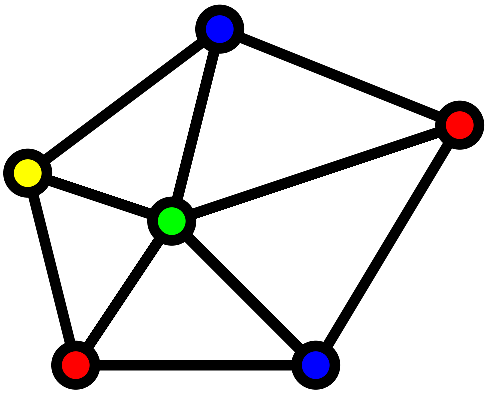
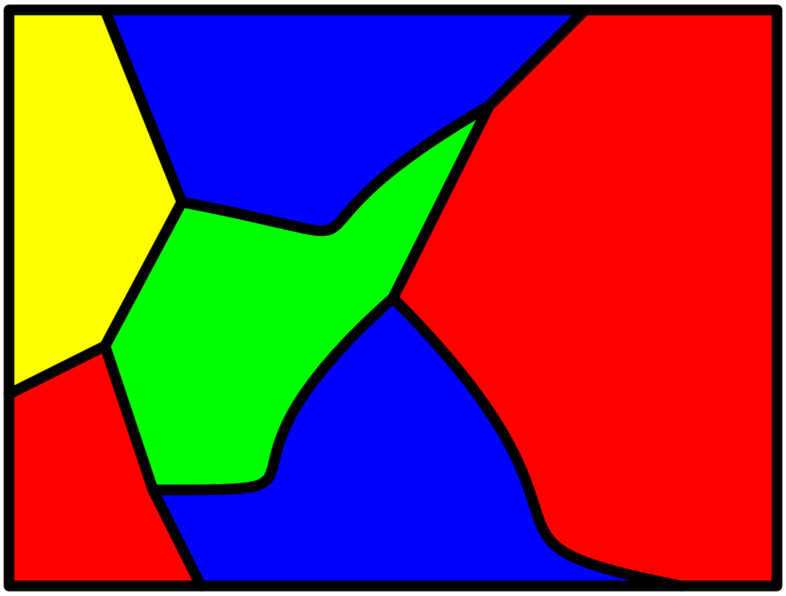
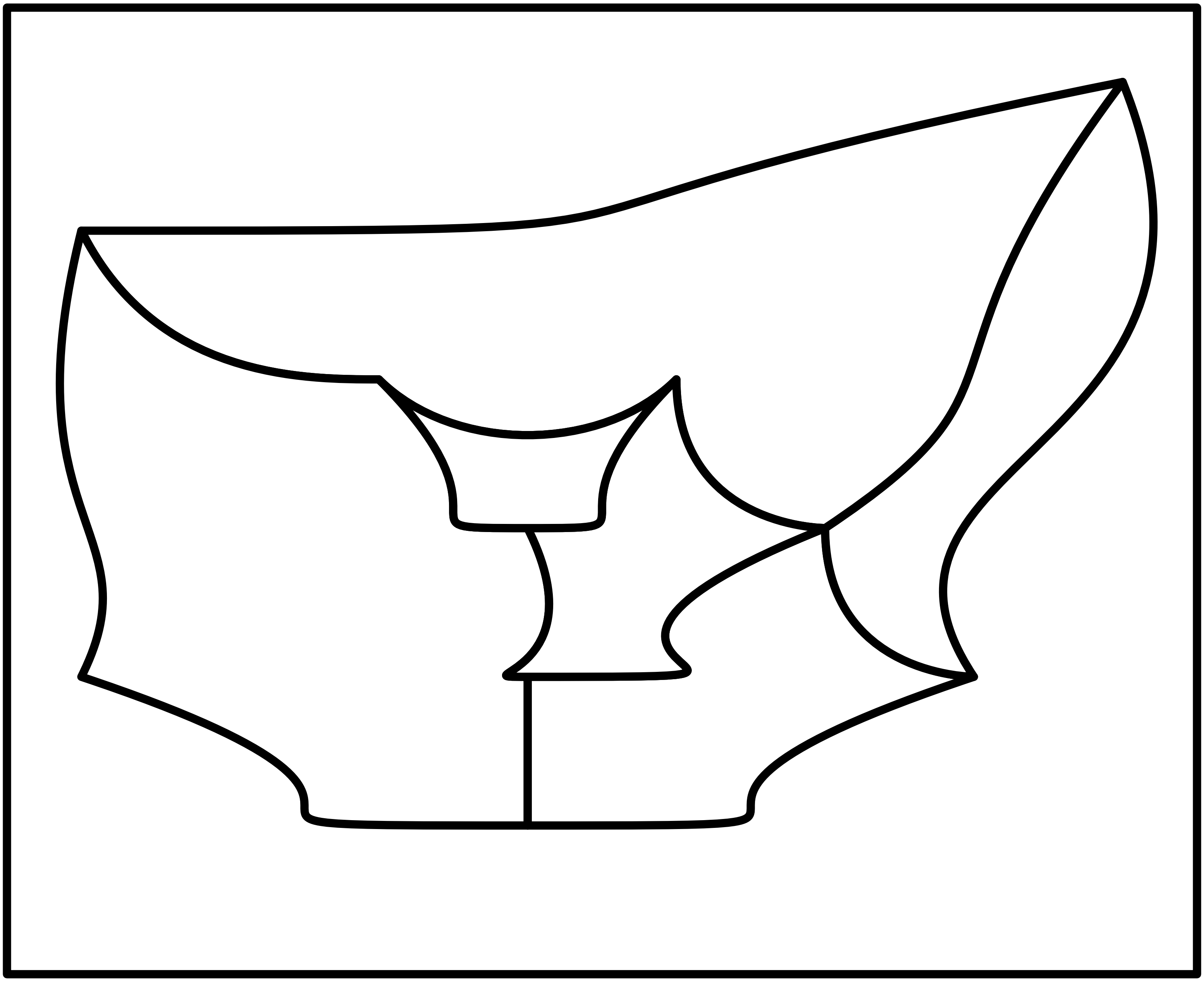
The Four Colour Theorem





Simon Ho
41509027
MATH 308-101
Fall 2005

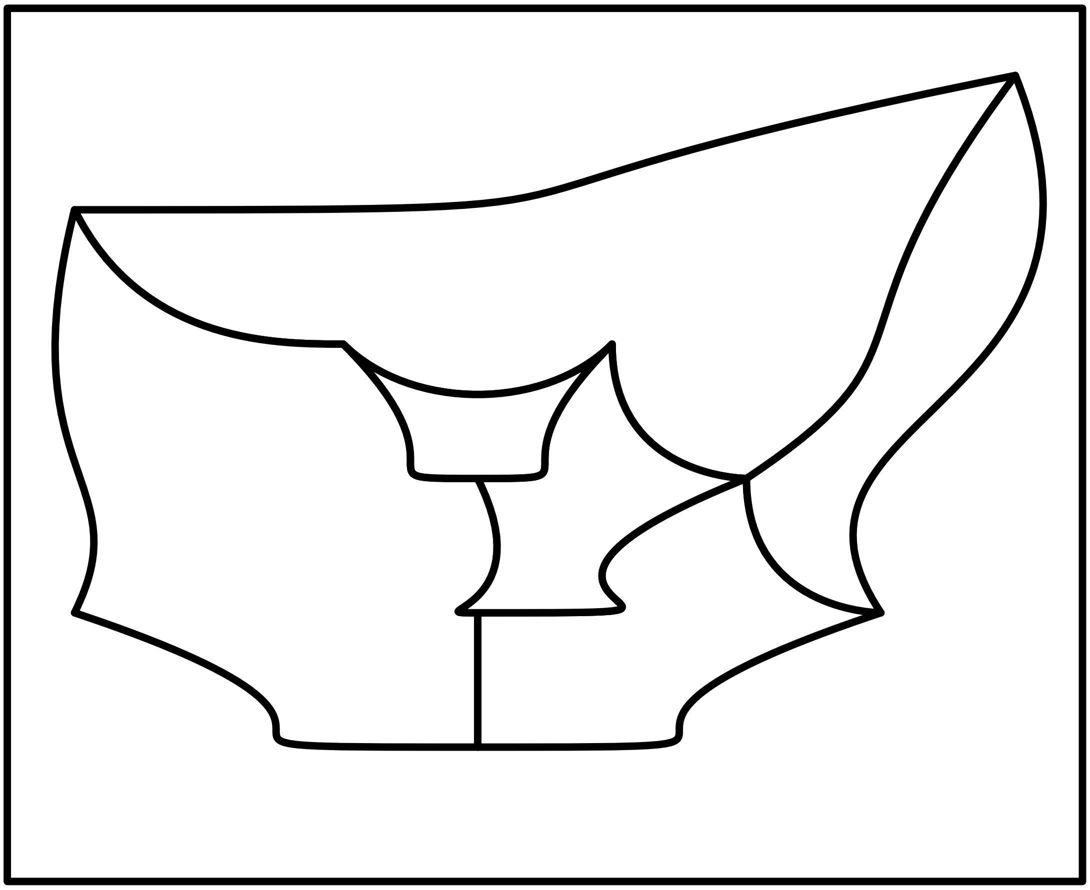


Given a map, what is the minimum number of different colours needed such that one can colour the map with no neighbouring countries having the same colour, where each country is contiguous and neighbouring countries must share a border and not just a point in common?

Theorem:

Not every map can be coloured using three different colours such that no neighbouring countries have the same colour.

Proof:



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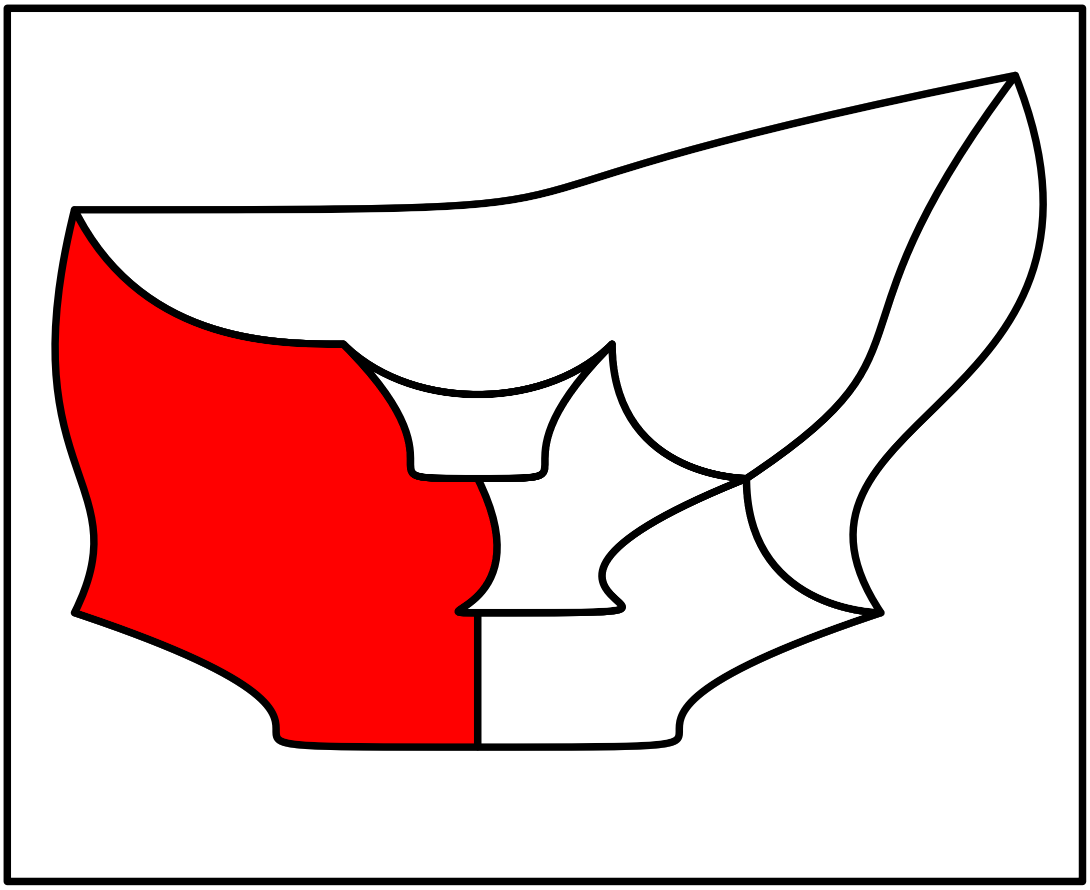

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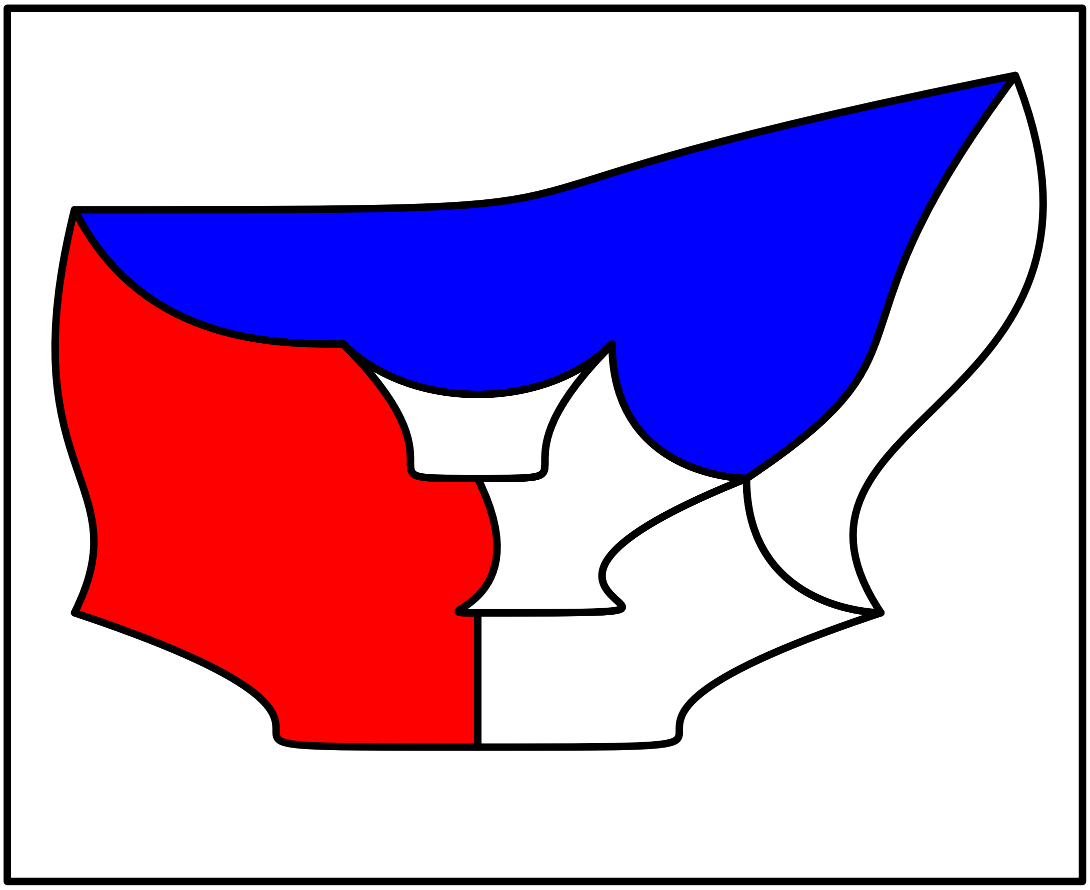

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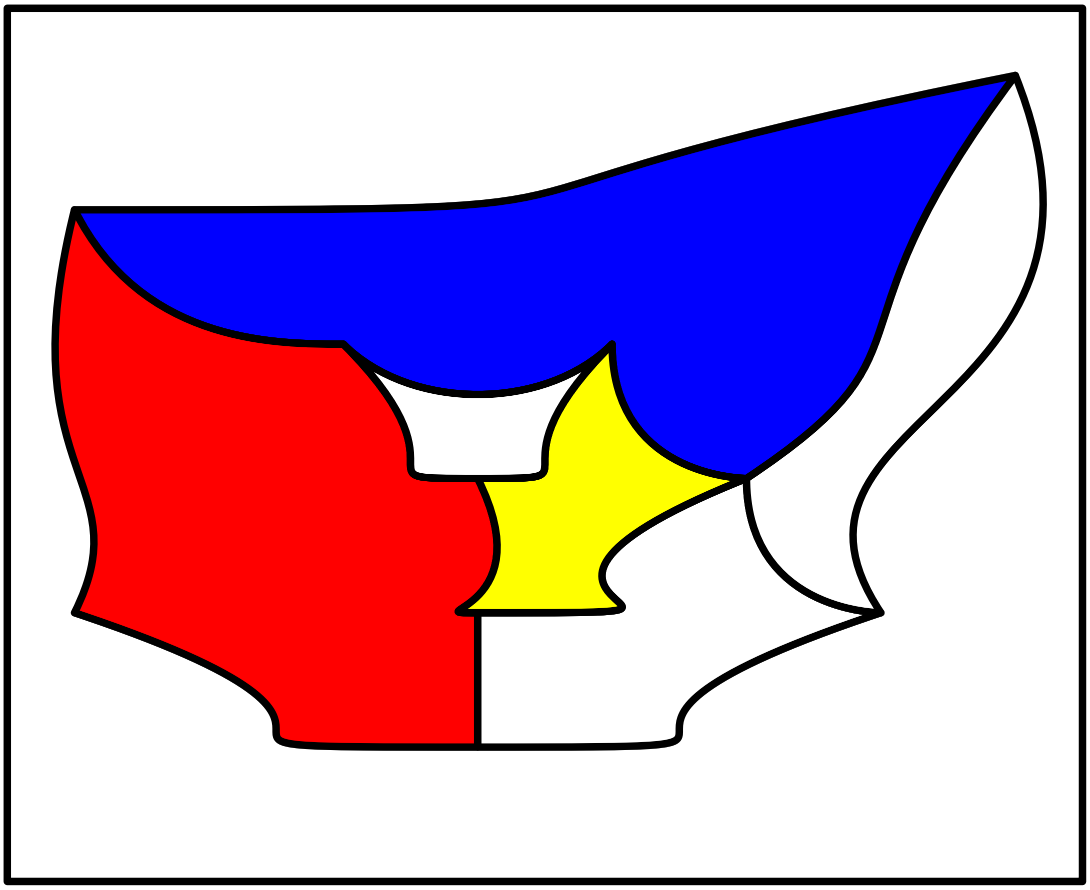

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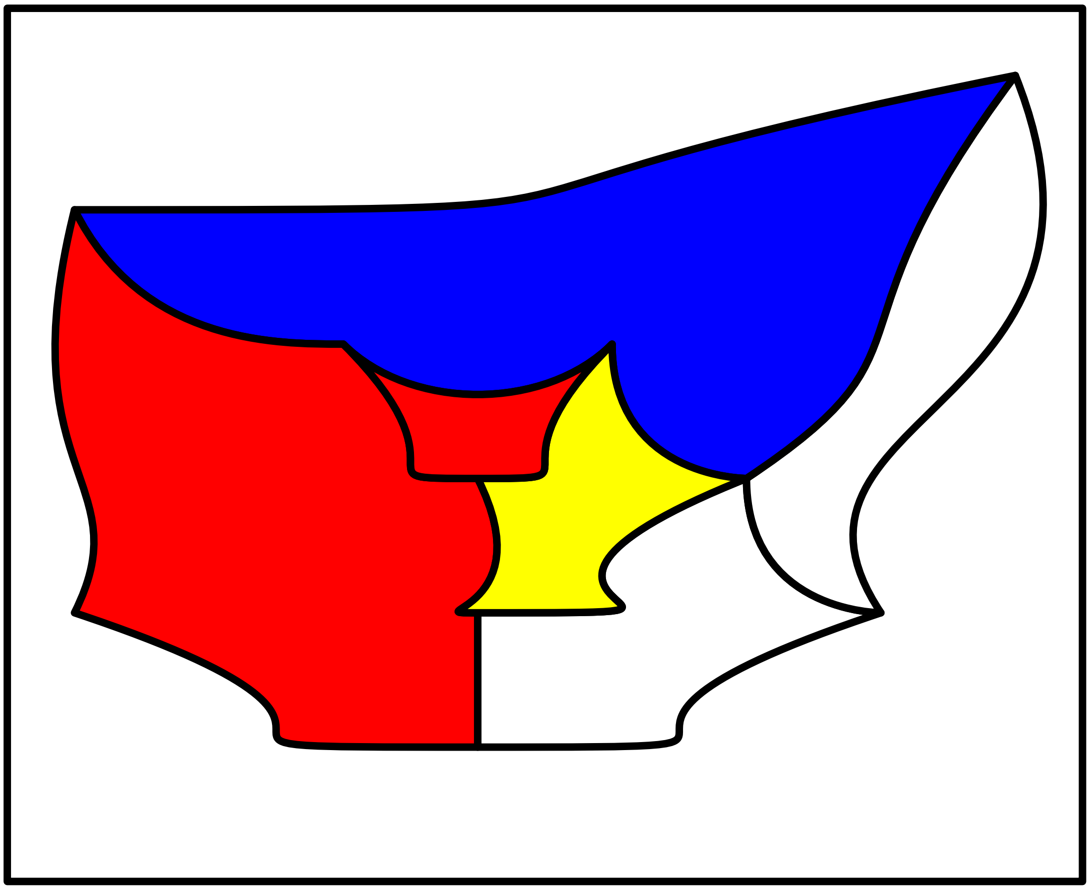

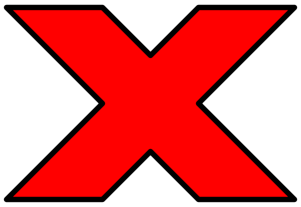

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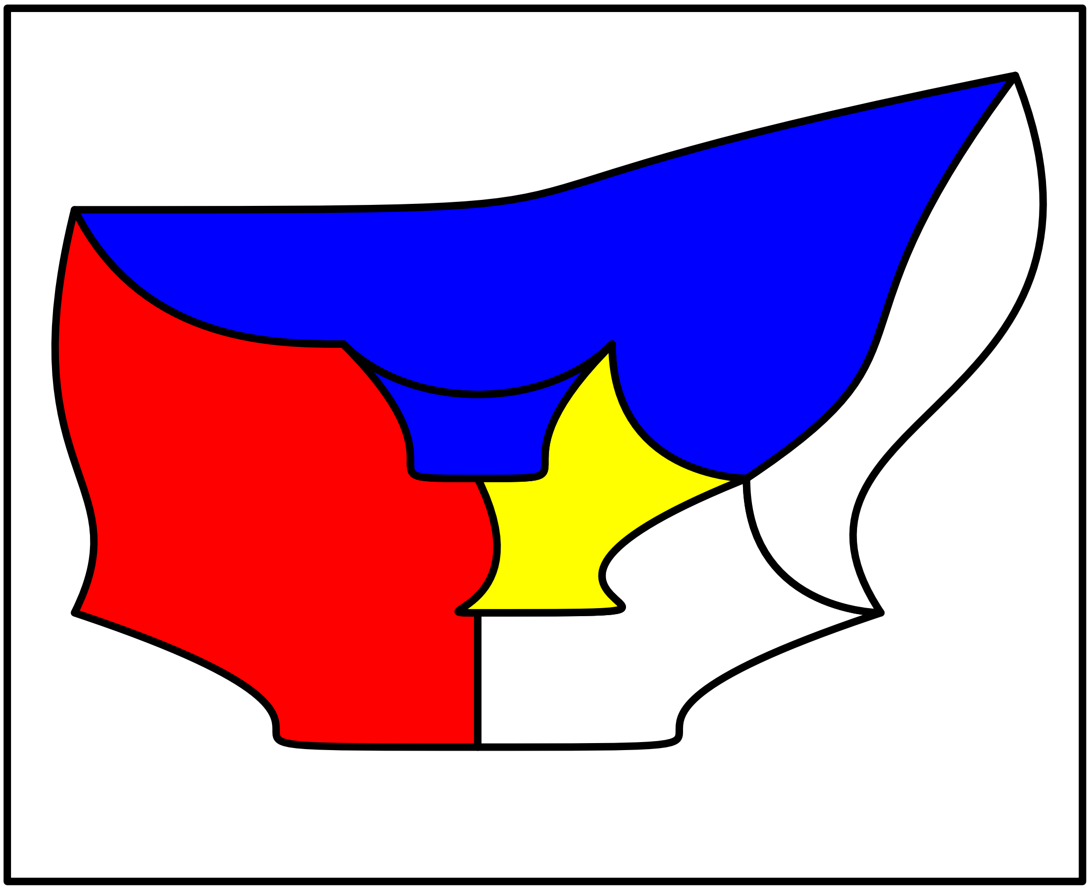

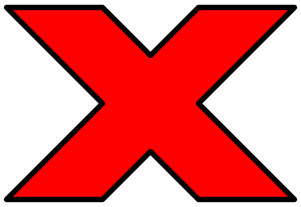

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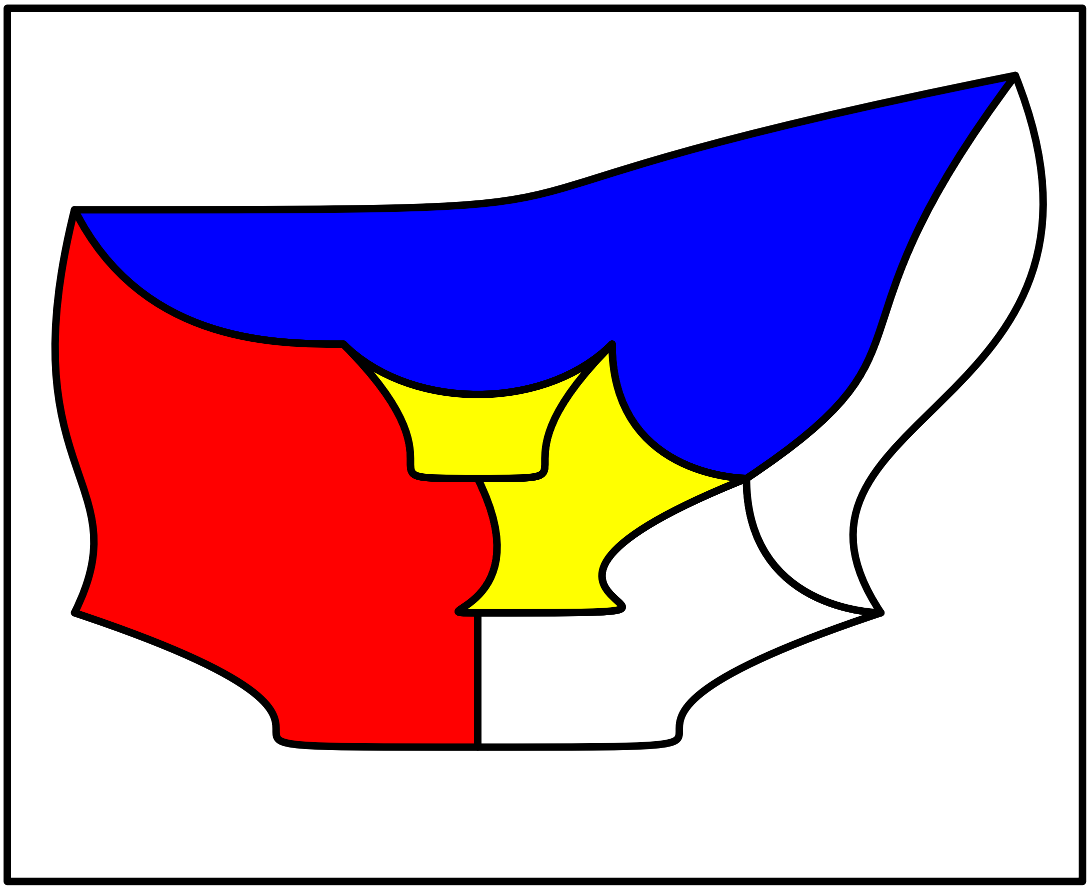

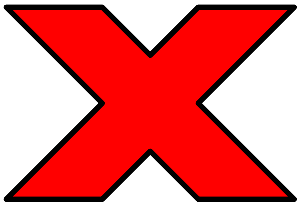

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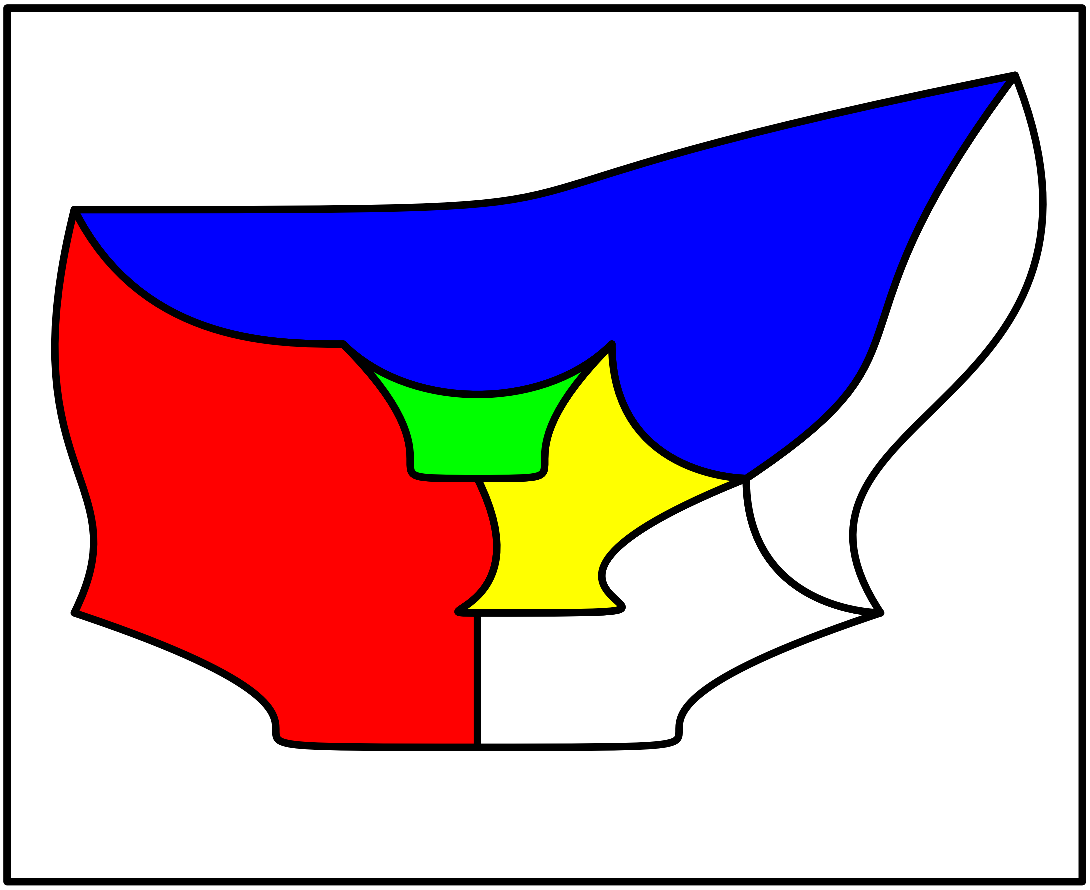

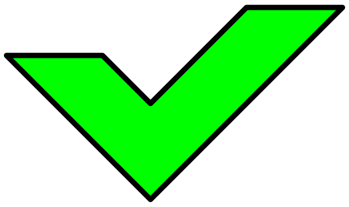

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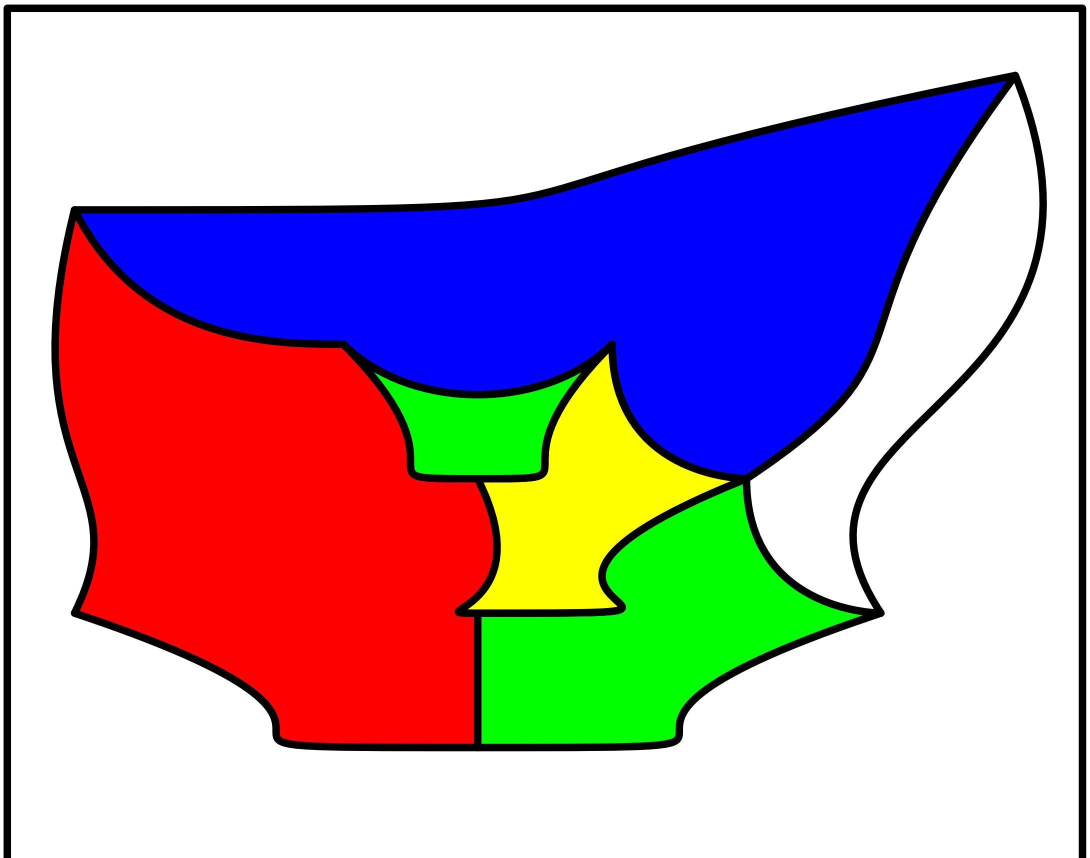

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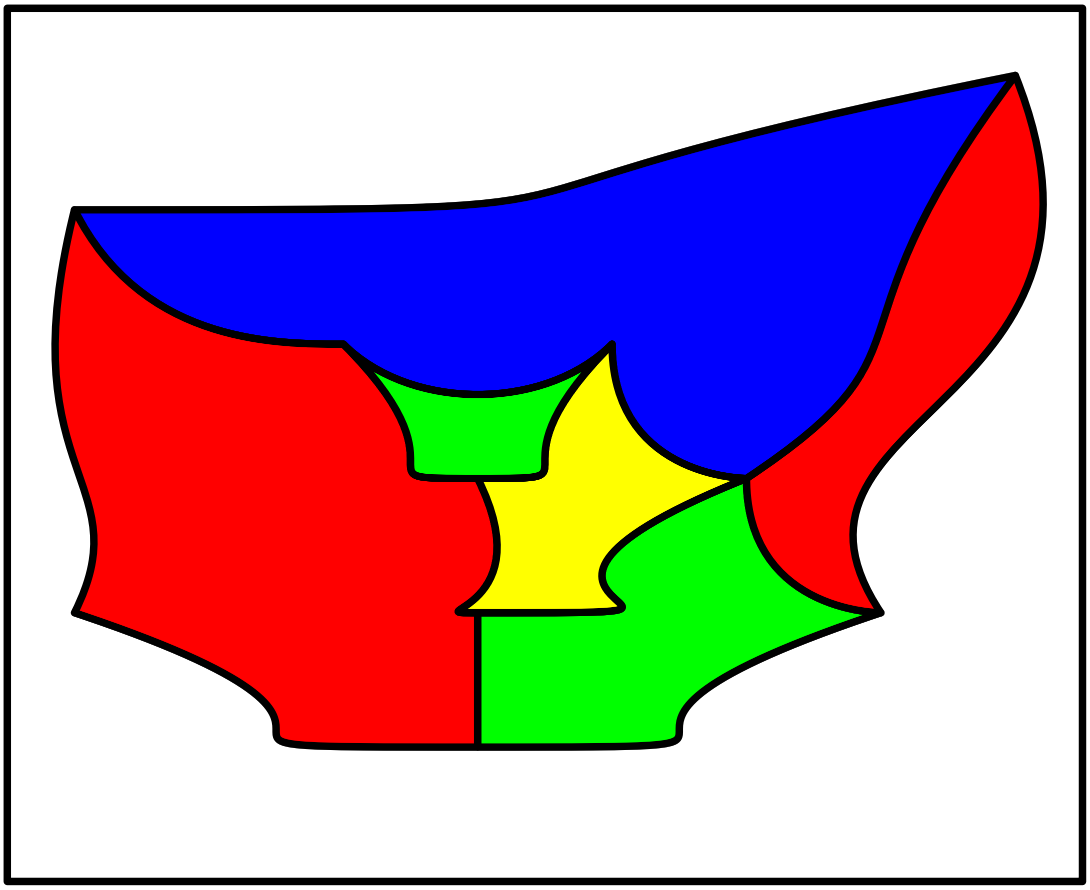
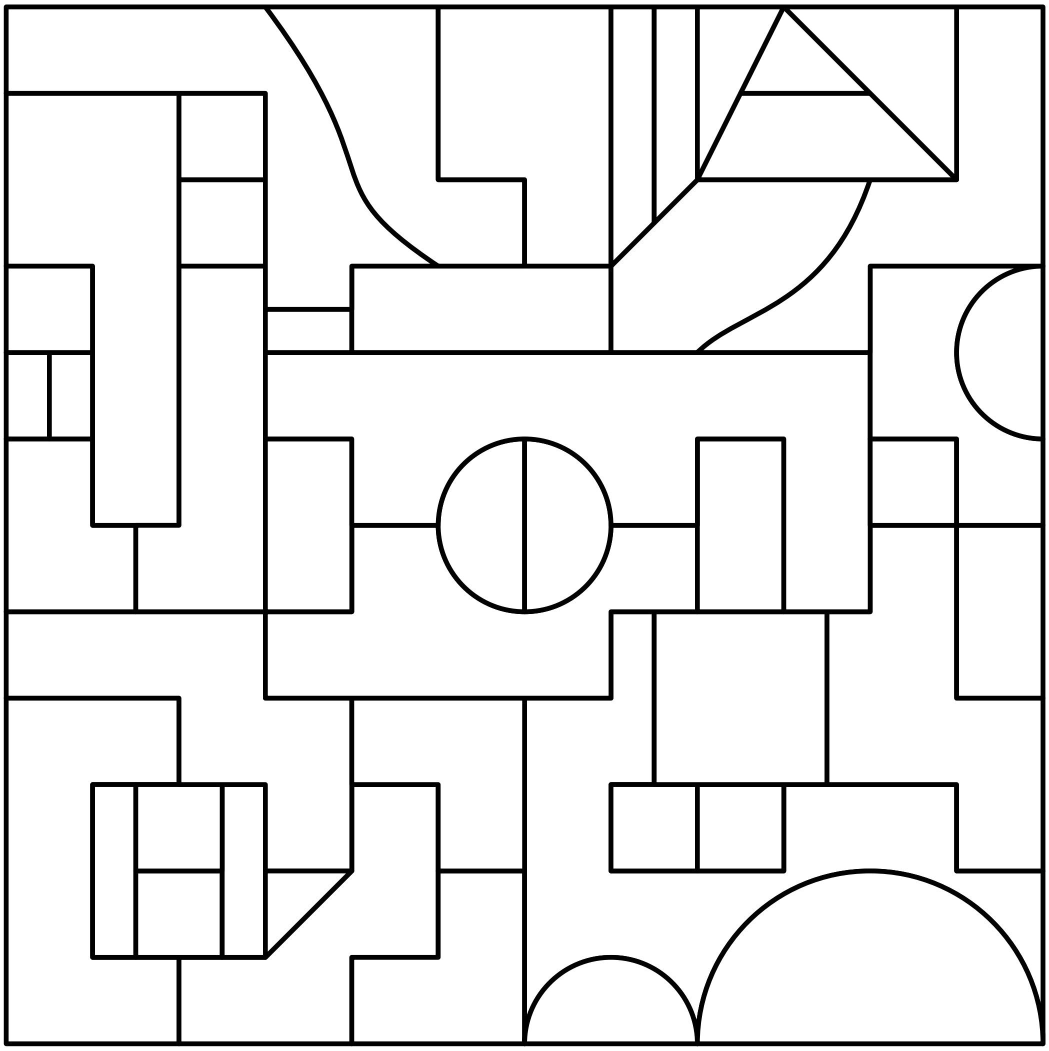
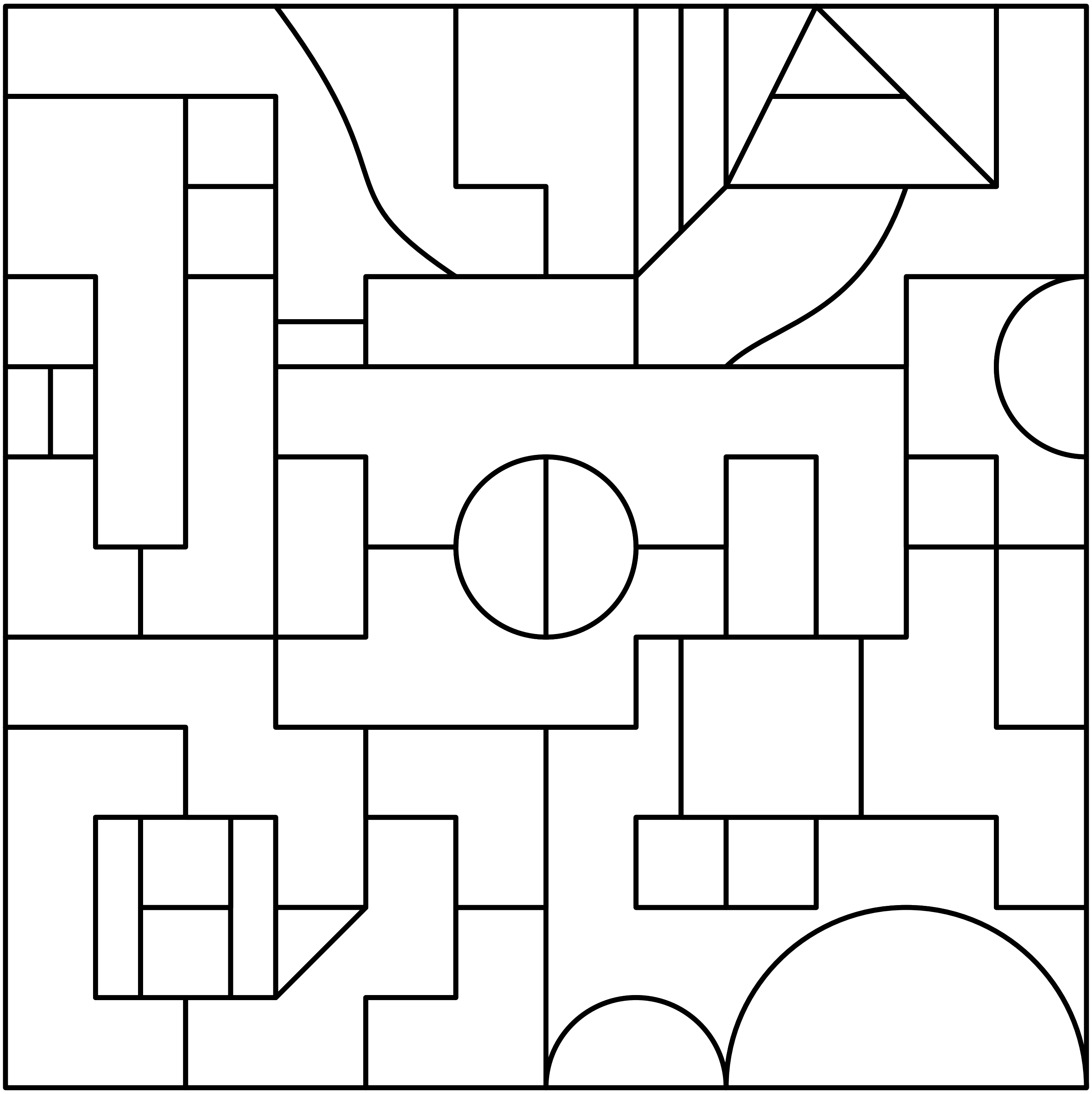


So, if using three colours is not enough, then using how many colours is enough?

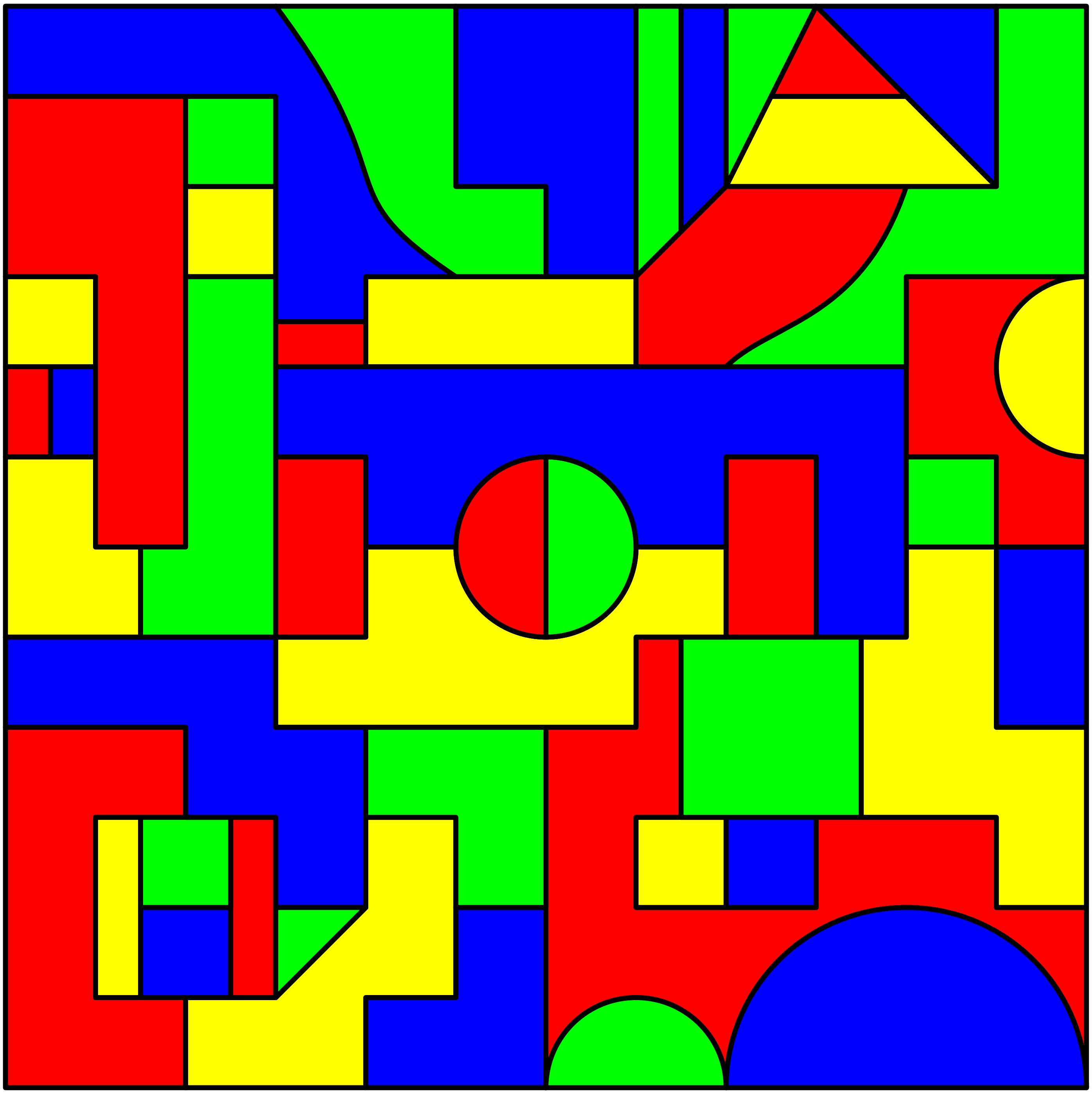
One may think that if one is given a very complicated map, then many colours are needed.

For example, consider this map.





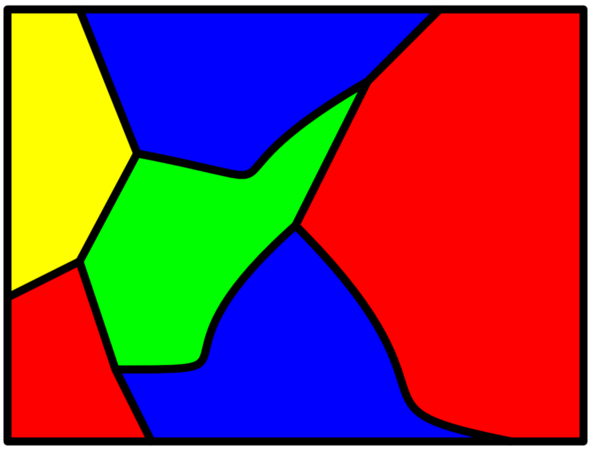
Are many colours needed to colour this map?

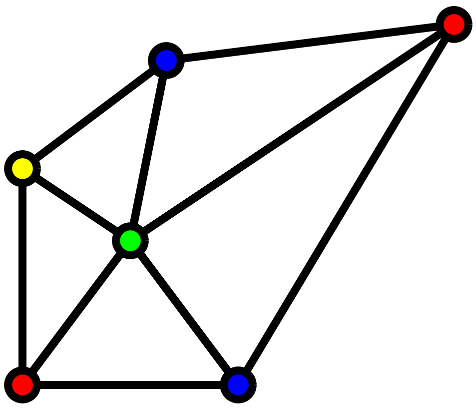


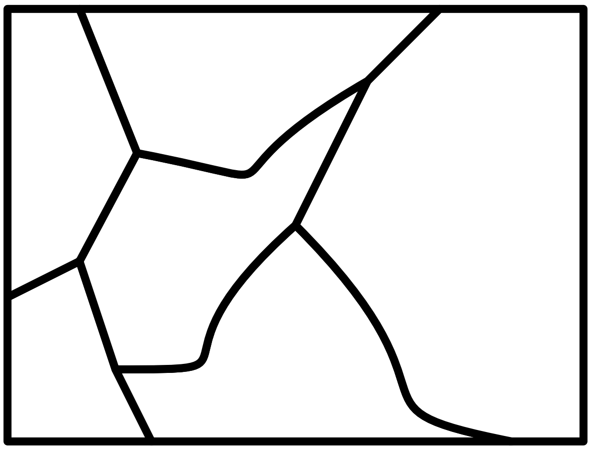
No, using four colours is enough to colour this map.

In fact, in 1852, Francis Guthrie conjectured that using four colours is enough to colour any map such that no neighbouring countries have the same colour.

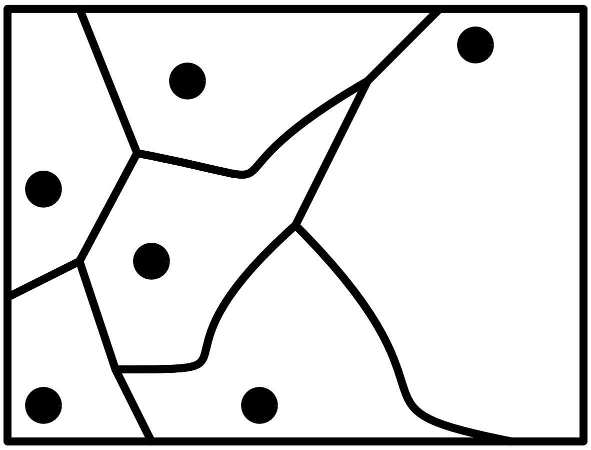
Formulating this statement in terms of graph theory, the vertices of every planar graph can be coloured using at most four colours such that no adjacent vertices have the same colour.







To convert a map into a graph, place a vertex on each region or country of the map.



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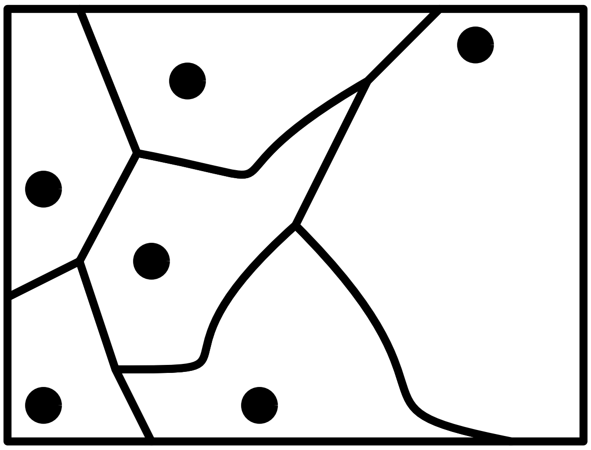
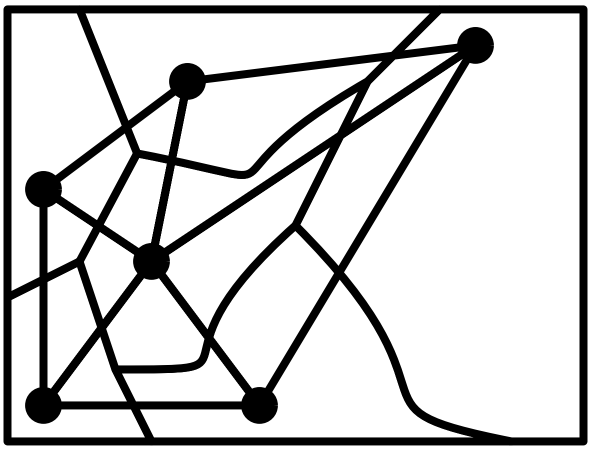


Then, join two vertices by an edge if and only if their corresponding regions or countries share a border and not just a point.



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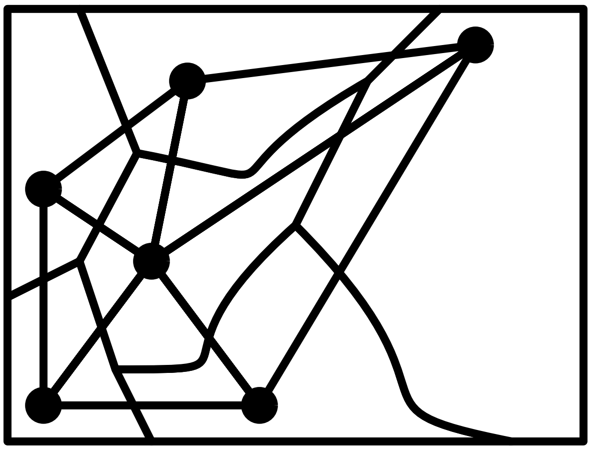
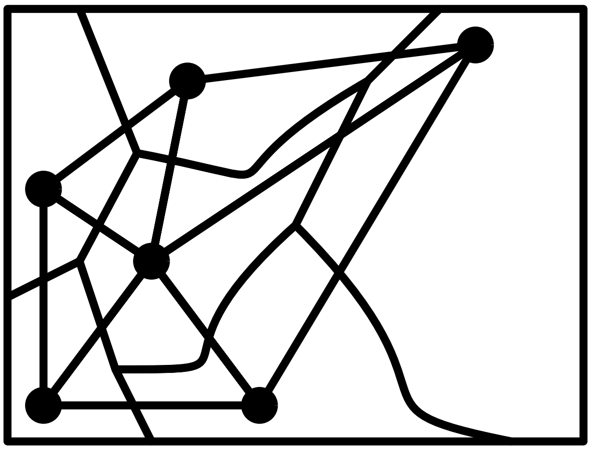
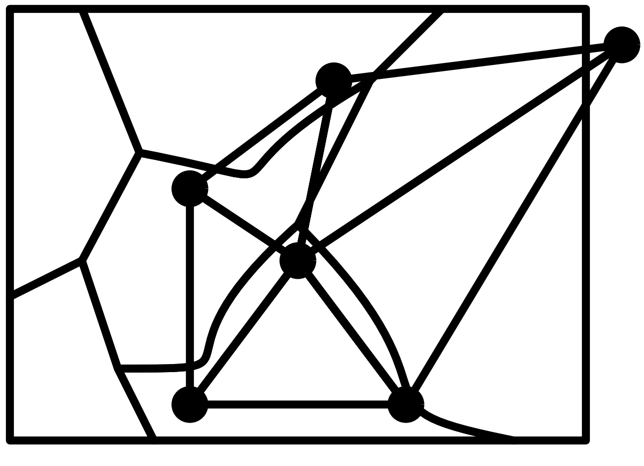
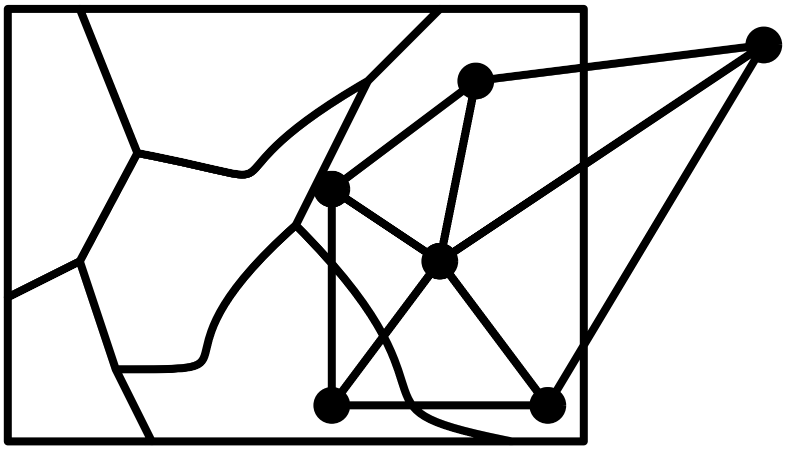
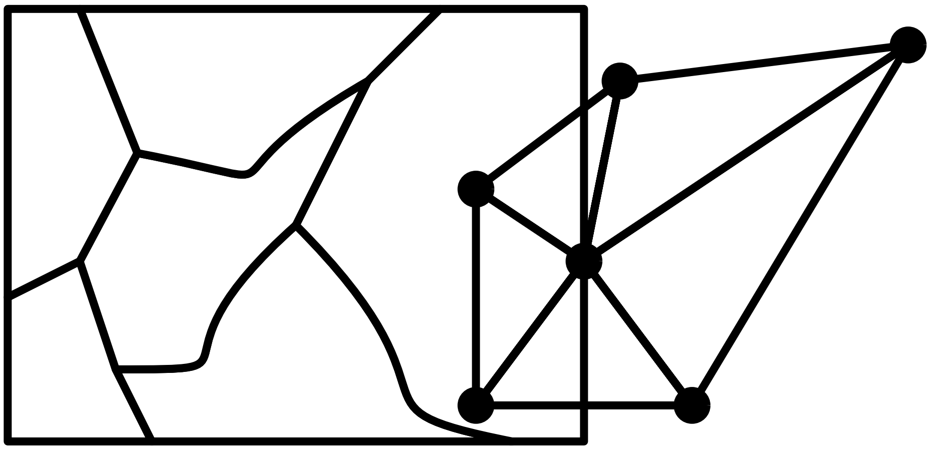


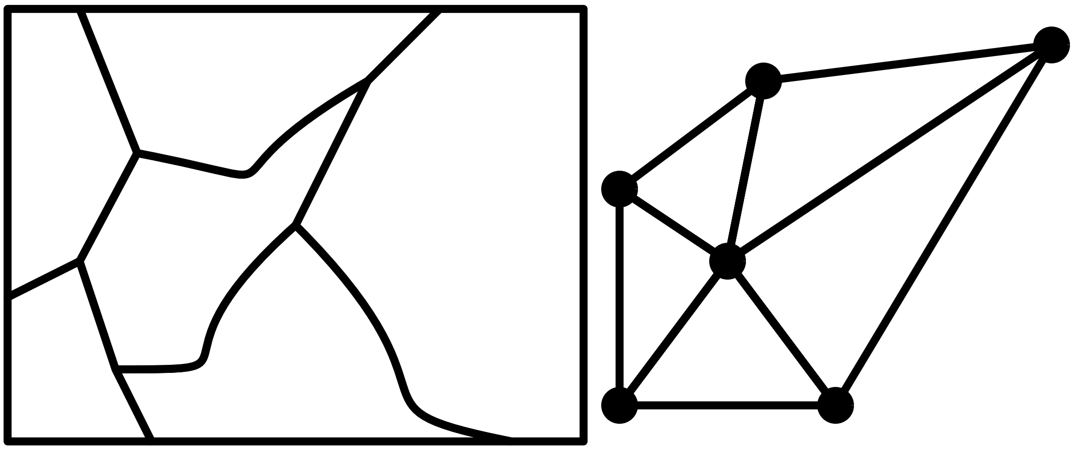
This results in a planar graph, that is, a graph where no edges cross each other.

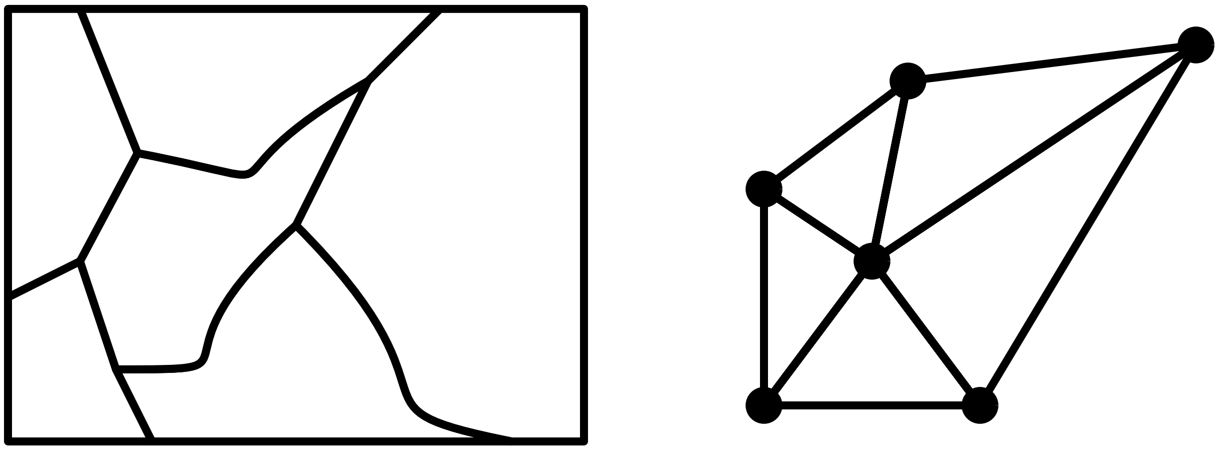


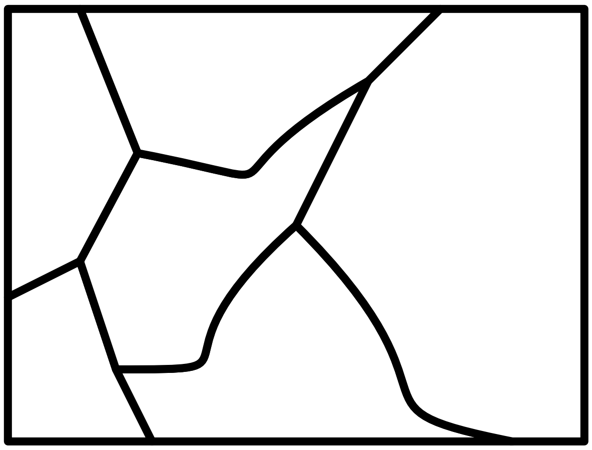


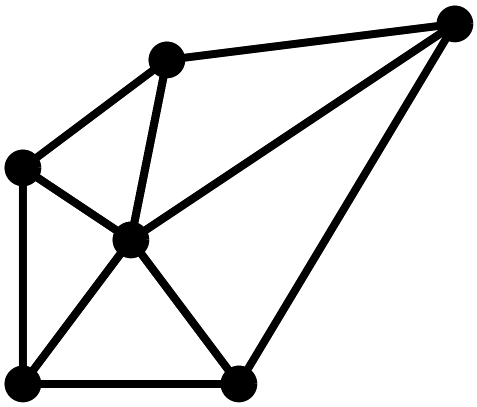


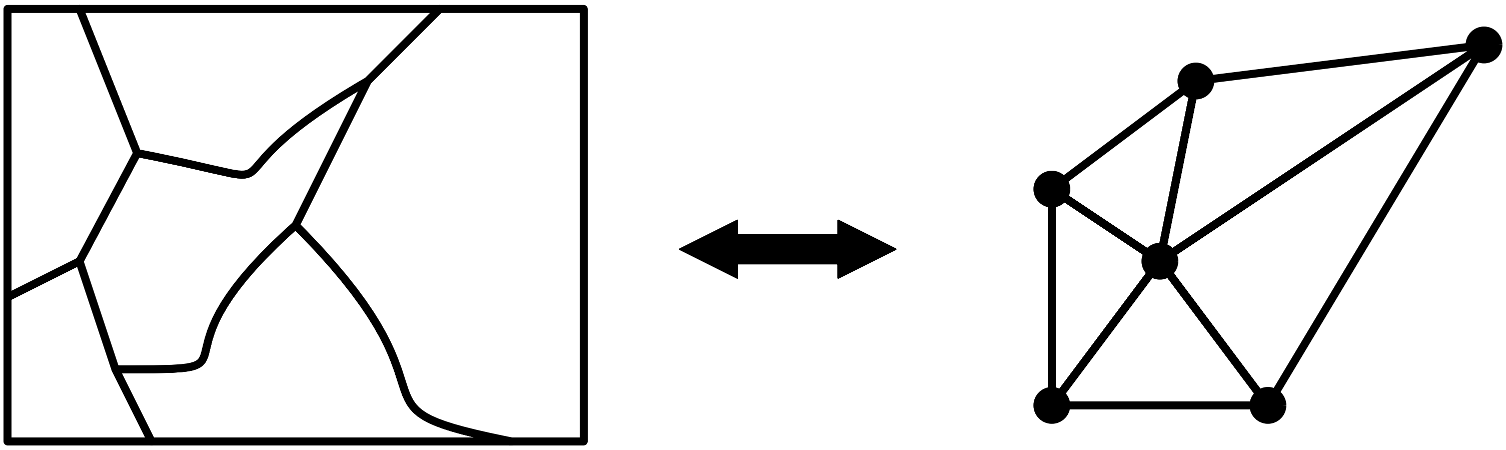


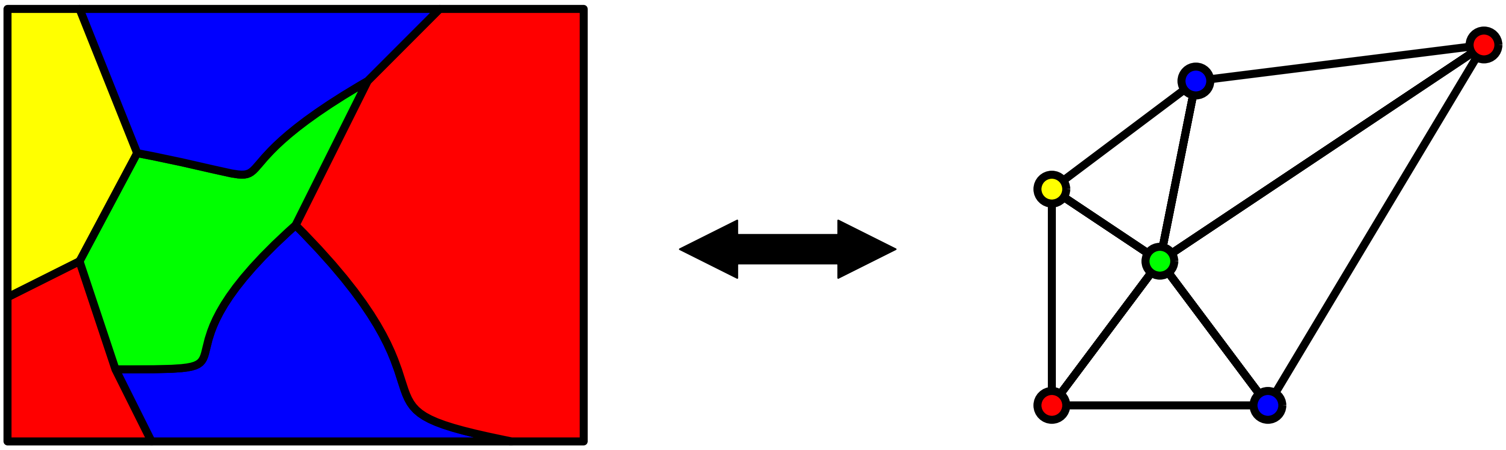












Theorem:

Every planar graph has at least one vertex of degree at most five.

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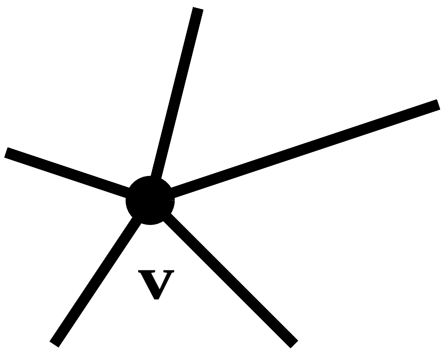
Definition: The degree of a vertex is the number of edges incident to it.

e.g.



The degree of vertex v is 5.
This is denoted deg(v) = 5.

Theorem:

Every planar graph has at least one vertex of degree at most five.

Proof:

Let there be a planar graph with v vertices.

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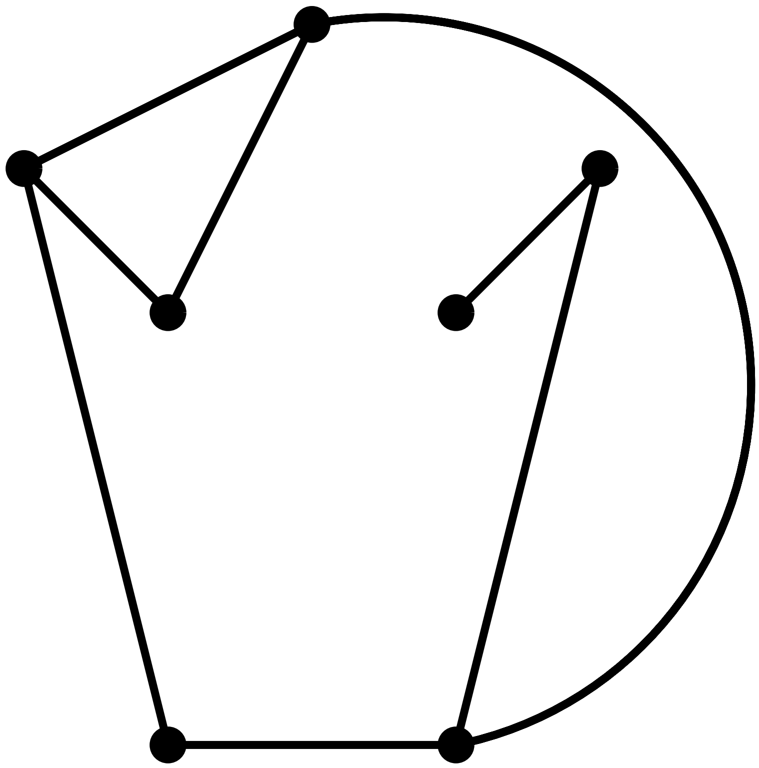

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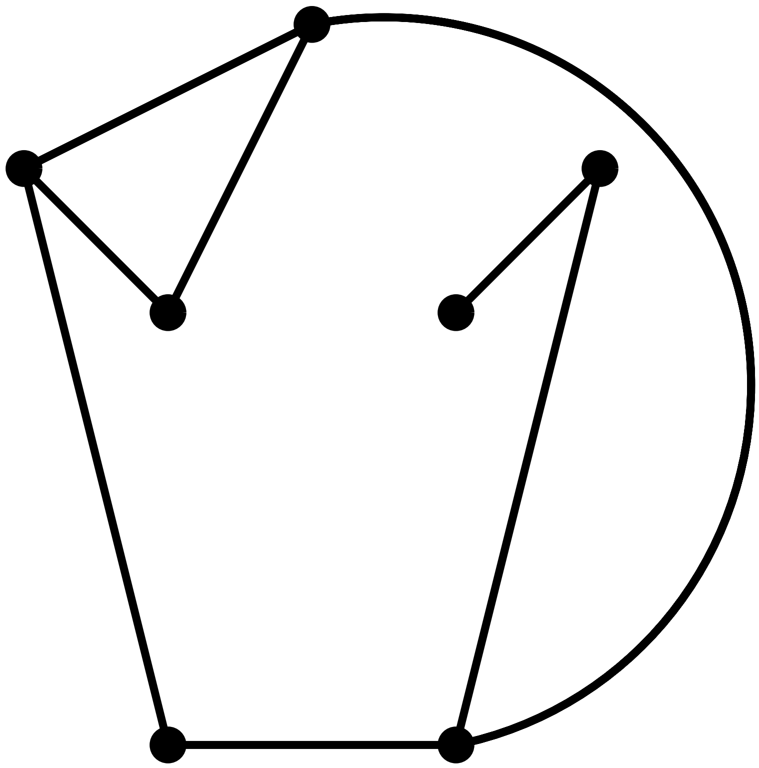
Proof:

Now, join non-adjacent vertices together such that the graph remains planar.

This produces a maximal planar graph.



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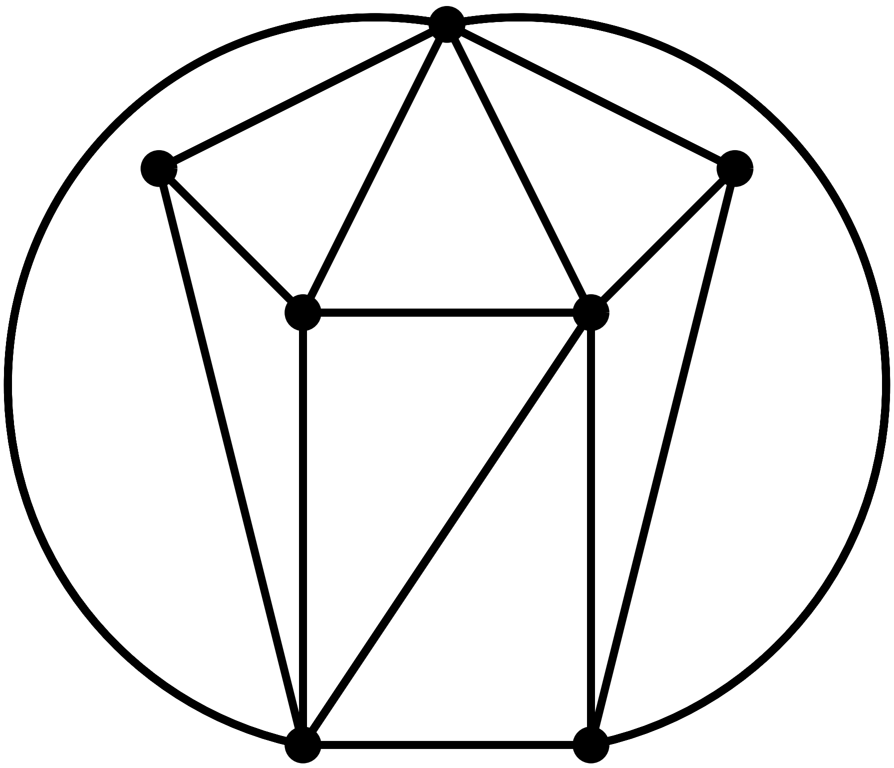
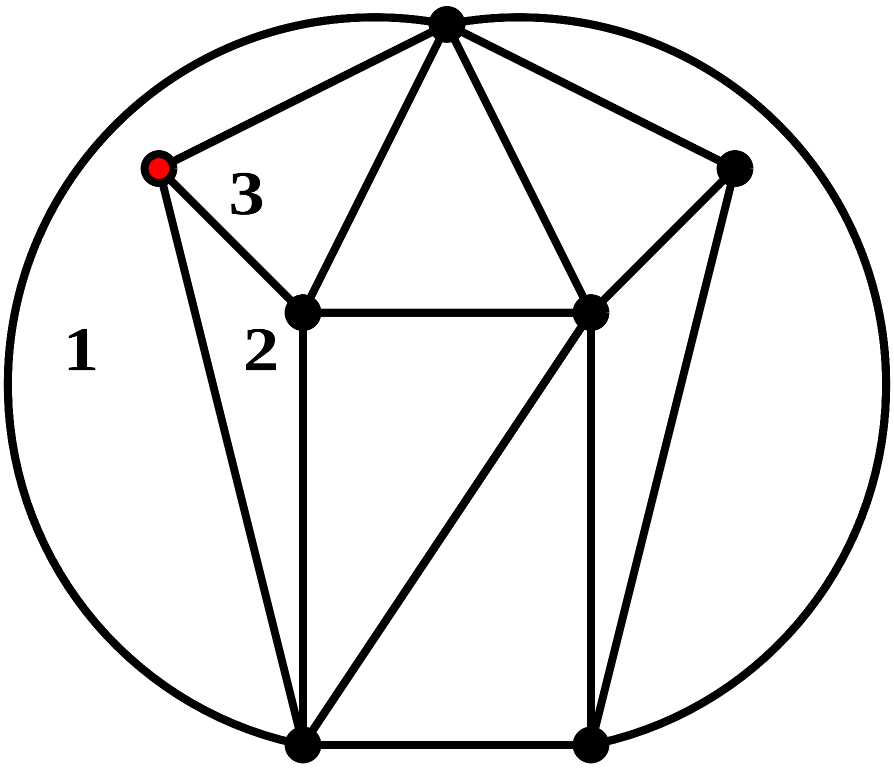


Theorem:

Every planar graph has at least one vertex of degree at most five.

Proof:

Now, there are at least 3 faces at each vertex.



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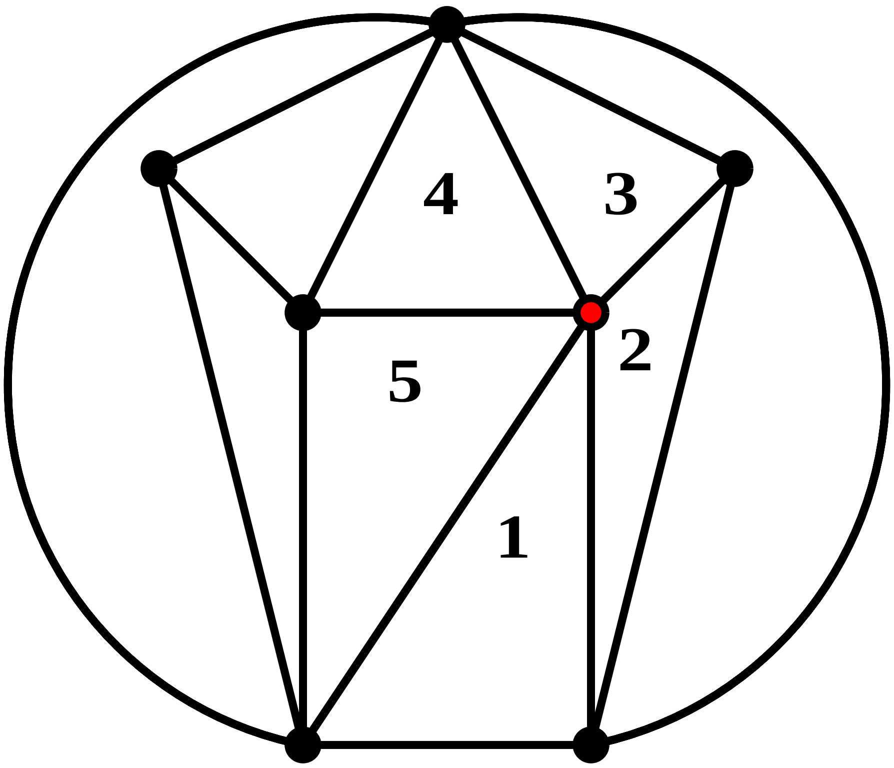

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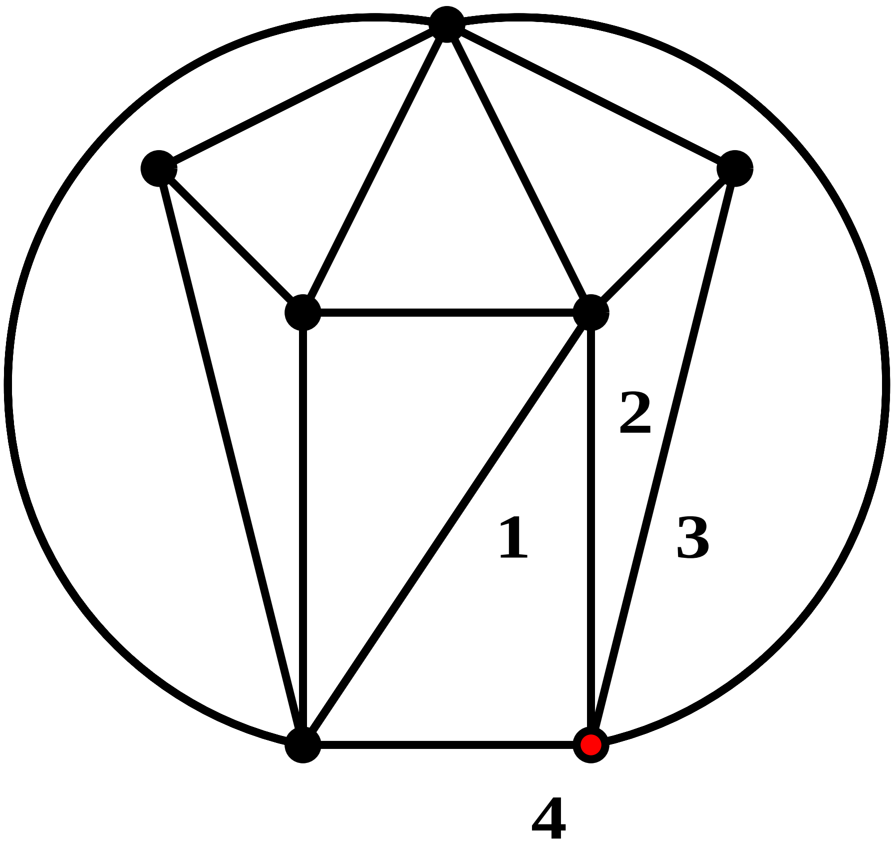


(The exterior also counts as a face.)

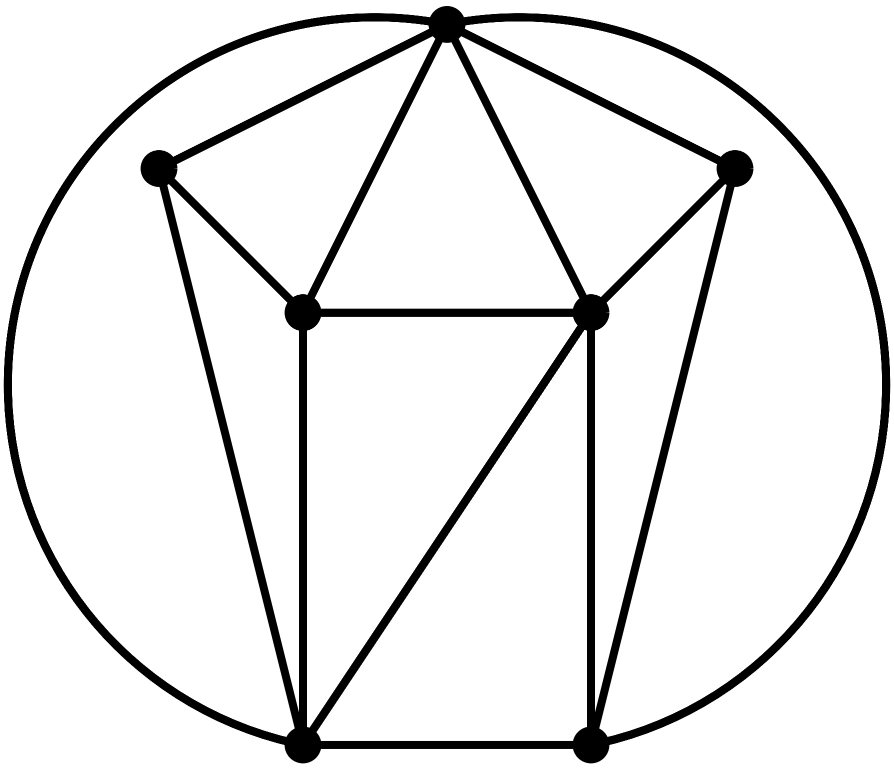
Theorem:

Every planar graph has at least one vertex of degree at most five.

Proof:

So, the number of edges, e, is at least three times the number of faces, f.

That is, 3f $\leq$ e.

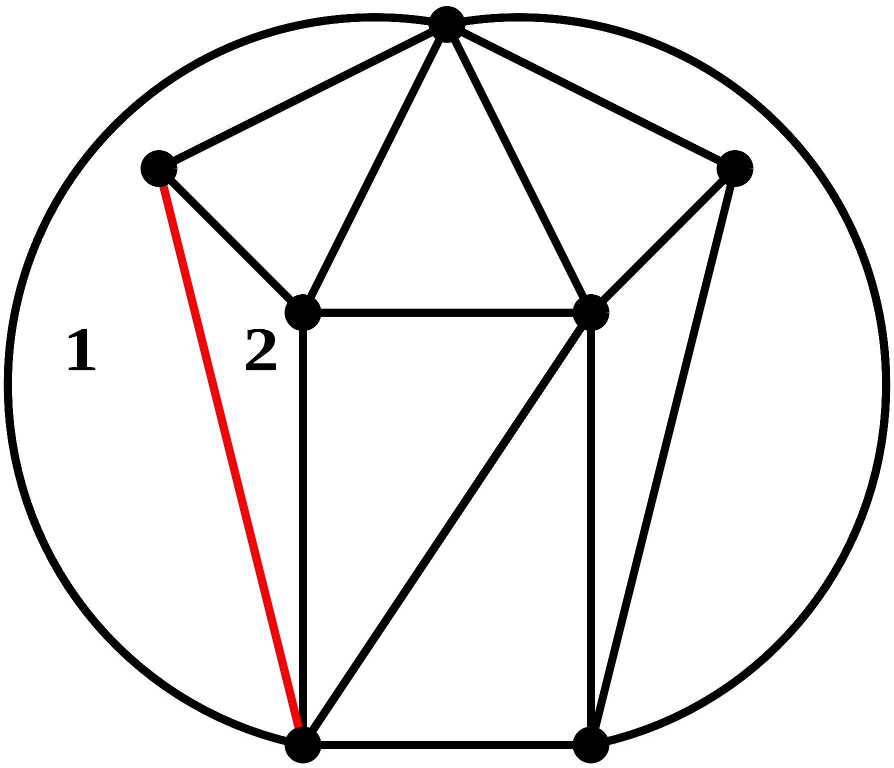


Theorem:

Every planar graph has at least one vertex of degree at most five.

Proof:

However, each edge is incident to two faces.



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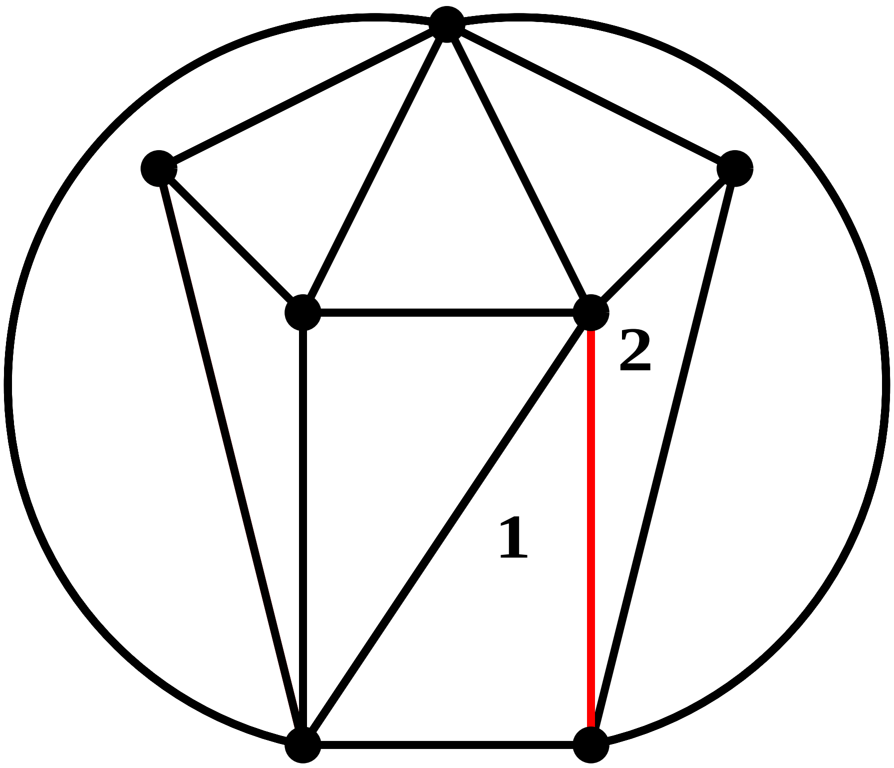

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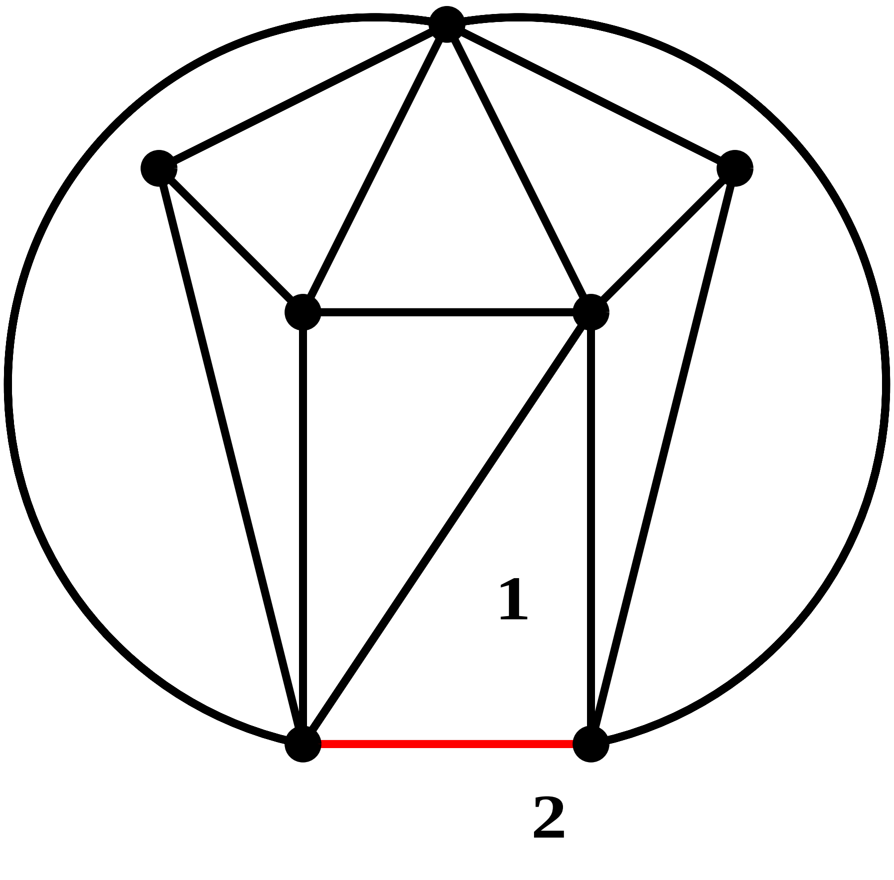


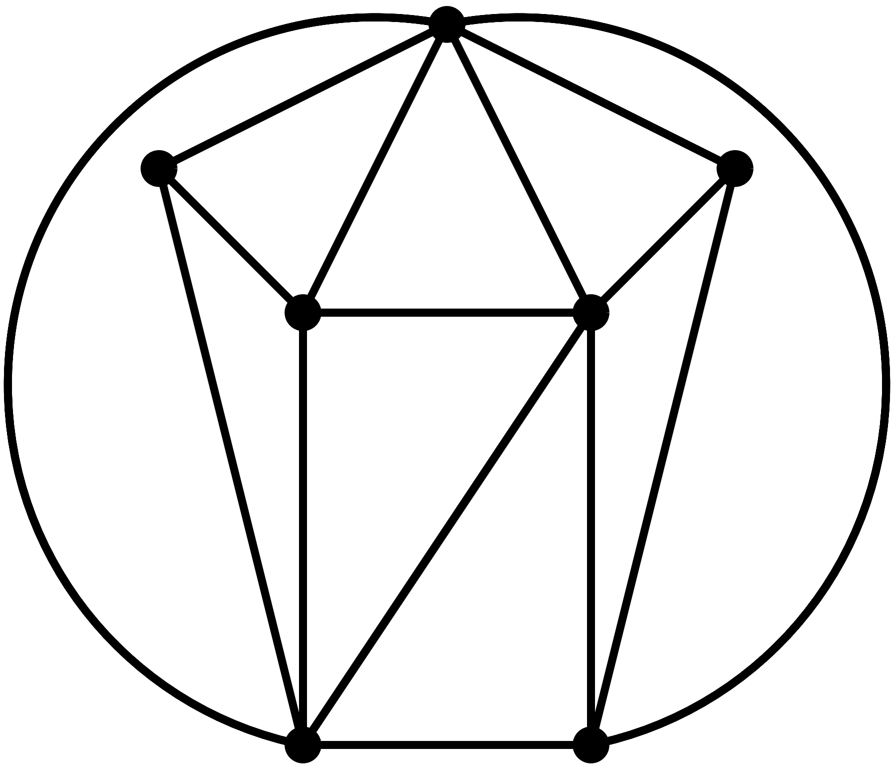
Theorem:

Every planar graph has at least one vertex of degree at most five.

Proof:

So, actually, $3f / 2 \leq e$.

This is true for every planar graph.



Theorem:

Every planar graph has at least one vertex of degree at most five.

Proof:

Now, suppose that not every planar graph has at least one vertex of degree at most five.

Then, there exists a planar graph such that every vertex has degree at least six.

Theorem:

Every planar graph has at least one vertex of degree at most five.

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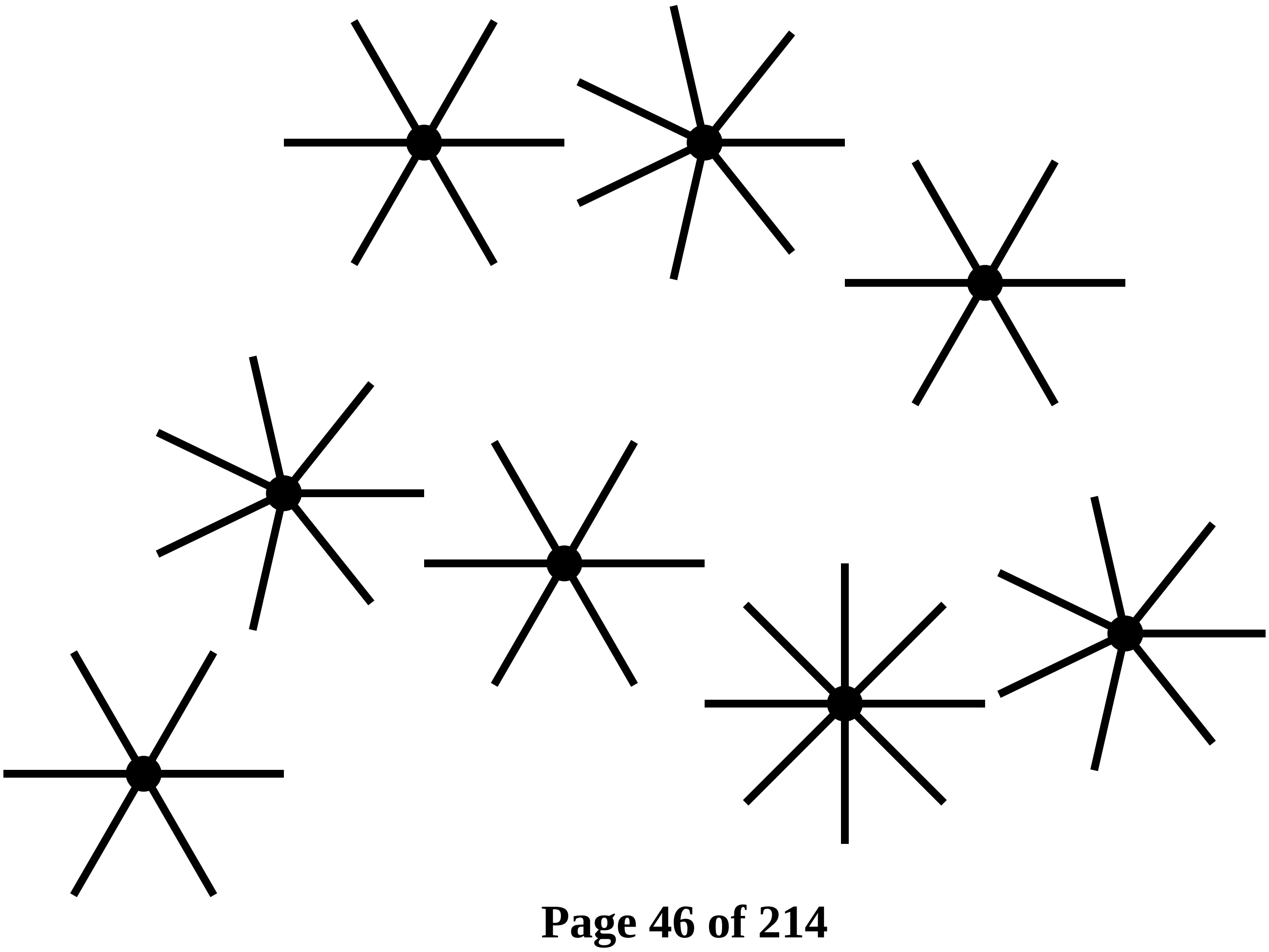

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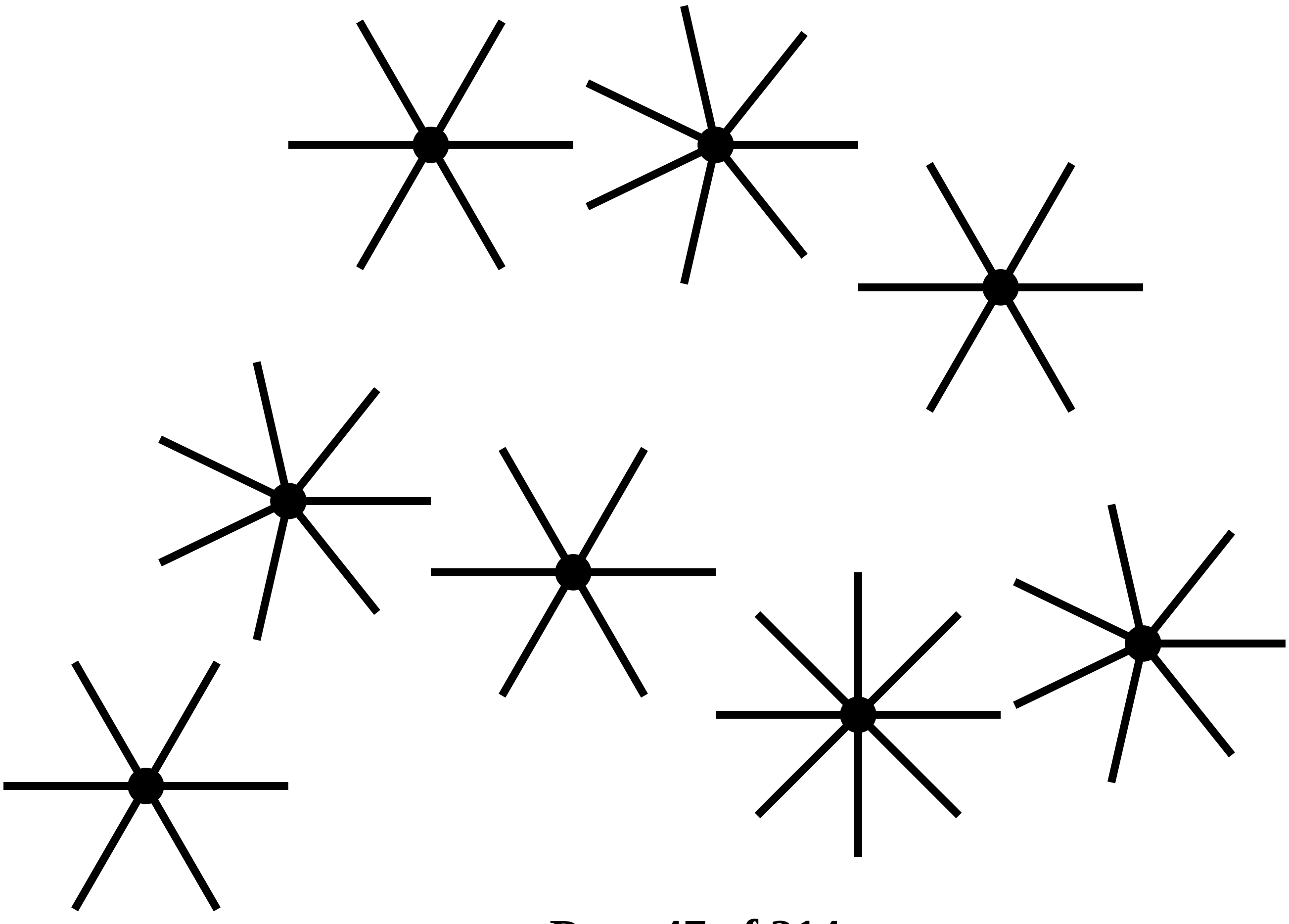
There are at least six edges incident to each vertex.



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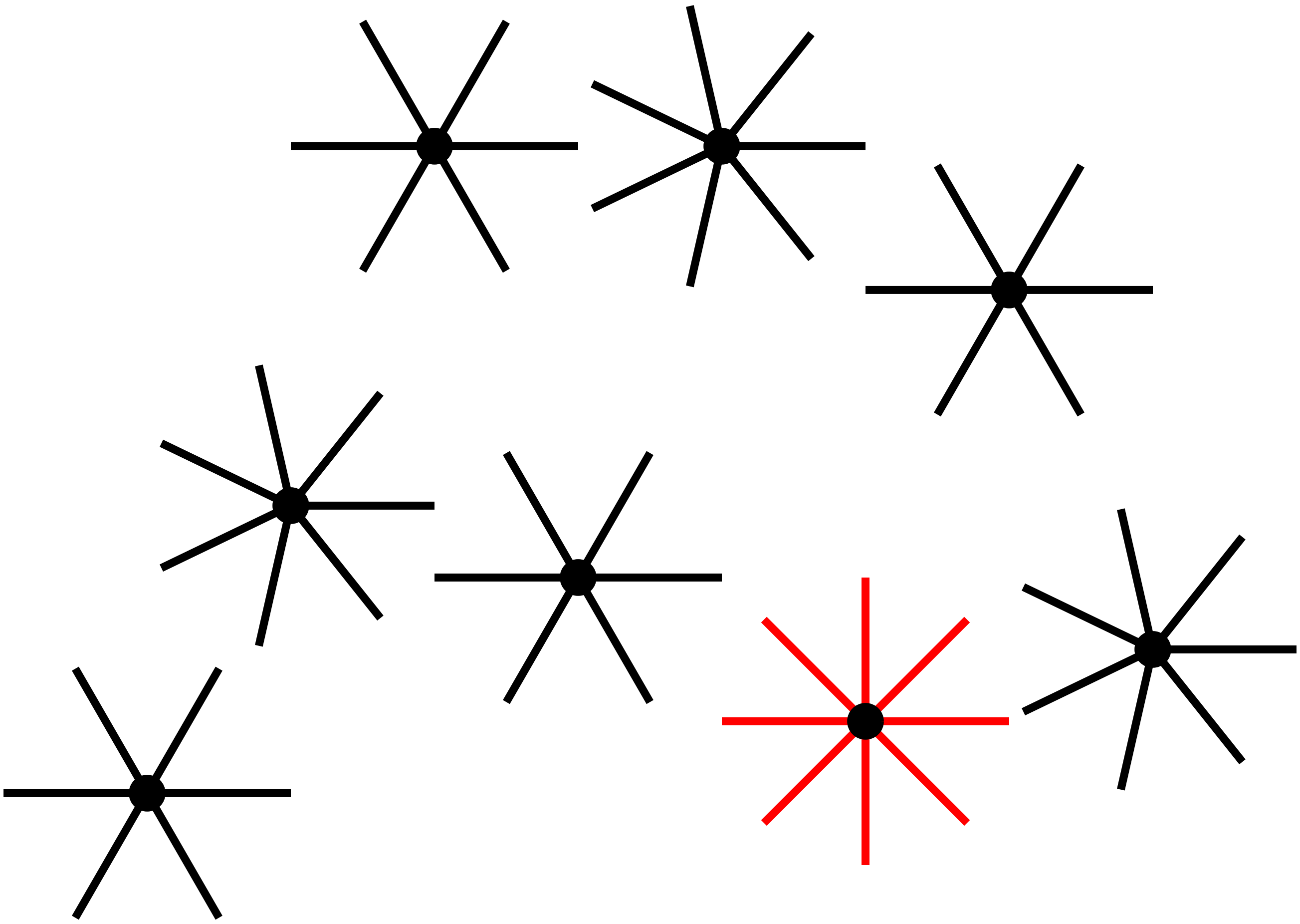

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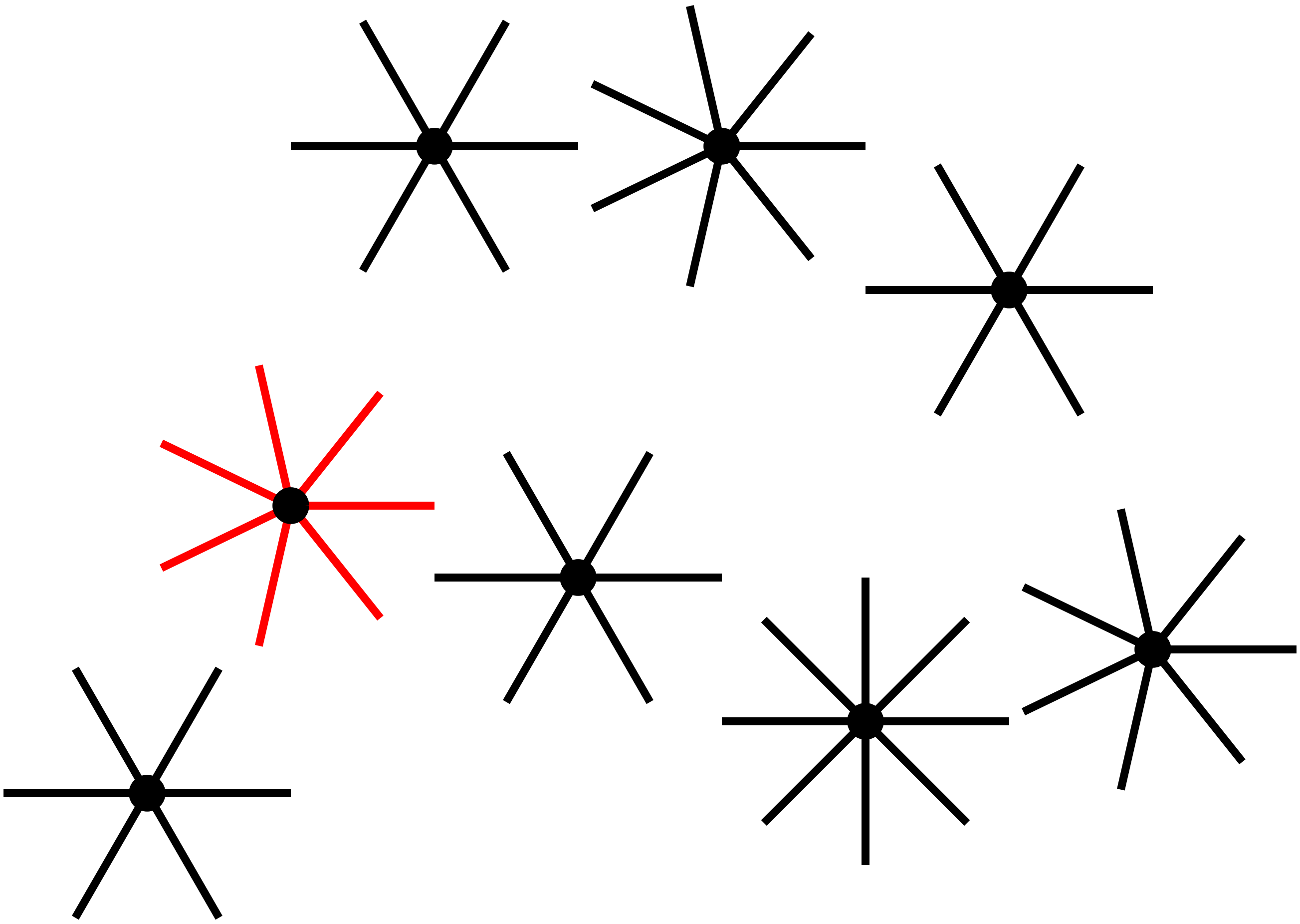

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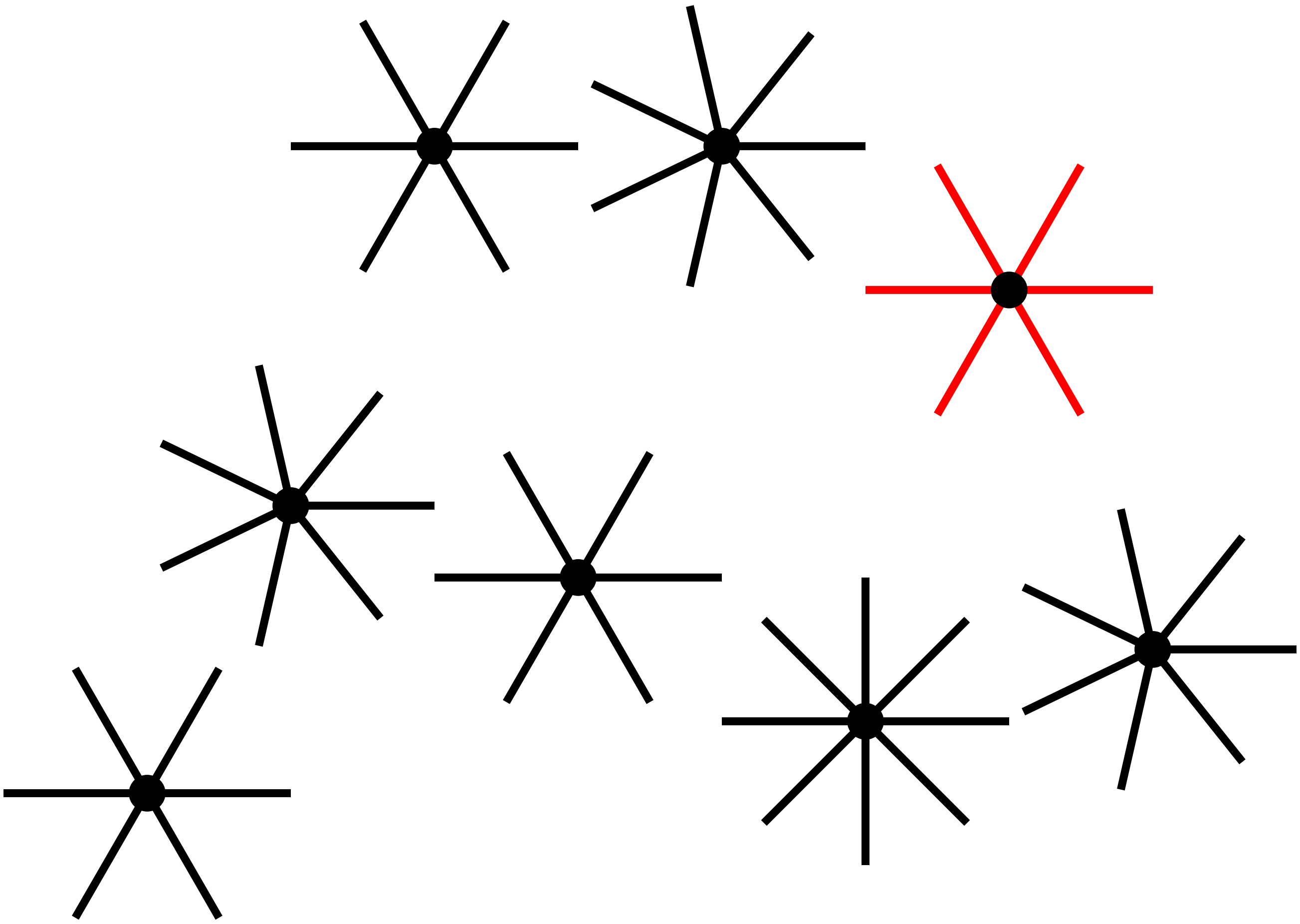

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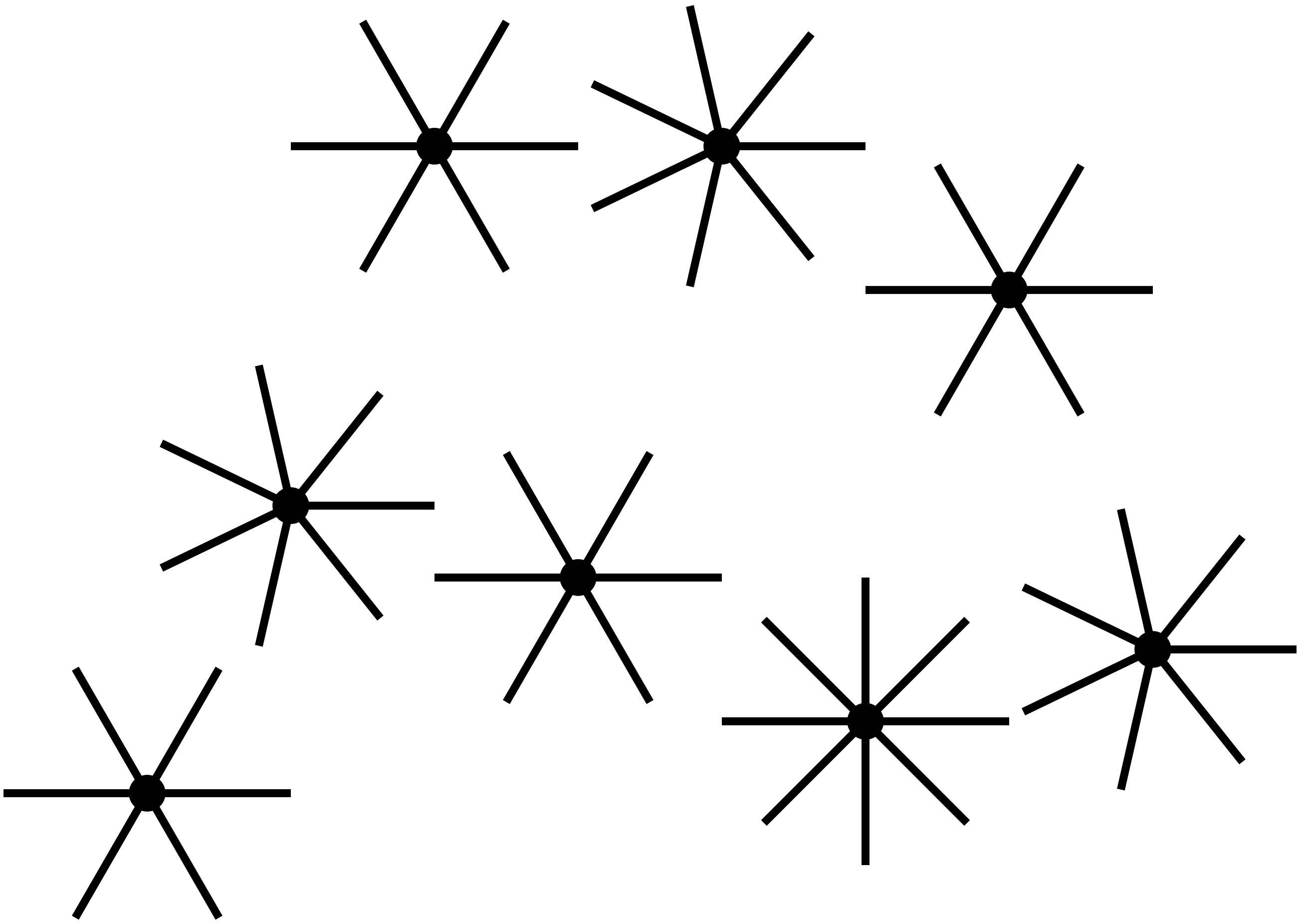
So, the number of edges, e, is at least six times the number of vertices, v.

That is, $6v \leq e$.



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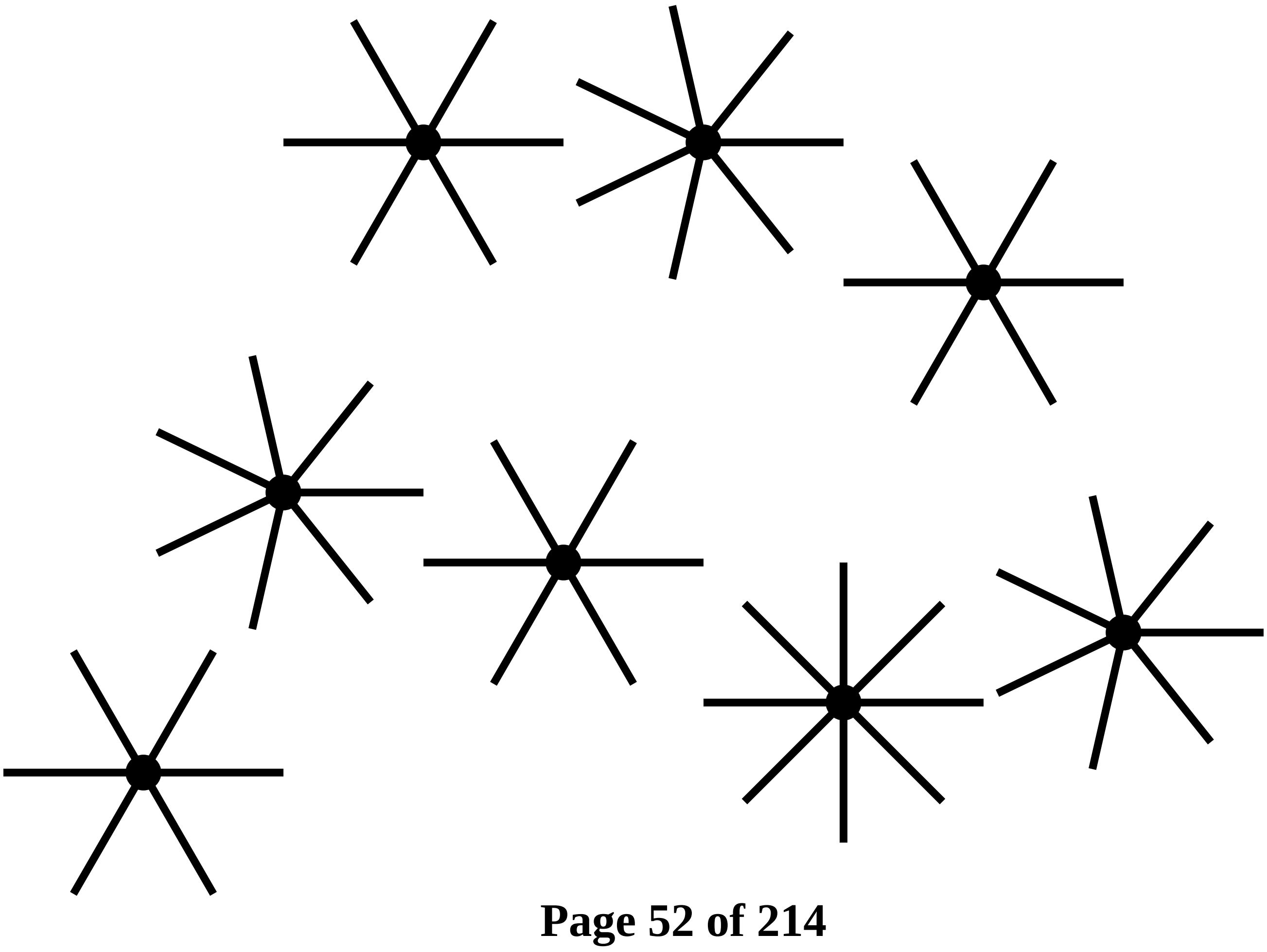
However, each edge is incident to two vertices.

So, actually, $6v / 2 = 3v \leq e$.



Theorem:

Every planar graph has at least one vertex of degree at most five.

Proof:

So, $3v \leq e \iff v \leq e / 3$
and $3f / 2 \leq e \iff f \leq 2e / 3$.

Thus, $f - e + v \leq (2e / 3) - e + (e / 3)$
$\iff f - e + v \leq 0$

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But, recall Euler's formula, $f - e + v = 2$, for planar graphs and polyhedra.

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$\iff f - e + v \leq 0$**

But, recall Euler's formula, $f - e + v = 2$, for planar graphs and polyhedra.

**So, $f - e + v \leq 0$
$\iff 2 \leq 0$**

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$\iff f - e + v \leq 0$

But, recall Euler's formula, $f - e + v = 2$, for planar graphs and polyhedra.

So, $f - e + v \leq 0$
$\iff 2 \leq 0$

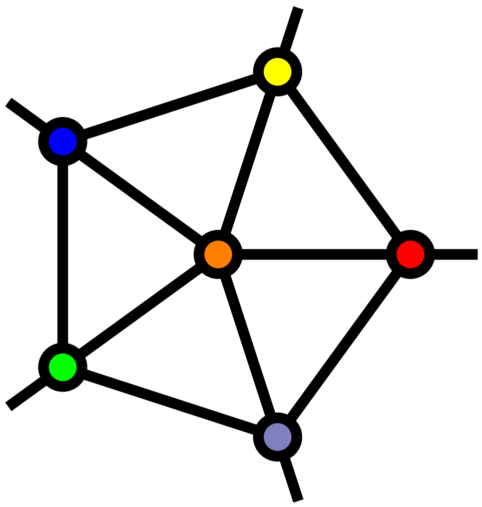
Impossible!

Thus, every planar graph has at least one vertex of degree at most five.

This theorem that was just proven is very important.

It can be used to prove a weaker statement that every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

This is called the six colour theorem.



The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Proof by induction on the number of vertices, n.

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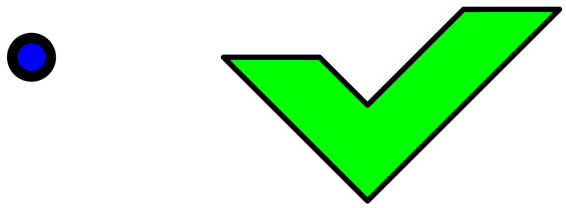
Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=1

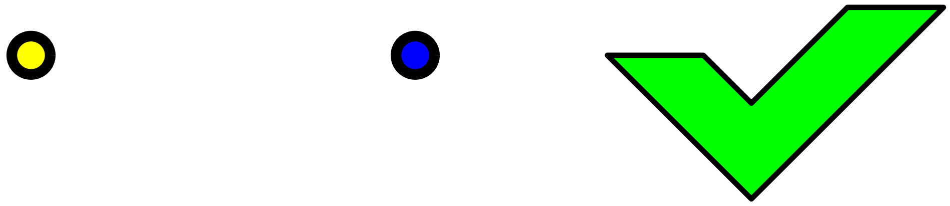


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=2

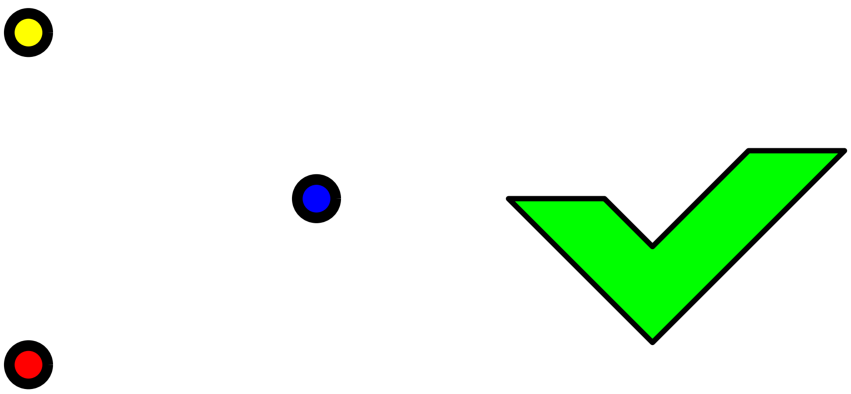


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=3

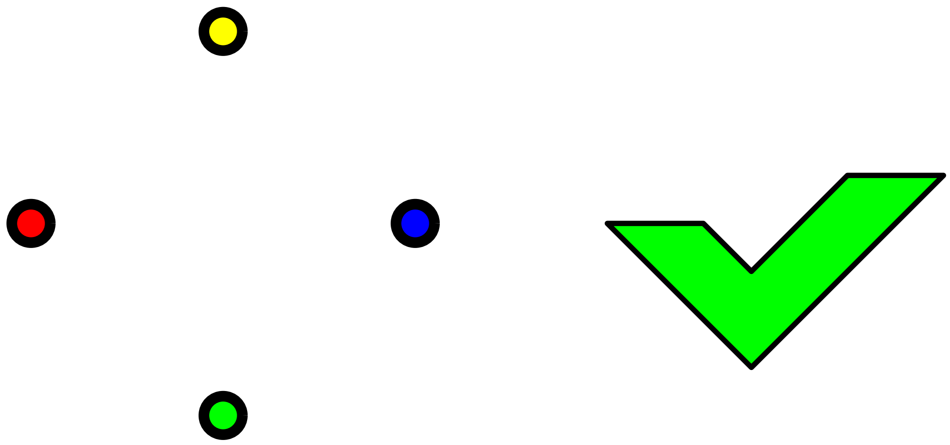


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=4

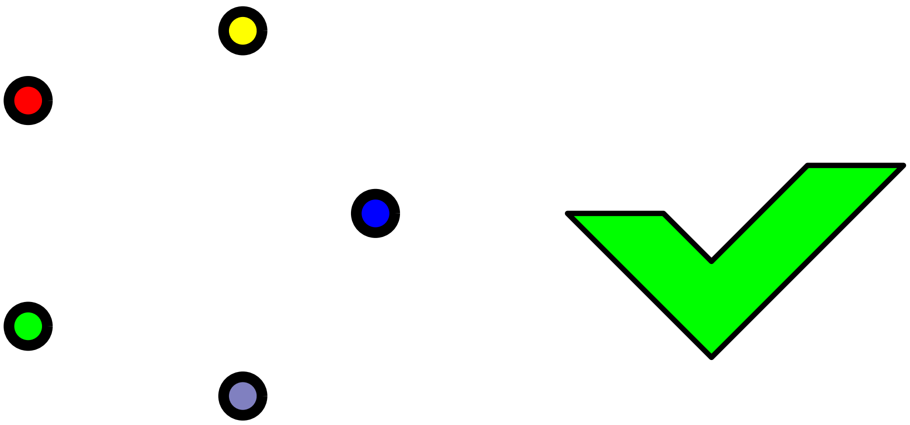


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=5

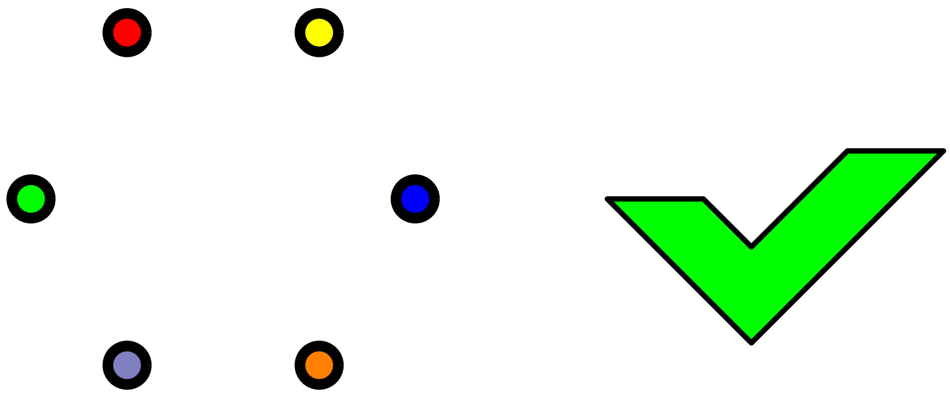


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=6



The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Induction Step:

Suppose that every planar graph with n vertices can be coloured using at most six colours.

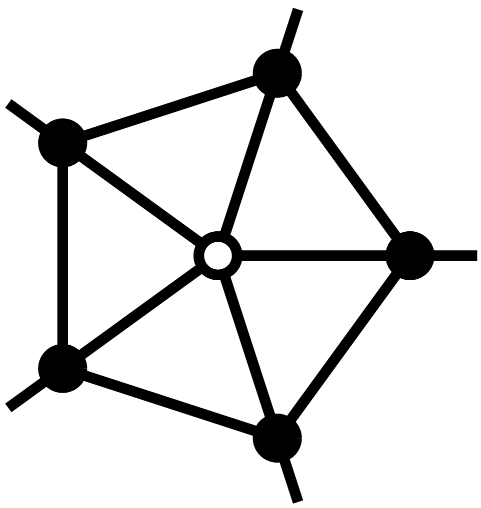
The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Let there be a planar graph with n + 1 vertices.

Then, by the previous theorem, there exists a vertex of degree at most five.

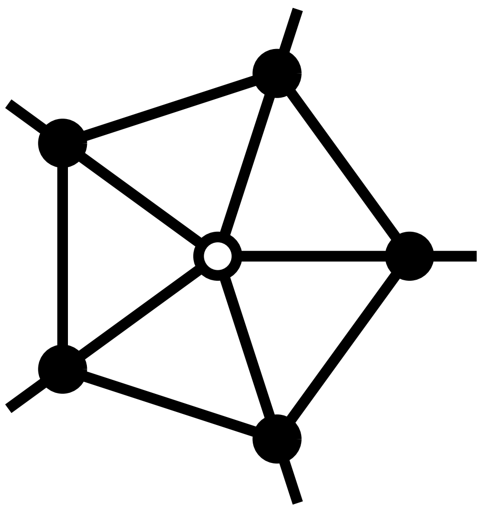


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

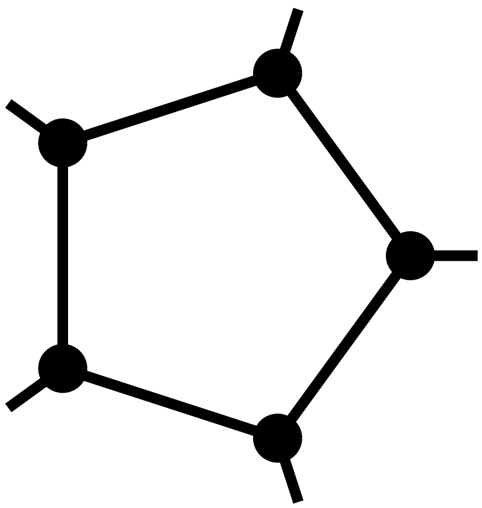
Remove that vertex from the graph.



The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

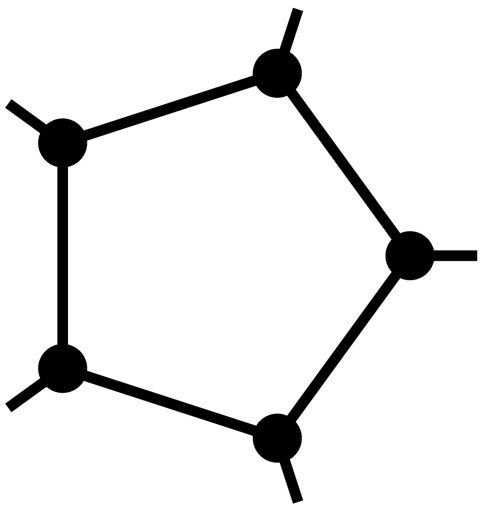


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Now, the remaining graph has n vertices.

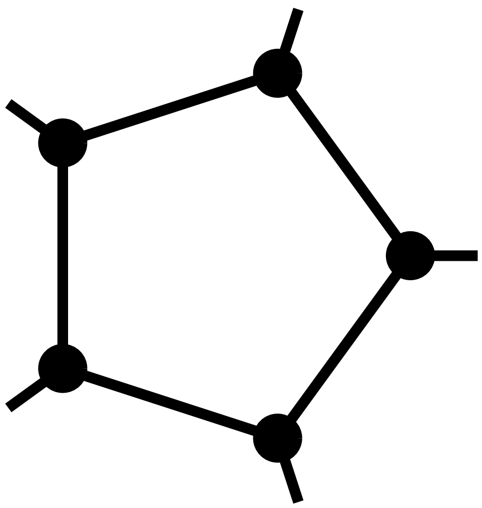


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

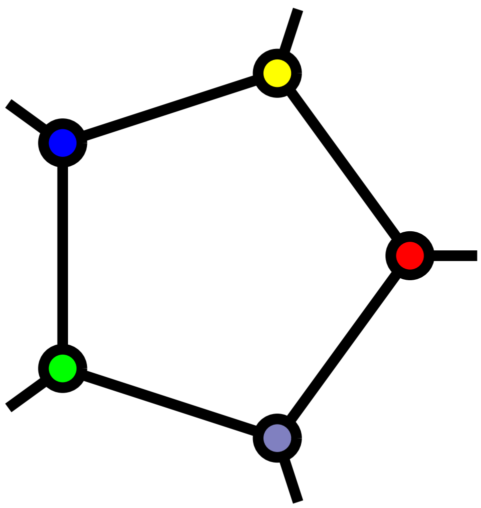
By the induction hypothesis, this graph can be coloured using at most six colours.



The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

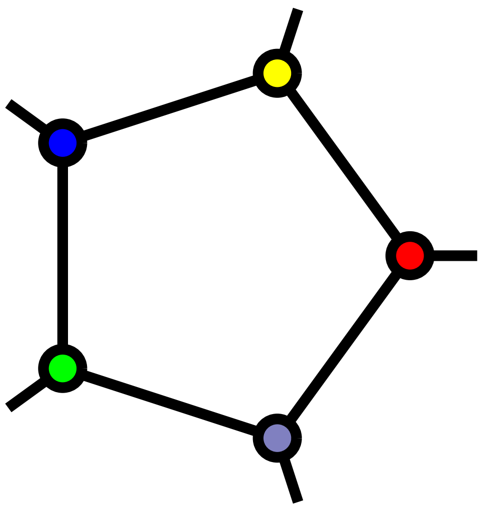


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

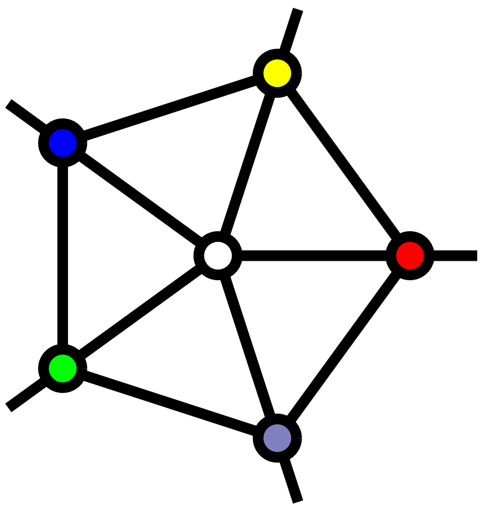
Place the removed vertex back.



The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:



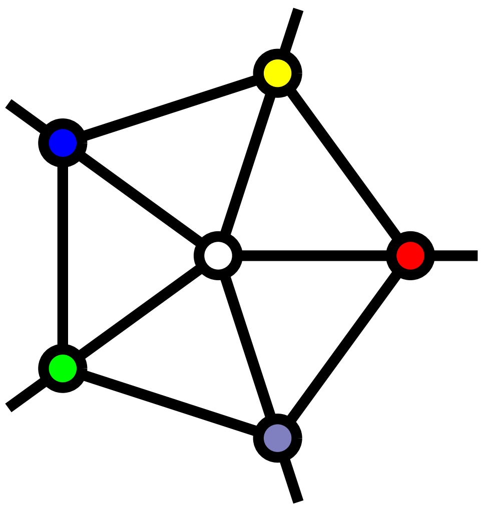
The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Colour that vertex with a colour that is different from the colours of the vertices adjacent to it.

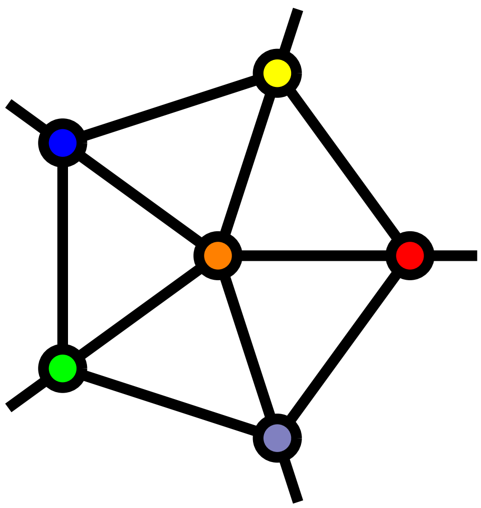
There are at most five adjacent vertices so such a colour exists.



The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

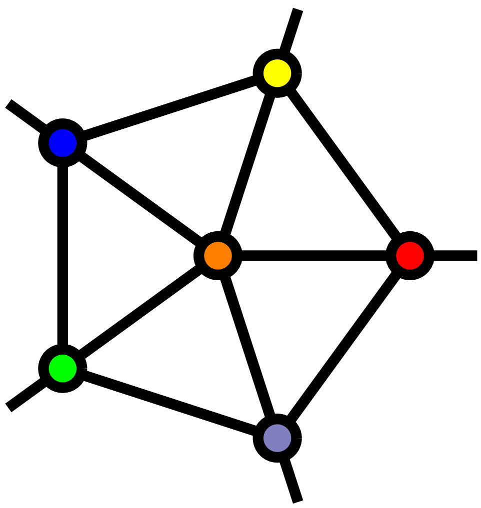


The Six Colour Theorem:

Every planar graph can be coloured using at most six colours such that no adjacent vertices have the same colour.

Proof:

Thus, every planar graph can be coloured using at most six colours.

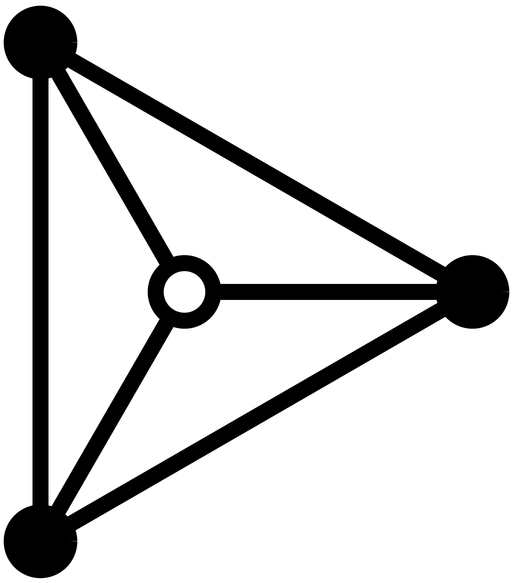


In 1879, Alfred Kempe used the theorem that every planar graph has at least one vertex of degree at most five to try to prove the four colour theorem.

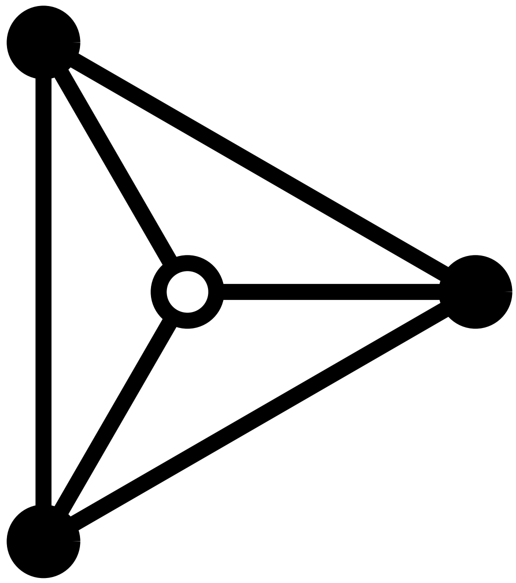
He gave a method for colouring any planar graph using four colours.

1. **Find a vertex with degree at most five.**

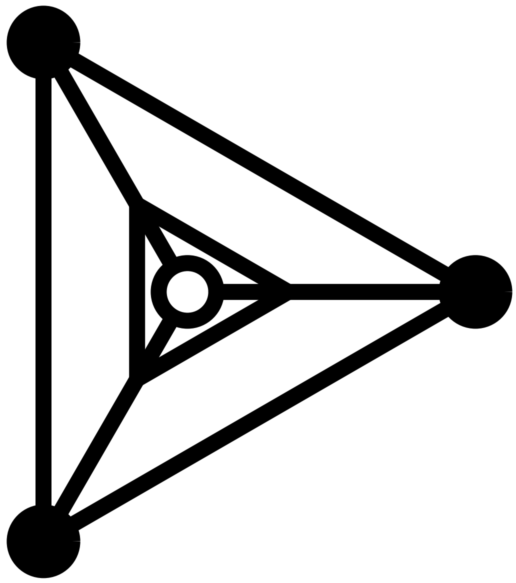
1. **Find a vertex with degree at most five.**



2. Place a patch over this vertex.

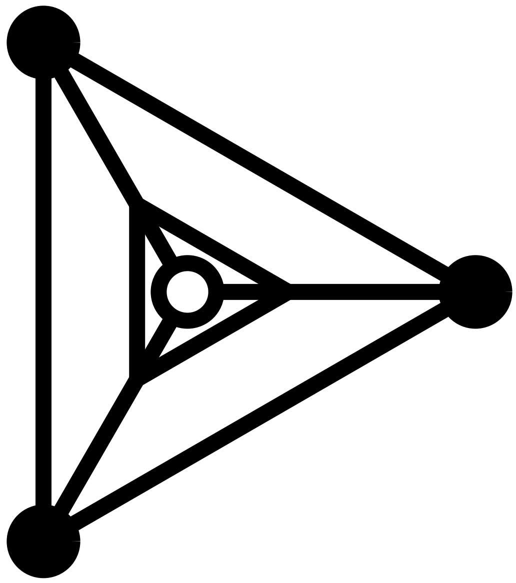


2. Place a patch over this vertex.

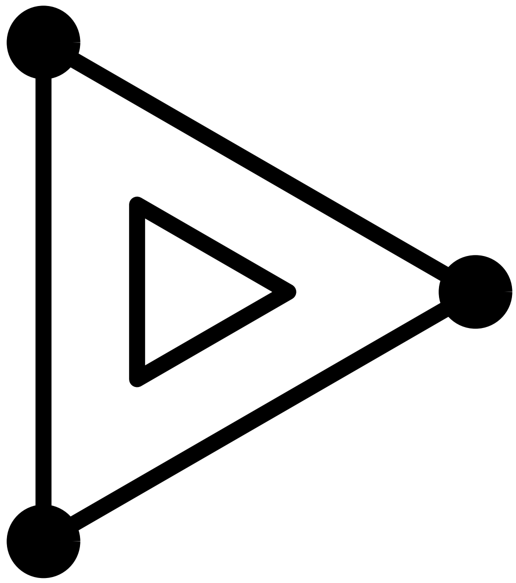


(Place a triangular patch since it has degree 3.)

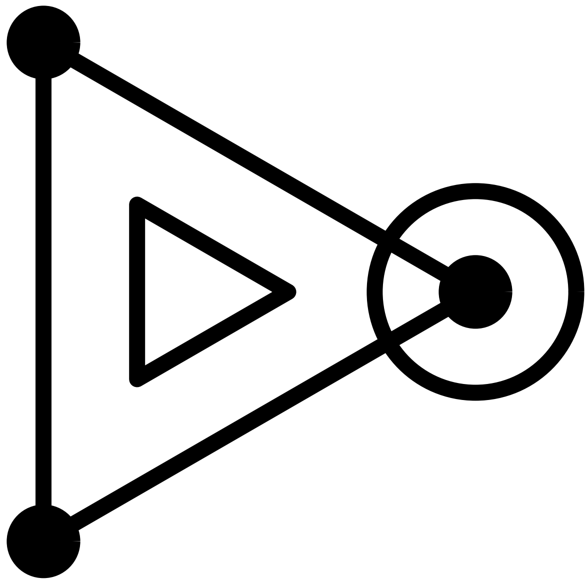
3. Remove this vertex and its edges.



3. **Remove this vertex and its edges.**

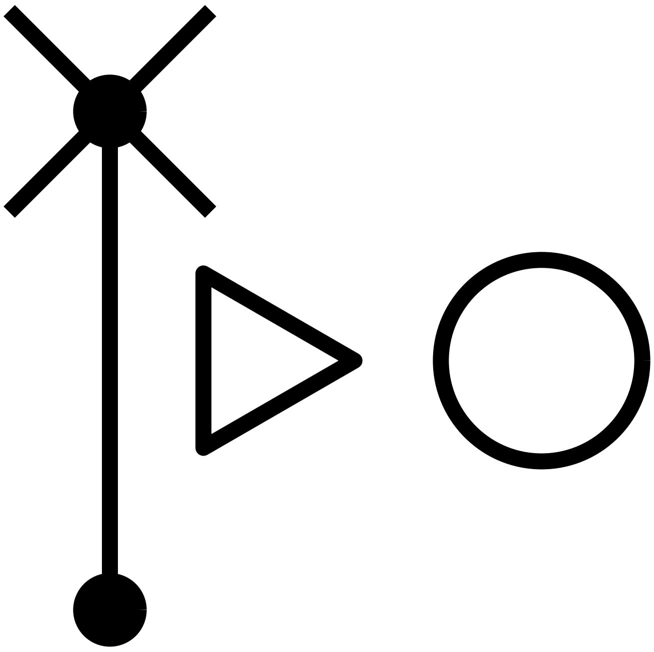


4. **Repeat the above steps until one vertex remains.**



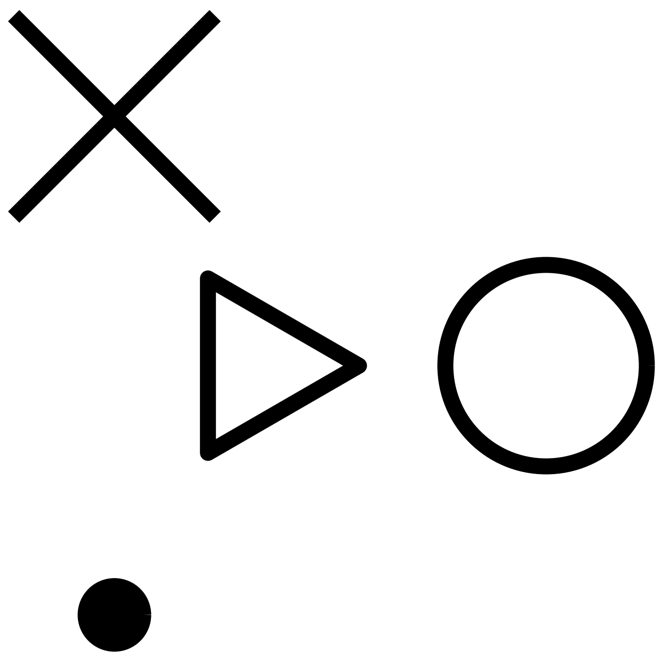
(Place a circular patch for degree 2.)

4. **Repeat the above steps until one vertex remains.**

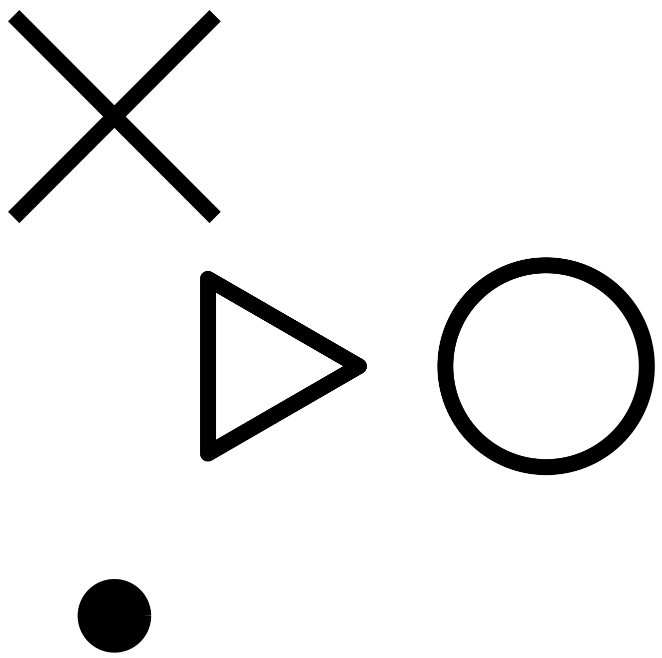


(Place a cross for degree 1.)

4. **Repeat the above steps until one vertex remains.**



5. **Colour the remaining vertex with any one of the four colours.**



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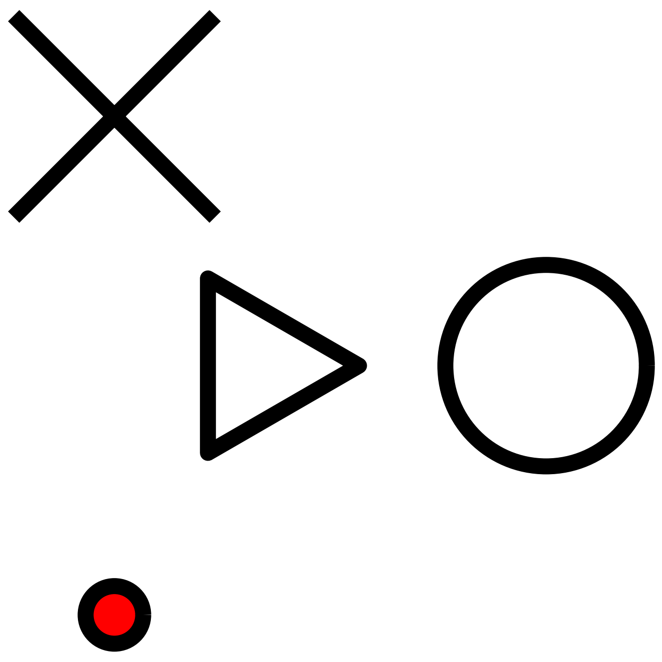
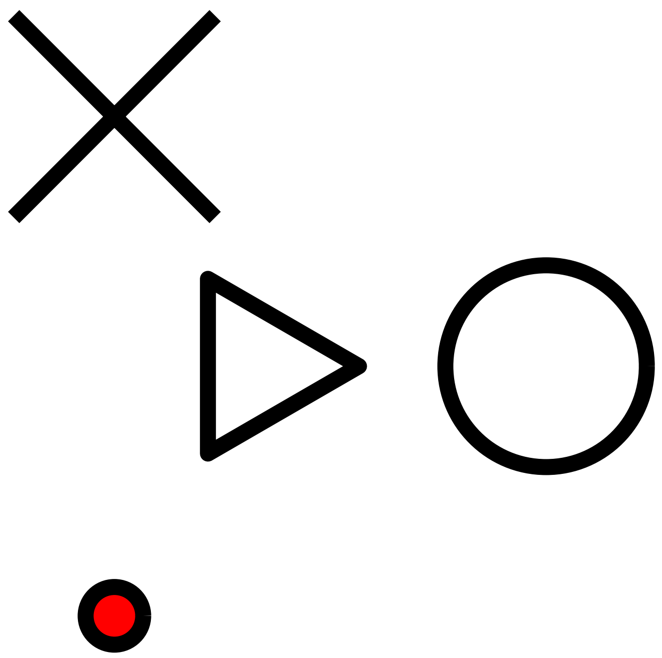


6. **Reverse the above steps by placing the vertex and its edges back and by removing the patch.**

   Colour the vertex with any colour such that no adjacent vertices have the same colour.



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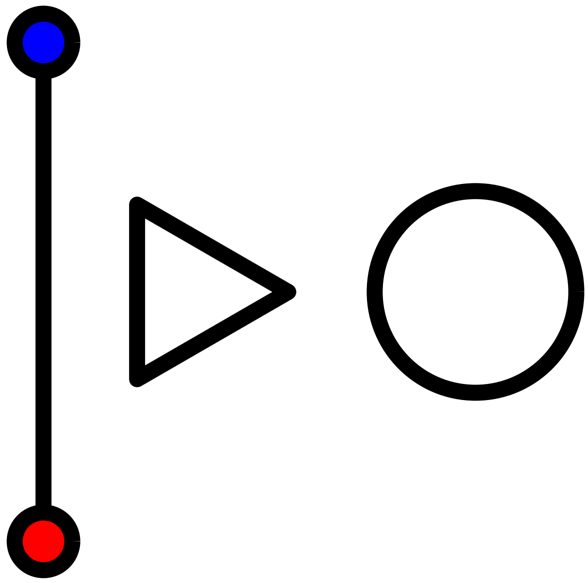

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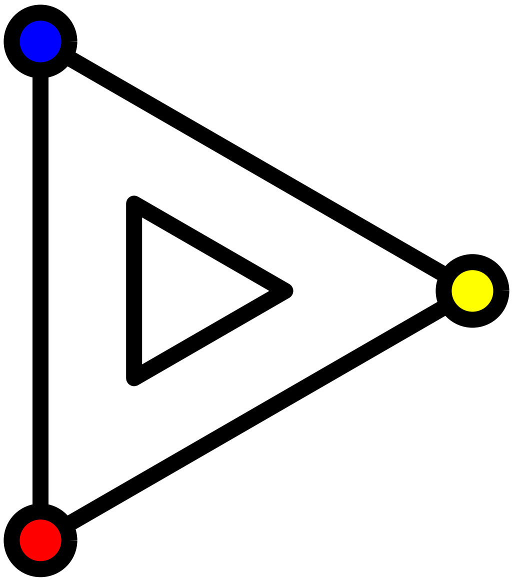

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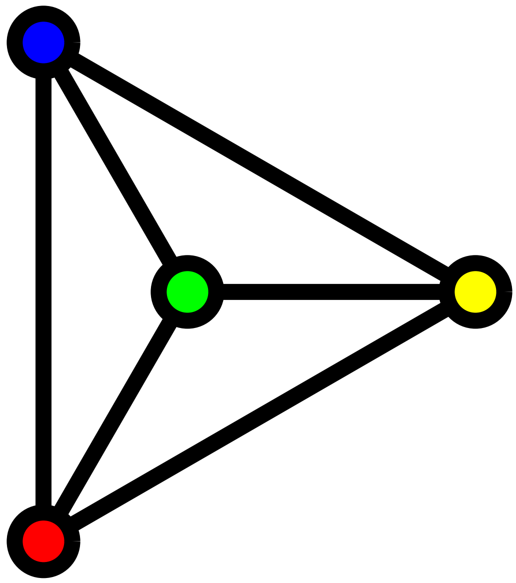
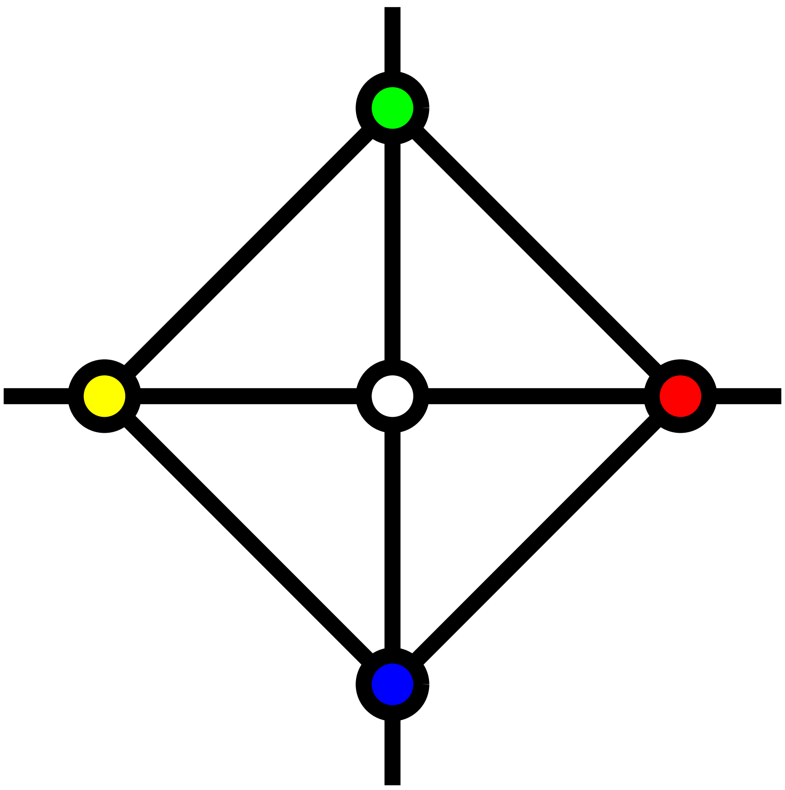


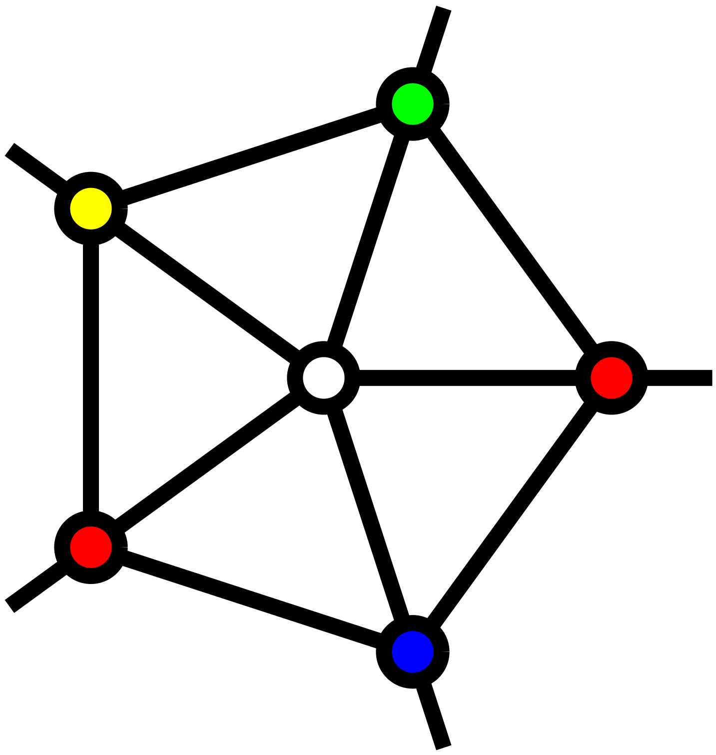
But, is there always a colour available when a vertex is put back?

What if these situations occur?

How does one colour the centre vertex?



Case A



Case B

Kempe used a method to deal with these situations.

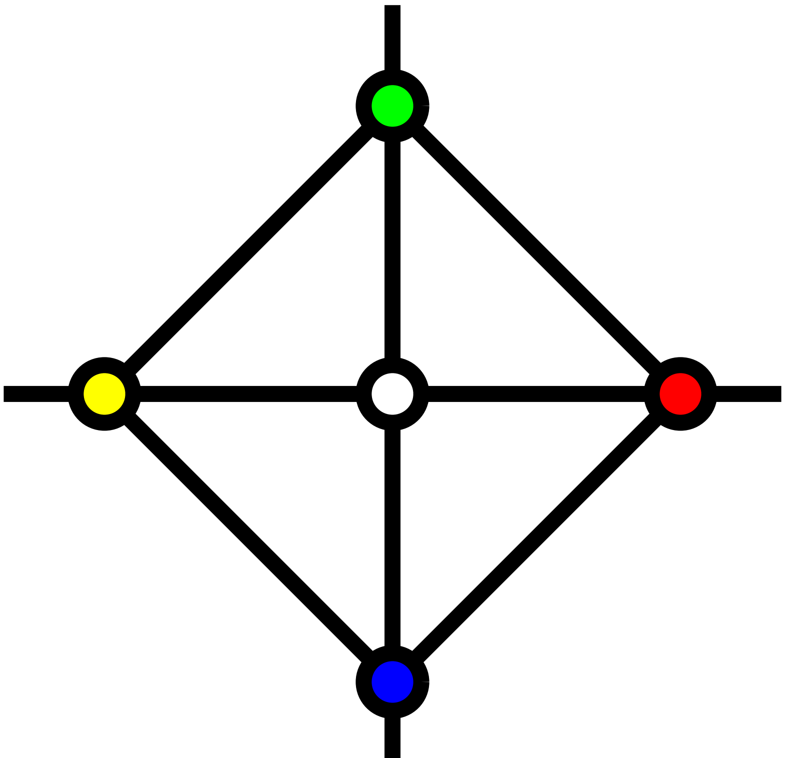
This method is now known as the method of Kempe chains.

Kempe used a method to deal with these situations.

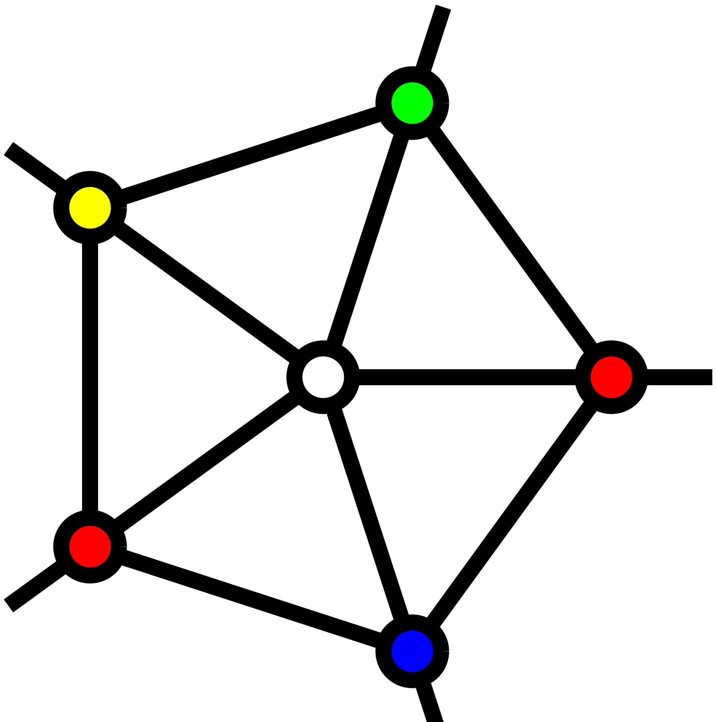
This method is now known as the method of Kempe chains.

Look at the colours of the adjacent vertices and choose two colours that are not adjacent to each other.

For both cases, choose, say, green and blue.



Case A

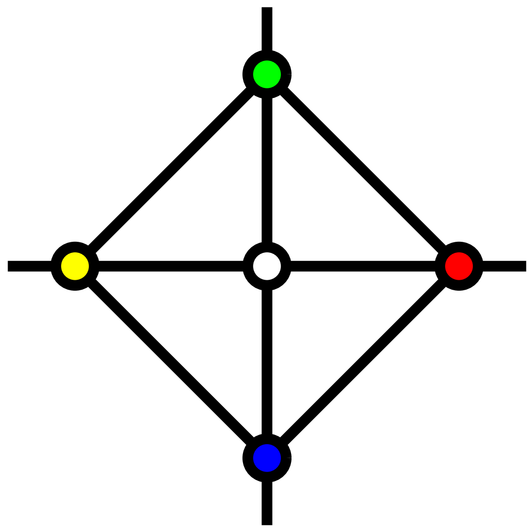


Case B

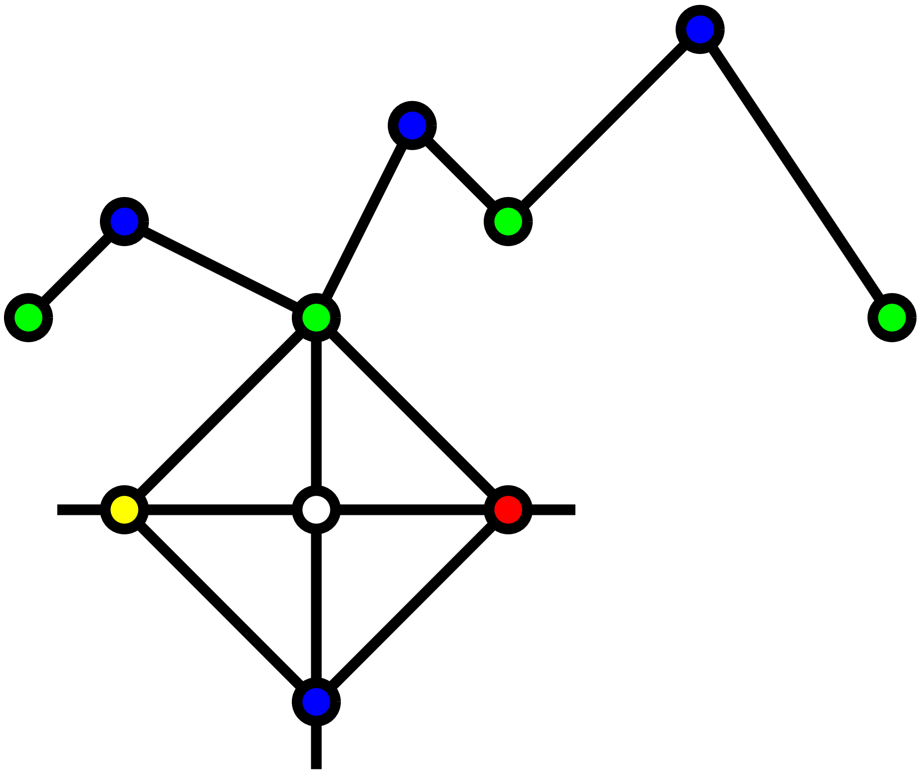
Case A:

Case A:

Here, the green vertex is the start of a green-blue chain of vertices.

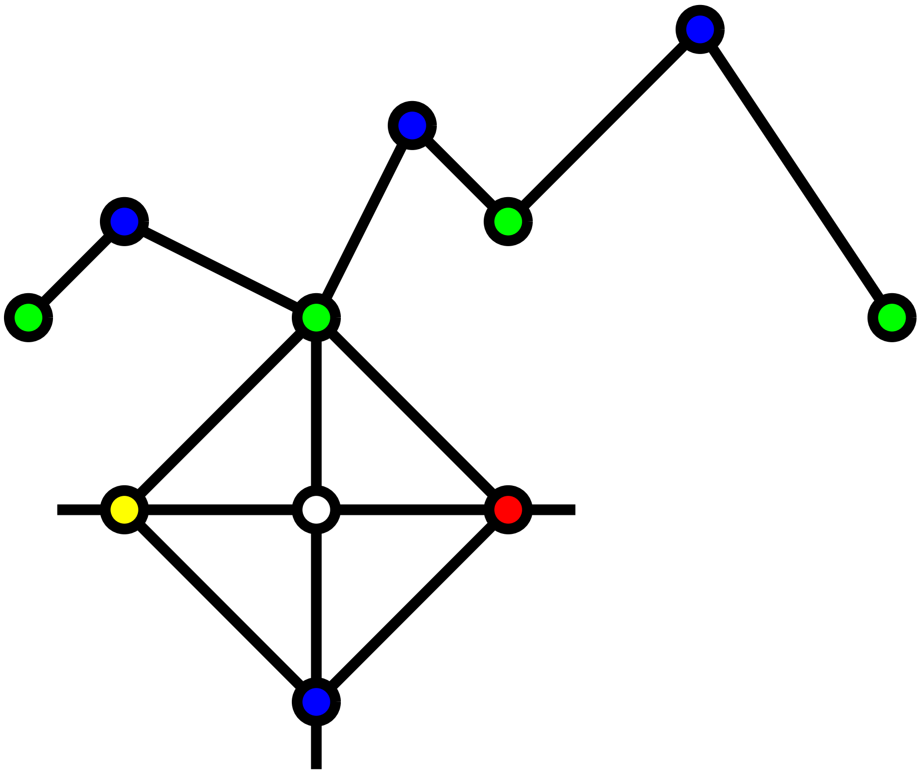


Case A:

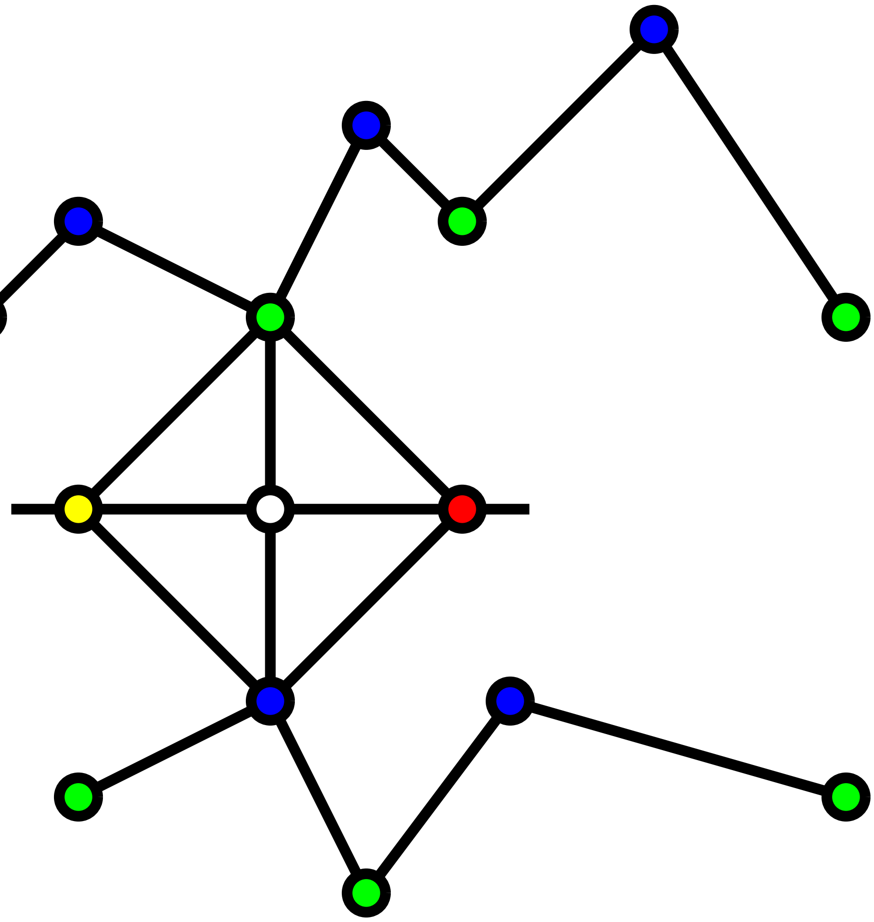


Case A:

The blue vertex is also the start of a green-blue chain of vertices.

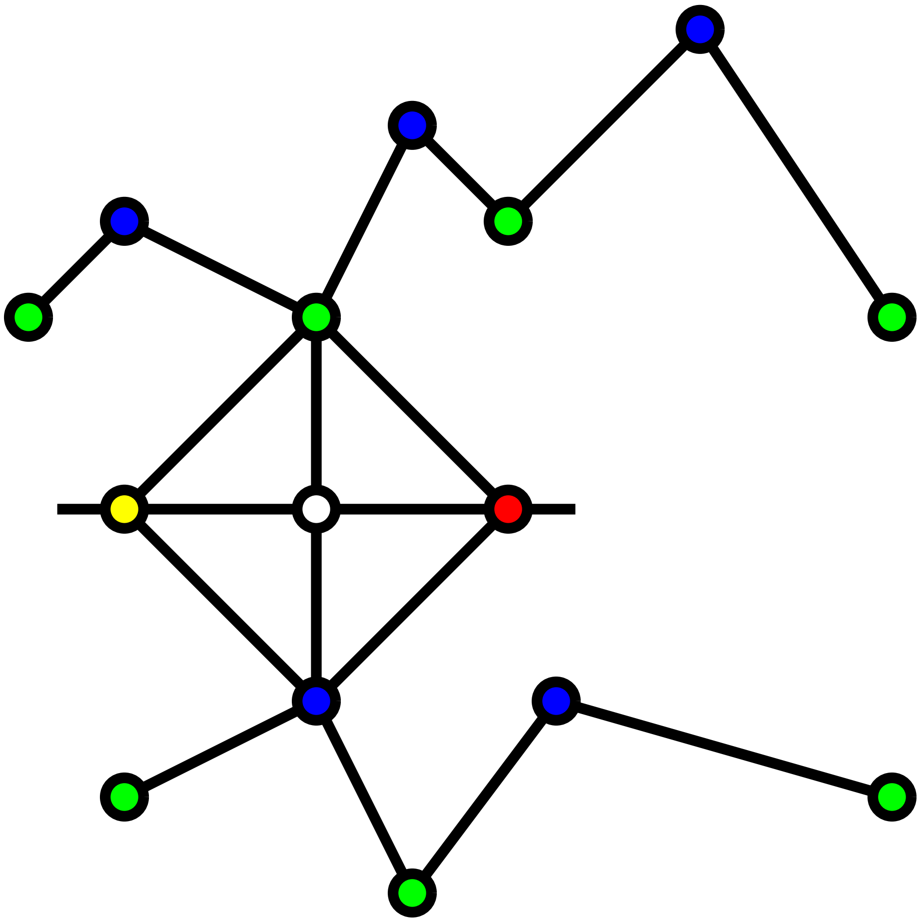


Case A:

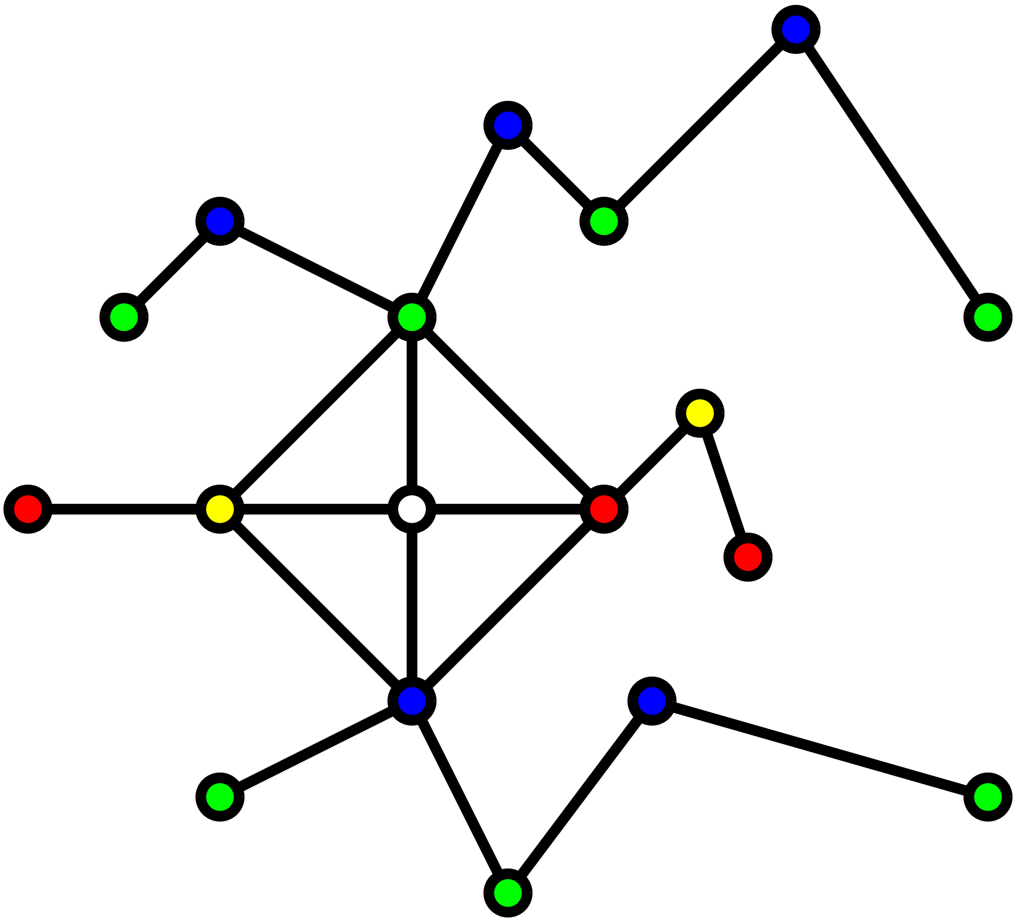


Case A:

Similarly, the red and yellow vertices start red-yellow chains of vertices.

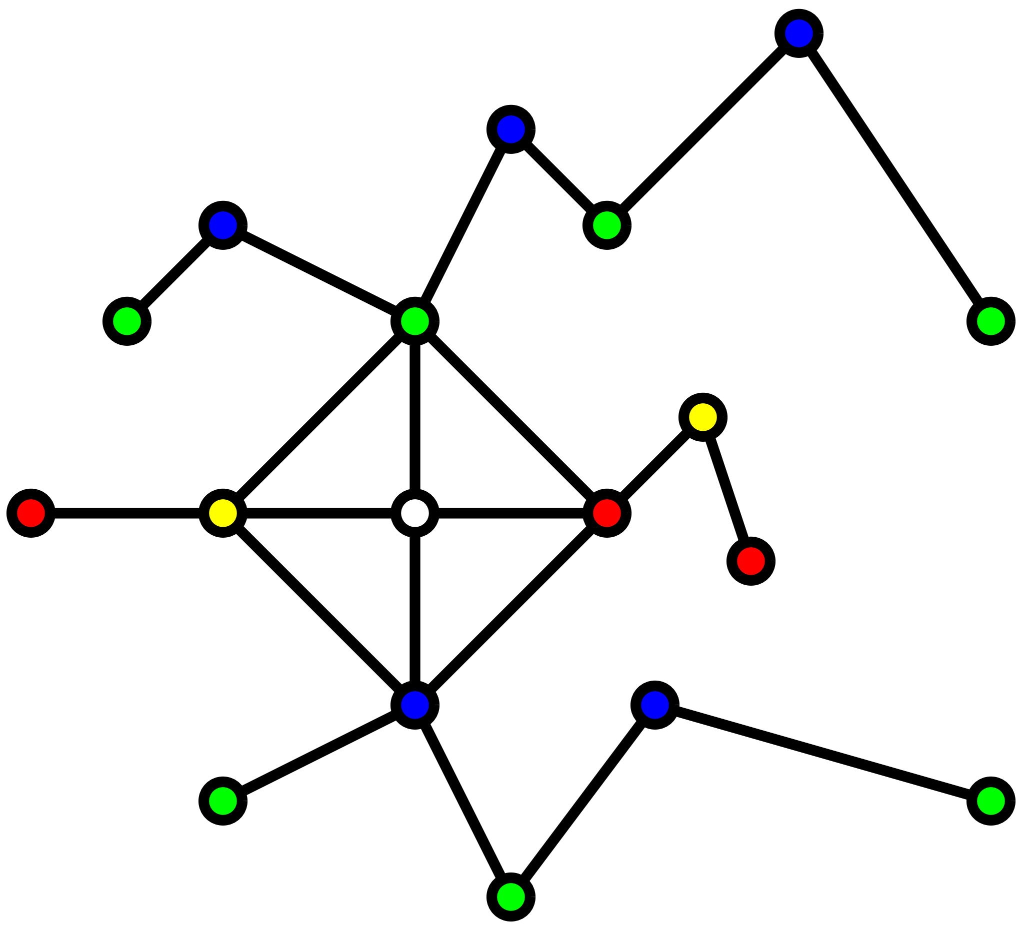


Case A:



Case A:

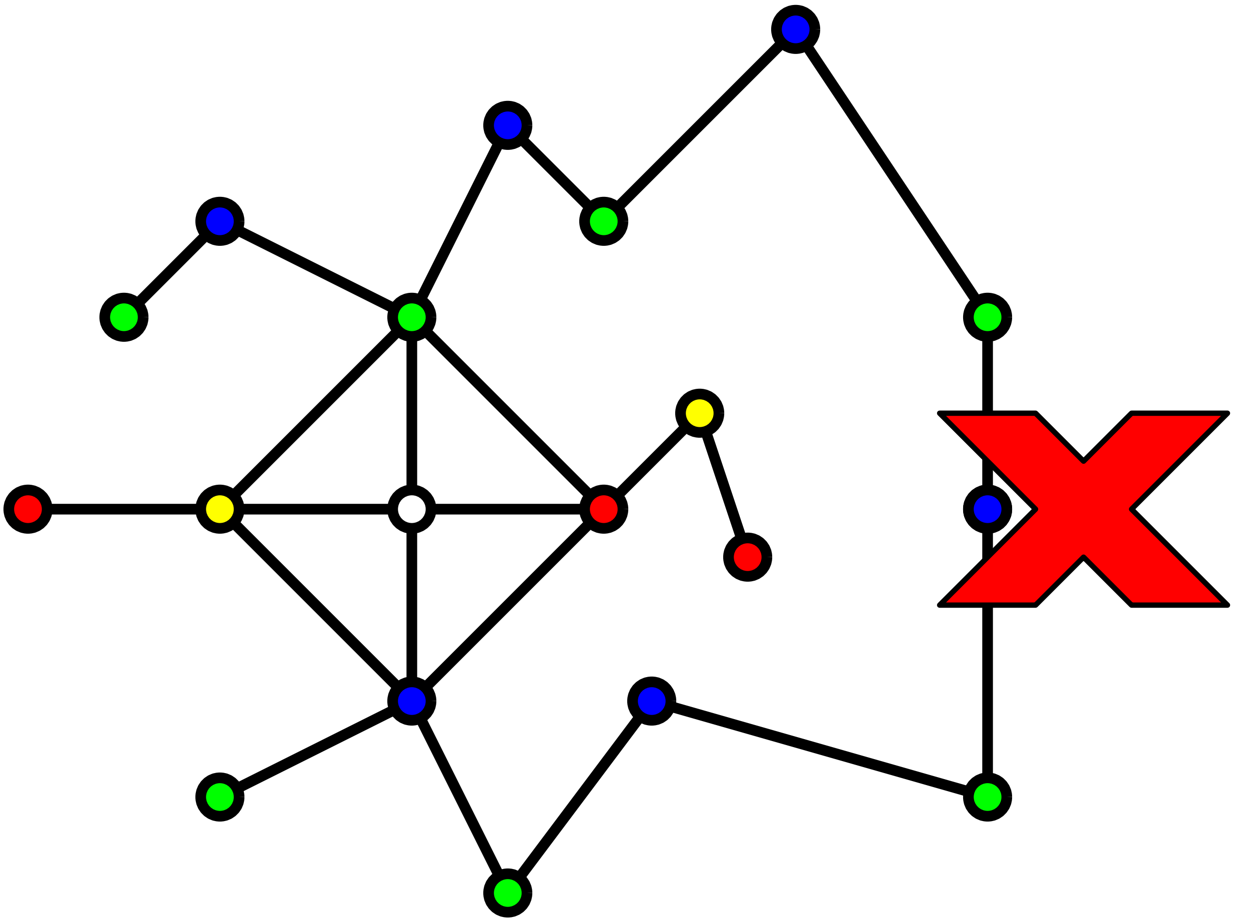
Now, there are two cases.



Case A:

Now, there are two cases.

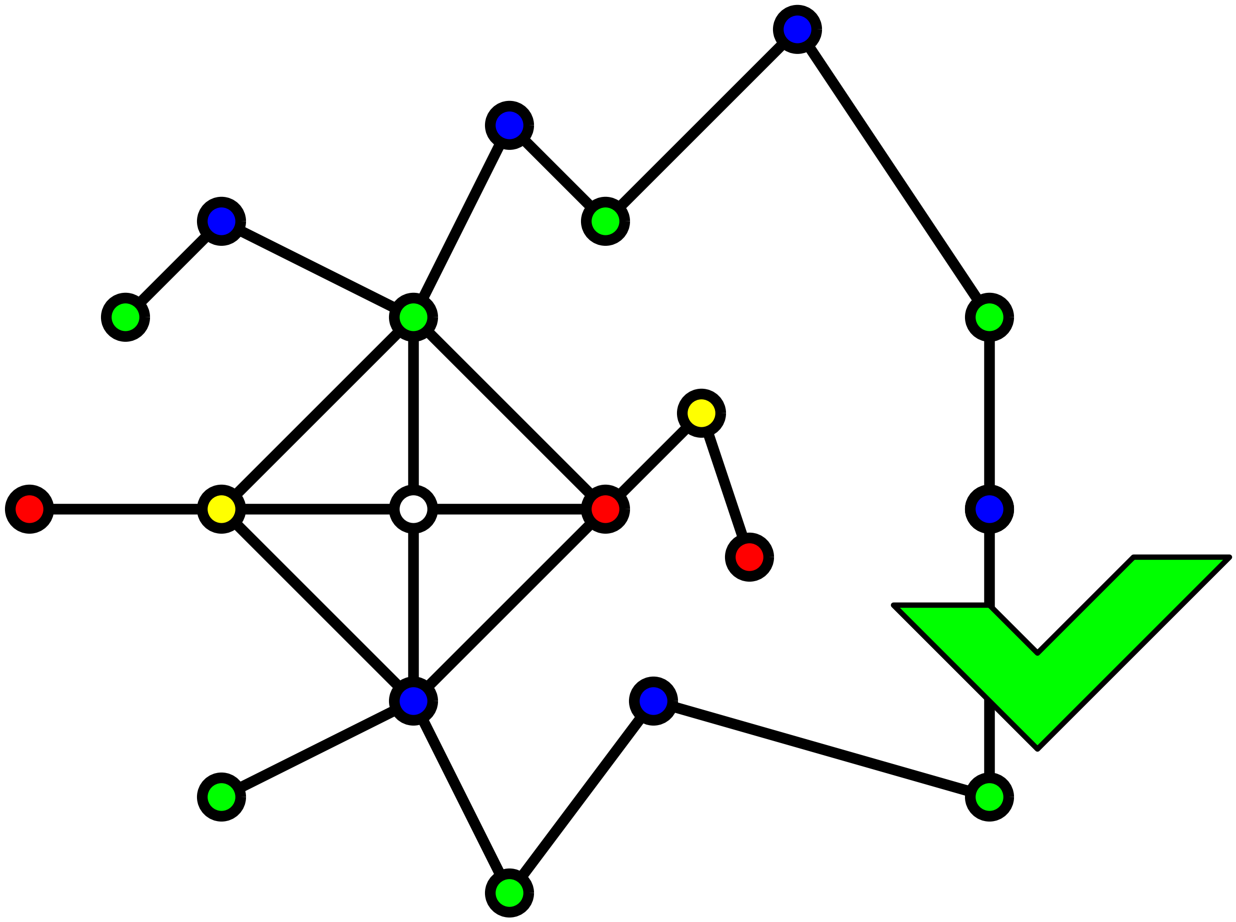
Case 1: The green-blue chain on top does not link up with the green-blue chain on the bottom.



Case A:

Now, there are two cases.

Case 2: The green-blue chain on top does link up with the green-blue chain on the bottom.



Case A:

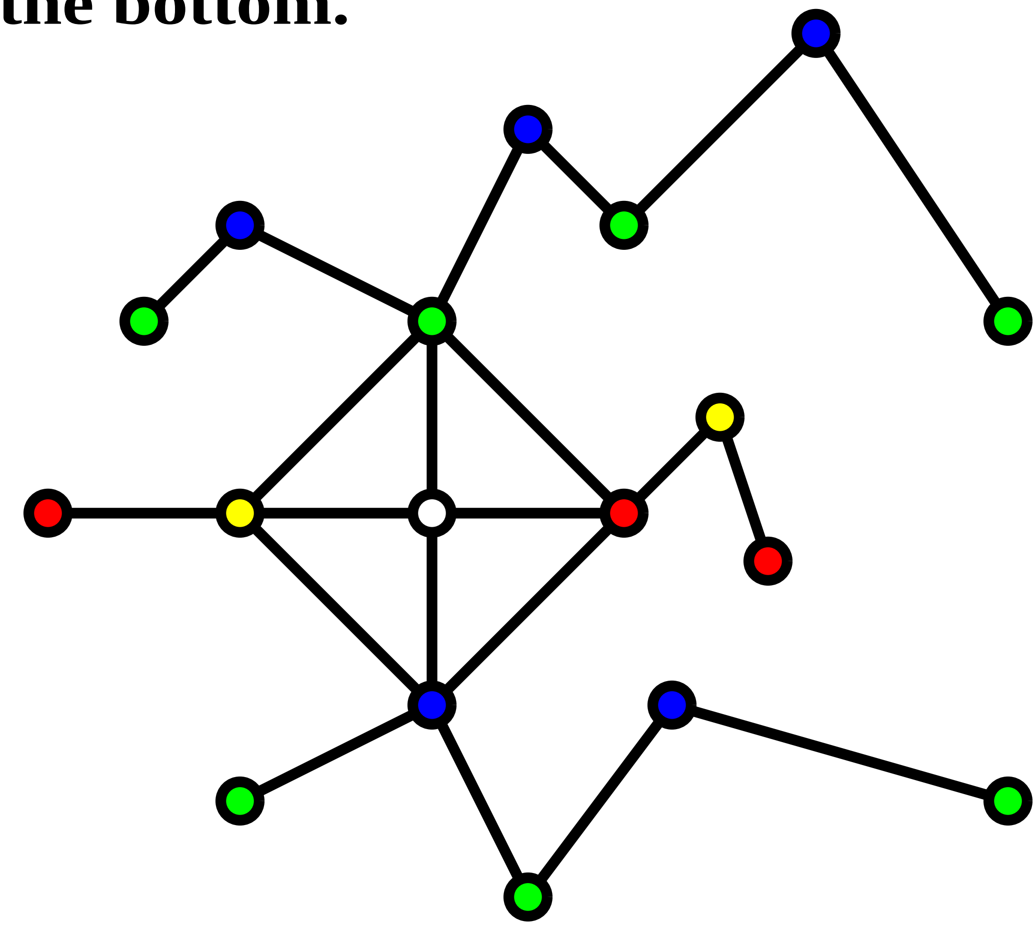
Case 1:

Case A:

Case 1:

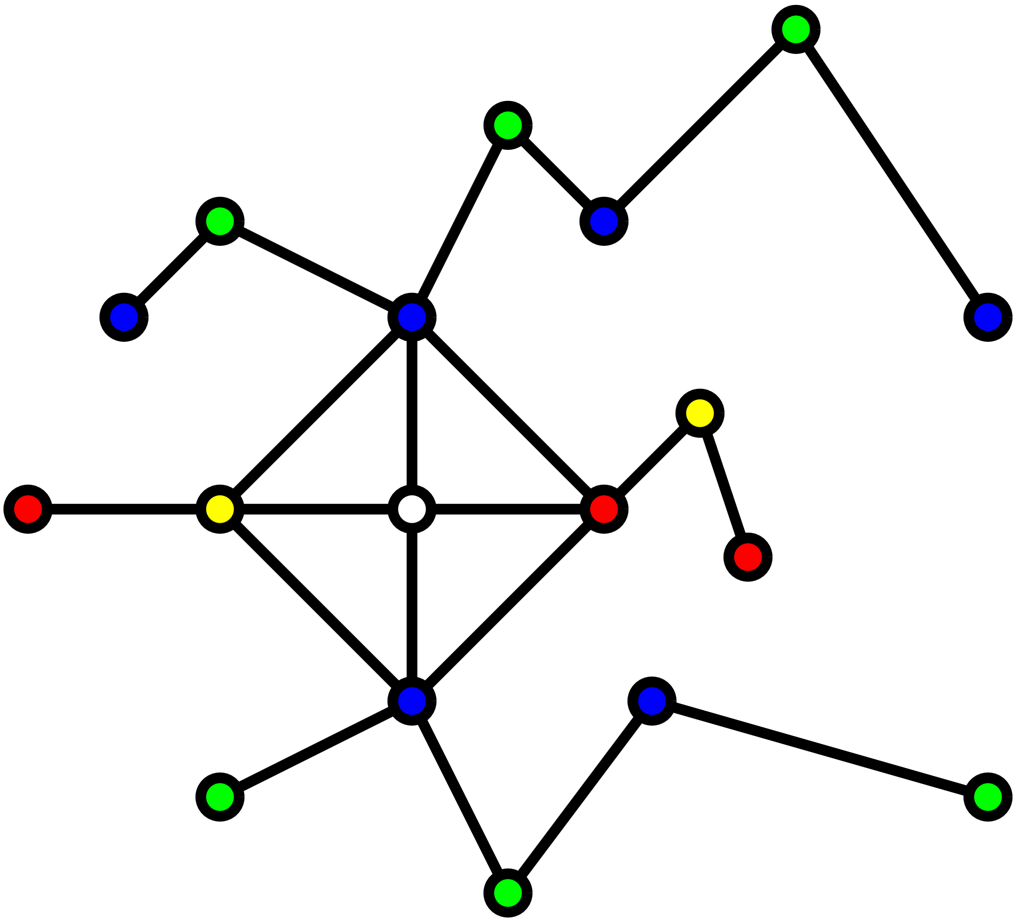
Since the green-blue chain on top does not link up with the green-blue chain on the bottom, interchange the colours green and blue in the green-blue chain on top.

This will not affect the green-blue chain on the bottom.



Case A:

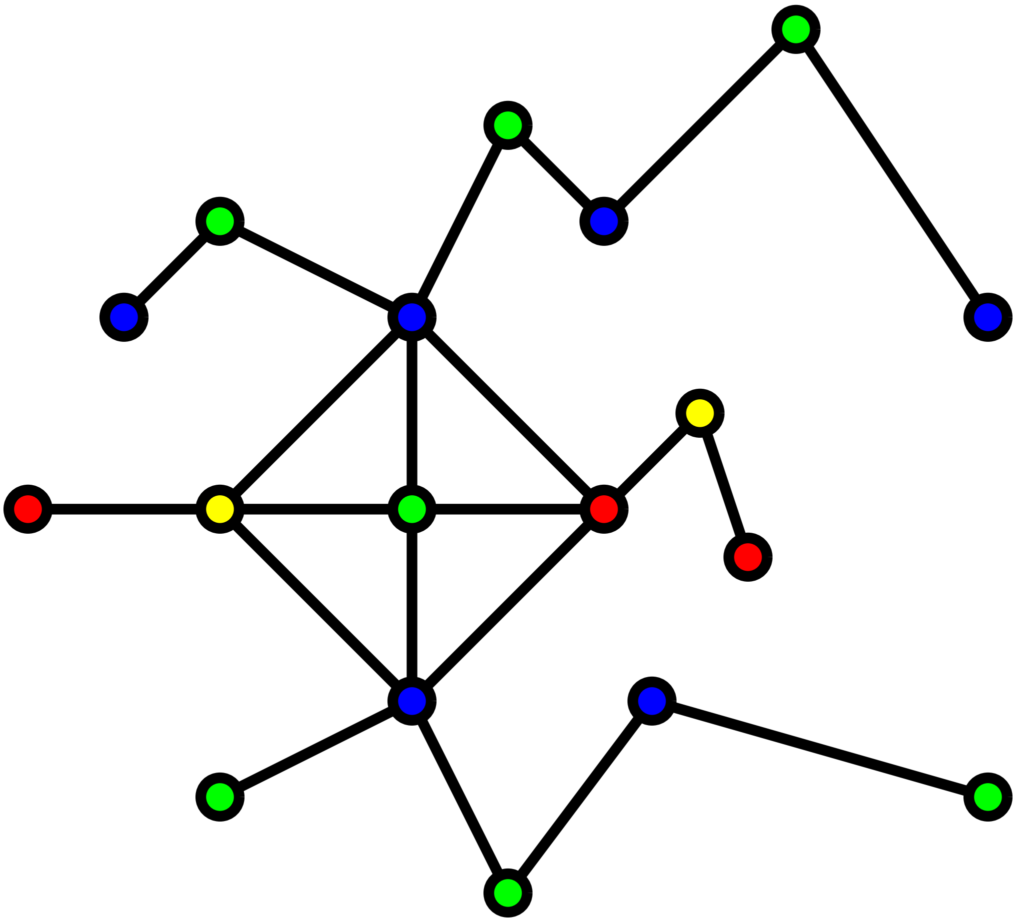
Case 1:



Case A:

Case 1:

Now, the centre vertex can be coloured green.



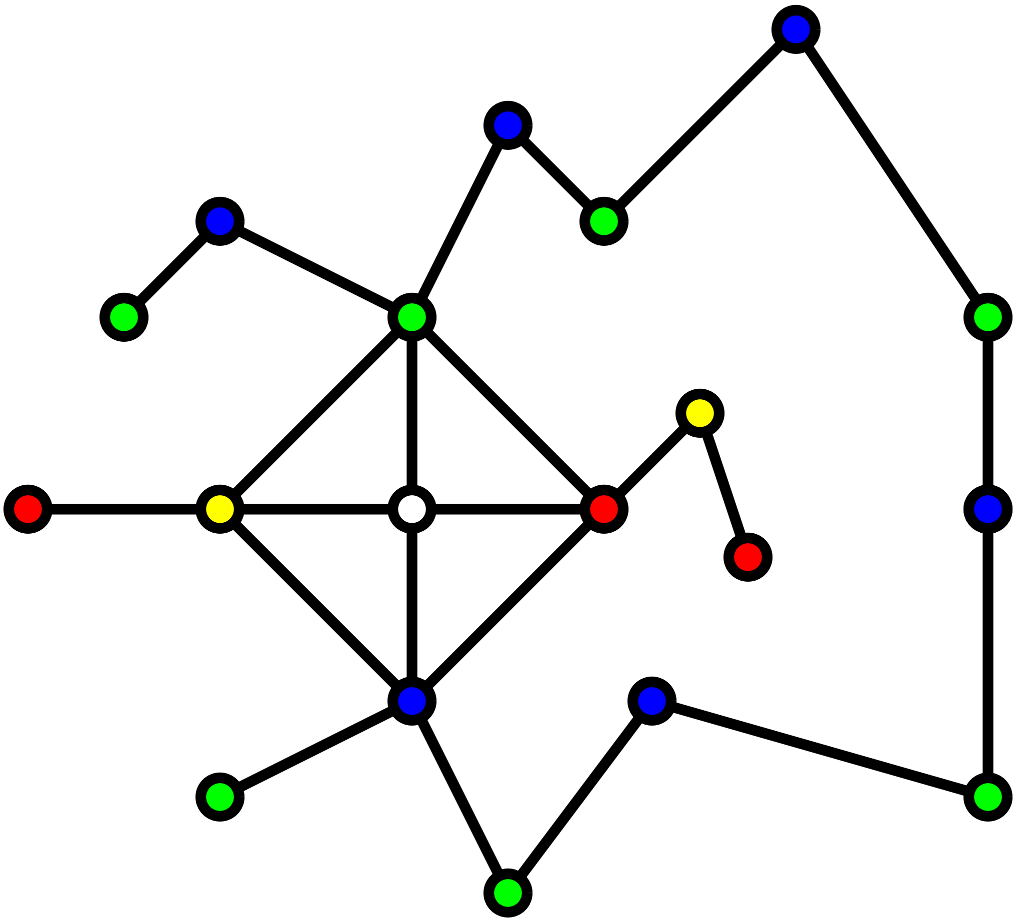
Case A:

Case 2:

Case A:

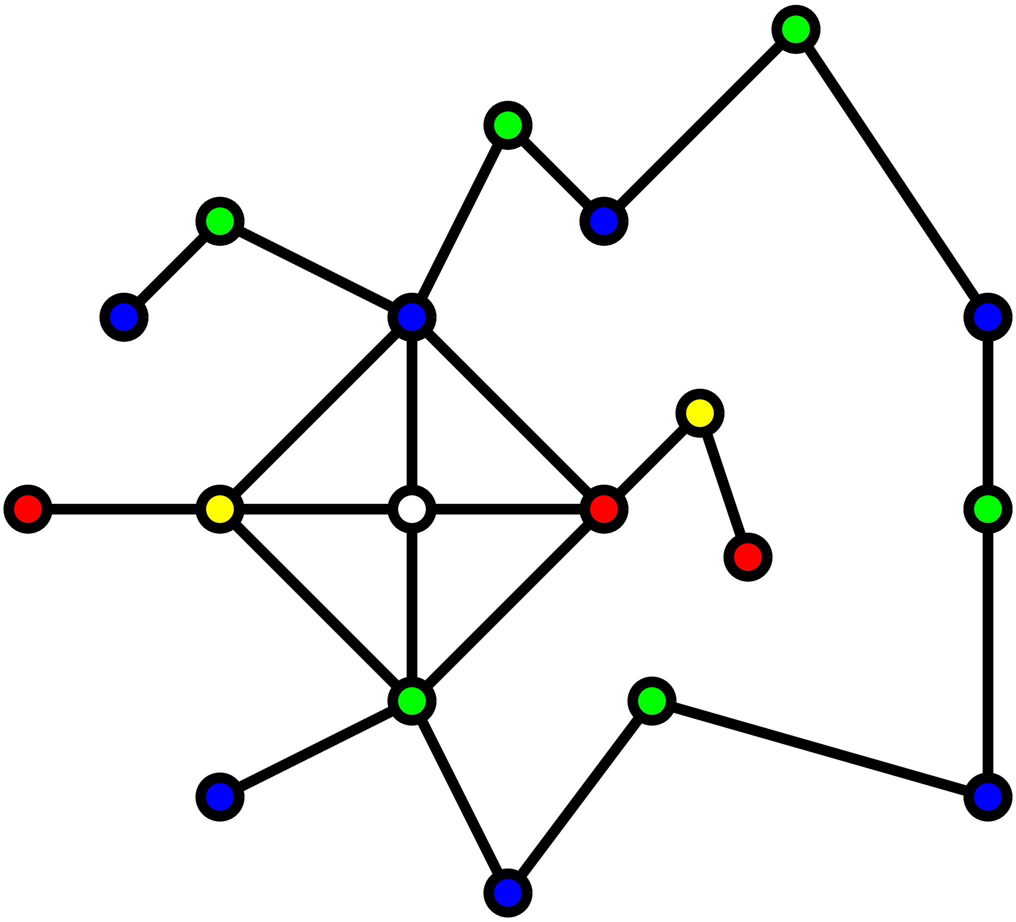
Case 2:

Since the green-blue chain on top links up with the green-blue chain on the bottom, interchanging the colours green and blue does nothing useful.



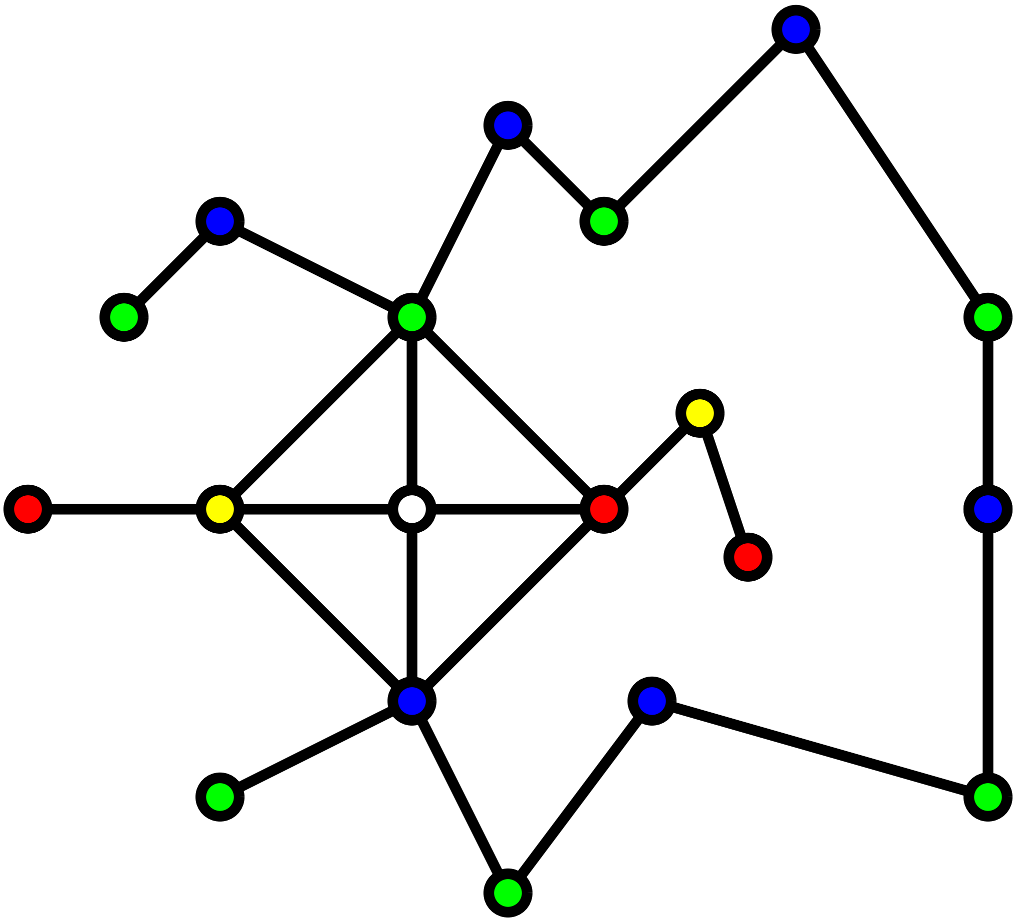
Case A:

Case 2:



Case A:

Case 2:

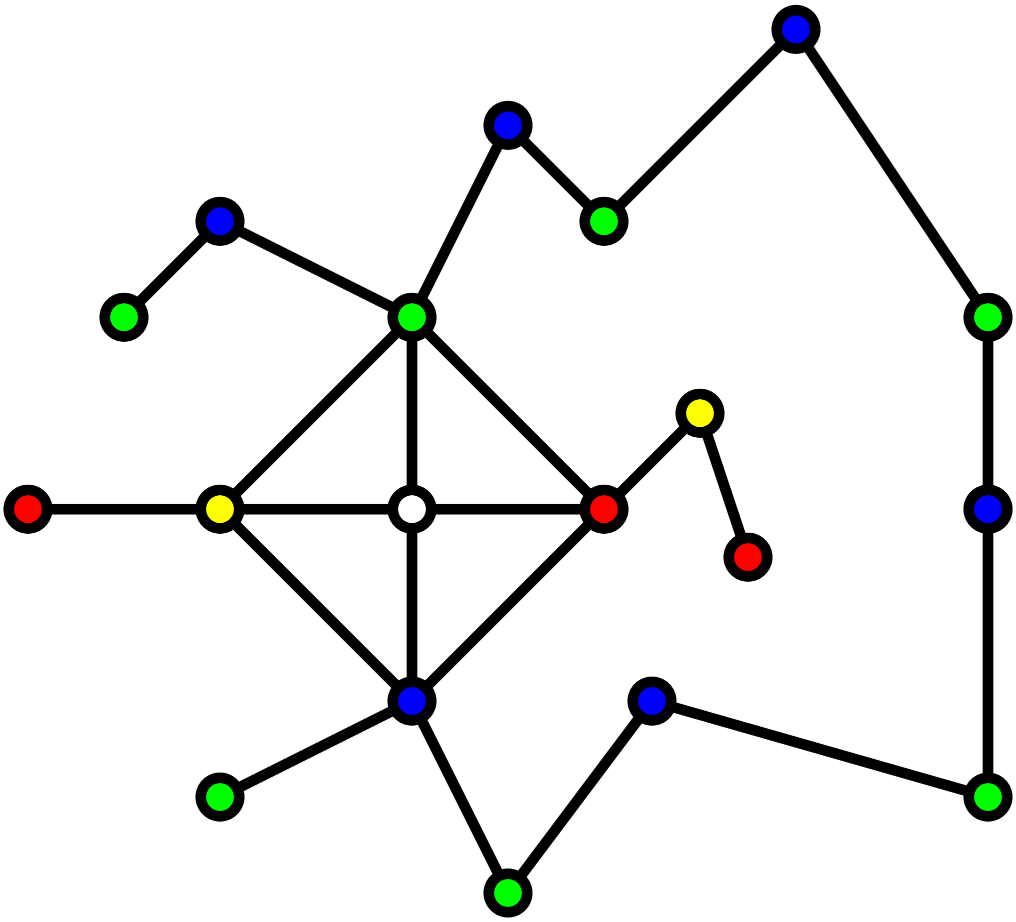


Case A:

Case 2:

So, look at the red-yellow chain inside the link.

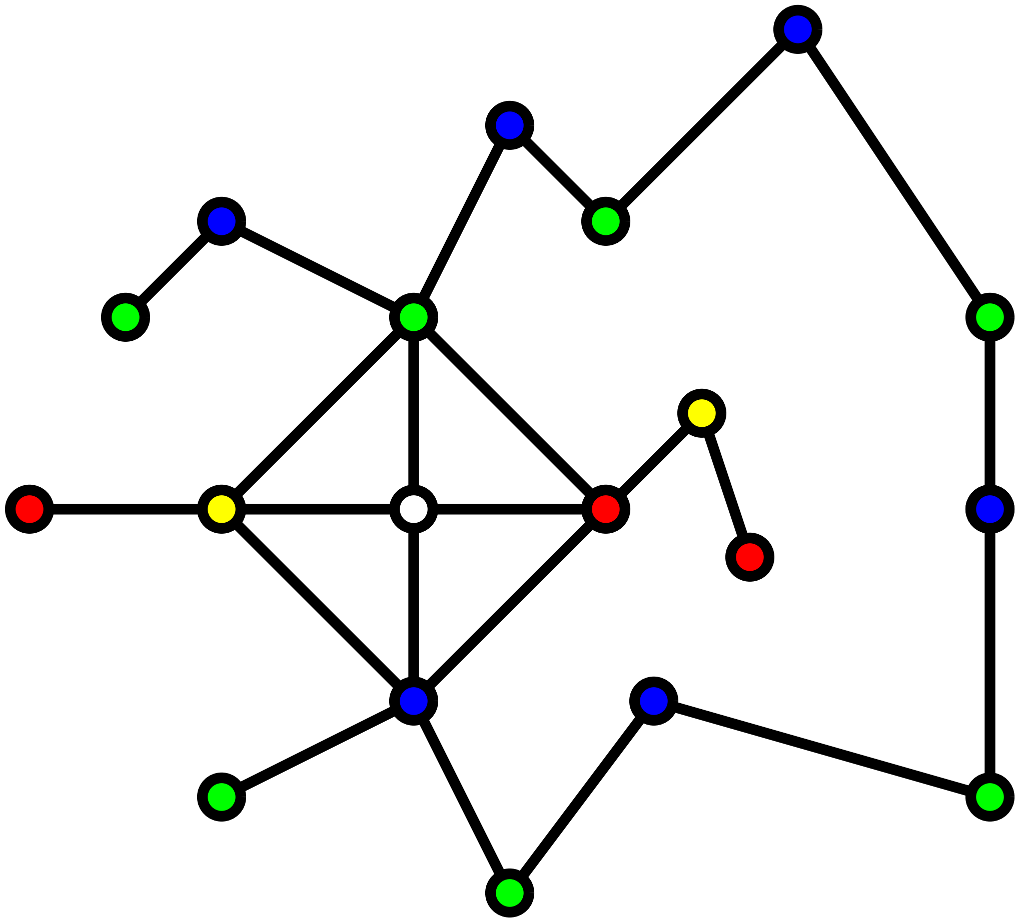
This red-yellow chain cannot link up with red-yellow chain on the left since the graph is planar.



Case A:

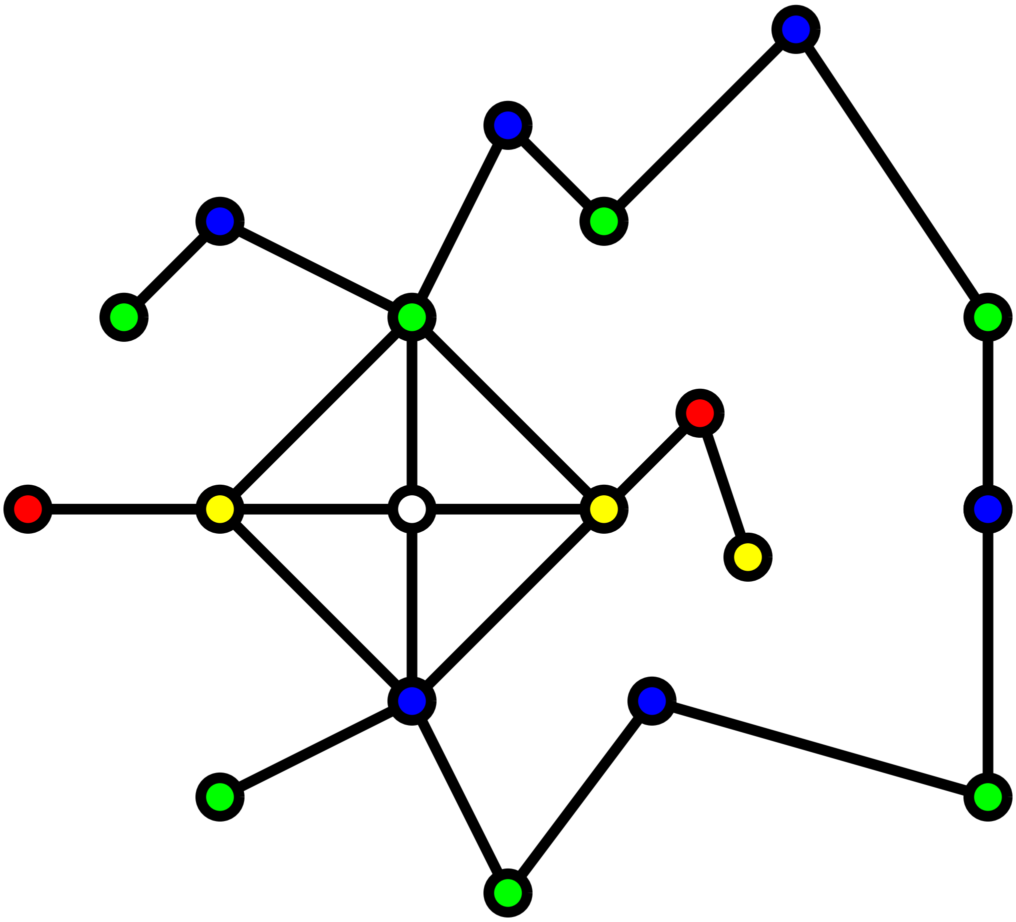
Case 2:

Interchange the colours red and yellow.



Case A:

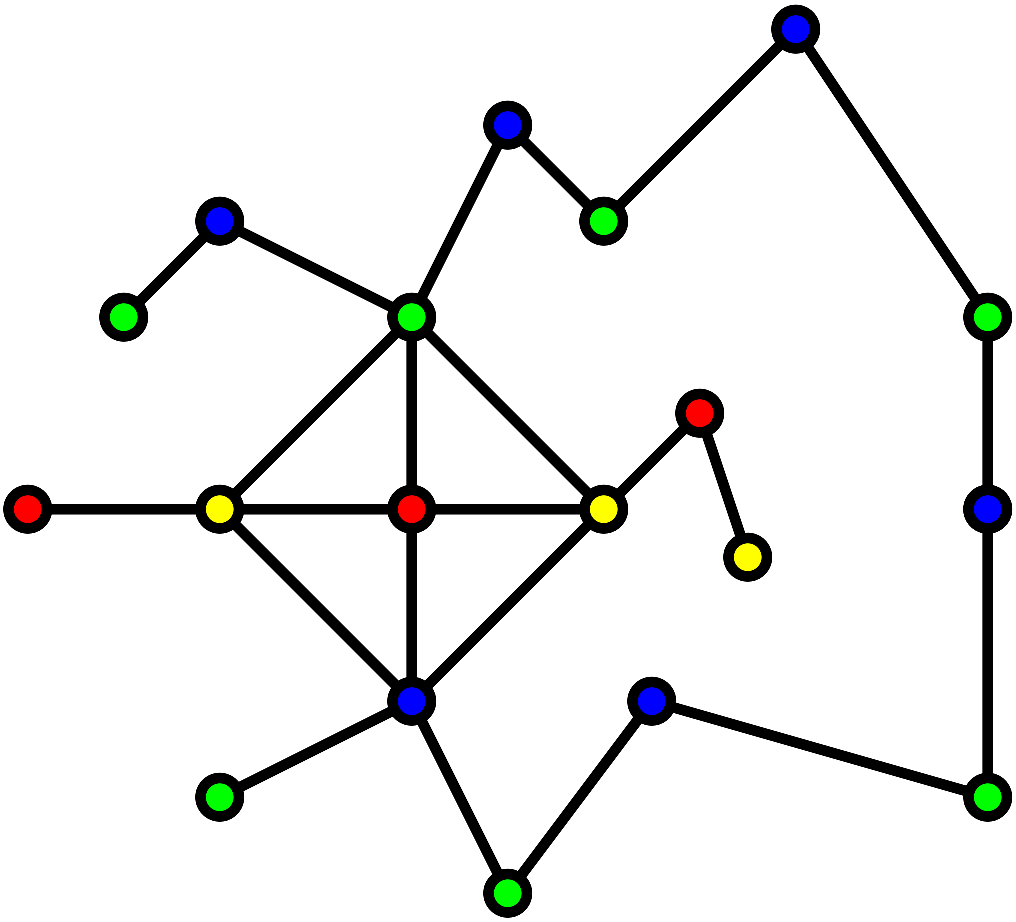
Case 2:



Case A:

Case 2:

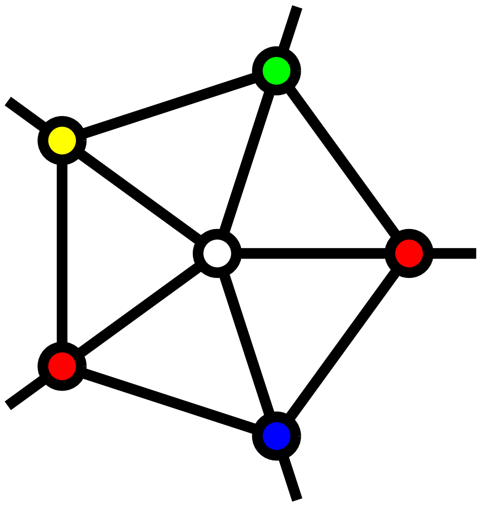
Now, the centre vertex can be coloured red.



Case B:

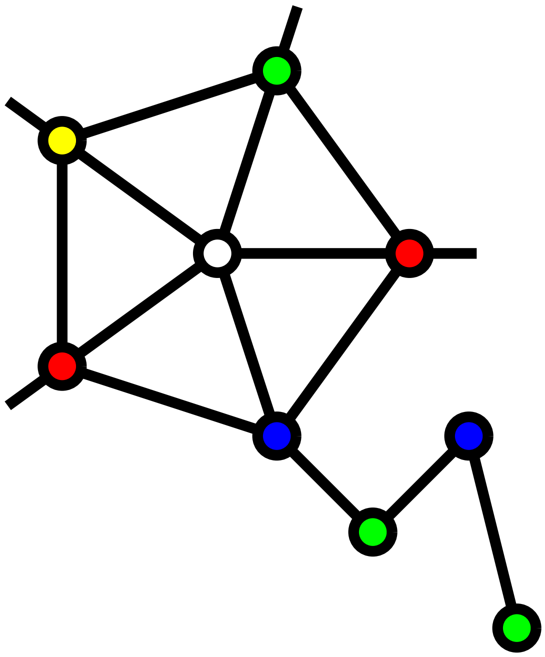
Case B:

Here, the blue vertex is the start of a green-blue chain.



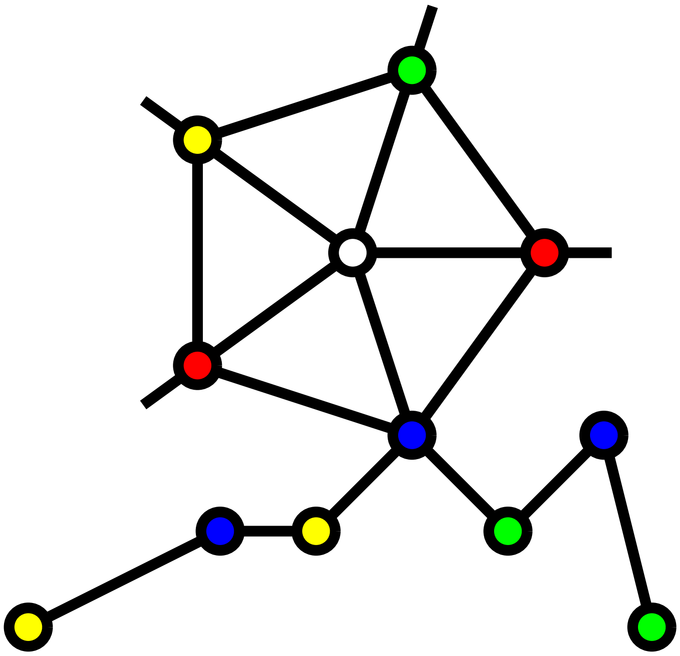
Case B:

Here, the blue vertex is the start of a green-blue chain.



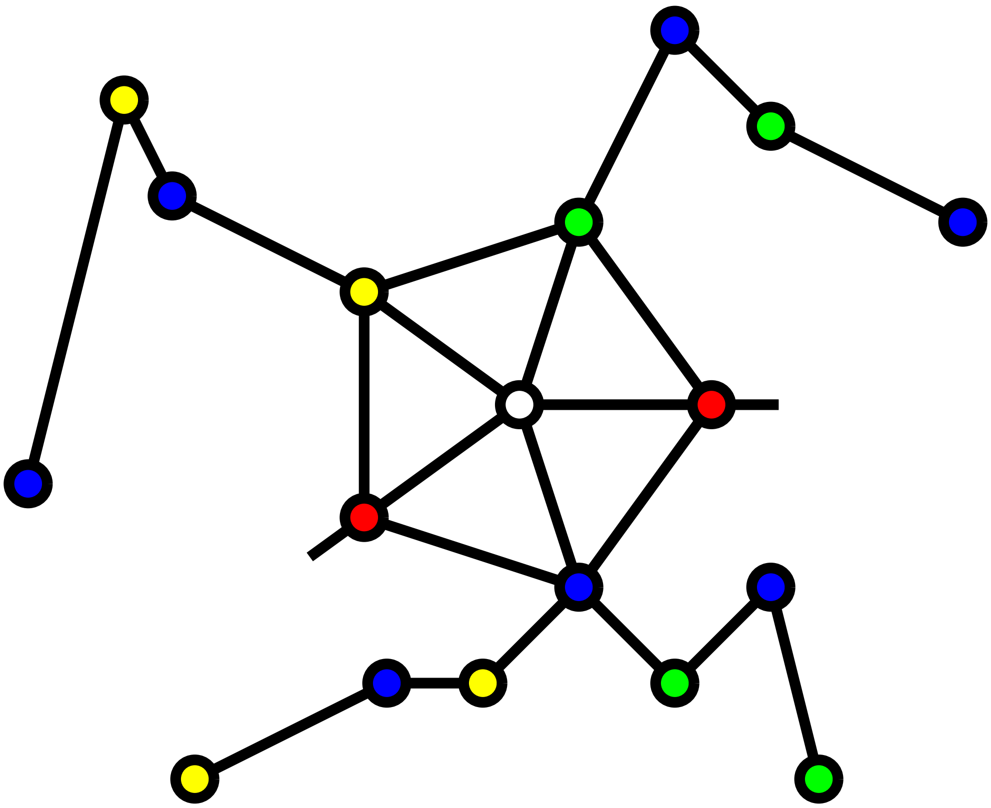
Case B:

The blue vertex is also the start of a yellow-blue chain.



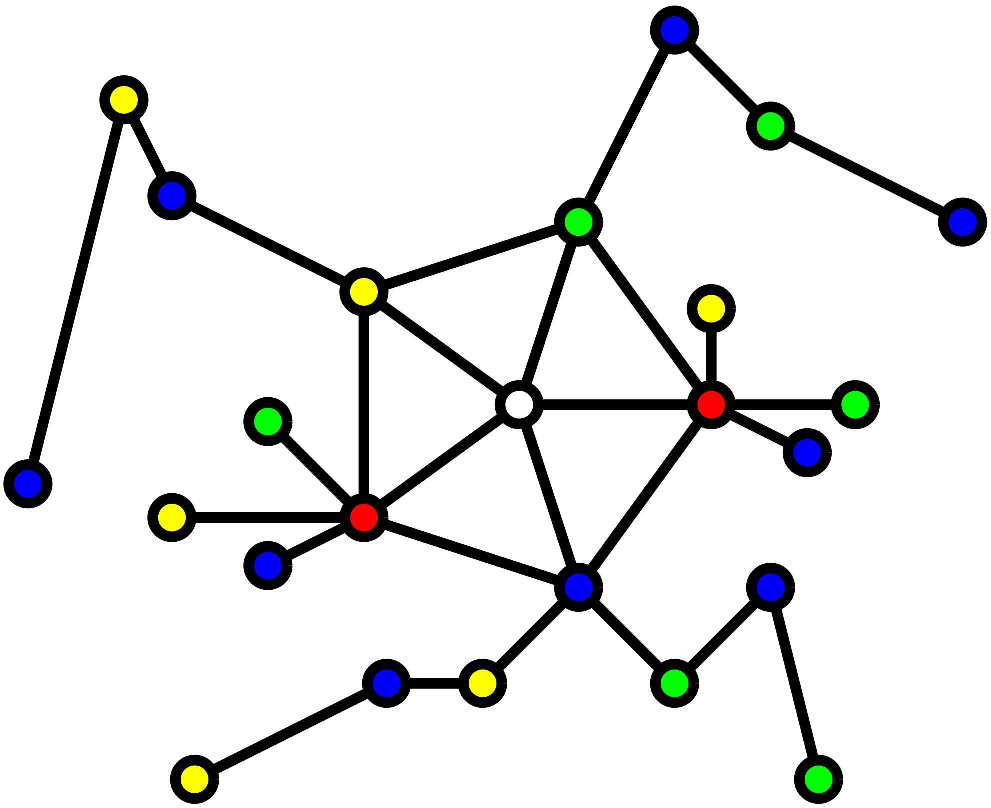
Case B:

The green vertex is the start of a green-blue chain and the yellow vertex is the start of a yellow-blue chain.



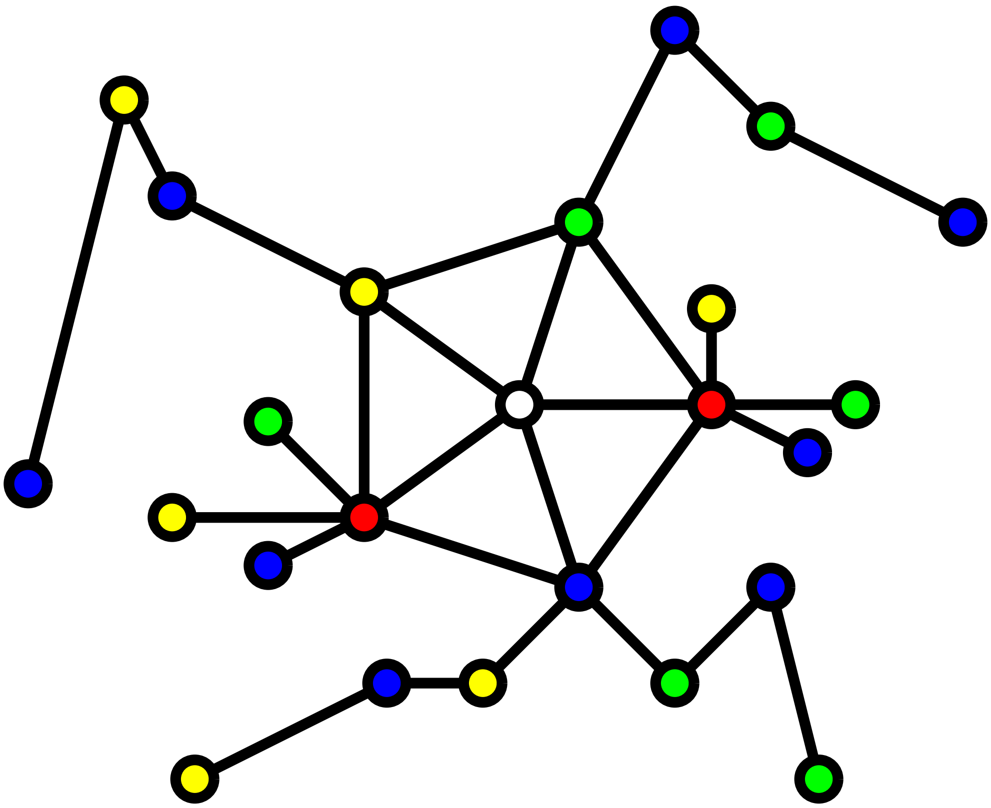
Case B:

The red vertices are the start of red-green, red-yellow and red-blue chains.



Case B:

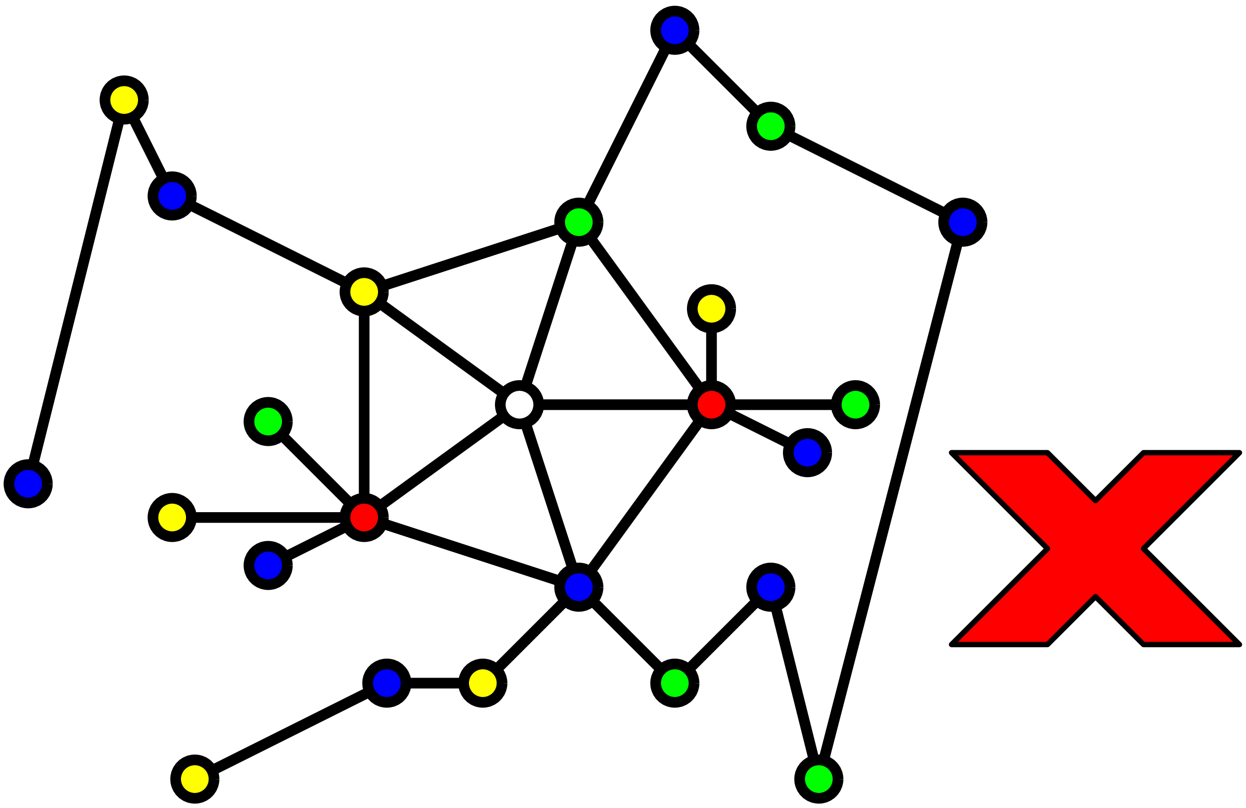
Now, there are two cases.



Case B:

Now, there are two cases.

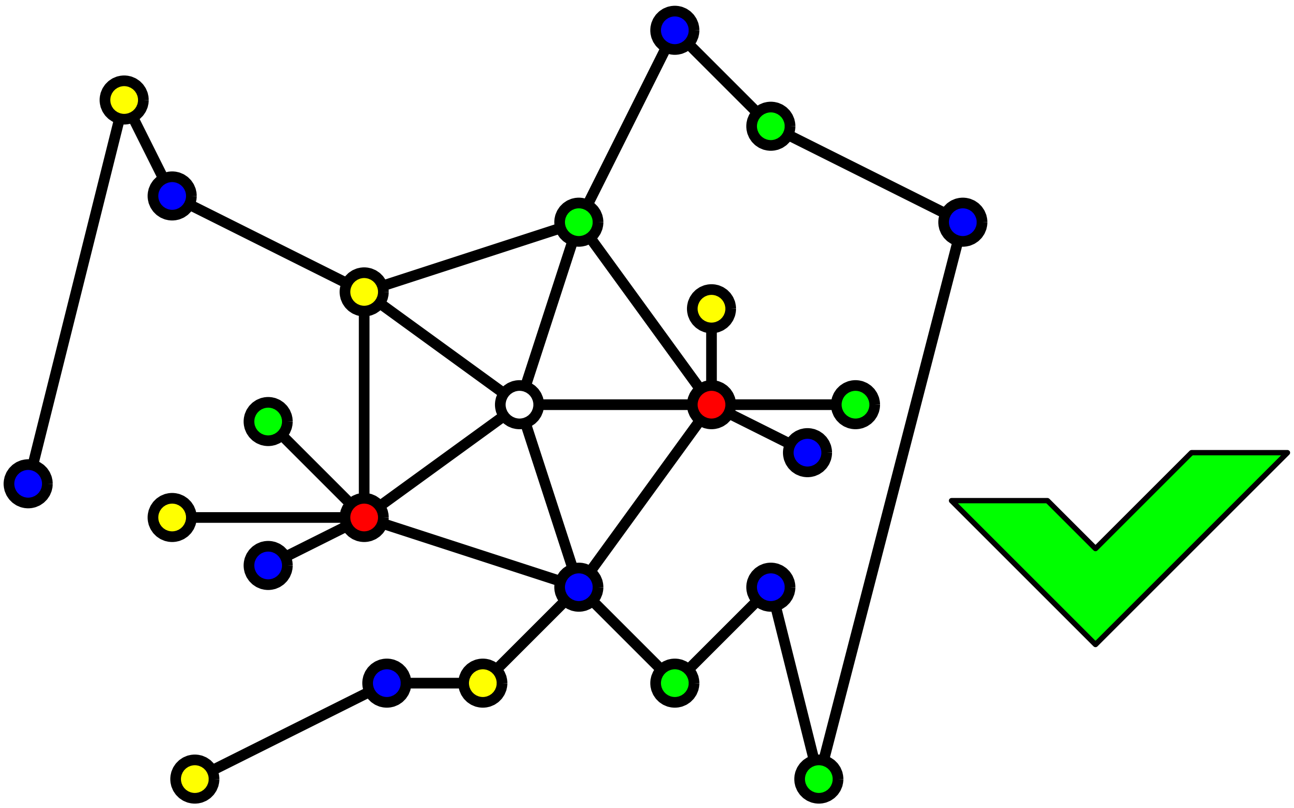
Case 1: The green-blue chain on top does not link up with the green-blue chain on the bottom.



Case B:

Now, there are two cases.

Case 2: The green-blue chain on top does link up with the green-blue chain on the bottom.



Case B:

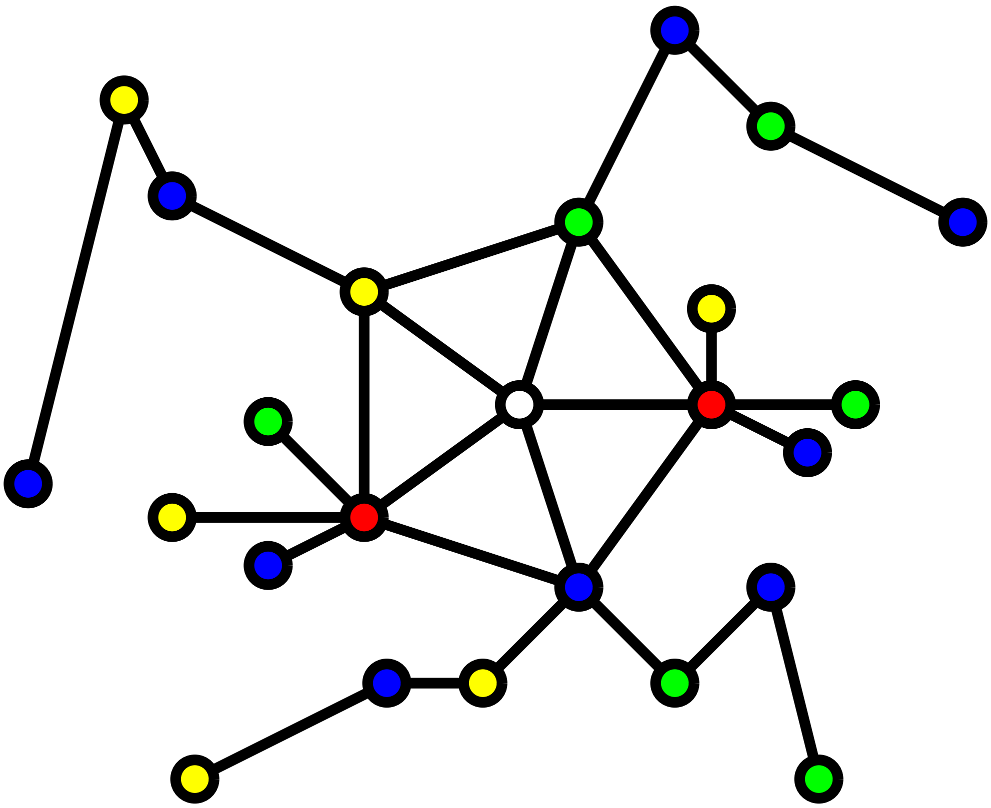
Case 1:

Case B:

Case 1:

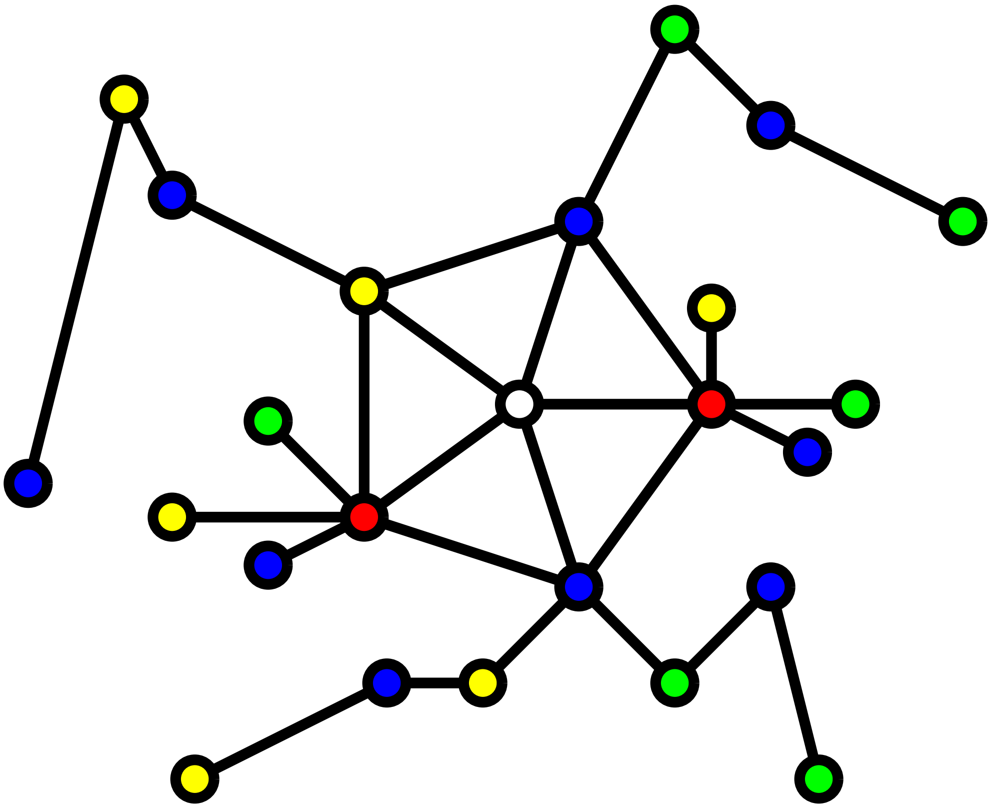
Since the green-blue chain on top does not link up with the green-blue chain on the bottom, interchange the colours green and blue in the green-blue chain on top.

This will not affect the green-blue chain on the bottom.



Case B:

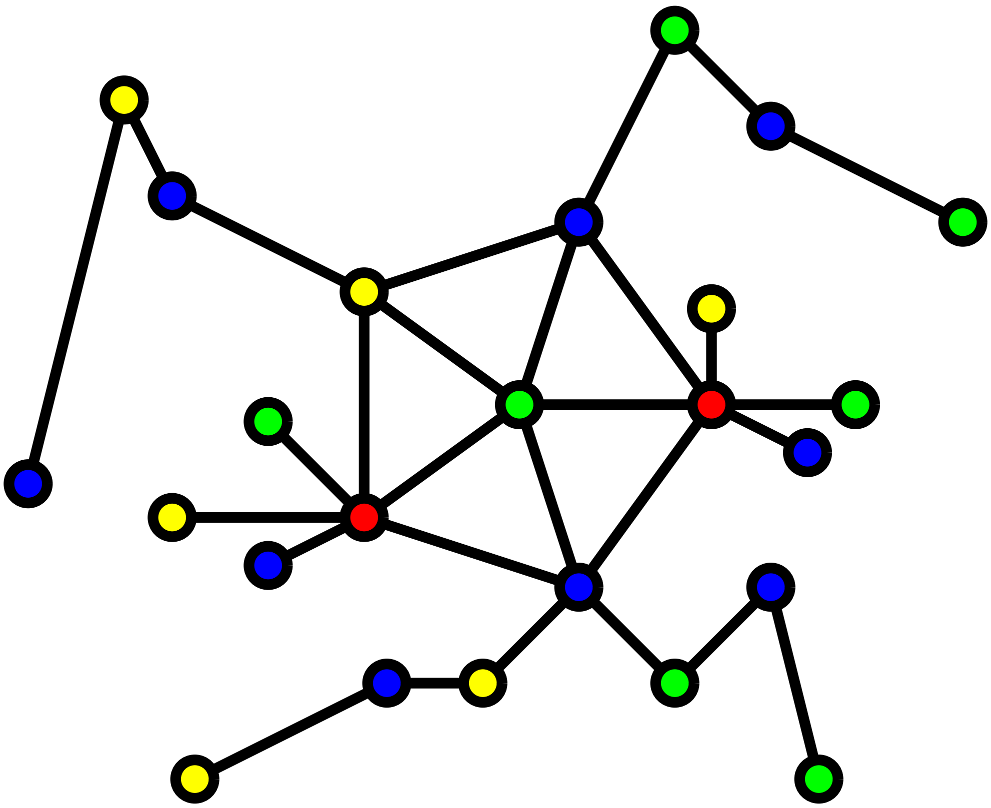
Case 1:



Case B:

Case 1:

Now, the centre vertex can be coloured green.



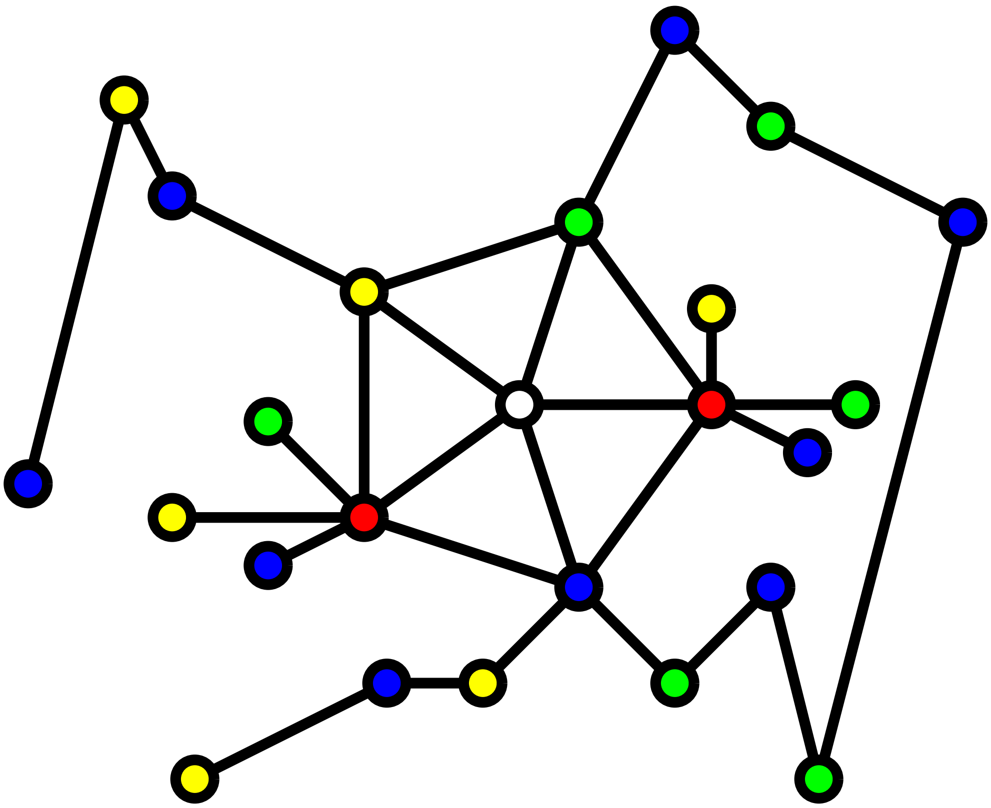
Case B:

Case 2:

Case B:

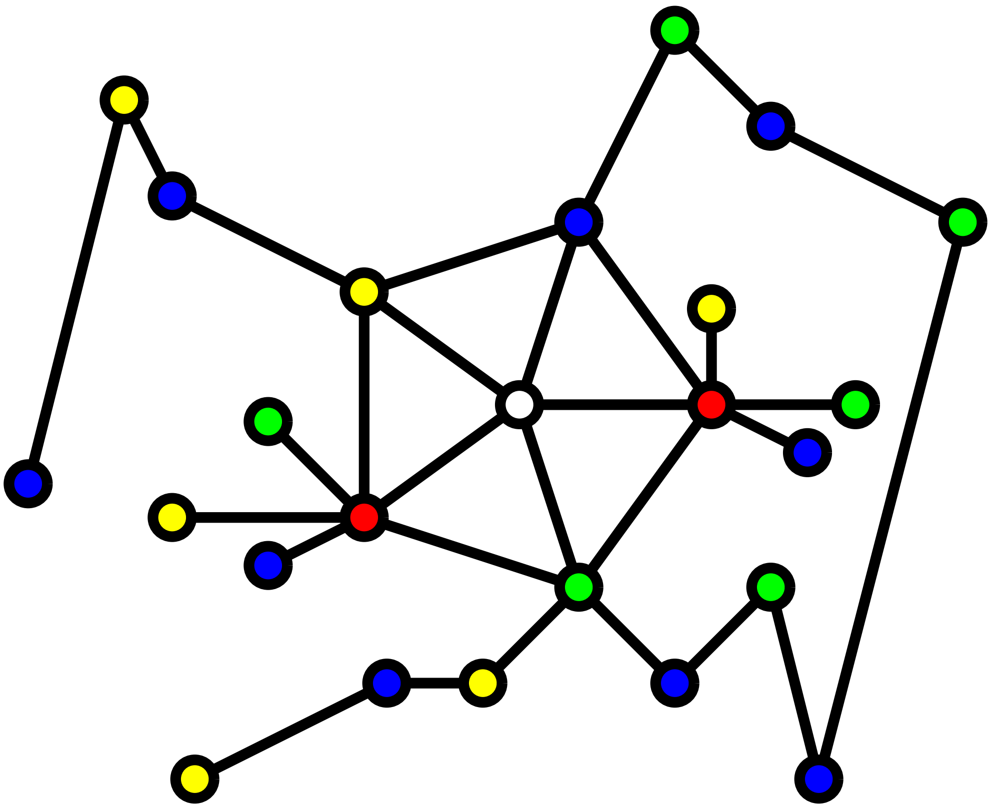
Case 2:

Since the green-blue chain on top links up with the green-blue chain on the bottom, interchanging the colours green and blue does nothing useful.



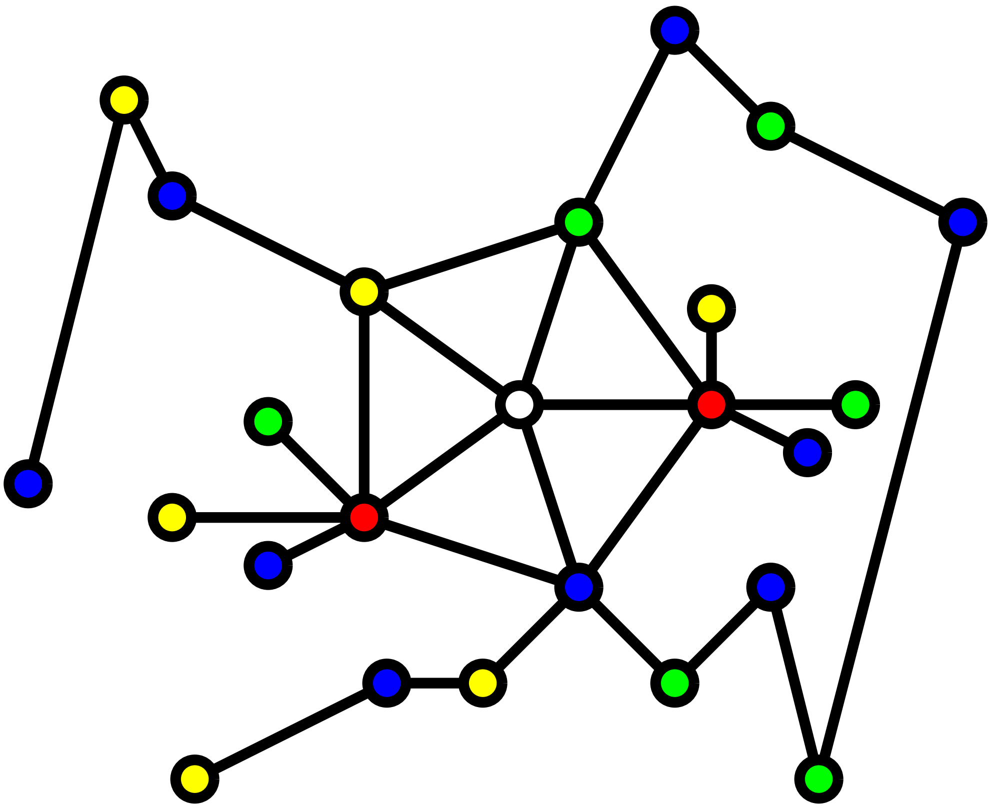
Case B:

Case 2:



Case B:

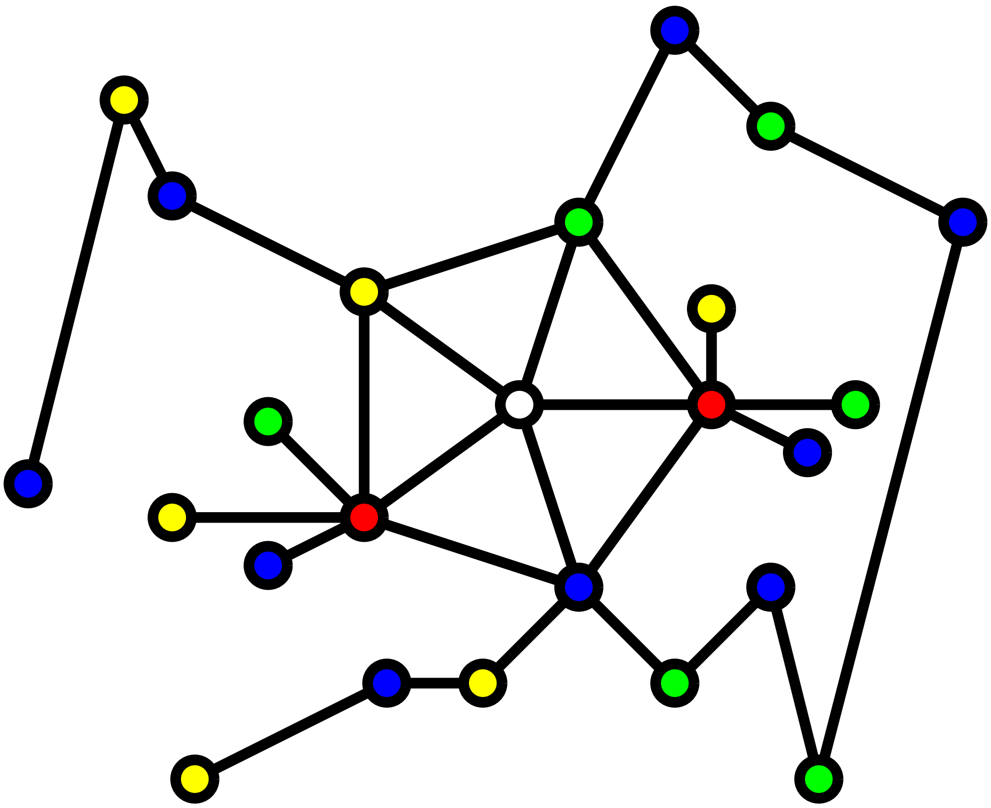
Case 2:



Case B:

Case 2:

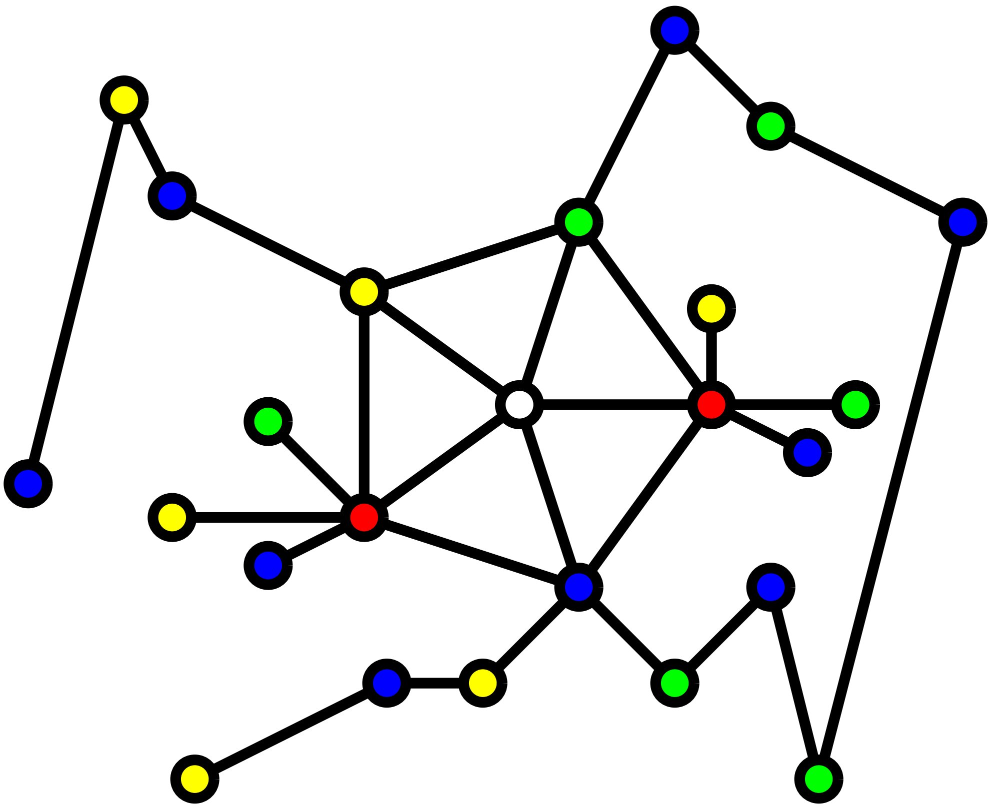
So, look at the yellow-blue chains.



Case B:

Case 2:

Now, there are two cases.

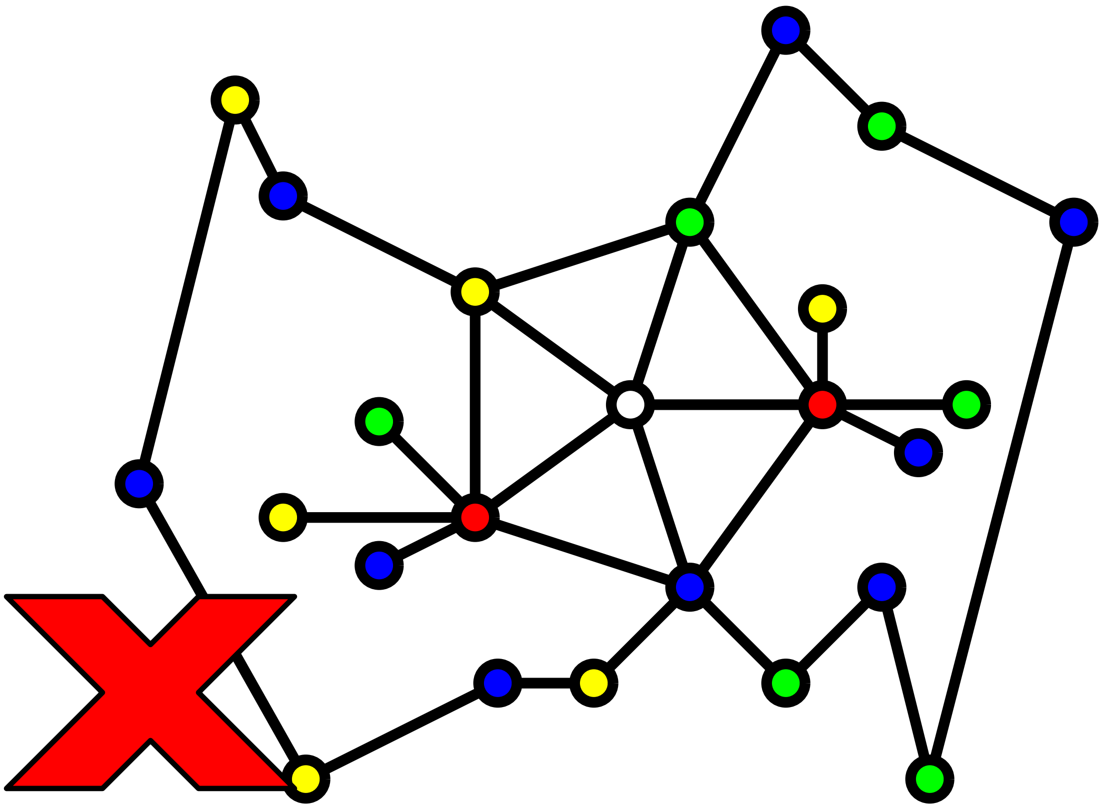


Case B:

Case 2:

Now, there are two cases.

Case i: The yellow-blue chain on top does not link up with the yellow-blue chain on the bottom.

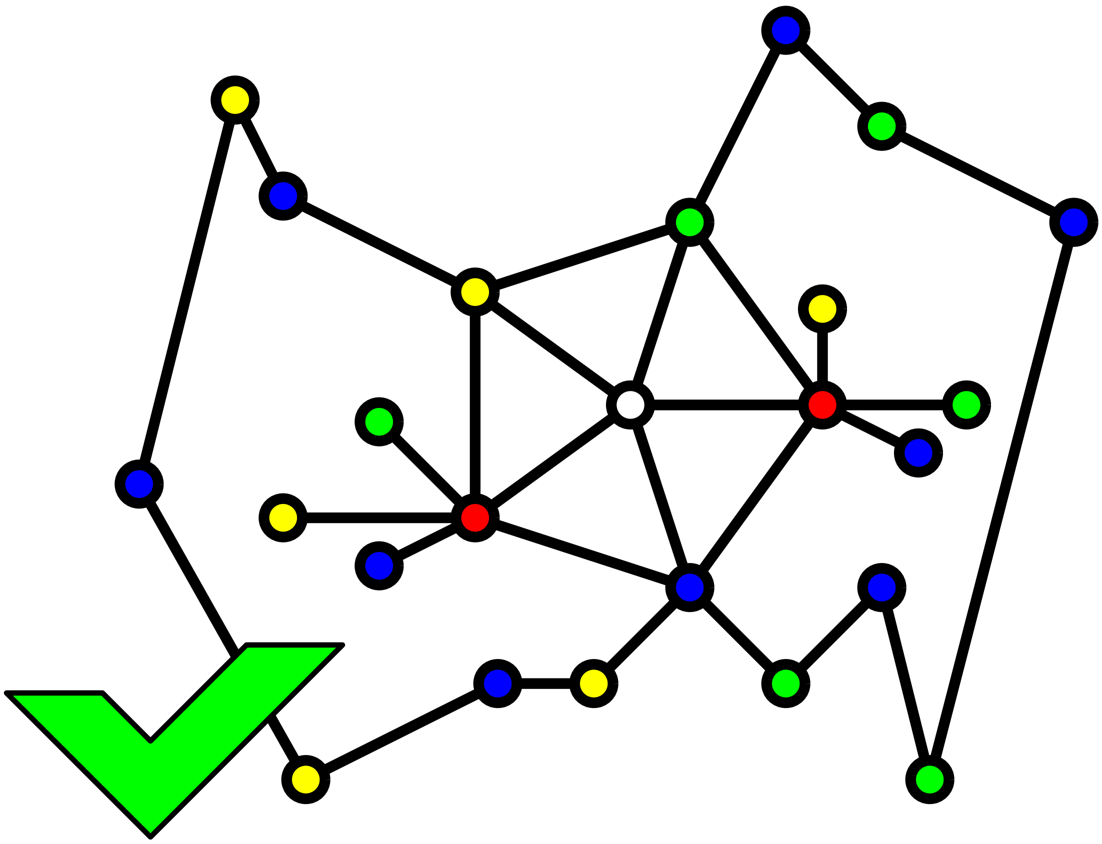


Case B:

Case 2:

Now, there are two cases.

Case ii: The yellow-blue chain on top does link up with the yellow-blue chain on the bottom.



Case B:

Case 2:

Case i:

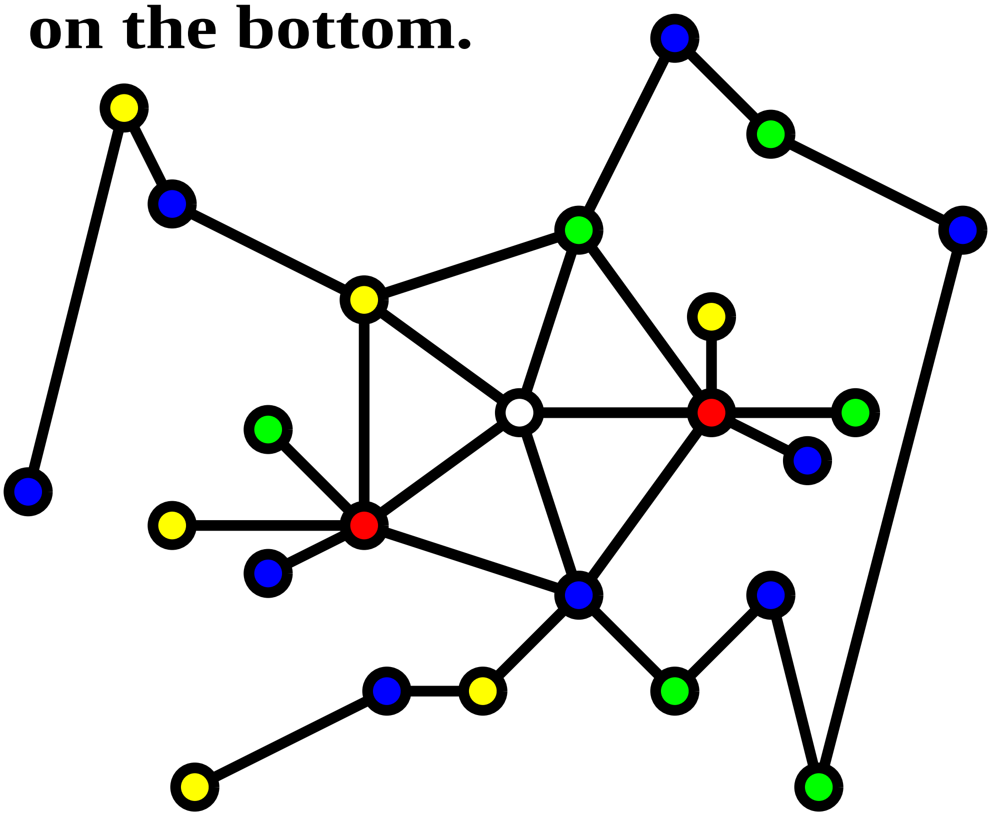
Case B:

Case 2:

Case i:

Since the yellow-blue chain on top does not link up with the yellow-blue chain on the bottom, interchange the colours yellow and blue in the yellow-blue chain on top.

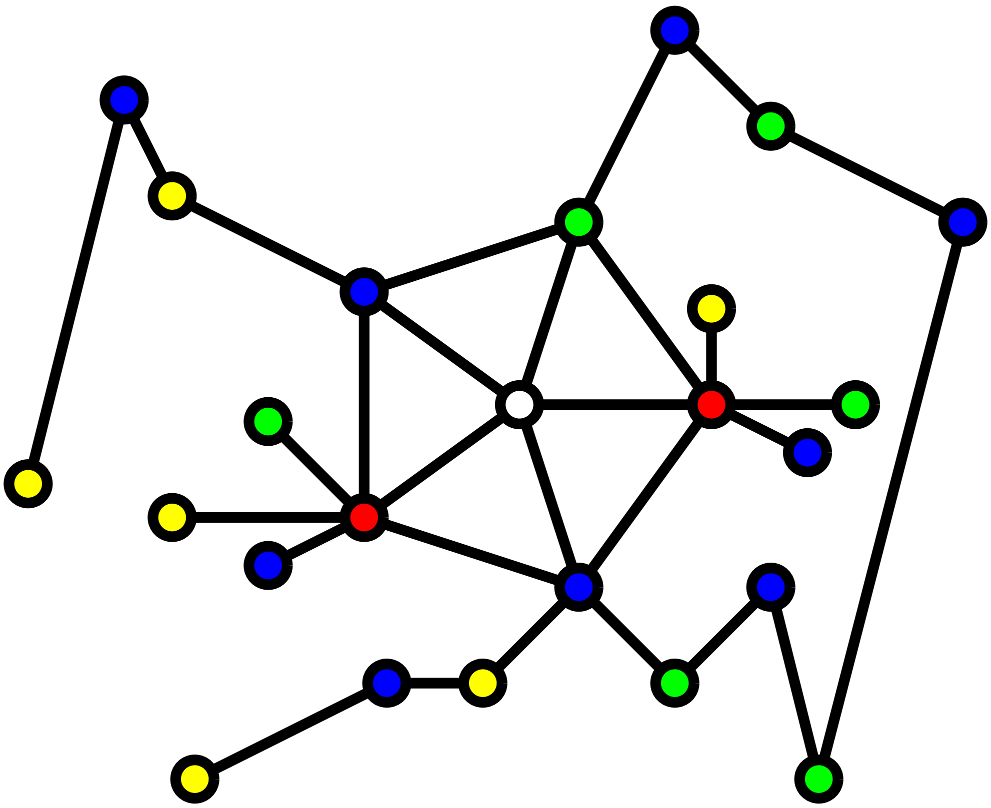
This will not affect the yellow-blue chain on the bottom.



Case B:

Case 2:

Case i:

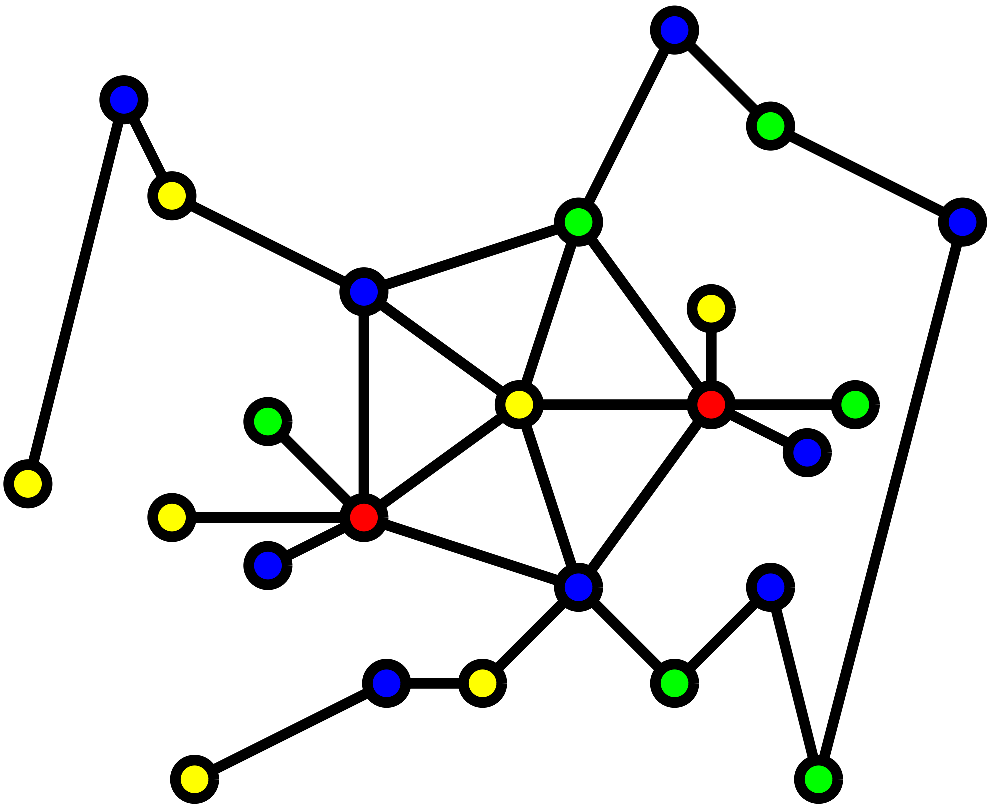


Case B:

Case 2:

Case i:

Now, the centre vertex can now be coloured yellow.



Case B:

Case 2:

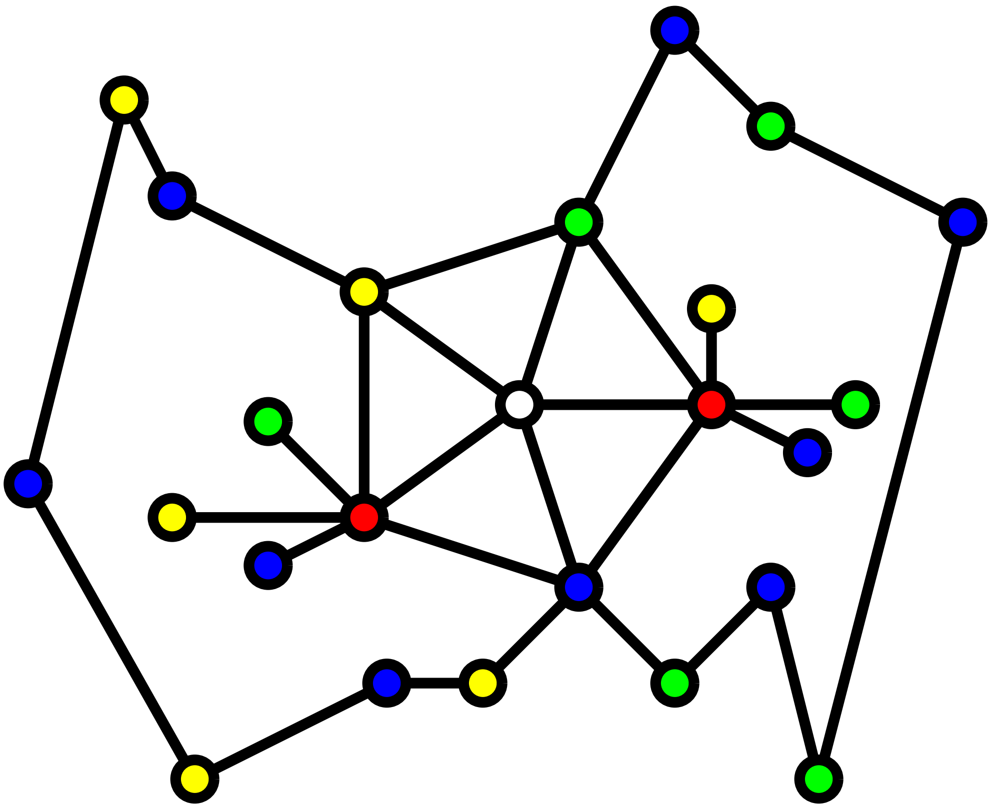
Case ii:

Case B:

Case 2:

Case ii:

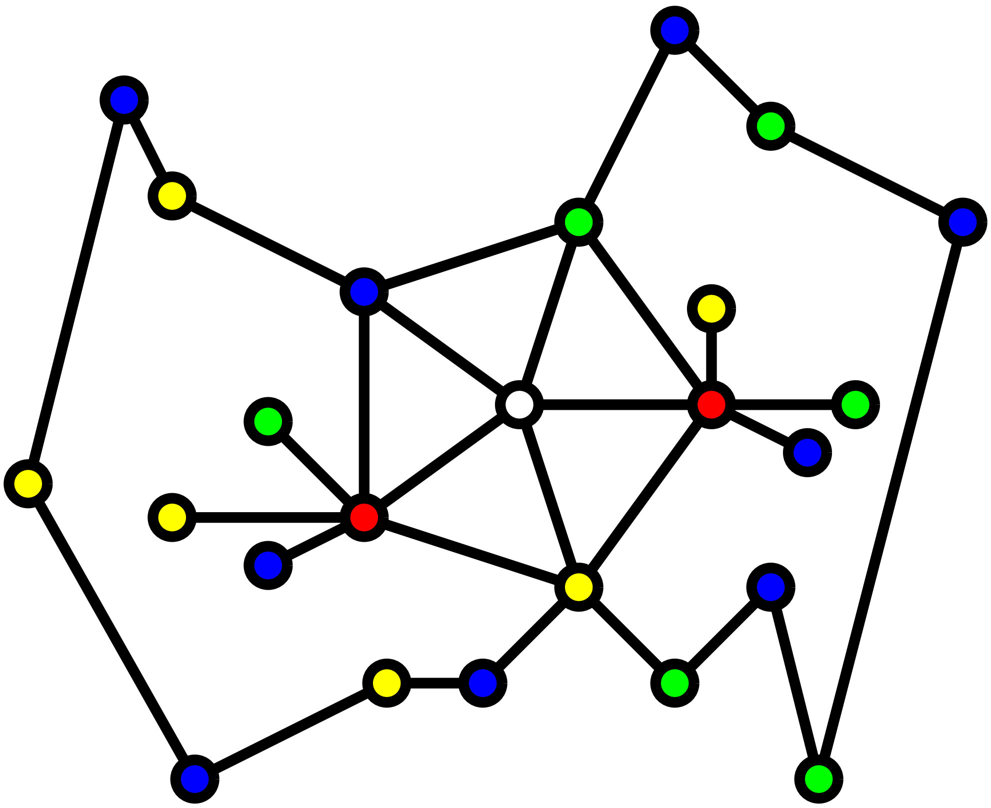
Since the yellow-blue chain on top links up with the yellow-blue chain on the bottom, interchanging the colours yellow and blue does nothing useful.



Case B:

Case 2:

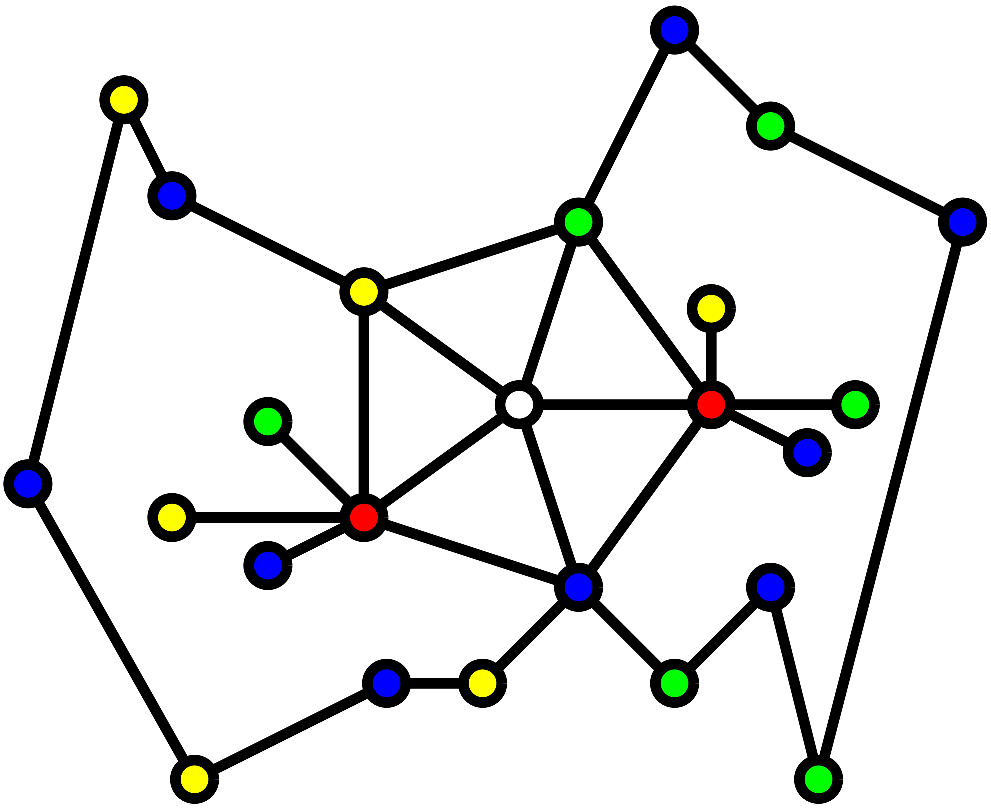
Case ii:



Case B:

Case 2:

Case ii:



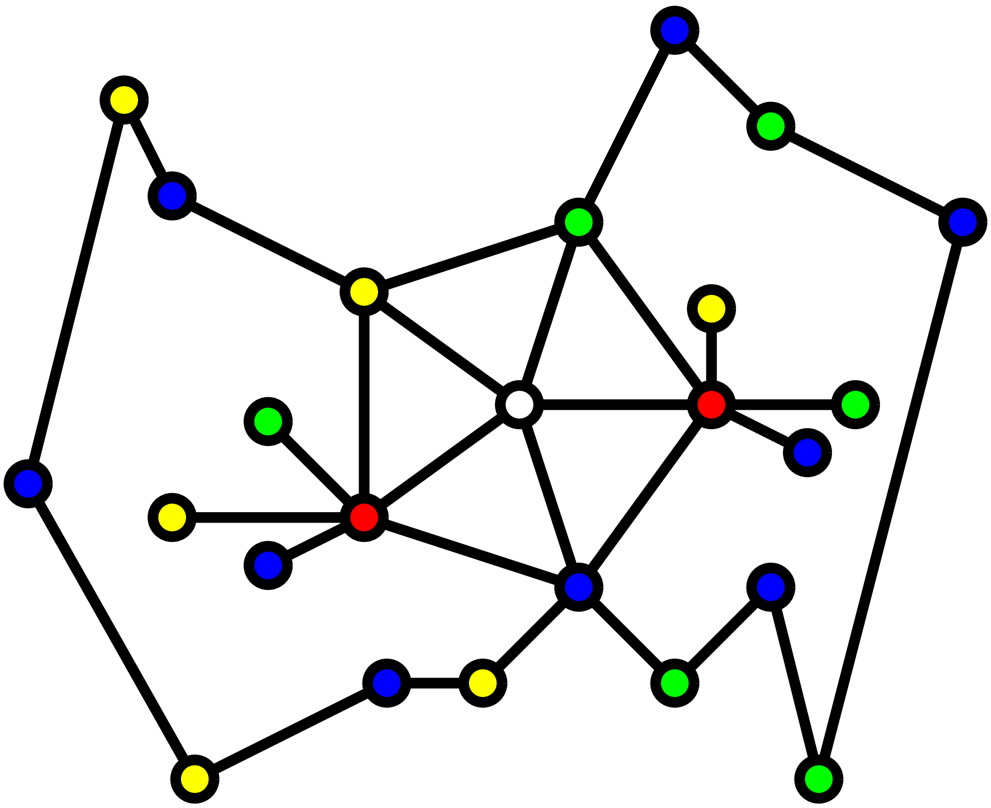
Case B:

Case 2:

Case ii:

So, look at the chains inside the left link.

The chains on the left cannot link up with the chains on the right since the graph is planar.



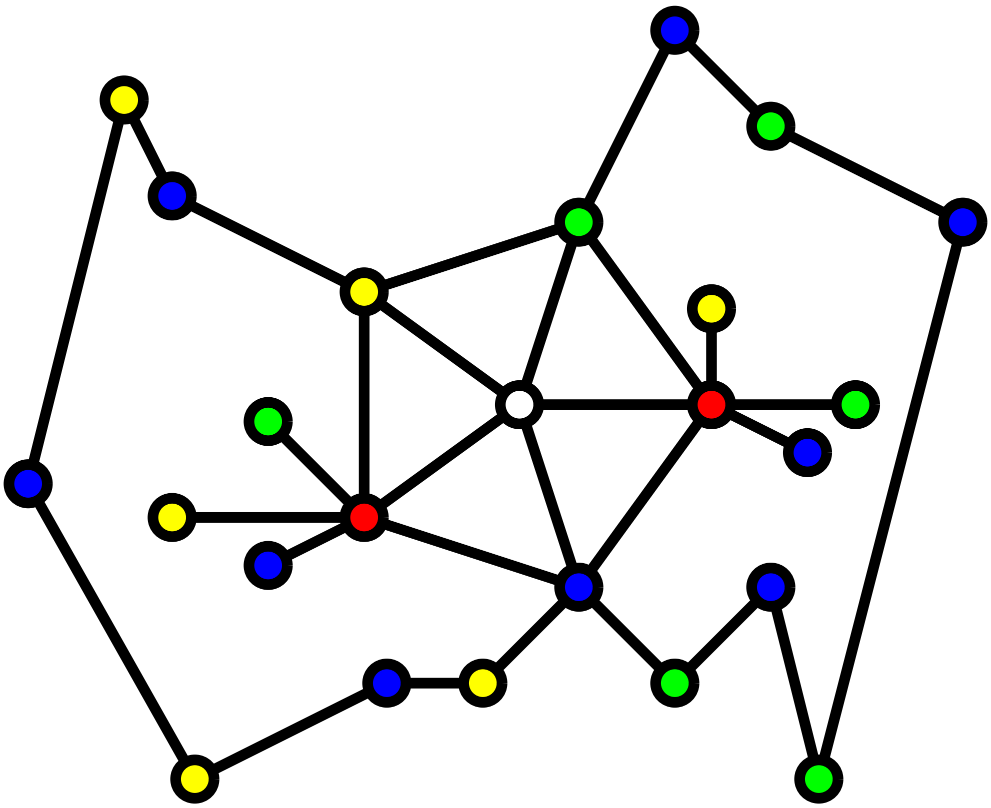
Case B:

Case 2:

Case ii:

Interchange the colours red and green.

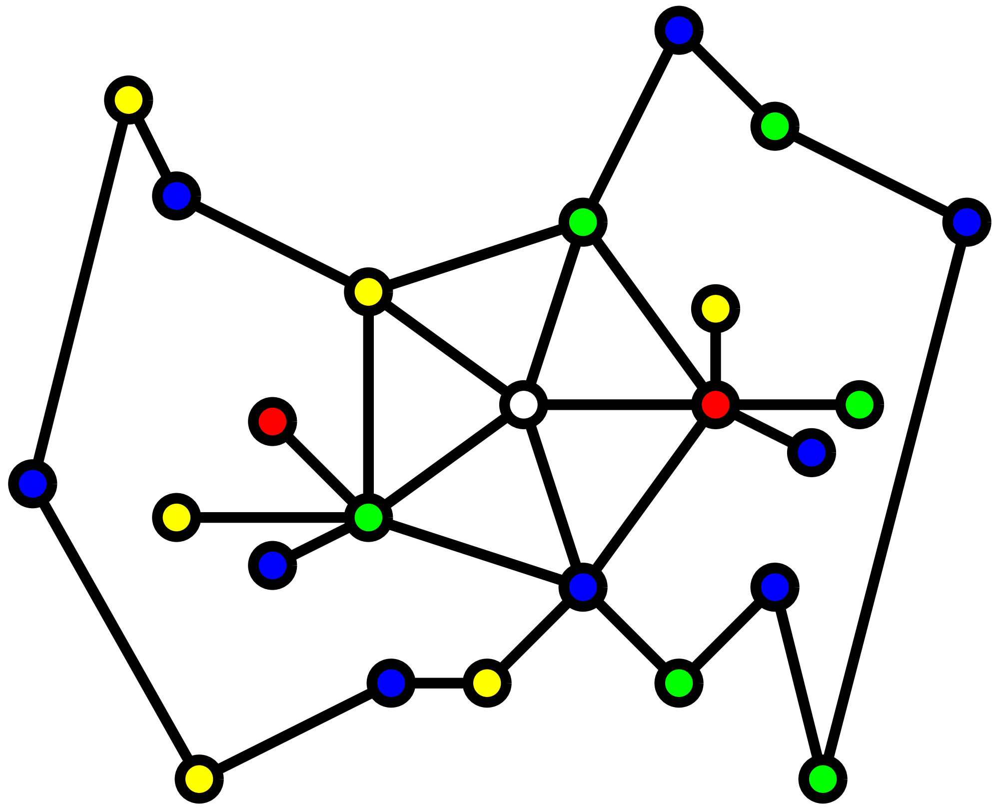
This will not affect the chains in the link on the right.



Case B:

Case 2:

Case ii:

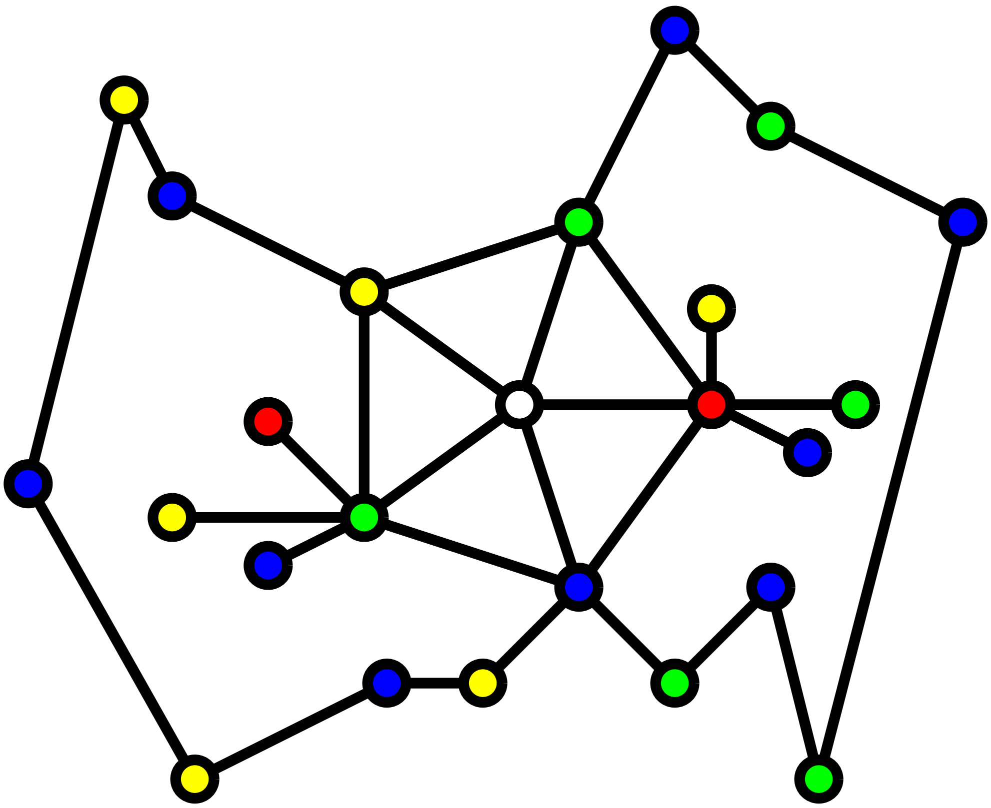


Case B:

Case 2:

Case ii:

However, the centre vertex can still not be coloured.



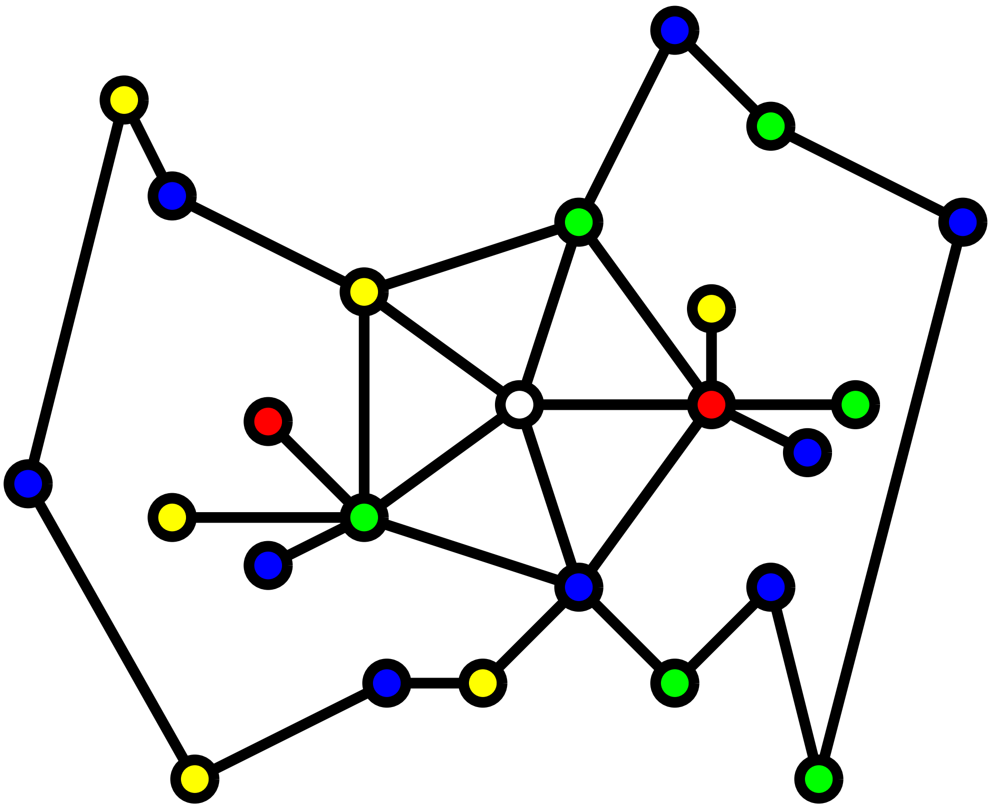
Case B:

Case 2:

Case ii:

So, look at the chains in the link on the right.

The chains on the right cannot link up with the chains on the left since the graph is planar.



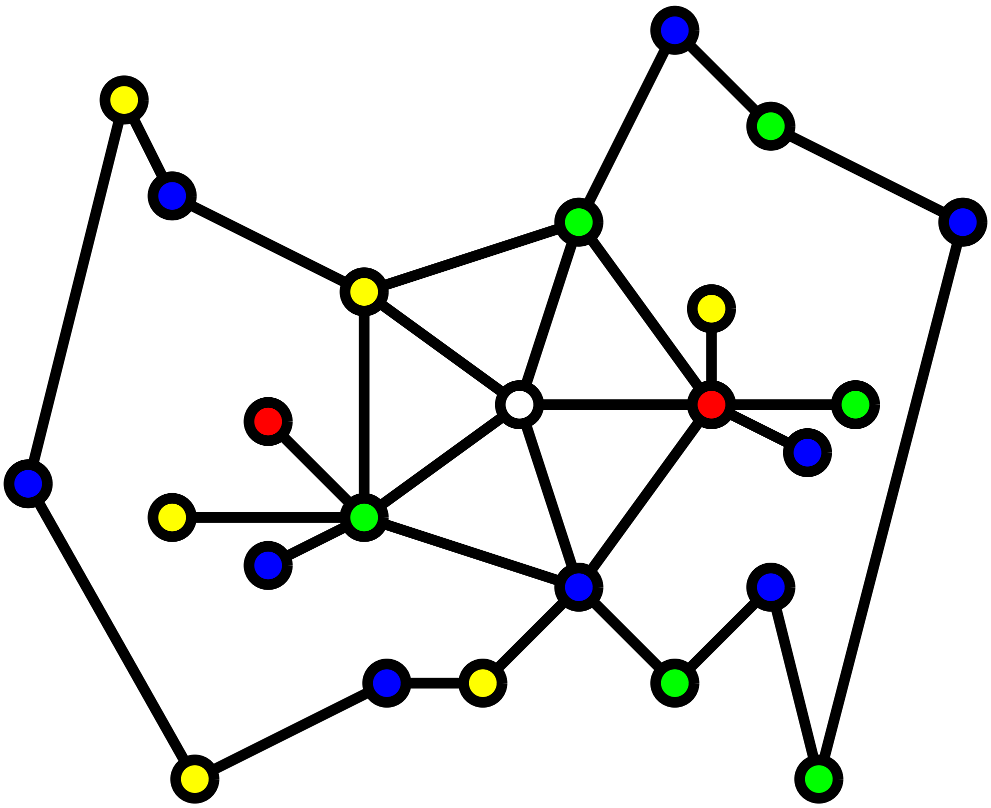
Case B:

Case 2:

Case ii:

Interchange the colours red and yellow.

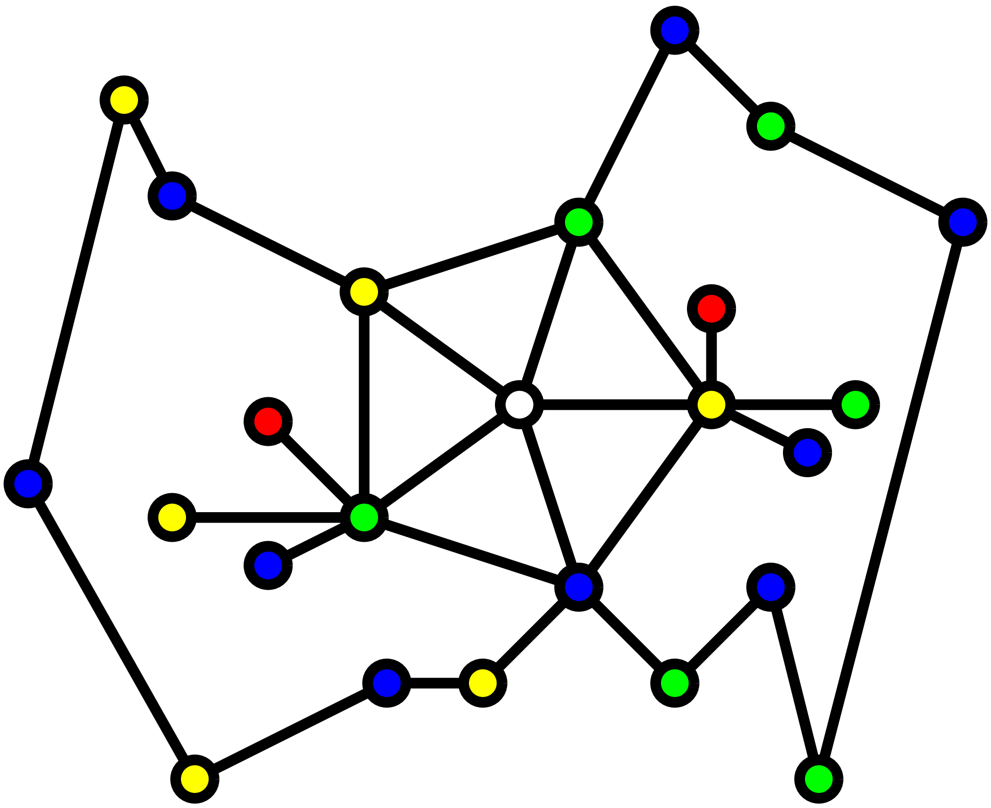
This will not affect the chains in the link on the left.



Case B:

Case 2:

Case ii:

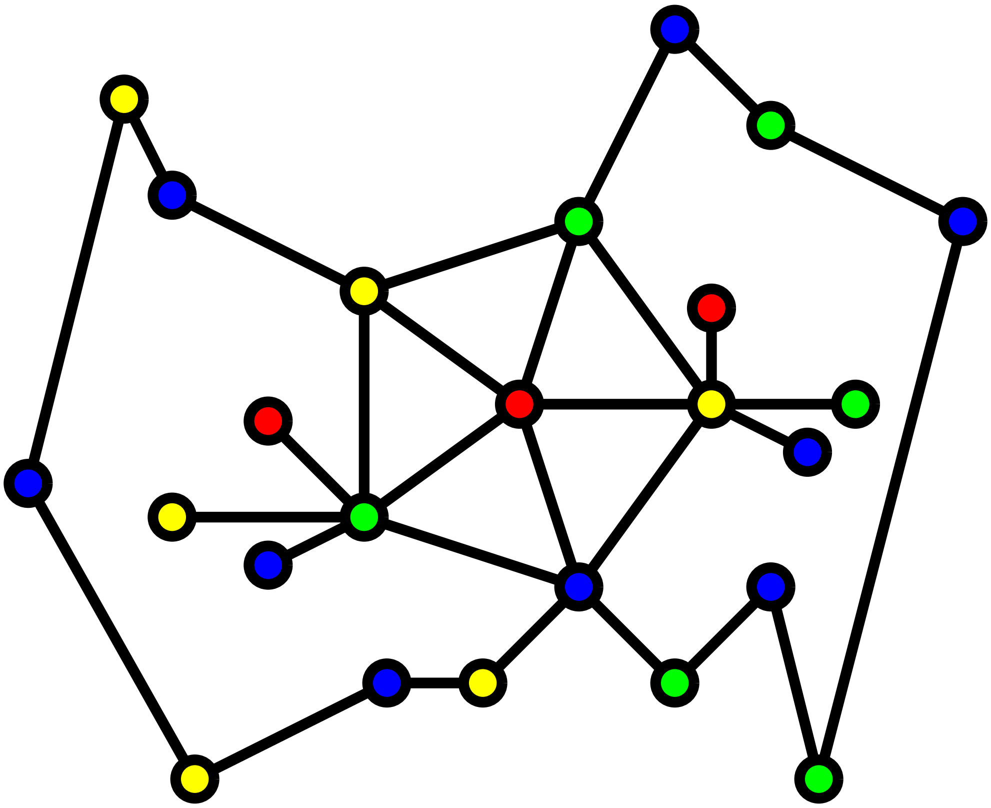


Case B:

Case 2:

Case ii:

Now, the centre vertex can be coloured red.



Thus, in 1879, Kempe proved the four colour theorem.

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Or, at least, he and the rest of the world thought he did for about 11 years.

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In 1890, Percy Heawood found a flaw in Kempe's proof.
He used a map of 25 regions to show that Kempe's proof failed.

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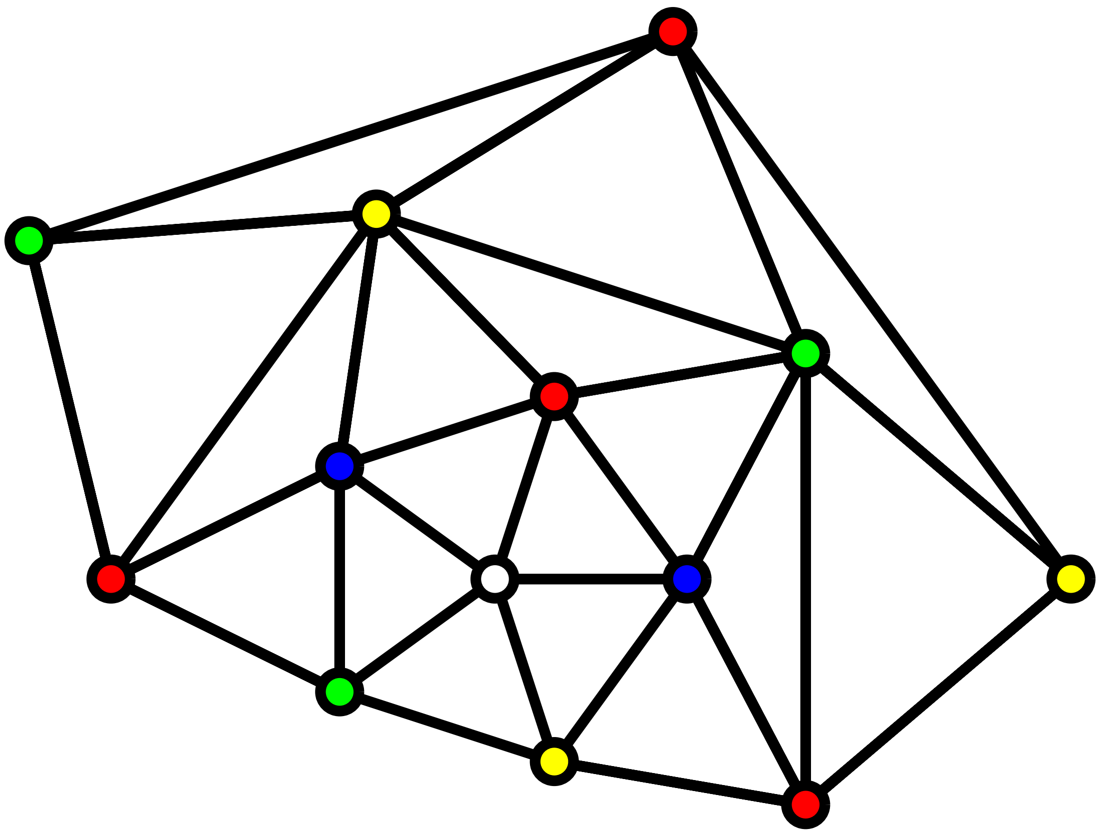
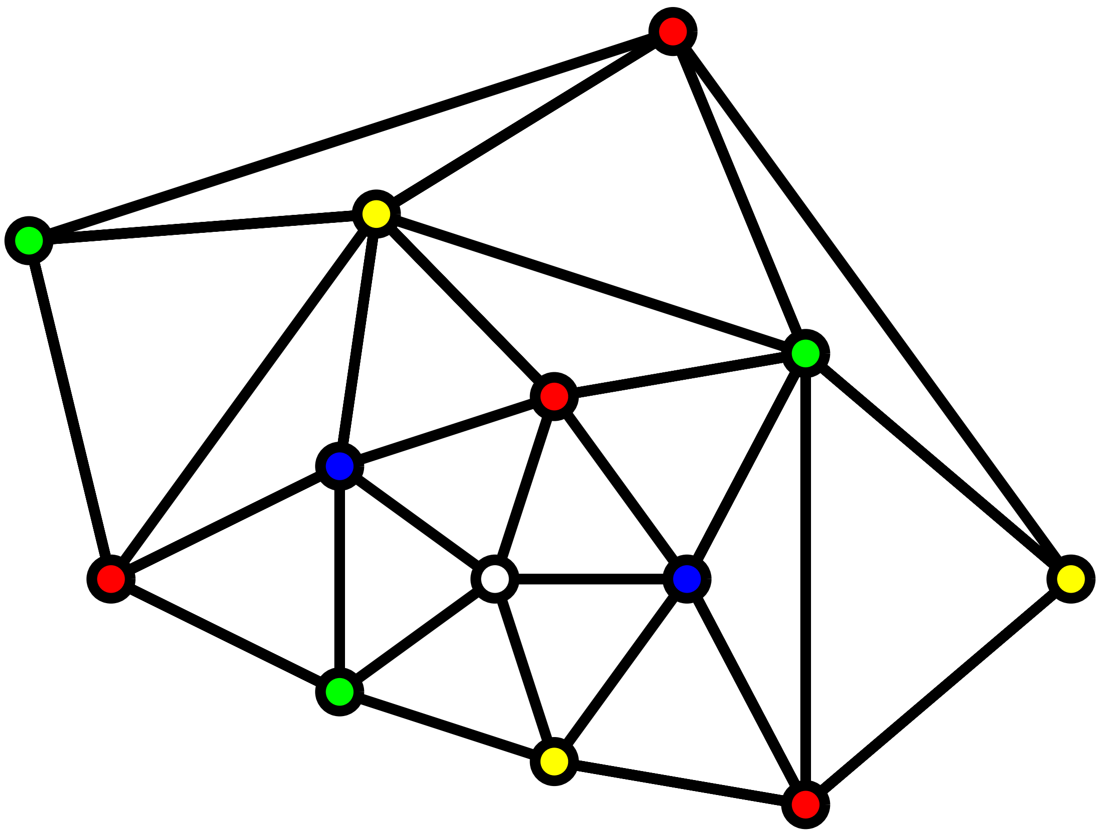
He used a map of 25 regions to show that Kempe's proof failed.

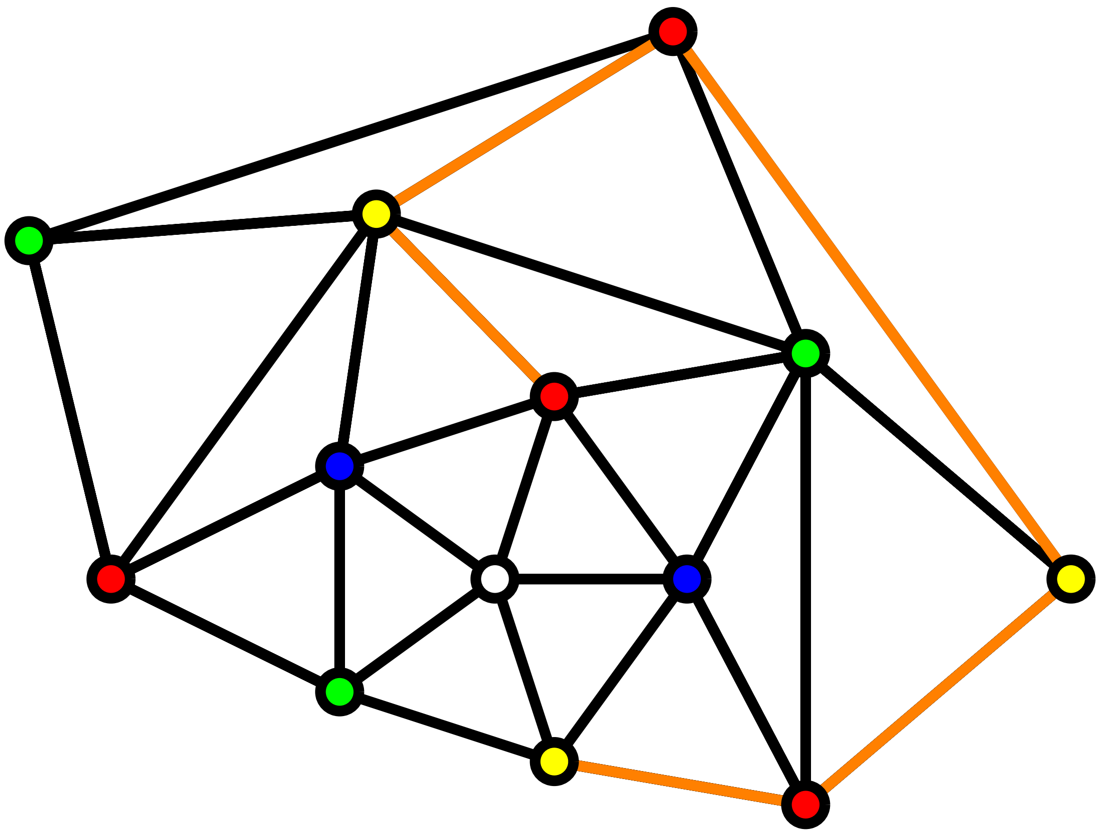
A much simpler example of how Kempe's proof failed is given by
Charles-Jean-Gustave-Nicolas
de la Vallee Poussin.

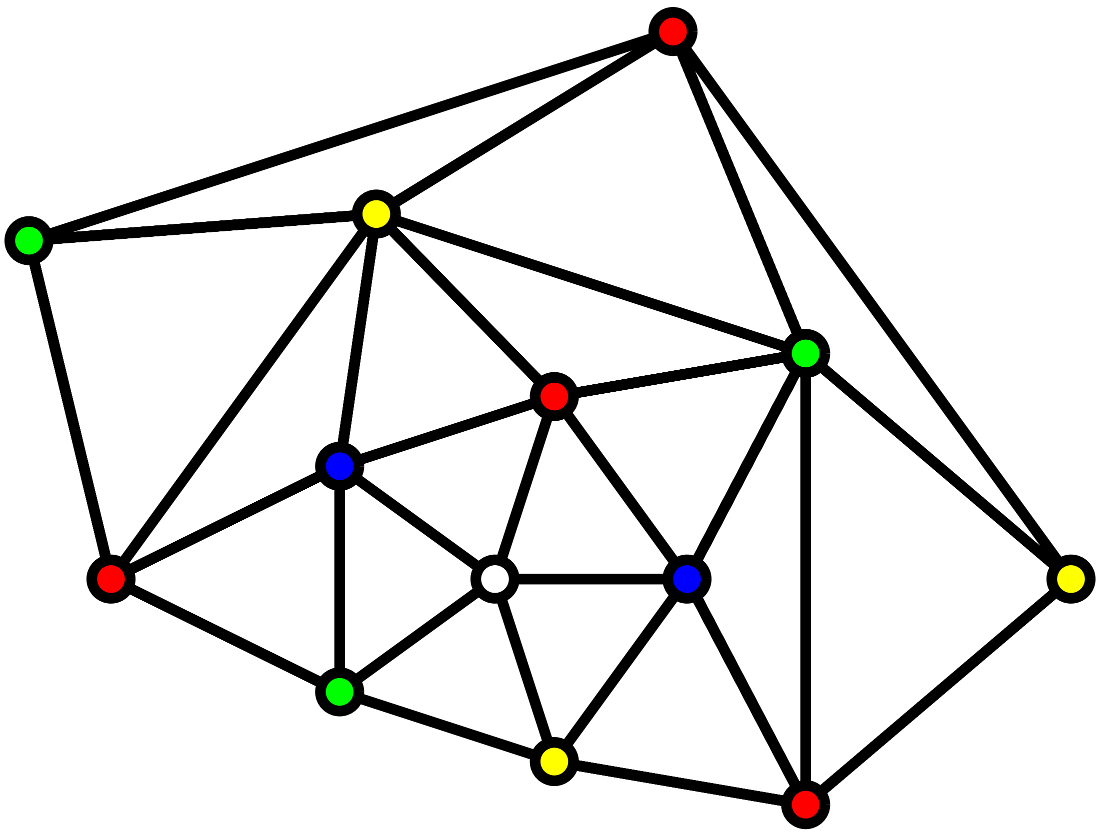


Here is the map’s corresponding graph.

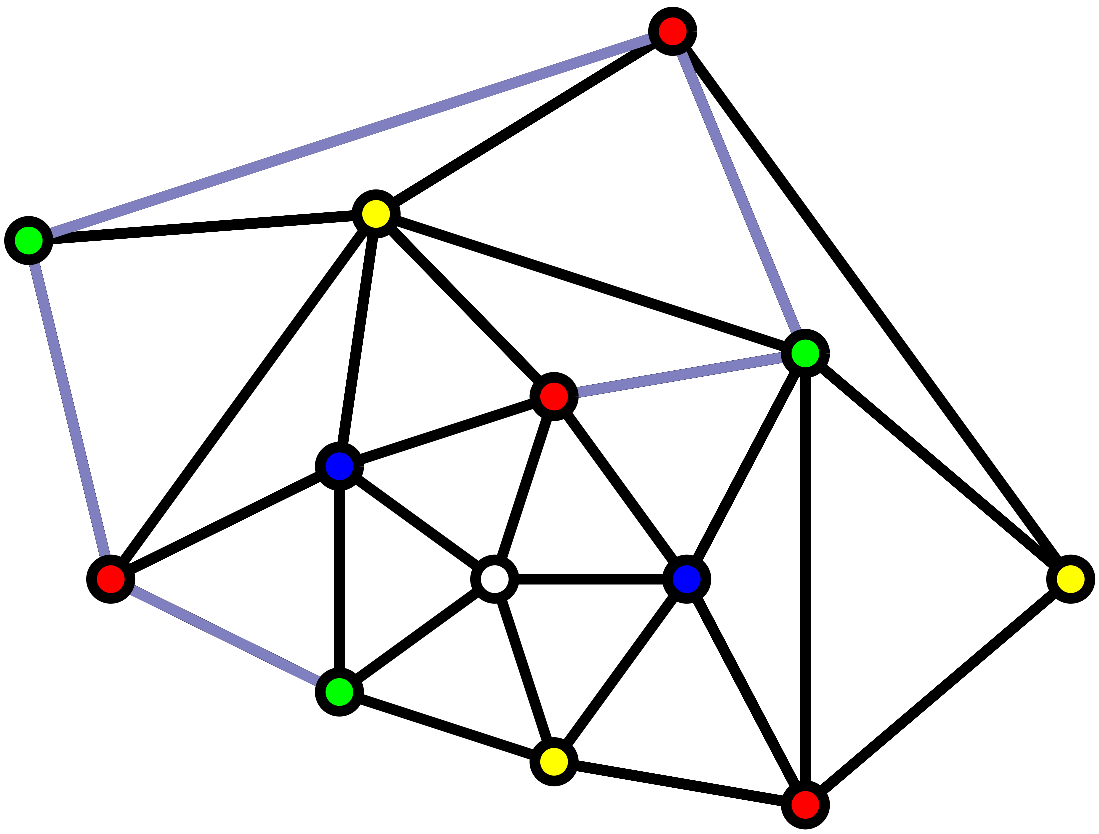


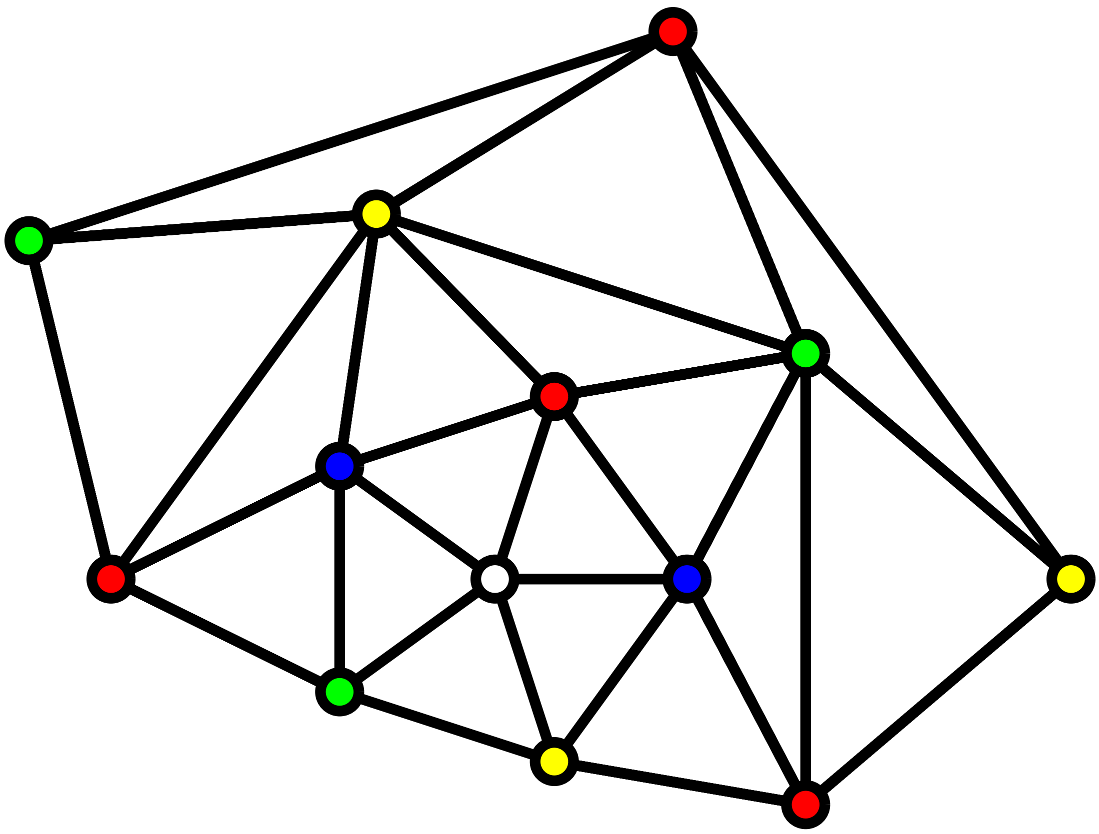
Notice that there is a red-yellow link.



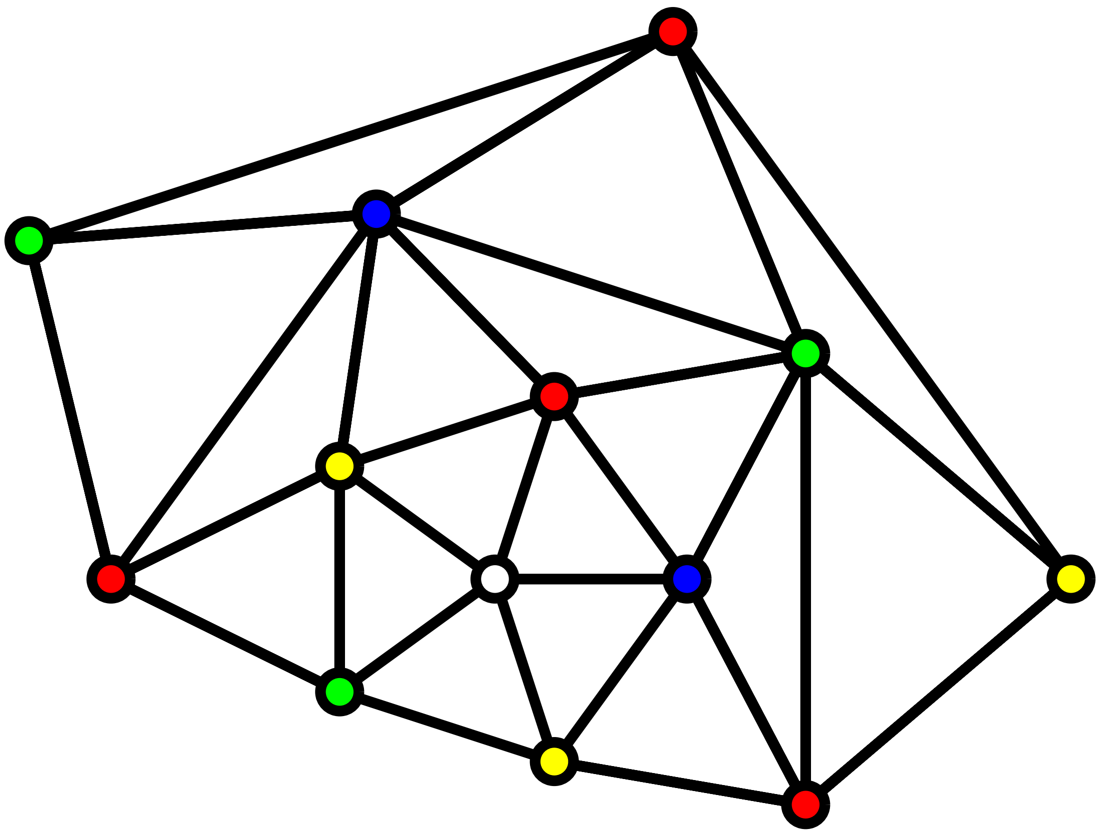


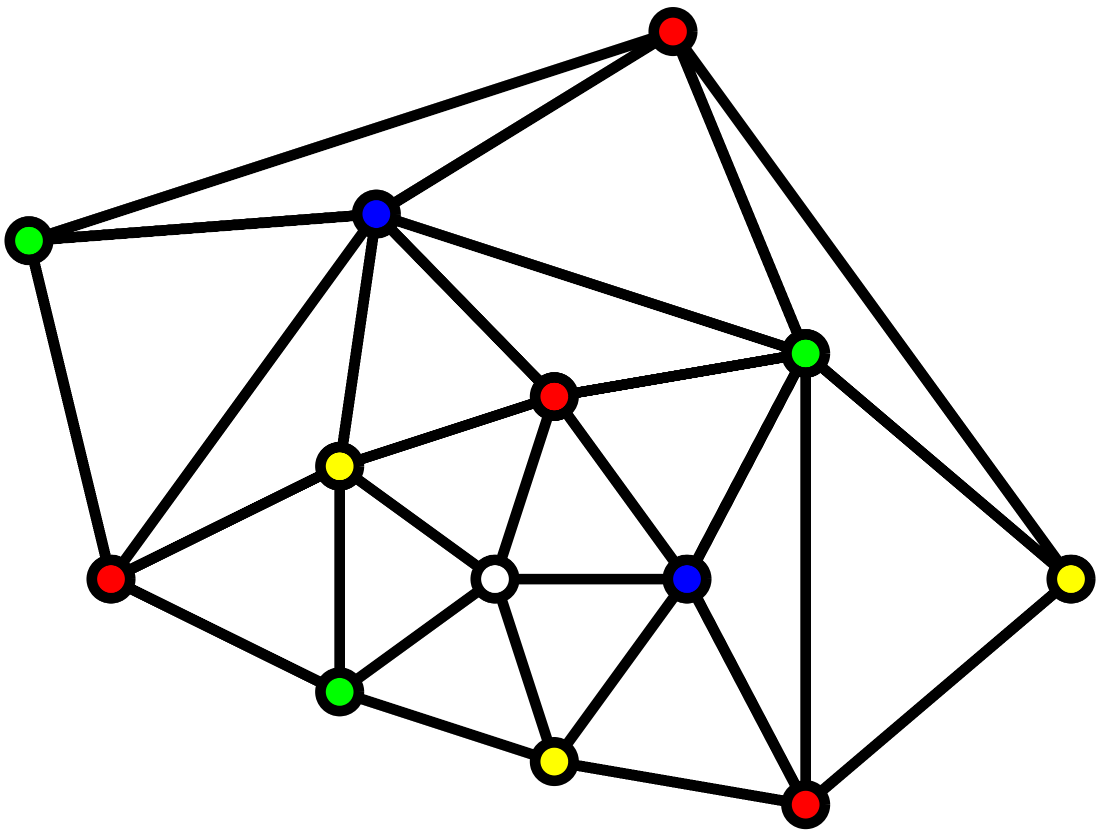
And notice that there is a red-green link.



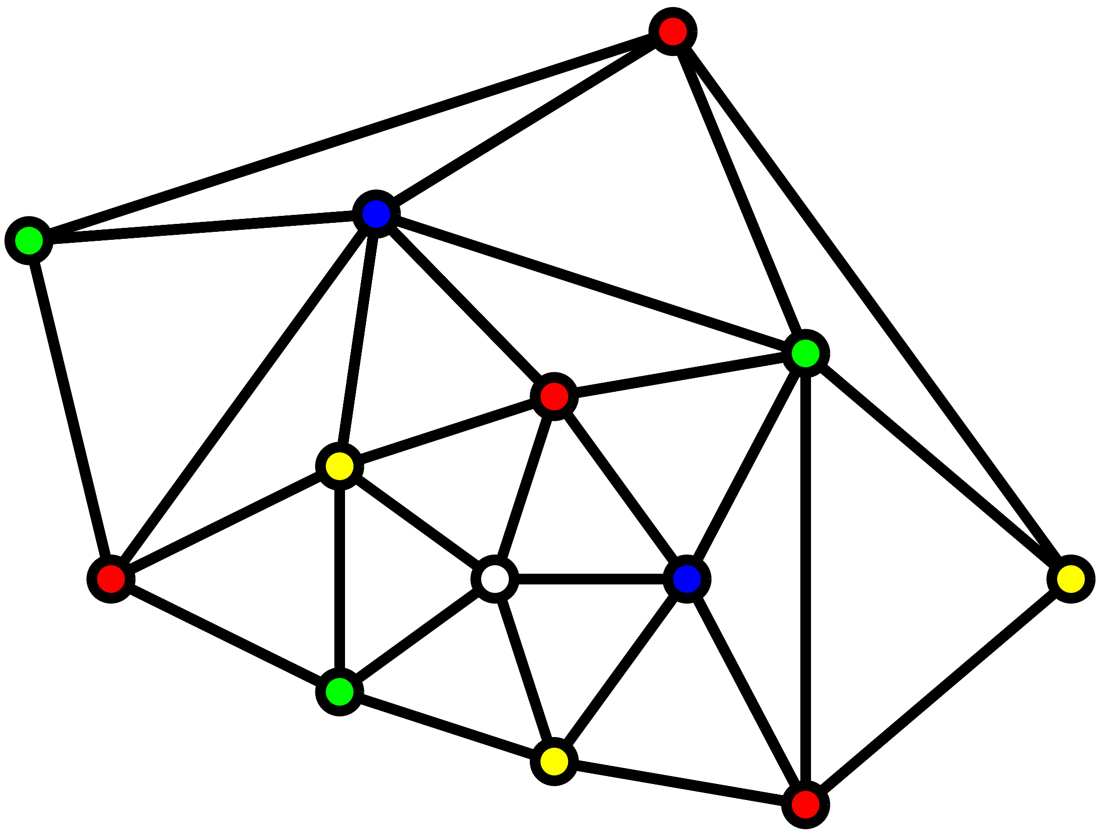


So, interchange the colours yellow and blue inside the left link.

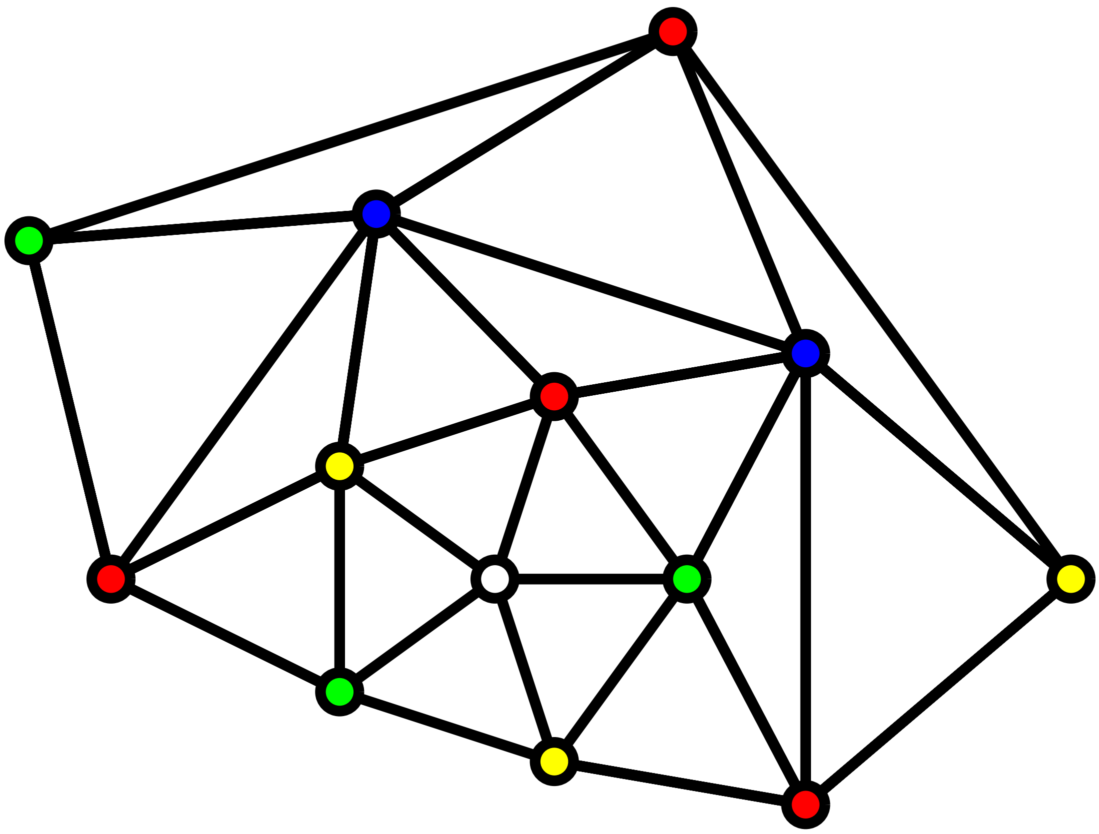


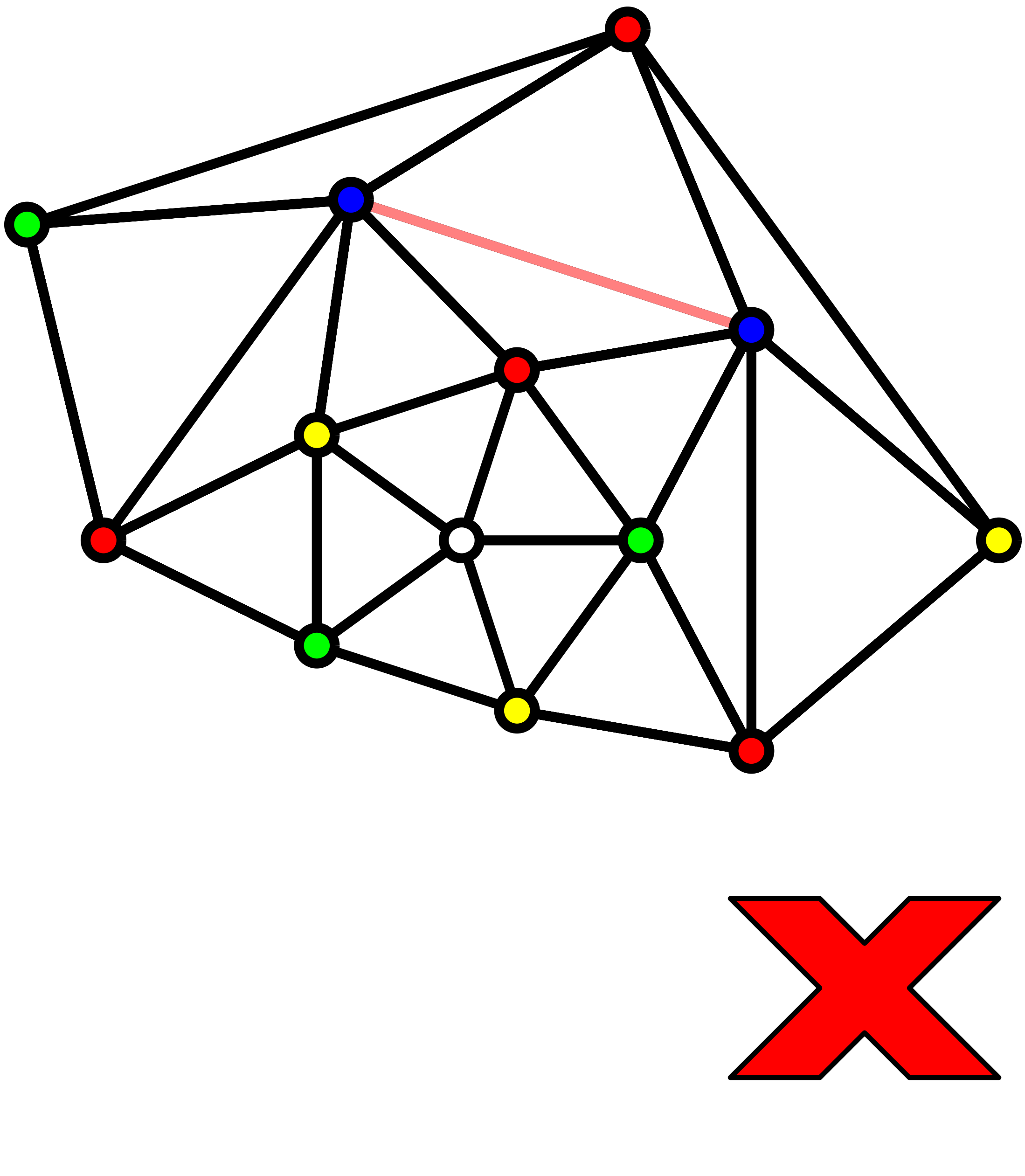


But, the centre vertex can still not be coloured.



So, interchange the colours blue and green inside the right link.

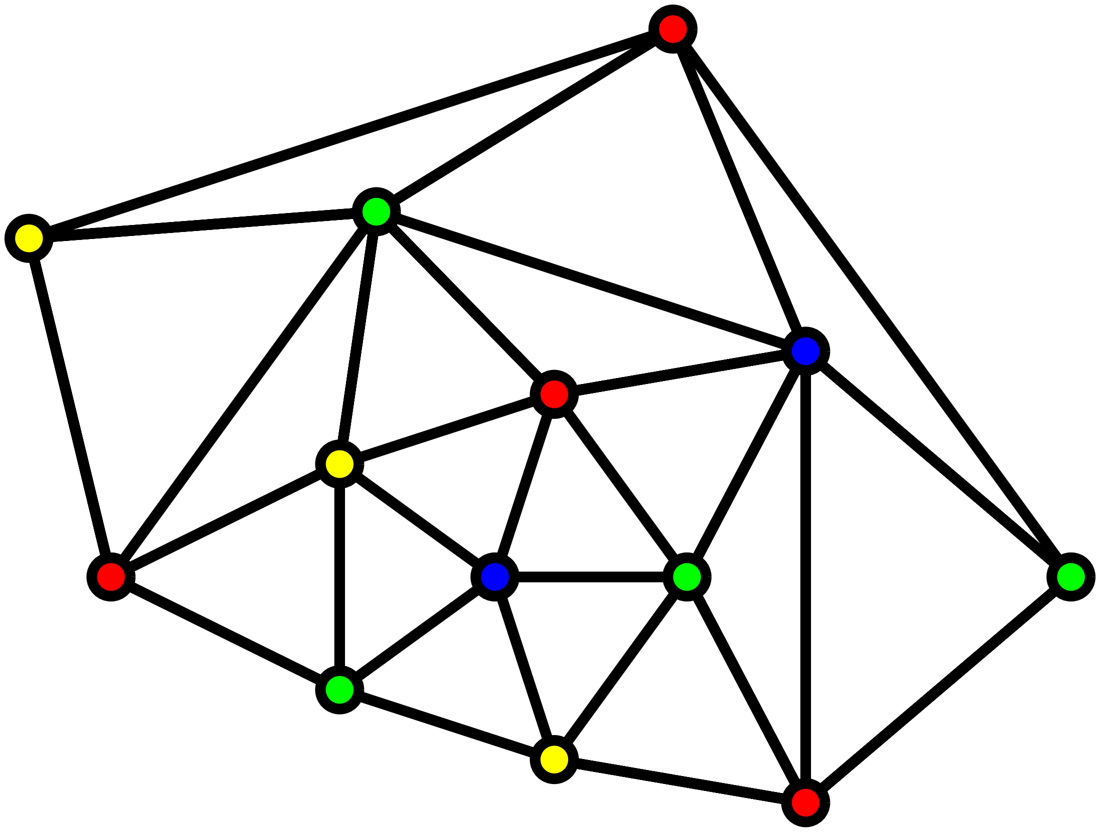




Thus, Kempe's proof is flawed.

But this does not mean that the four colour theorem is false.

The previous graph can certainly be coloured using four colours as shown below.



Even though Kempe’s proof is flawed, the method of Kempe chains is very useful.

It can be used to prove that every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

This is called the five colour theorem.

The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

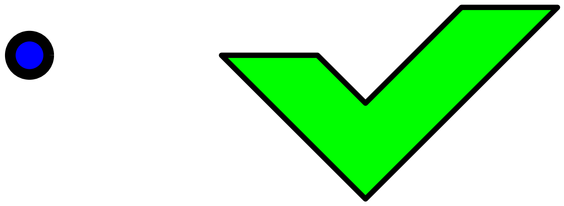
Proof by induction on the number of vertices, n.

The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=1

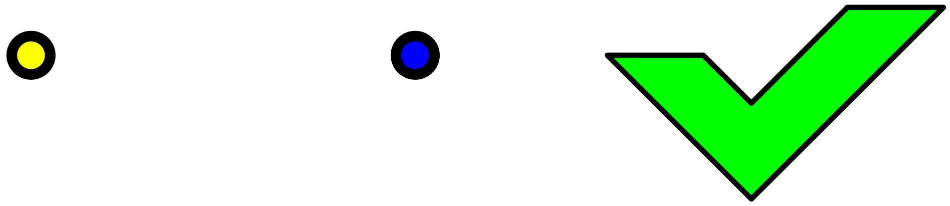


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=2

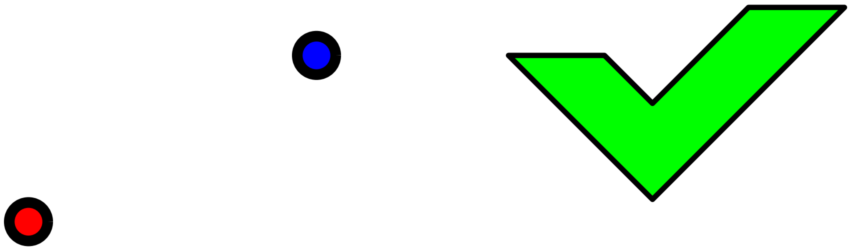


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=3

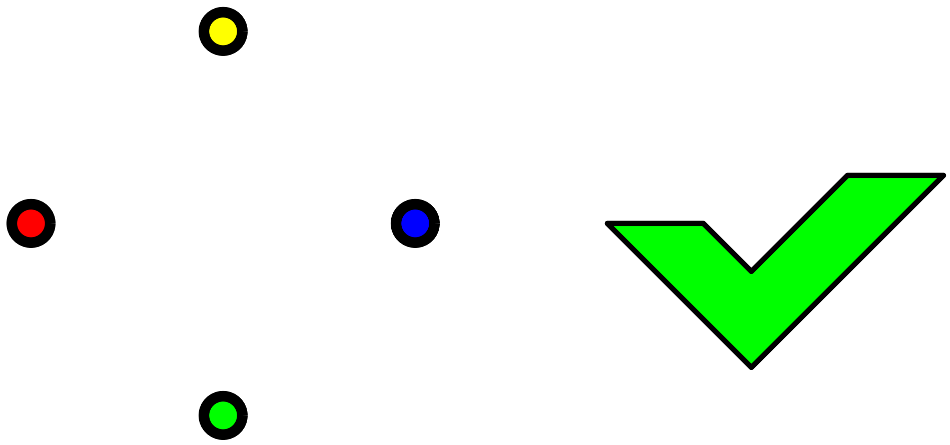


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=4

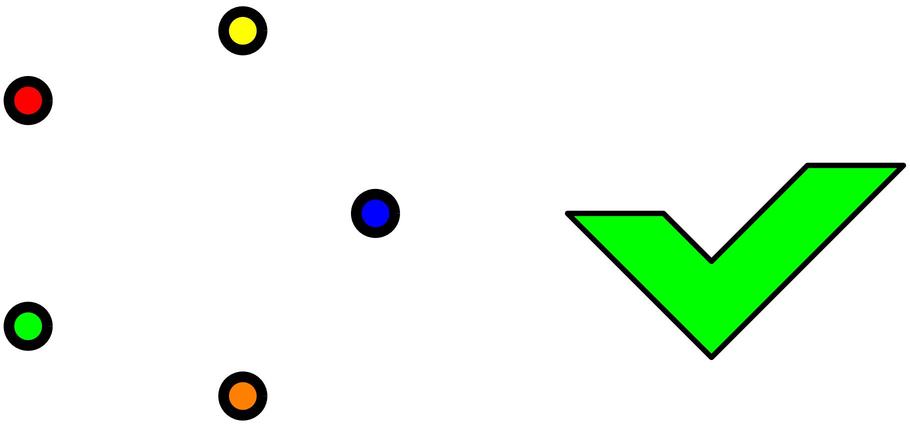


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Base Cases: n=5



The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Induction Step:

Suppose that every planar graph with n vertices can be coloured using at most five colours.

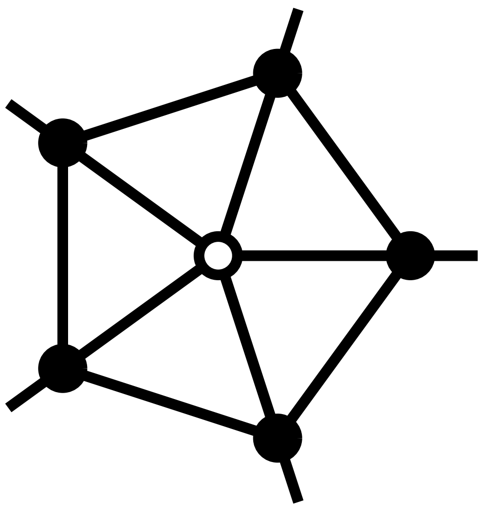
The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Let there be a planar graph with n + 1 vertices.

Then there exists a vertex of degree at most five.

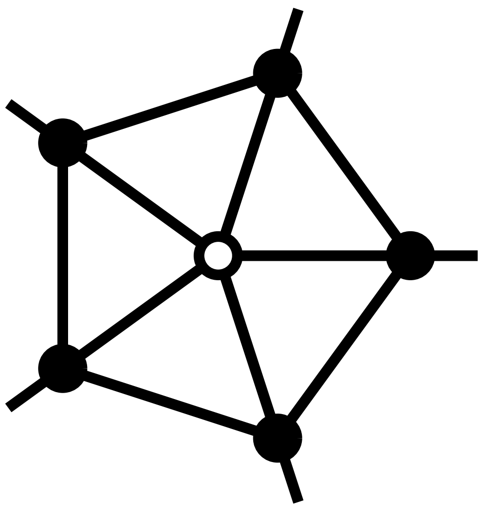


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Remove that vertex from the graph.



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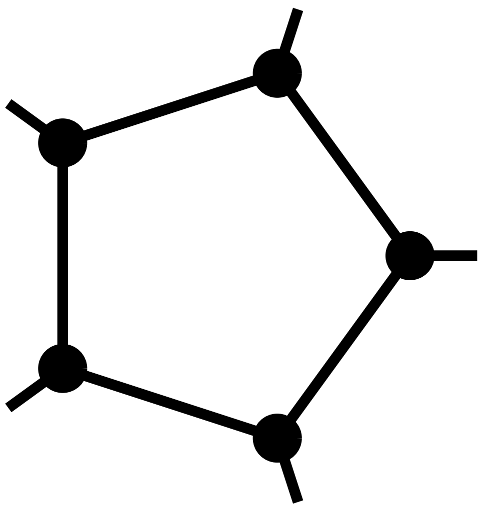
Proof:

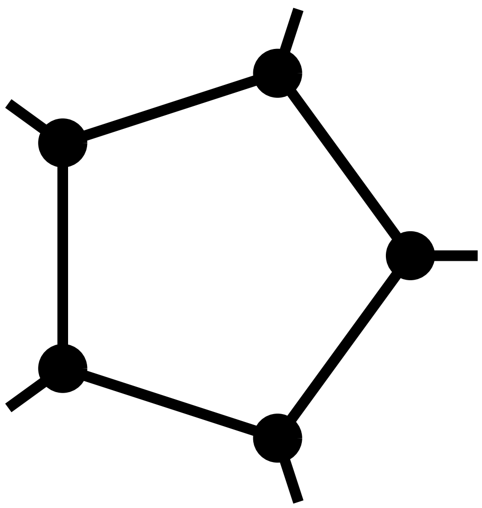


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Now, the remaining graph has n vertices.

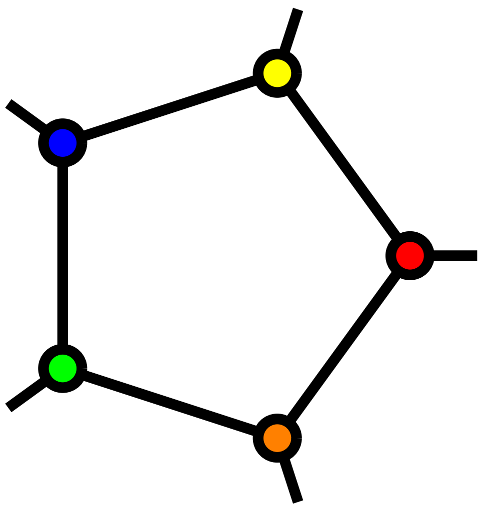


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

By the induction hypothesis, this graph can be coloured using at most five colours.

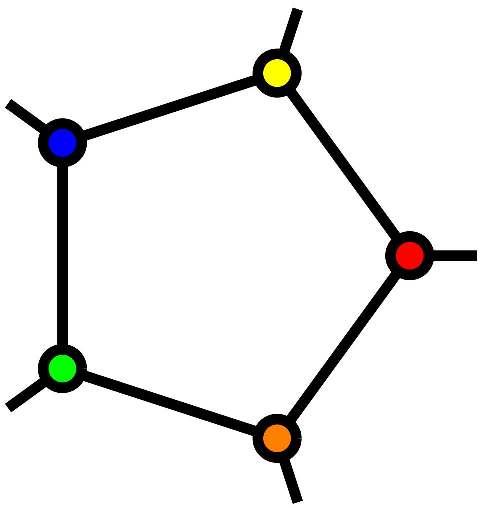


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

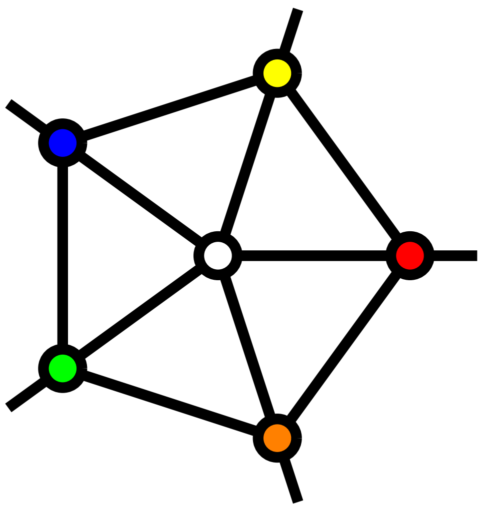
Place the removed vertex back.



The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

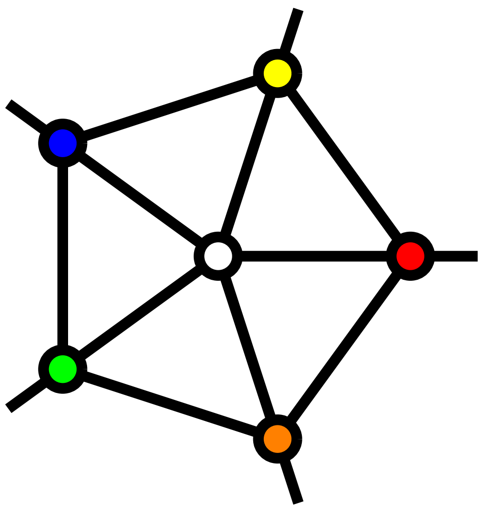


The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Now, apply the method of Kempe chains.



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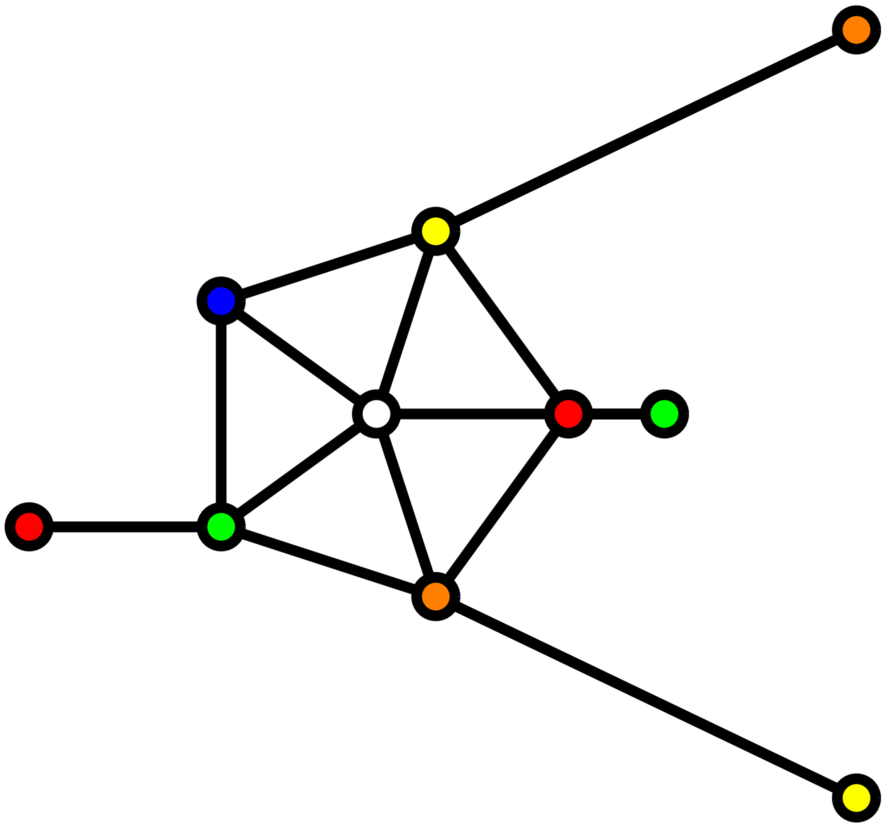
Proof:



The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

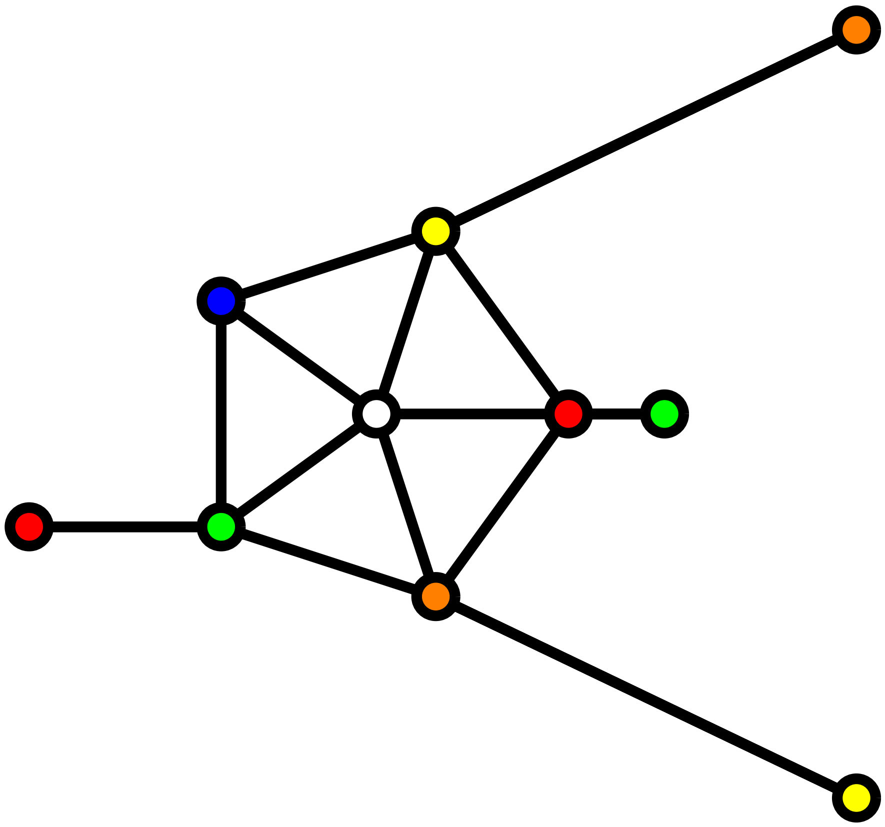
Case 1:

The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Case 1:



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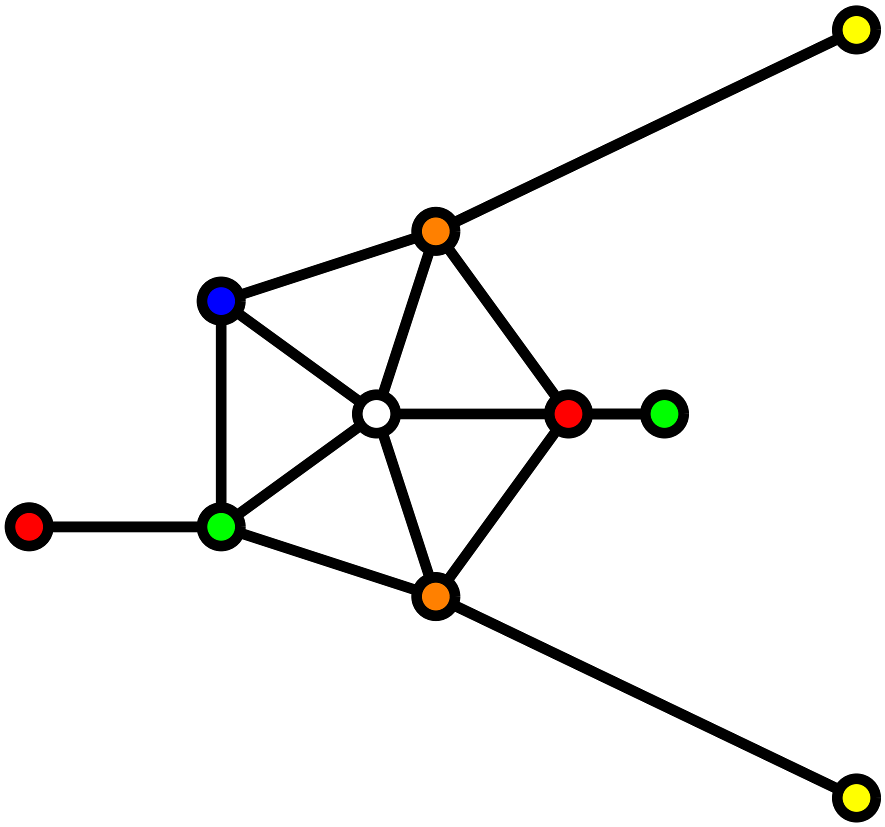

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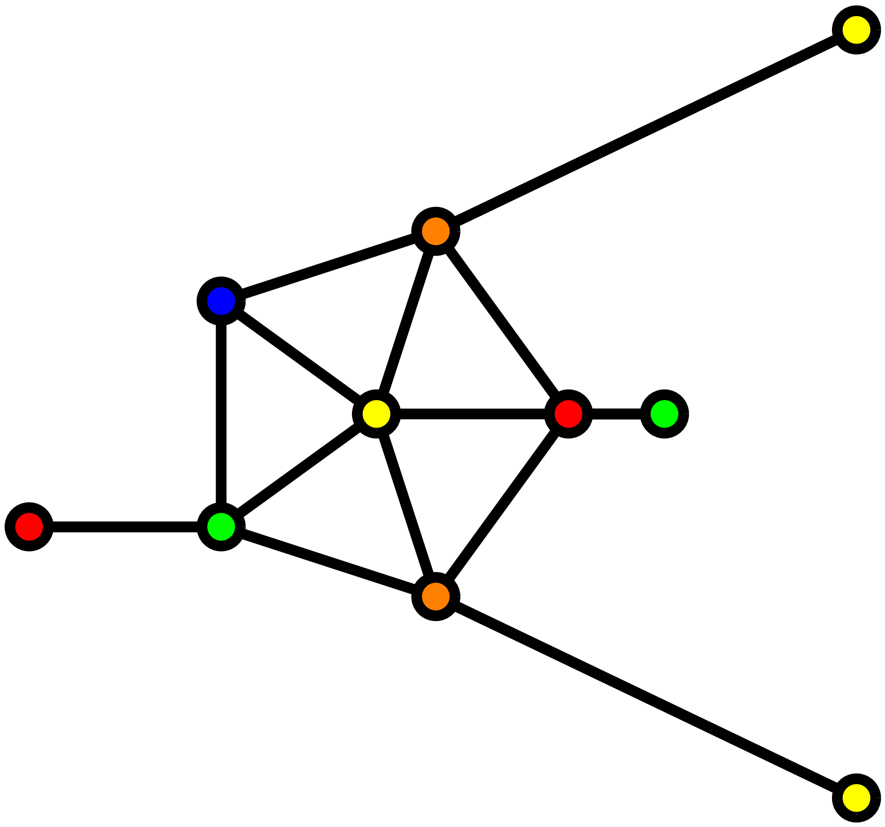
Case 1:



The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

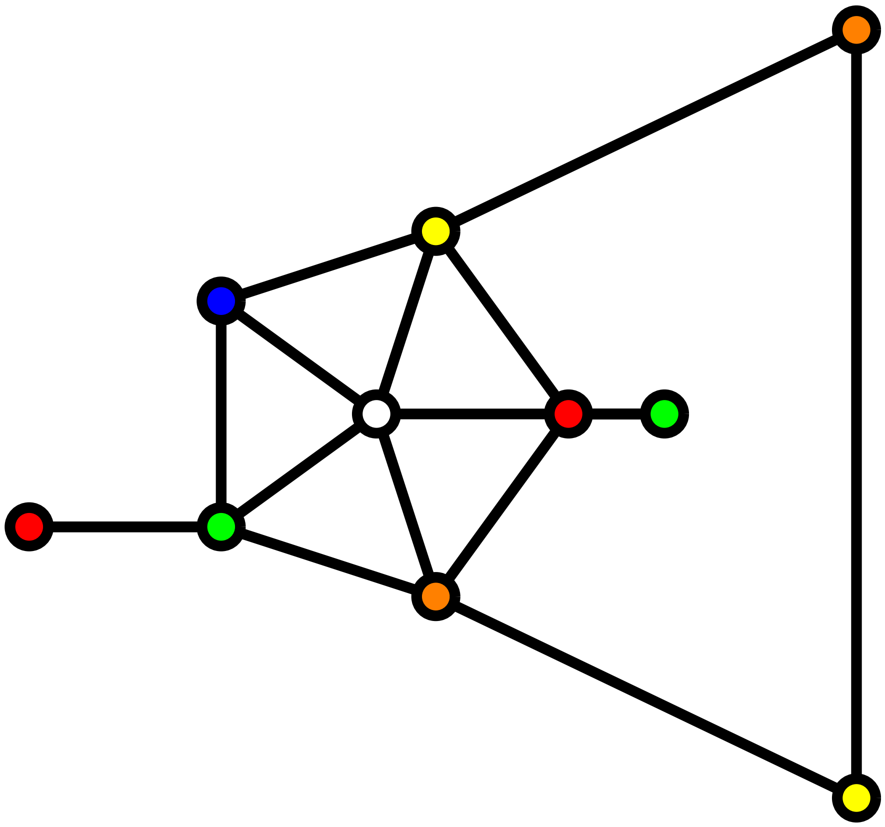
Case 2:

The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Case 2:



The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

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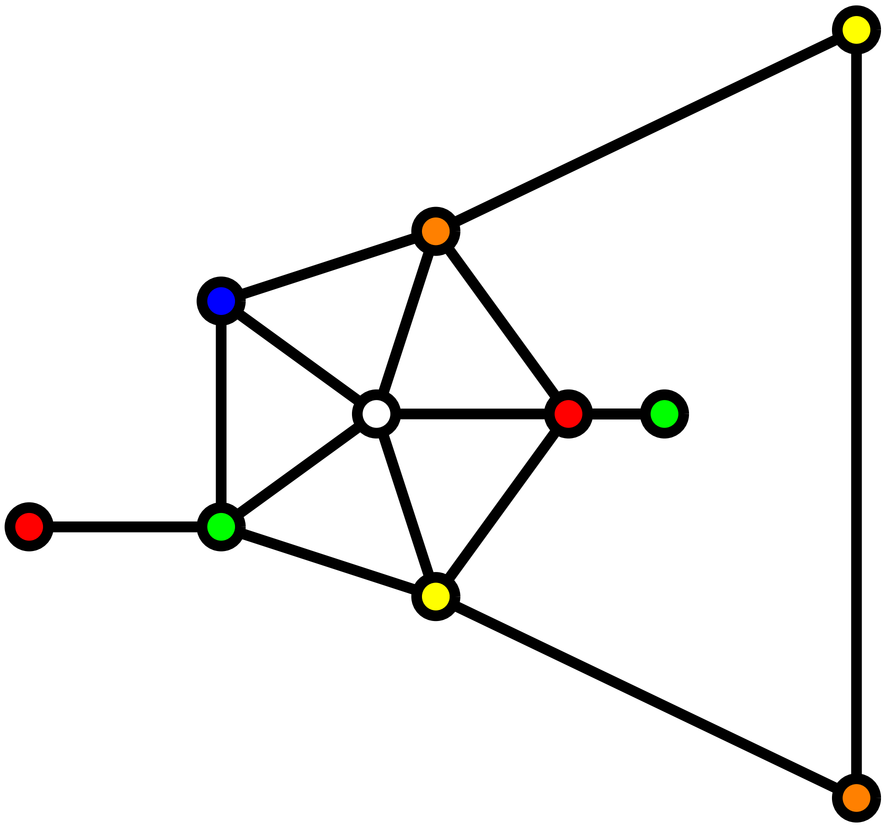

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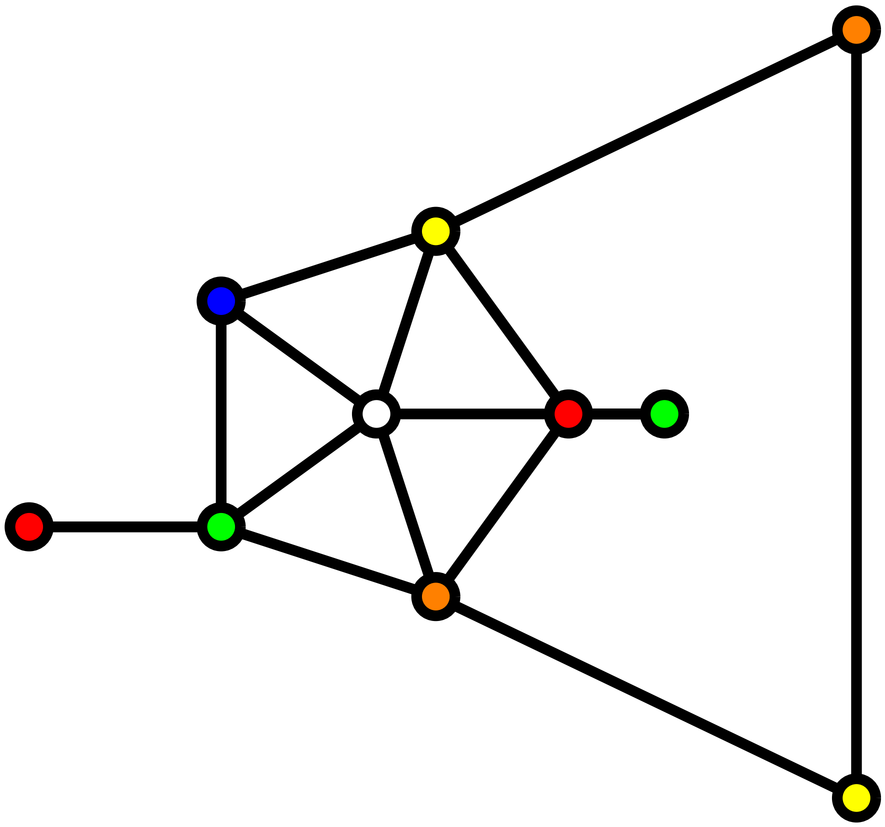

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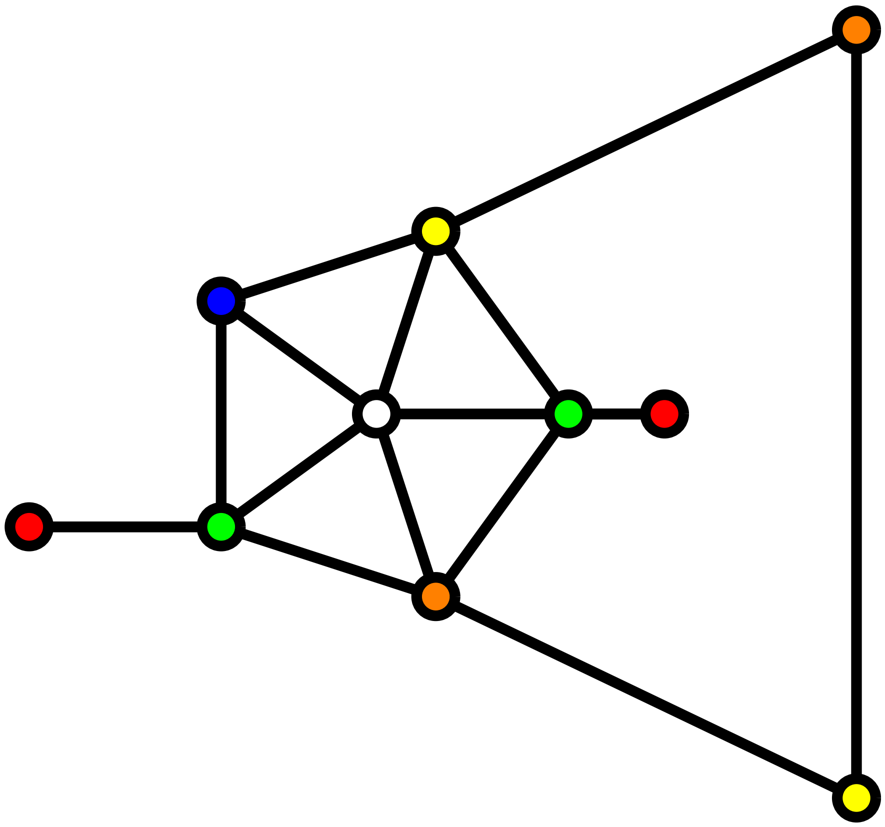

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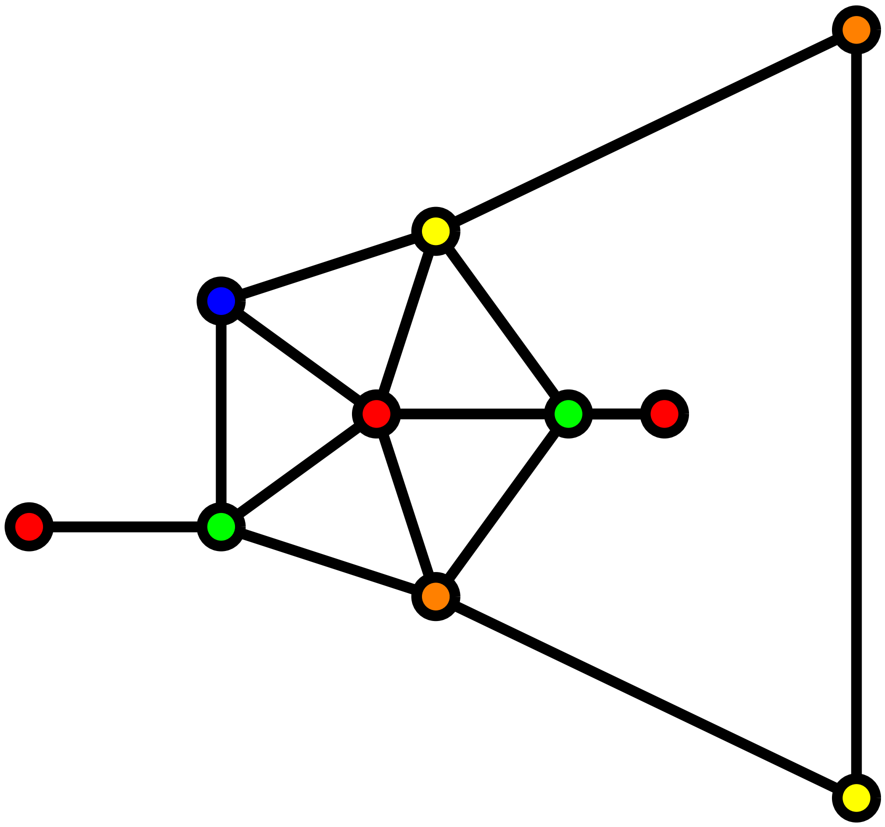

The Five Colour Theorem:

Every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

Proof:

Thus, every planar graph can be coloured using at most five colours such that no adjacent vertices have the same colour.

So, the six colour theorem is proved and the five colour theorem is proved.

But, how does one prove the four colour theorem?

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Recall that every planar graph has at least one vertex of degree at most five.

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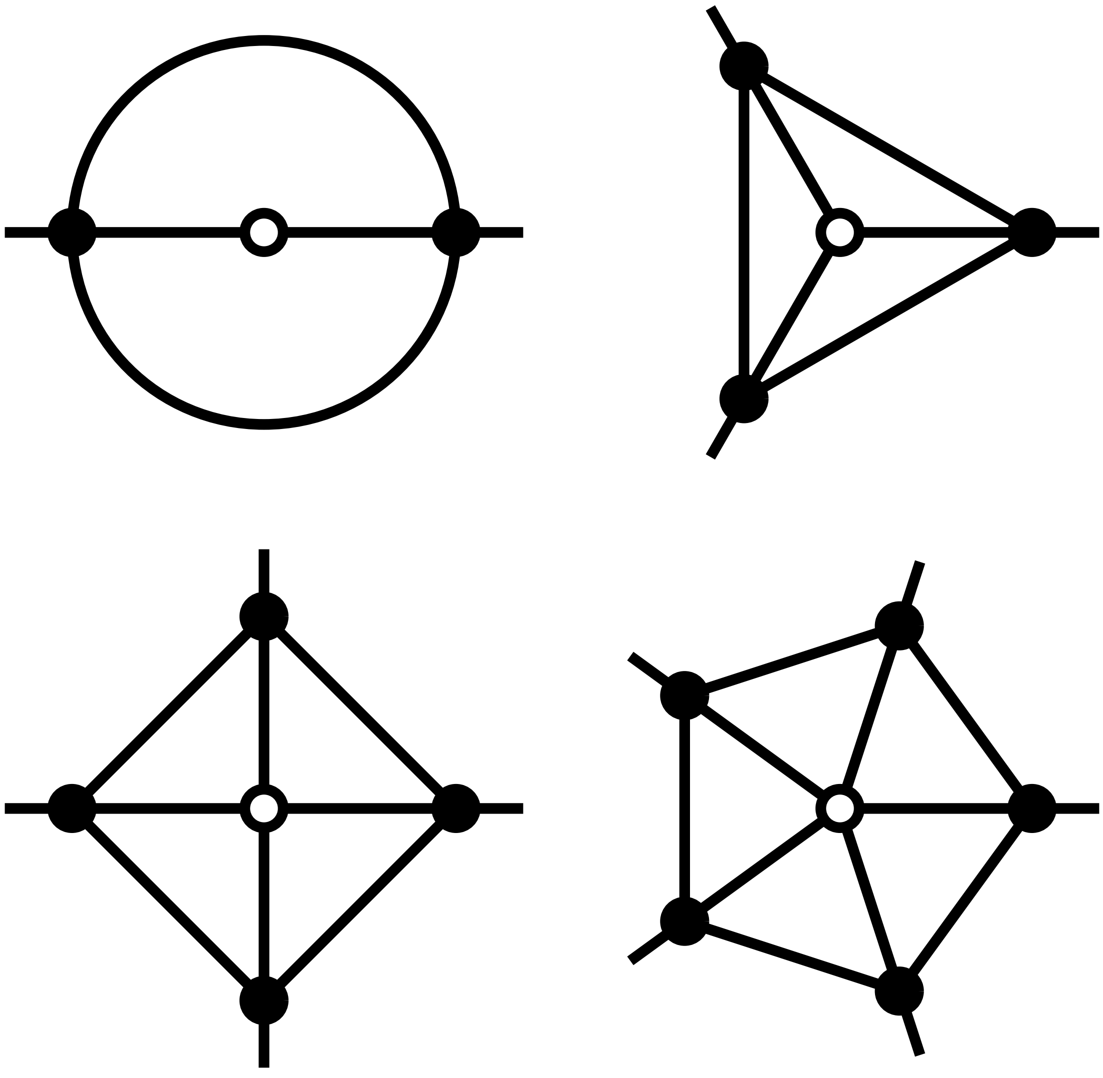
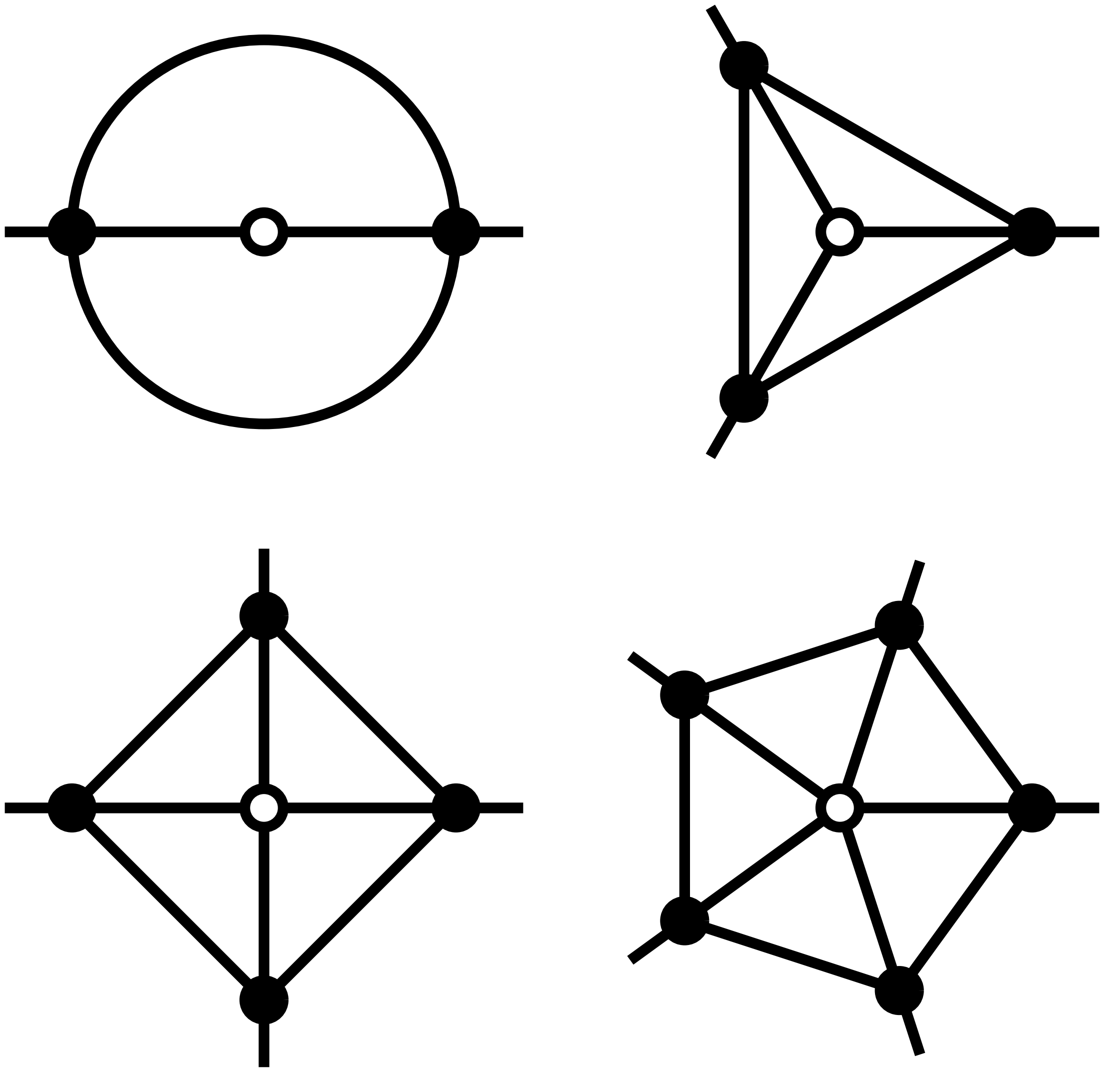
But, how does one prove the four colour theorem?

Recall that every planar graph has at least one vertex of degree at most five.

This means that every planar graph must contain at least one of the following:



This is called an unavoidable set.



Many other unavoidable sets exist.

A planar graph is irreducible if its vertices require at least five colours to colour and any graph with fewer vertices requires less than five colours to colour.

A planar graph is reducible if it is not contained in an irreducible graph.

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Thus, the four colour theorem implies that there are no irreducible graphs.

That is, there are no planar graphs that require at least five colours to colour its vertices.

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Thus, the four colour theorem implies that there are no irreducible graphs.

That is, there are no planar graphs that require at least five colours to colour its vertices.

Thus, the proof of the four colour theorem lies in finding an unavoidable set of reducible graphs.

In 1976, Kenneth Appel and Wolfgang Haken found an unavoidable set of 1936 reducible configurations, later reduced to 1482 reducible configurations.

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The computers ran for many hours and found that those reducible configurations were, indeed, reducible.

Thus, in 1976, the four colour theorem was proved.

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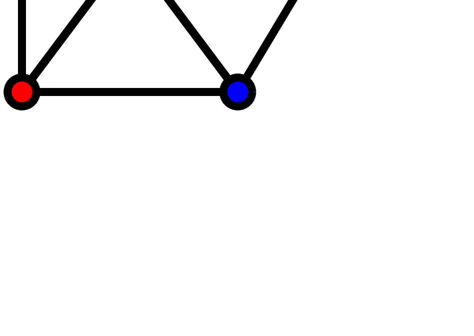
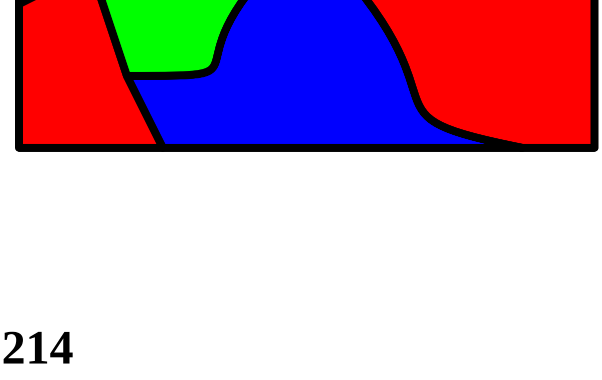
Many mathematicians reject this proof because of this, as the proof cannot be verified by hand.

But, nonetheless, the vertices of every planar graph can be coloured using at most four colours such that no adjacent vertices have the same colour and the contiguous countries of every map can be coloured using at most four colours such that no neighbouring countries have the same colour.





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