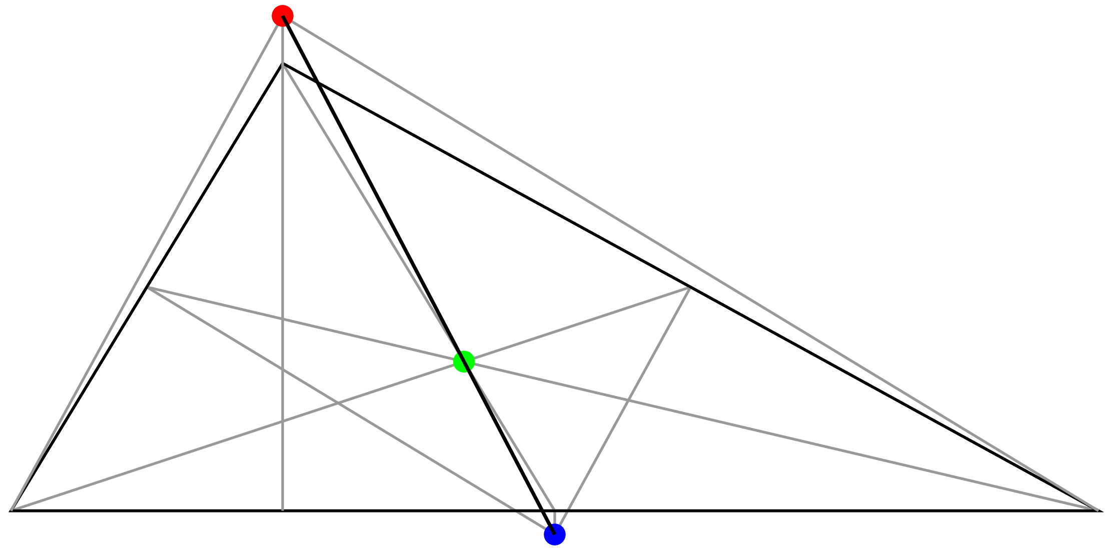

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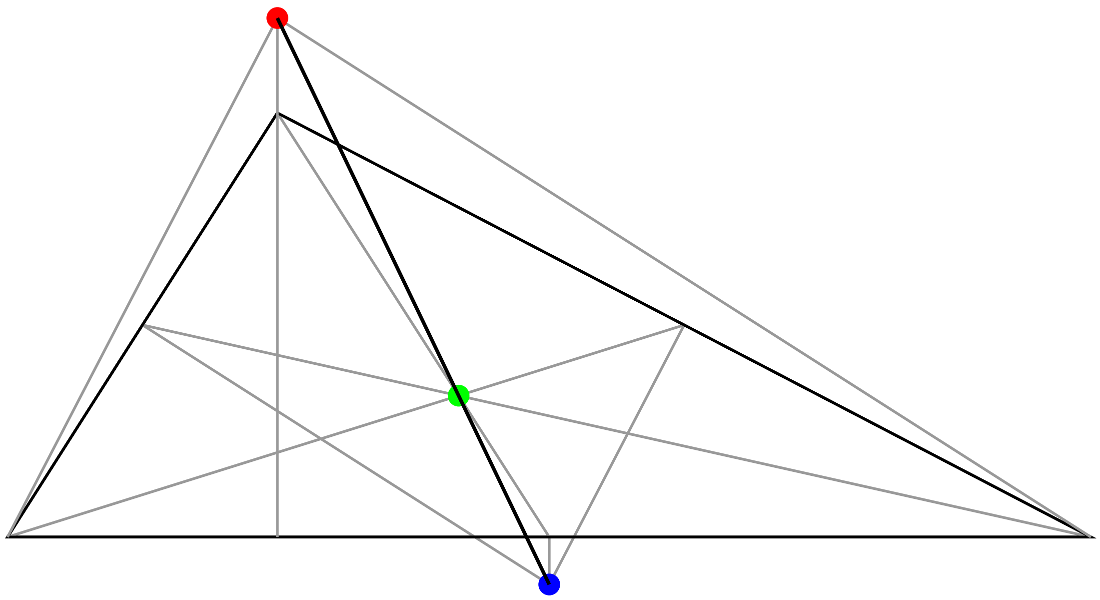

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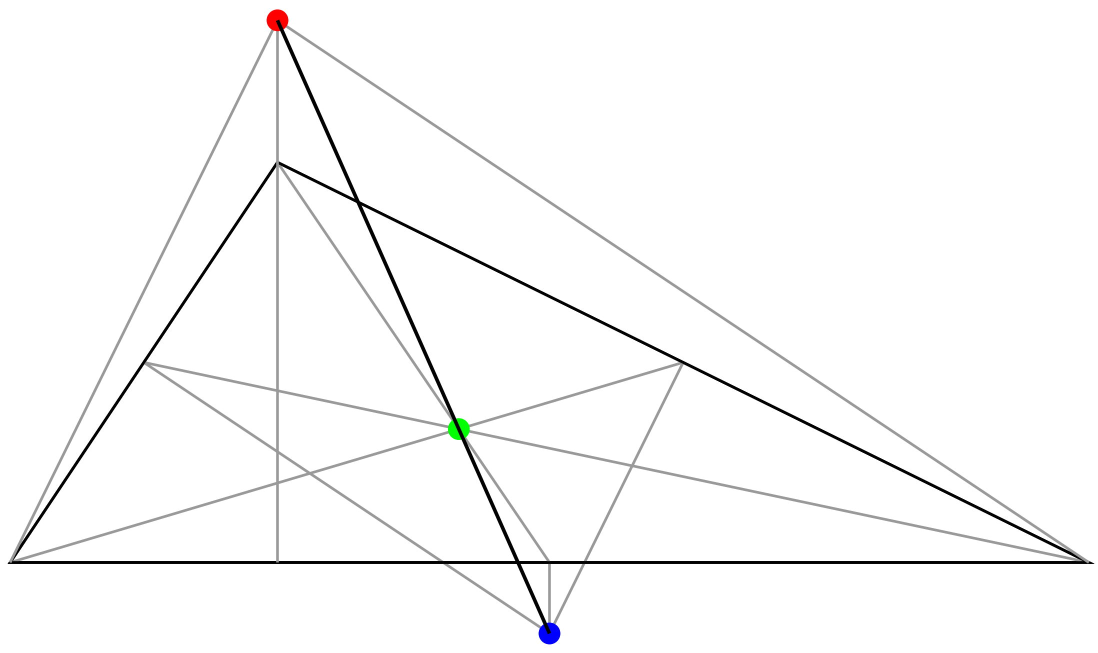

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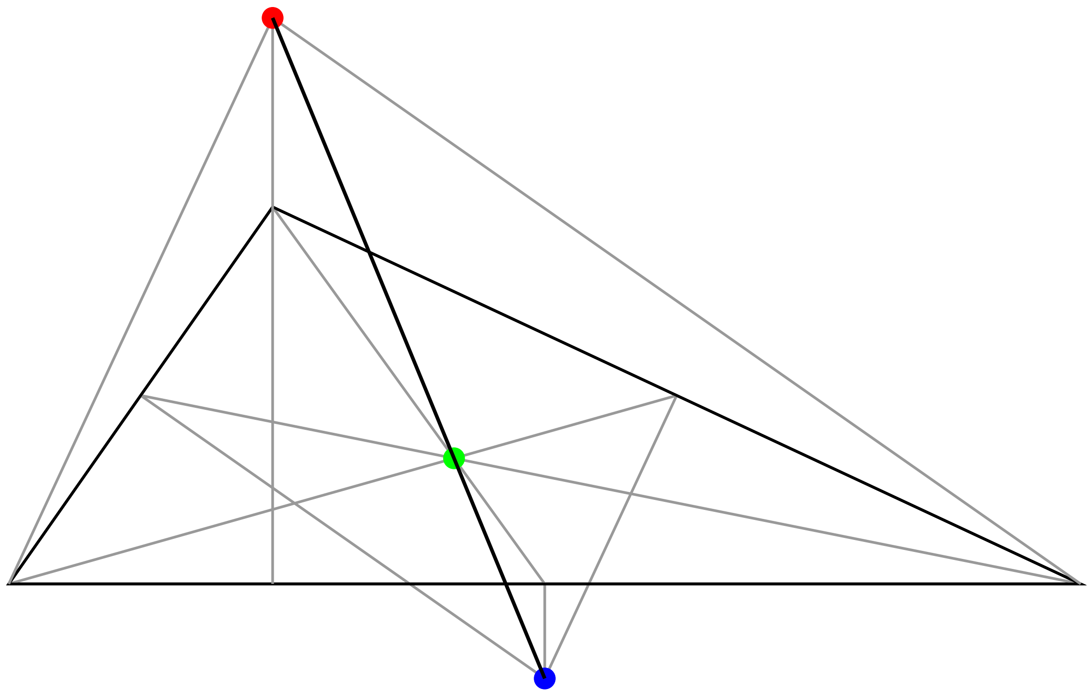

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~ The End ~

Raymond Wan (84605013)