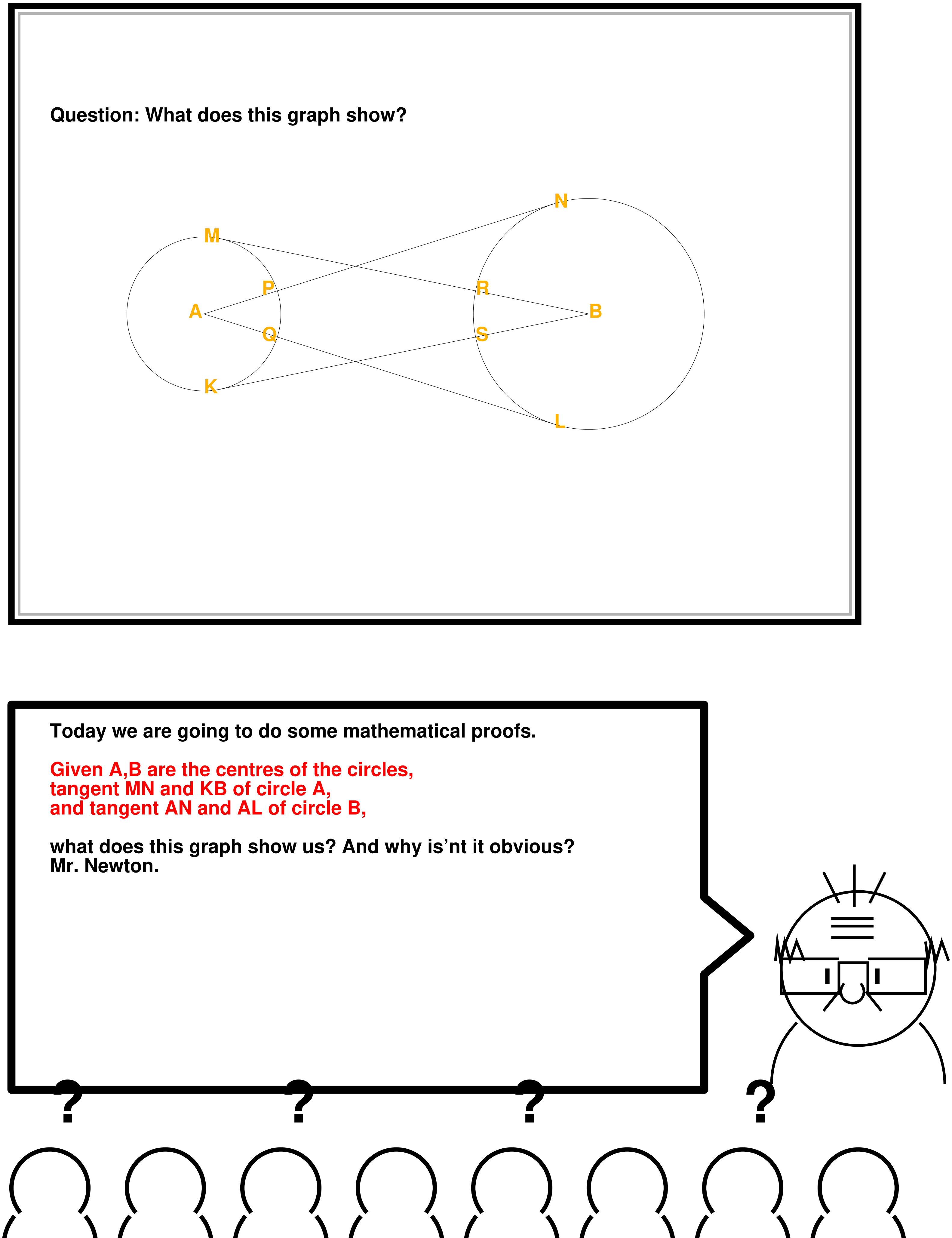
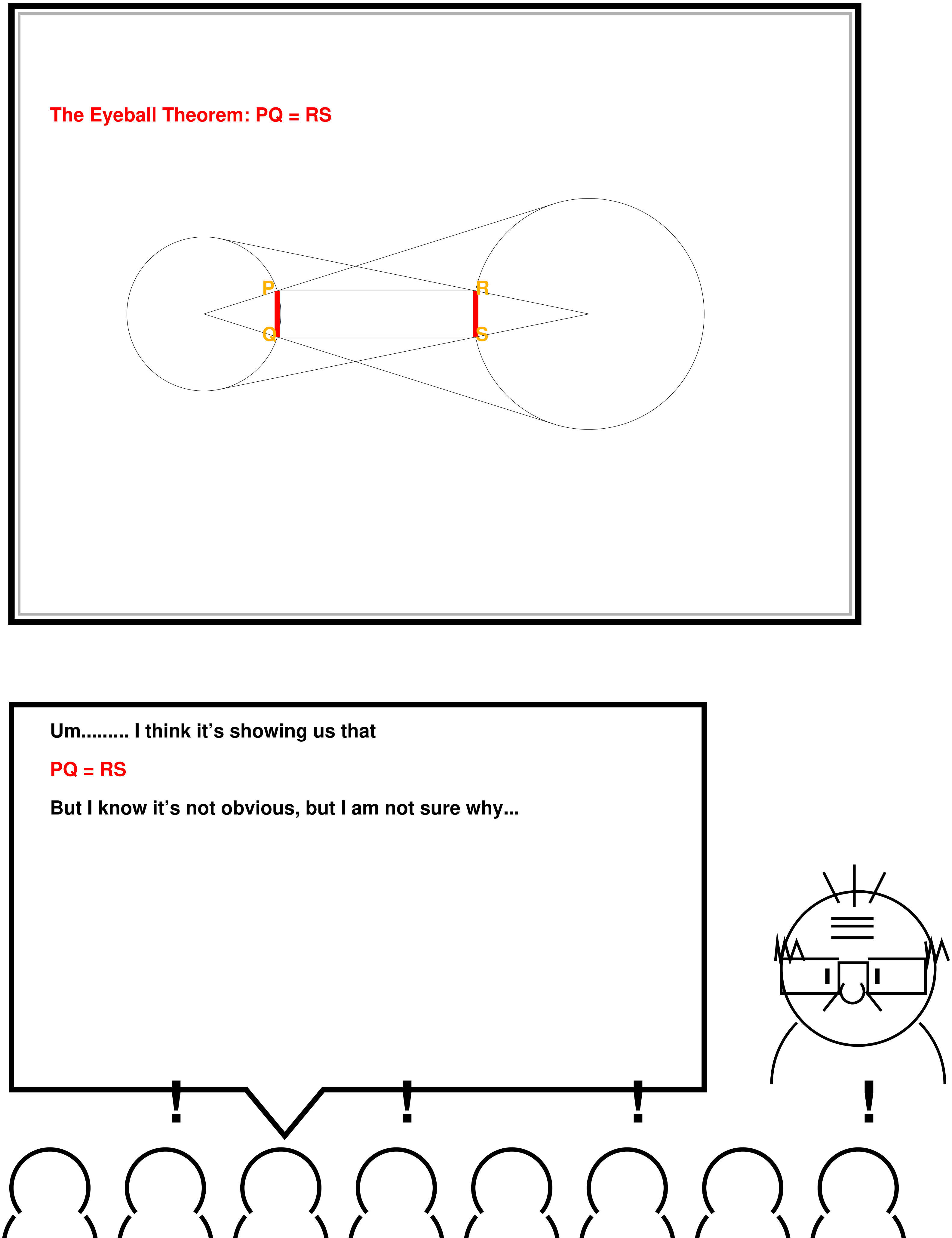
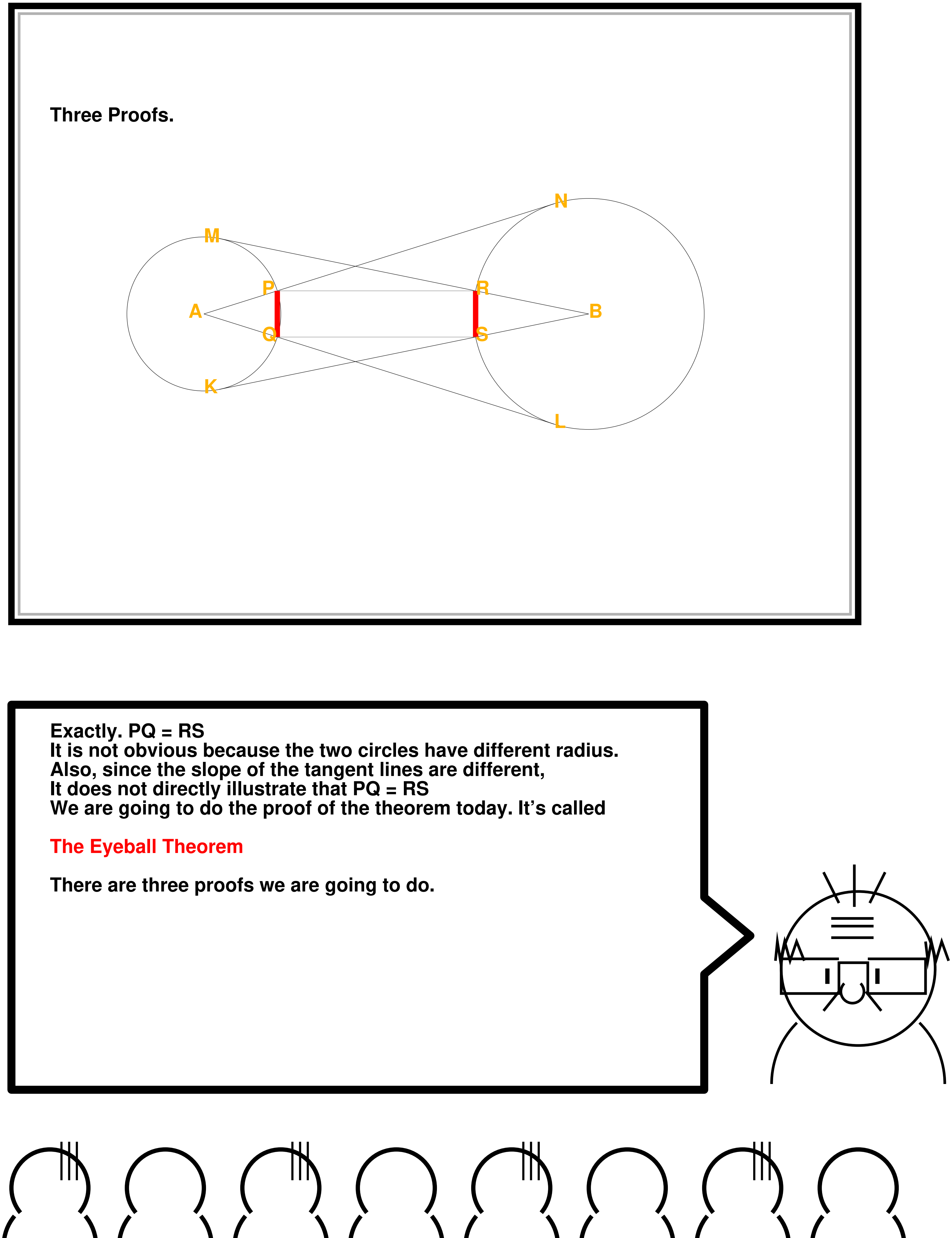
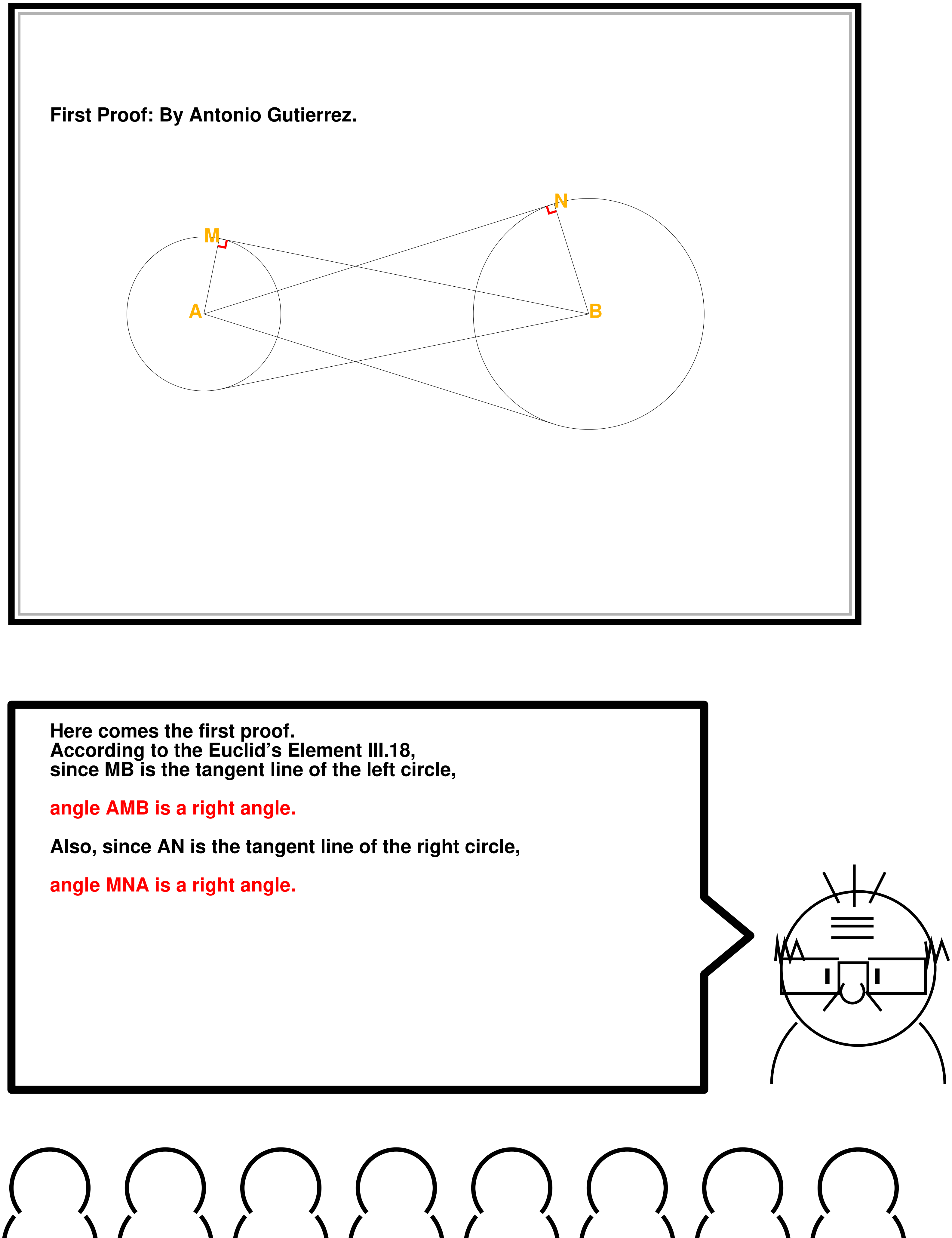


Question: What does this graph show?
N
M
P
R
A
B
Q
S
K
L
Today we are going to do some mathematical proofs.
Given A,B are the centres of the circles,
tangent MN and KB of circle A,
and tangent AN and AL of circle B,
what does this graph show us? And why is’nt it obvious?
Mr. Newton.
?
?
?
?

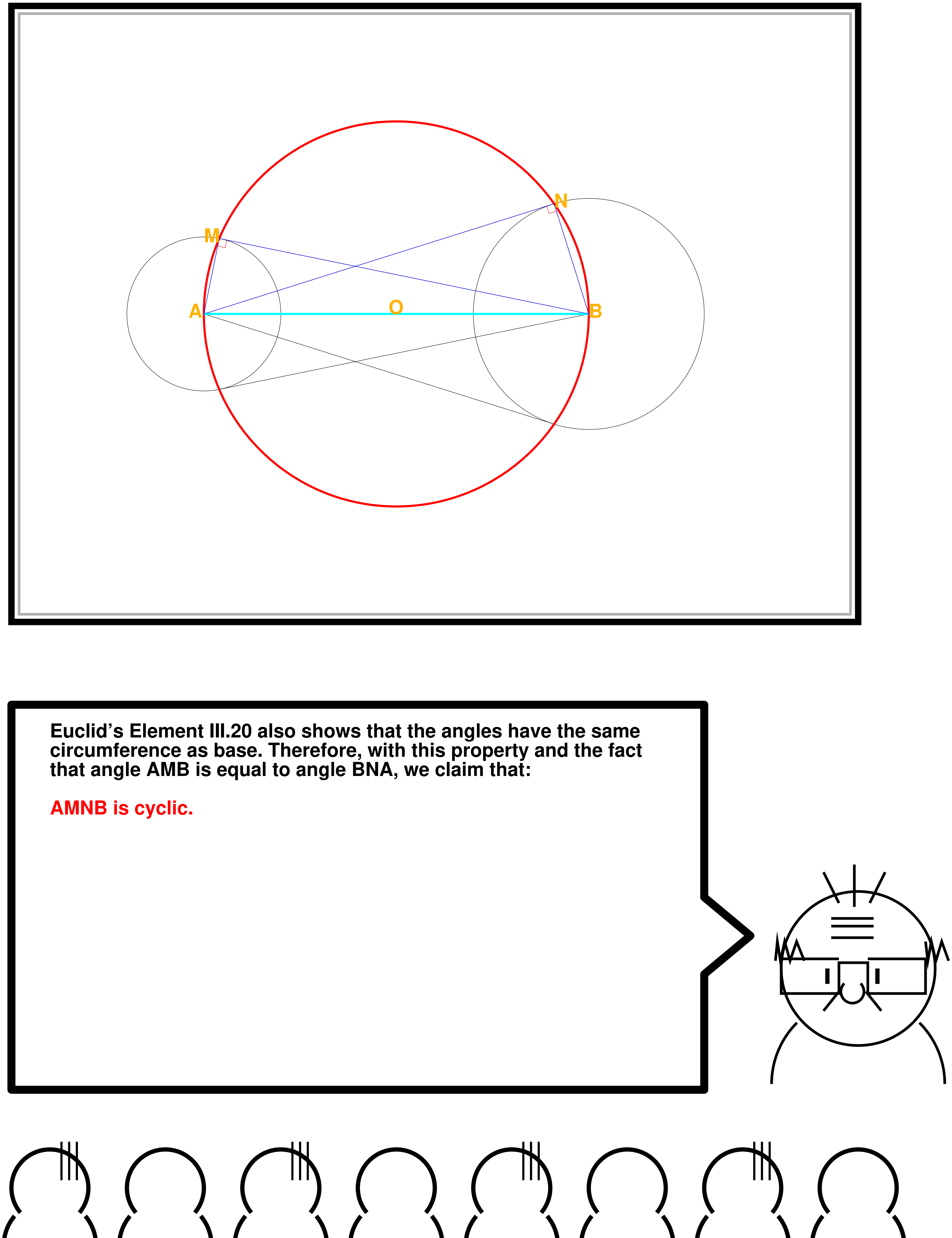


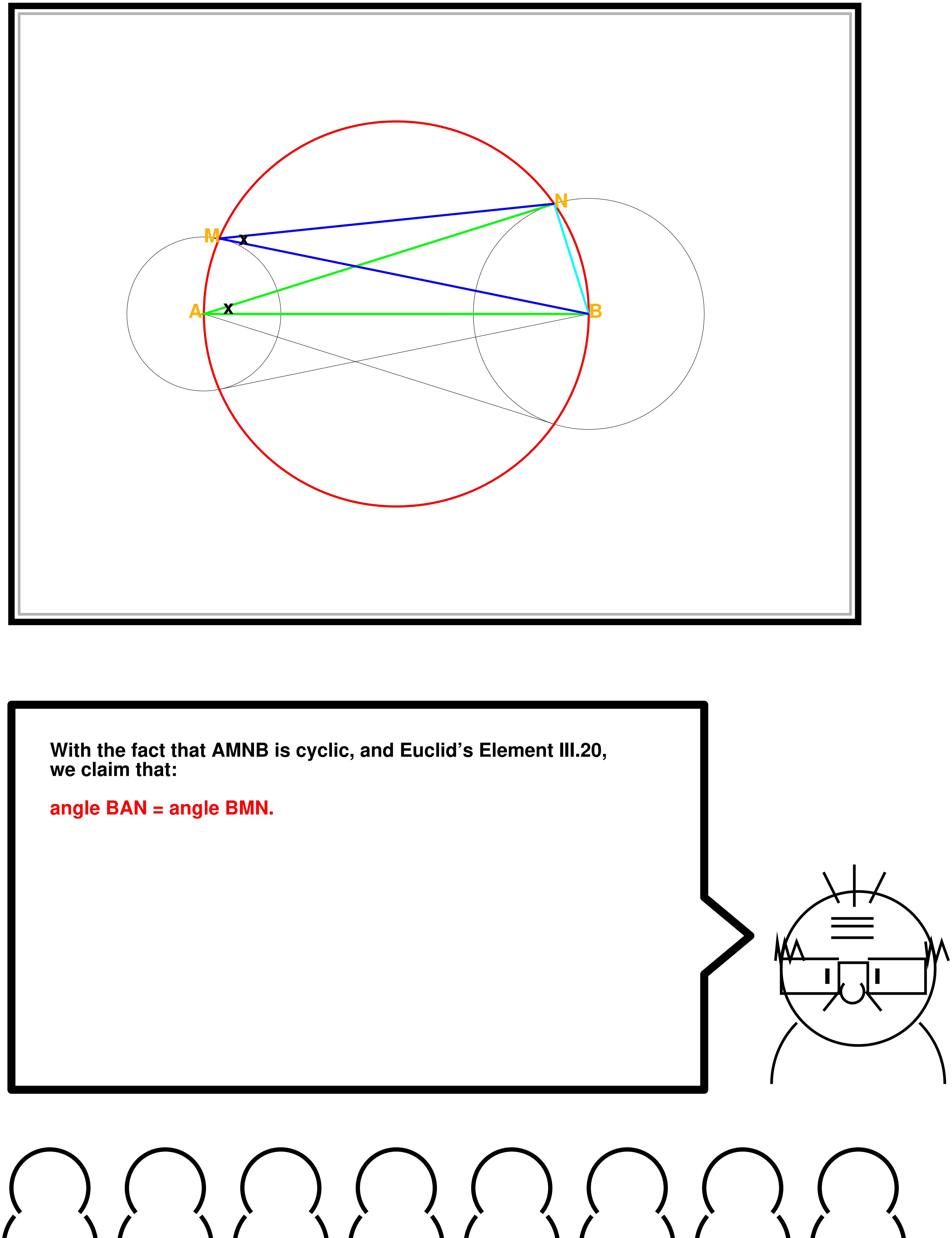
The Eyeball Theorem: PQ = RS
P
R
Q
S
Um……… I think it's showing us that
PQ = RS
But I know it's not obvious, but I am not sure why…
! ! ! !

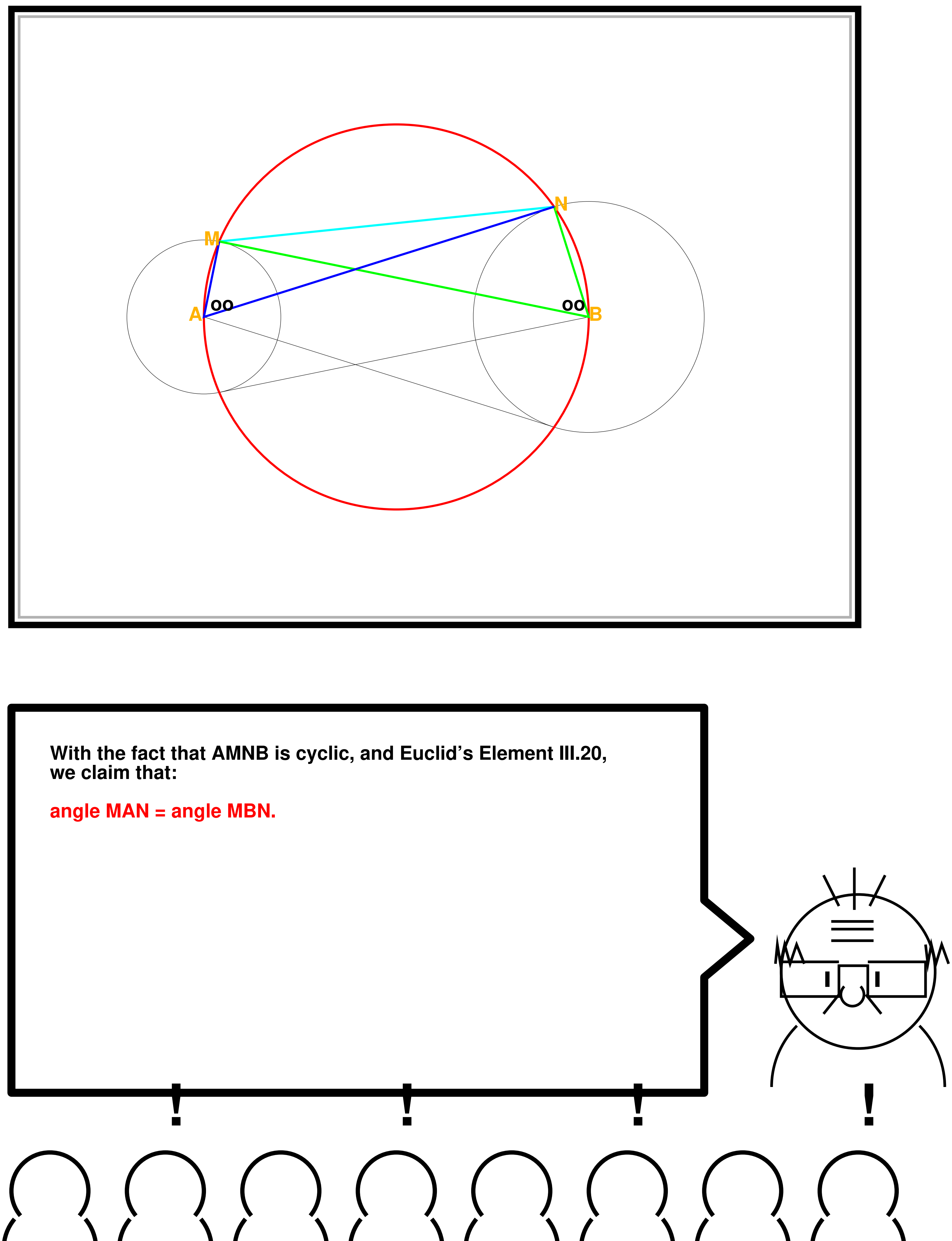

Three Proofs.
M
N
P
R
A
B
Q
S
K
L
Exactly. PQ = RS
It is not obvious because the two circles have different radius.
Also, since the slope of the tangent lines are different,
It does not directly illustrate that PQ = RS
We are going to do the proof of the theorem today. It’s called
The Eyeball Theorem
There are three proofs we are going to do.

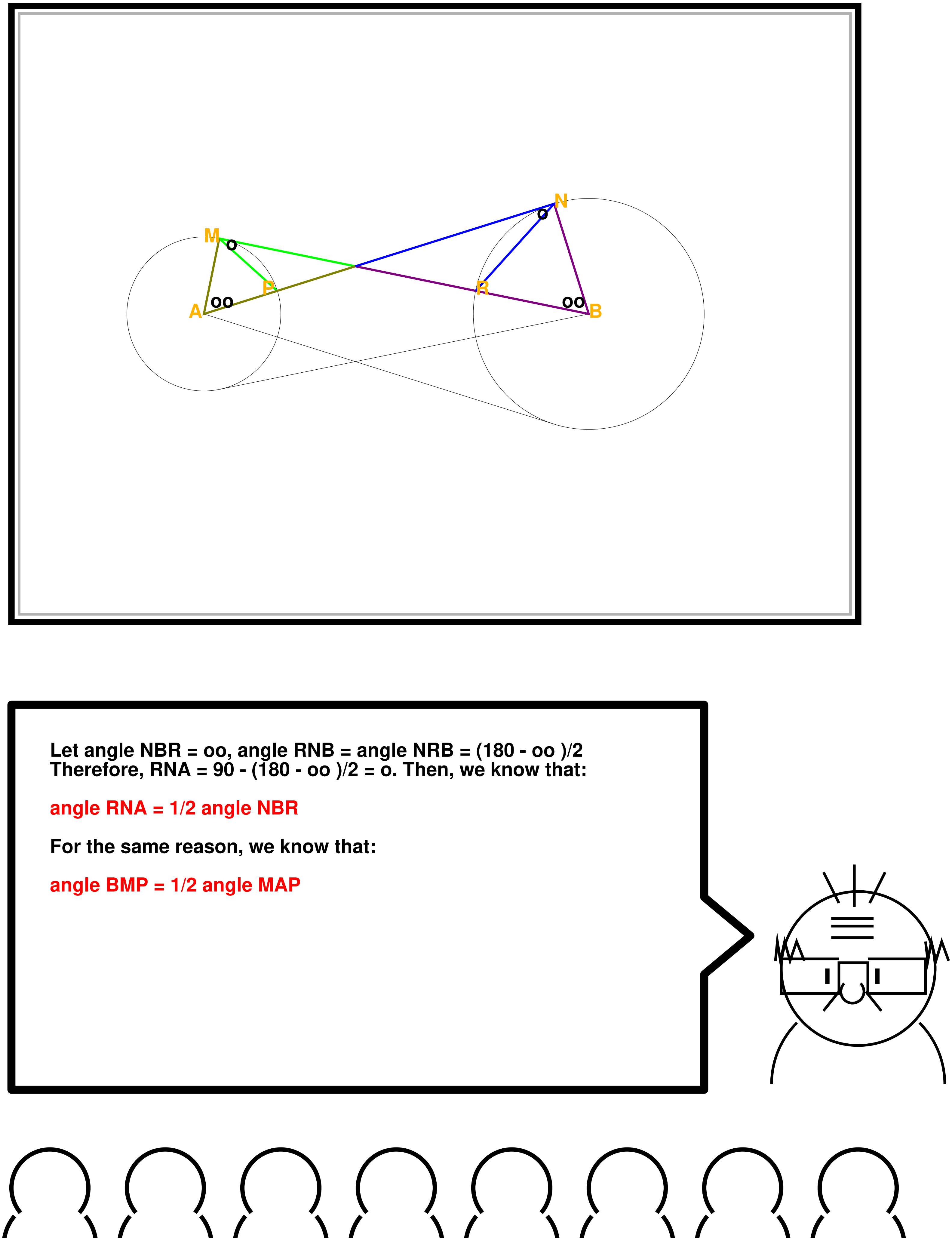


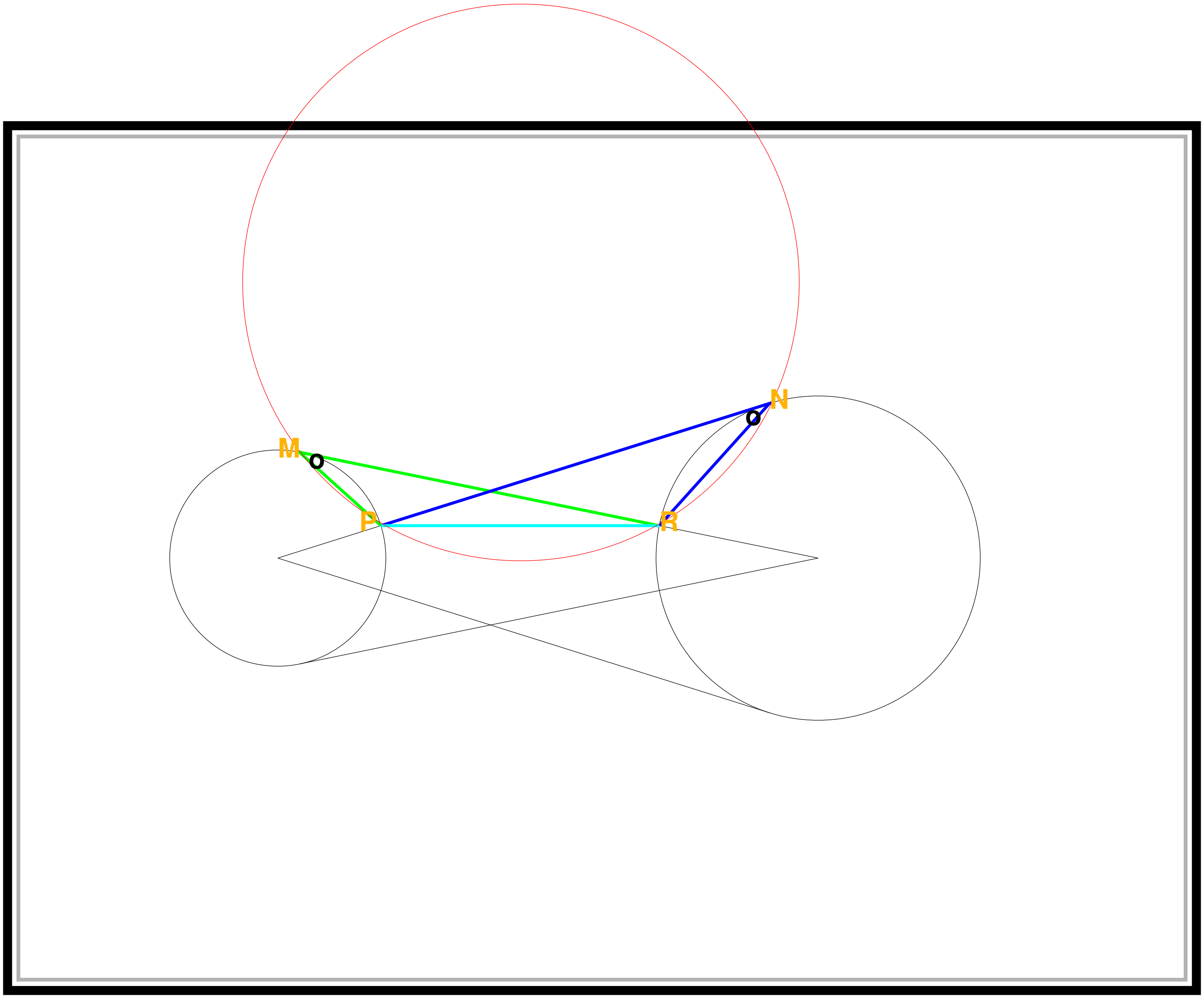
First Proof: By Antonio Gutierrez.
M
N
A
B
Here comes the first proof.
According to the Euclid's Element III.18,
since MB is the tangent line of the left circle,
angle AMB is a right angle.
Also, since AN is the tangent line of the right circle,
angle MNA is a right angle.


N
M
A
O
B
Euclid’s Element III.20 also shows that the angles have the same circumference as base. Therefore, with this property and the fact that angle AMB is equal to angle BNA, we claim that:
AMNB is cyclic.


M
N
A
B
x
x
With the fact that AMNB is cyclic, and Euclid’s Element III.20, we claim that:
angle BAN = angle BMN.

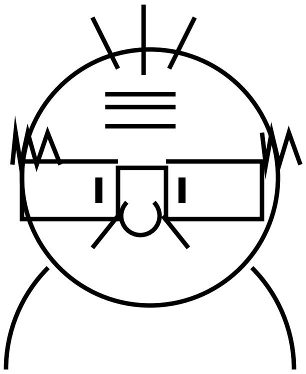

M
N
A
oo
oo
B
With the fact that AMNB is cyclic, and Euclid’s Element III.20,
we claim that:
angle MAN = angle MBN.
!
!
!
!

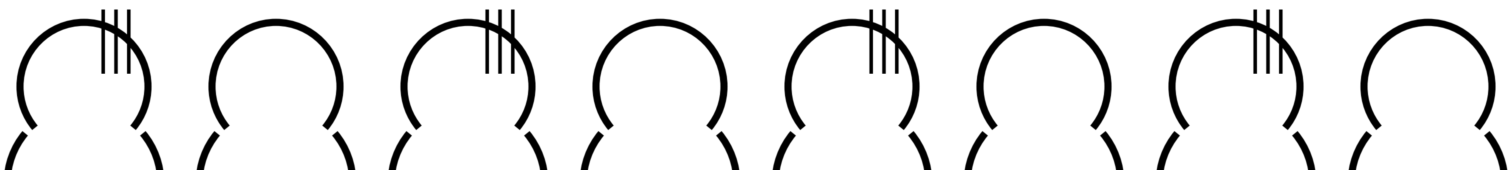

N
o
M
o
P
R
A
oo
oo
B
Let angle NBR = oo, angle RNB = angle NRB = (180 - oo)/2
Therefore, RNA = 90 - (180 - oo)/2 = o. Then, we know that:
angle RNA = 1/2 angle NBR
For the same reason, we know that:
angle BMP = 1/2 angle MAP

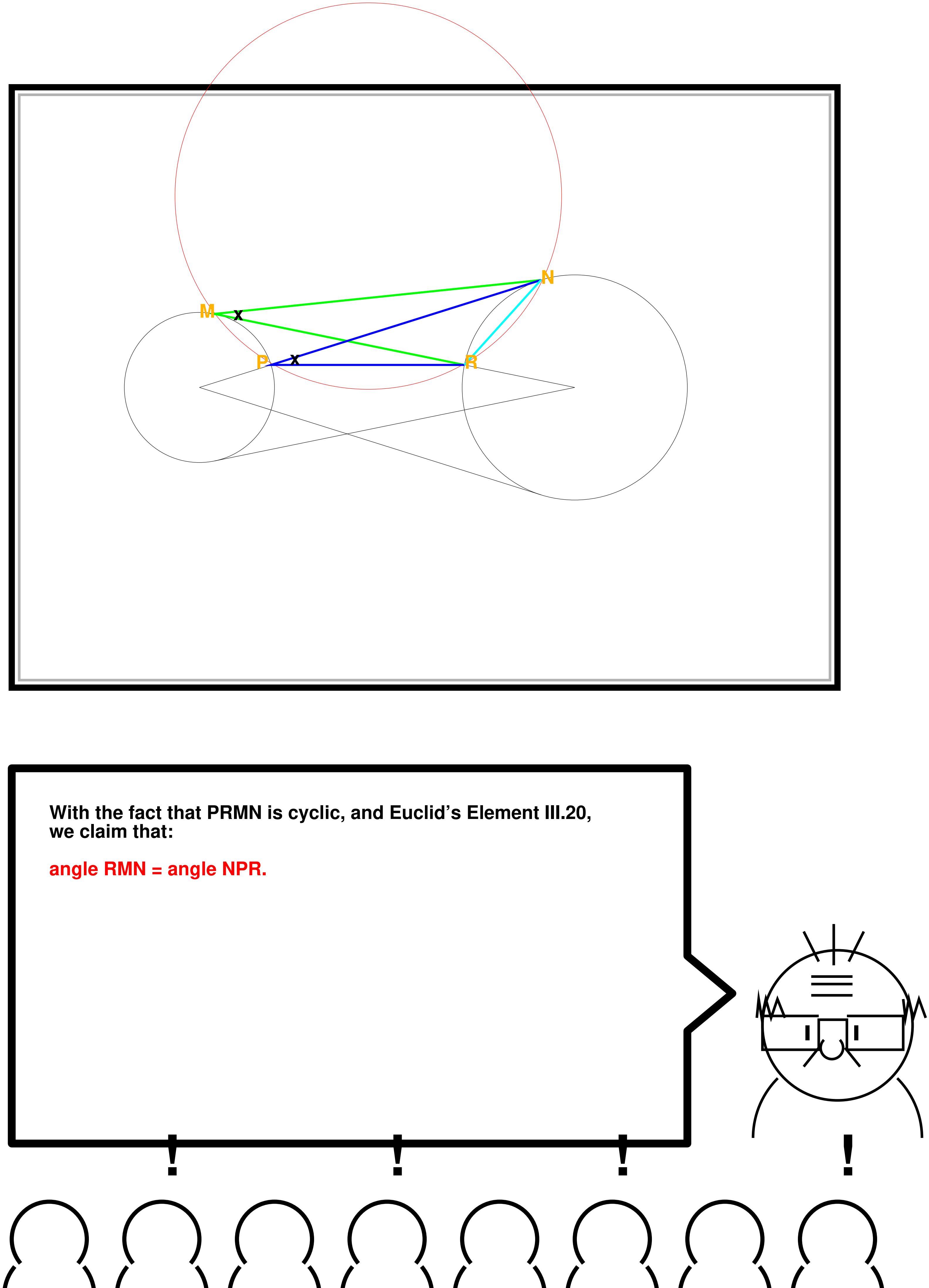


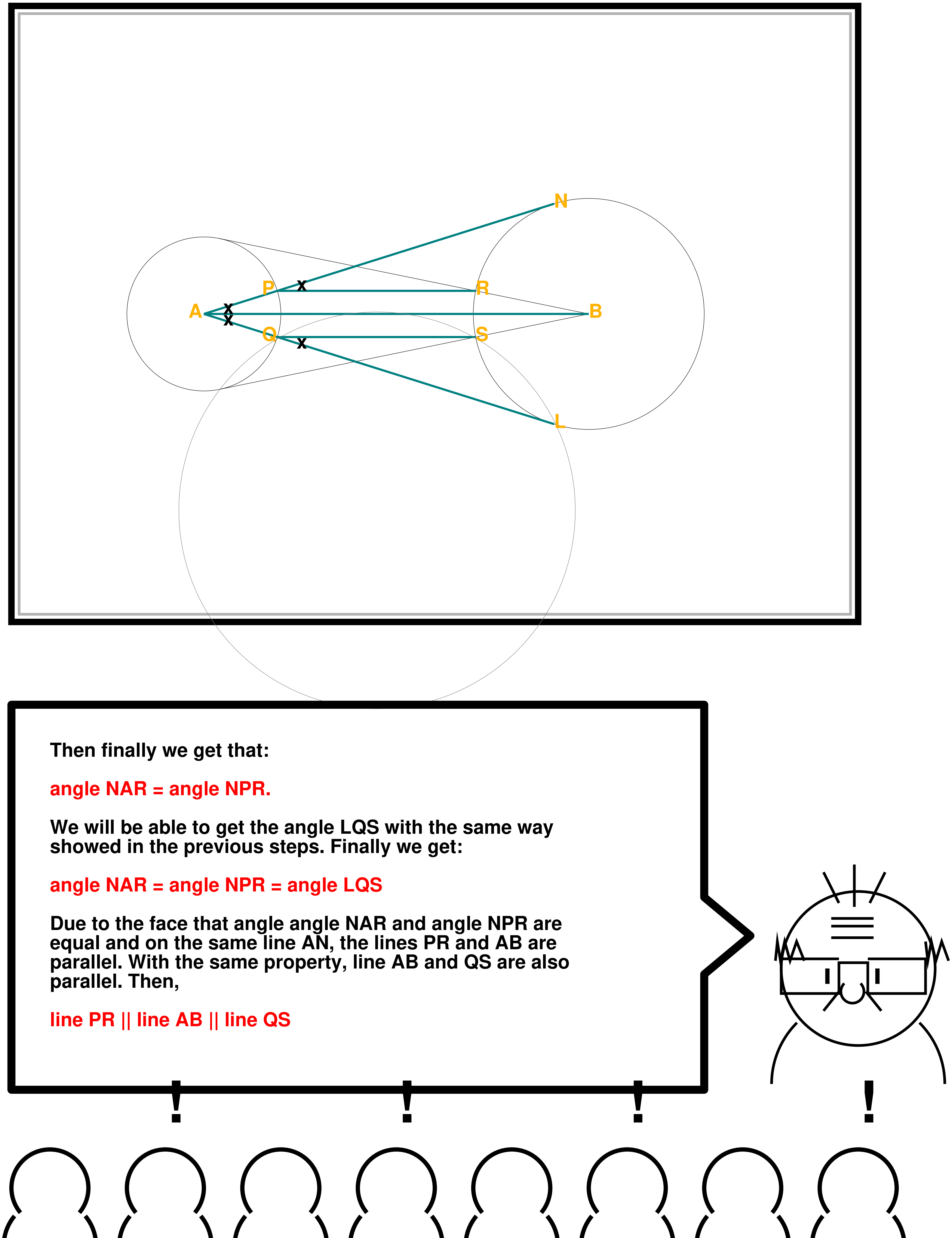
Because angle PMR is equal to angle RNP, and again we use Euclid's Element III.20, we claim that:

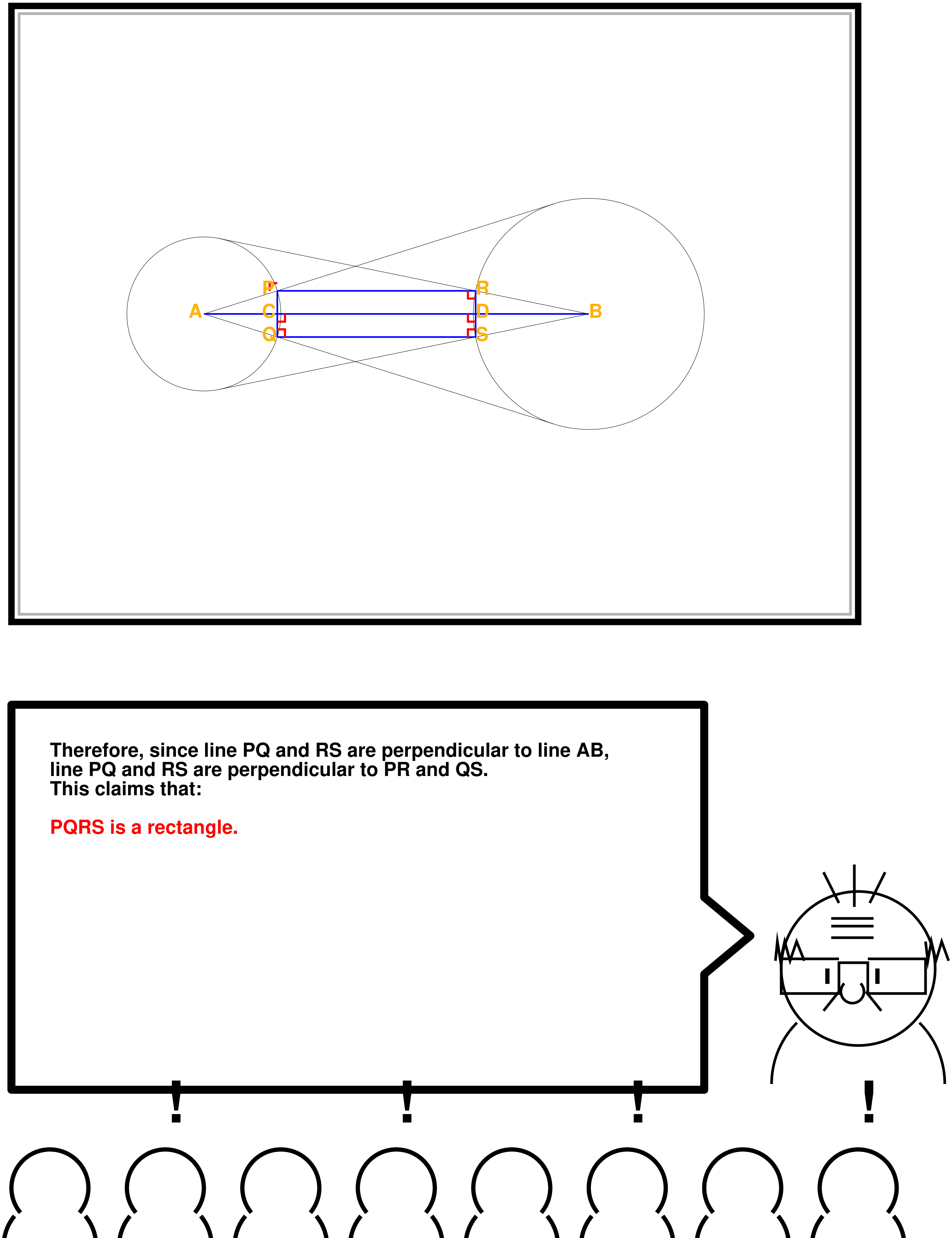
PMNR is cyclic.

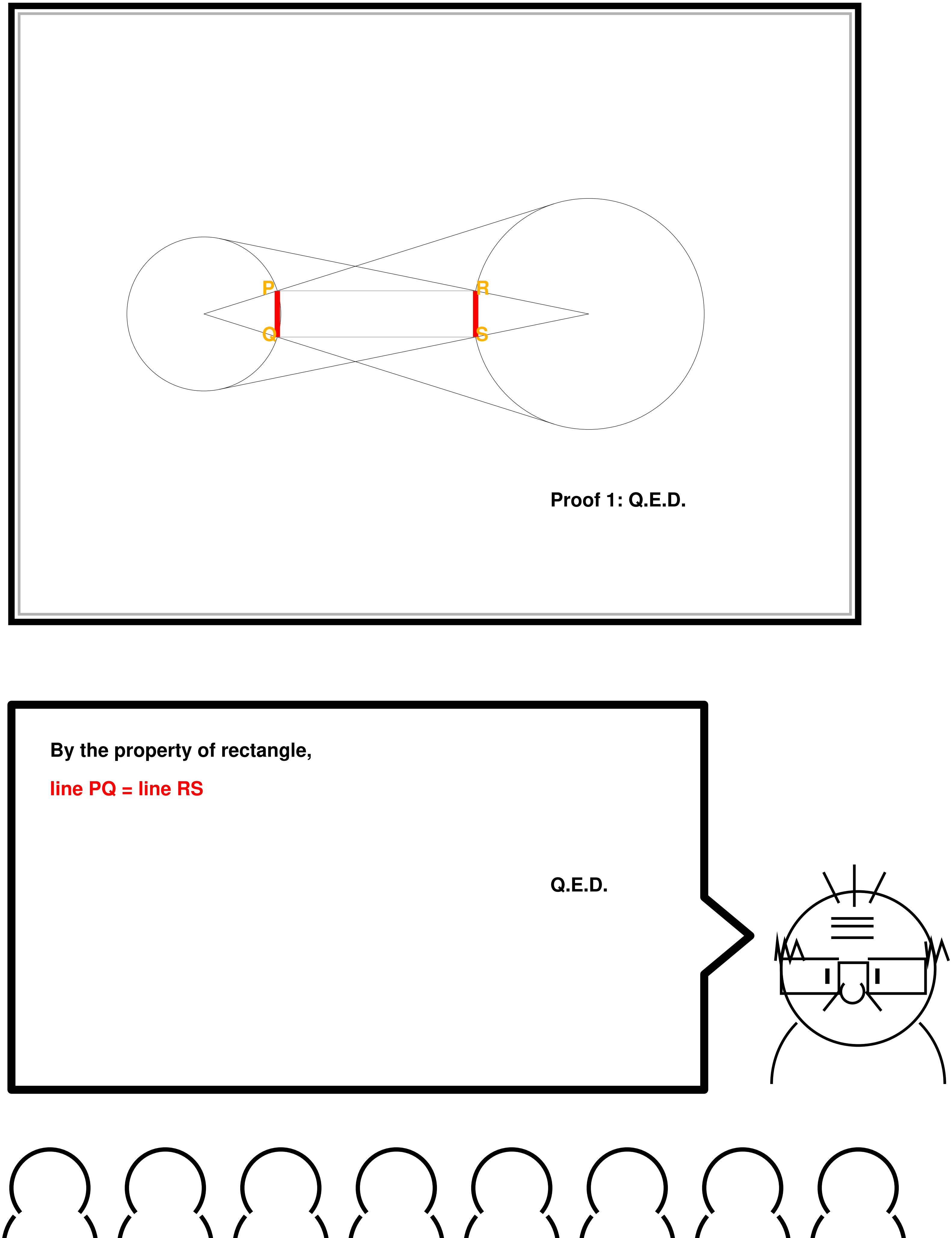


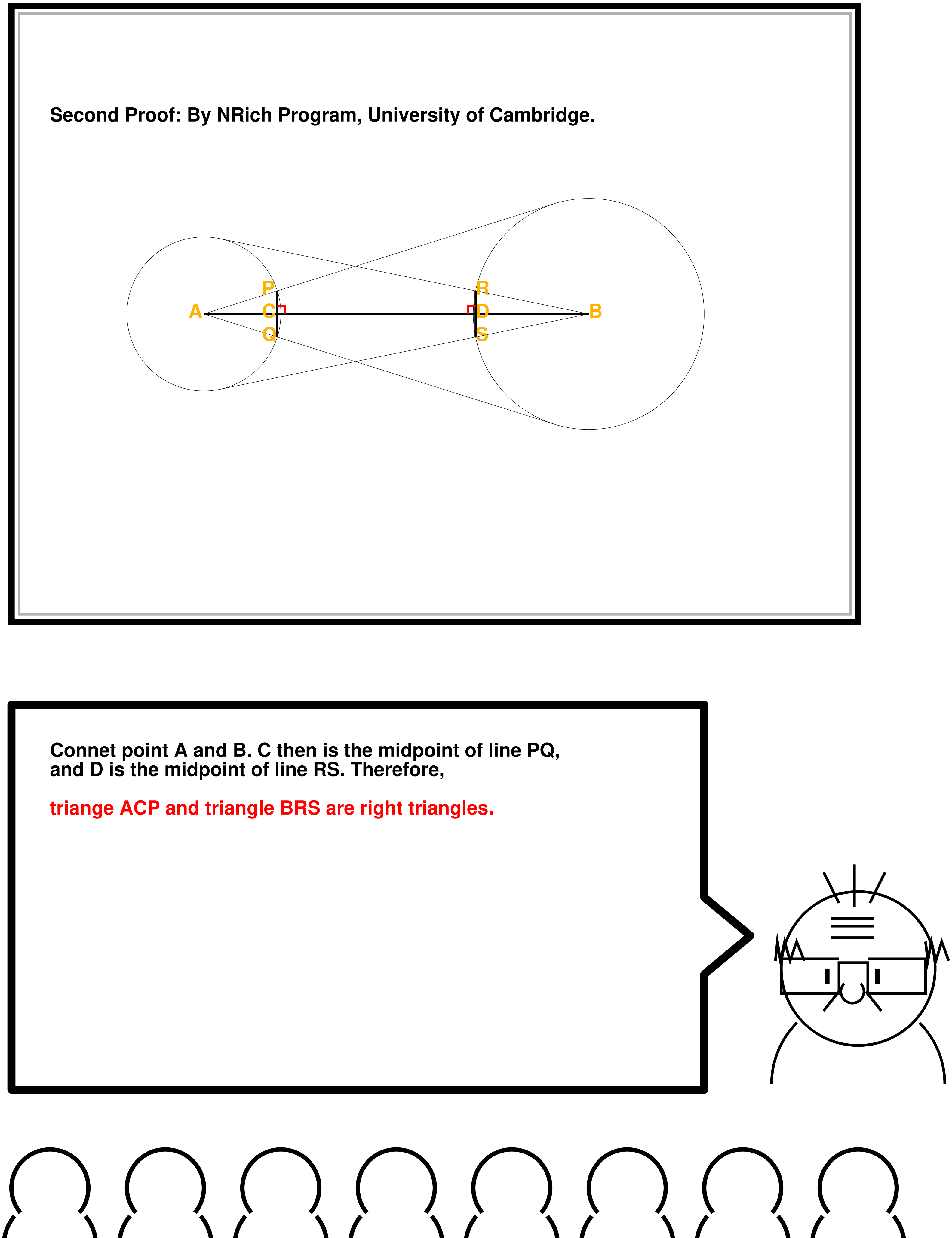


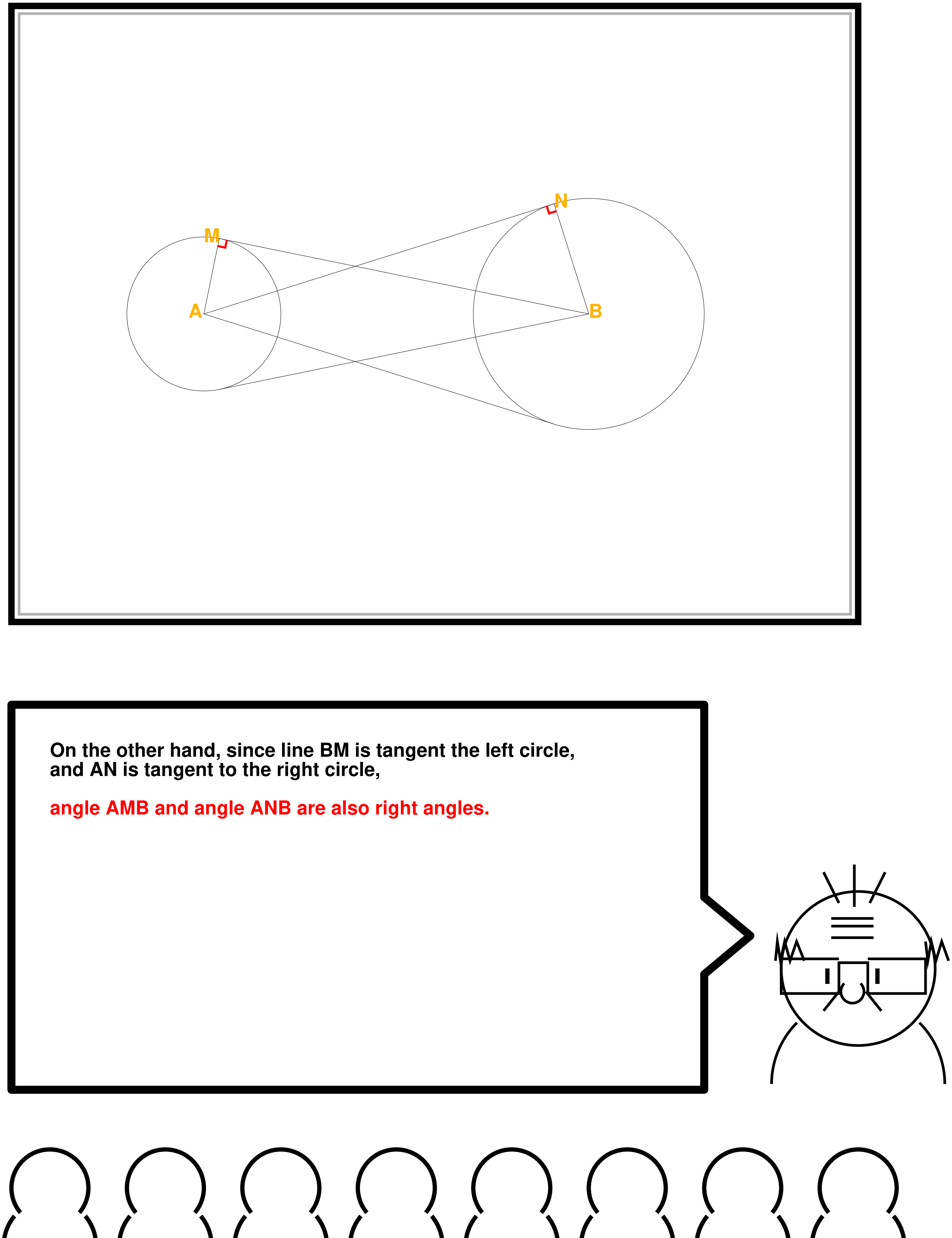

M
x
N
P
x
R
With the fact that PRMN is cyclic, and Euclid’s Element III.20, we claim that:
angle RMN = angle NPR.
!
!
!
!

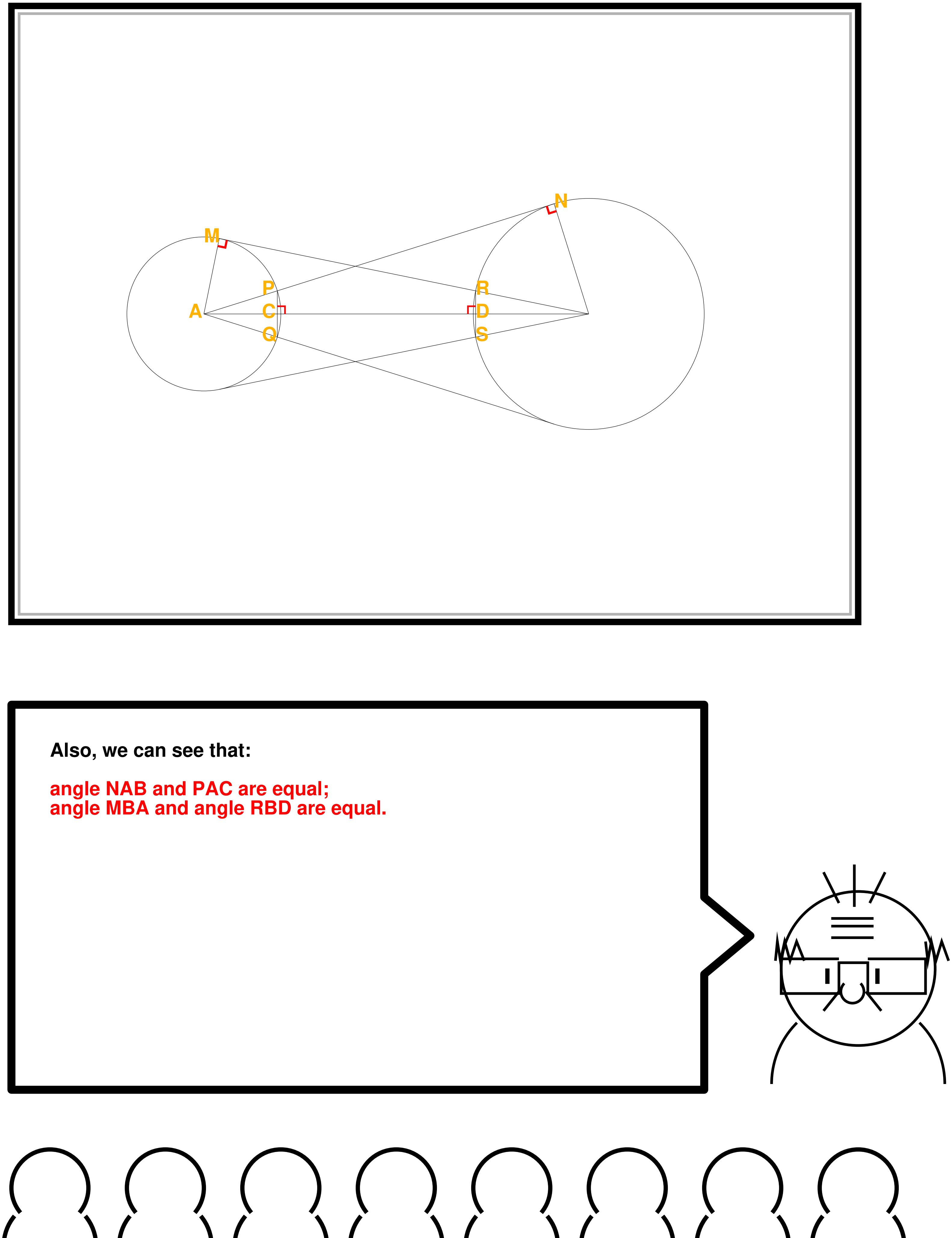

N
P
R
A
B
Q
S
L
x
x
x
x
Then finally we get that:
angle NAR = angle NPR.
We will be able to get the angle LQS with the same way showed in the previous steps. Finally we get:
angle NAR = angle NPR = angle LQS
Due to the face that angle angle NAR and angle NPR are equal and on the same line AN, the lines PR and AB are parallel. With the same property, line AB and QS are also parallel. Then,
line PR || line AB || line QS
!
!
!
!

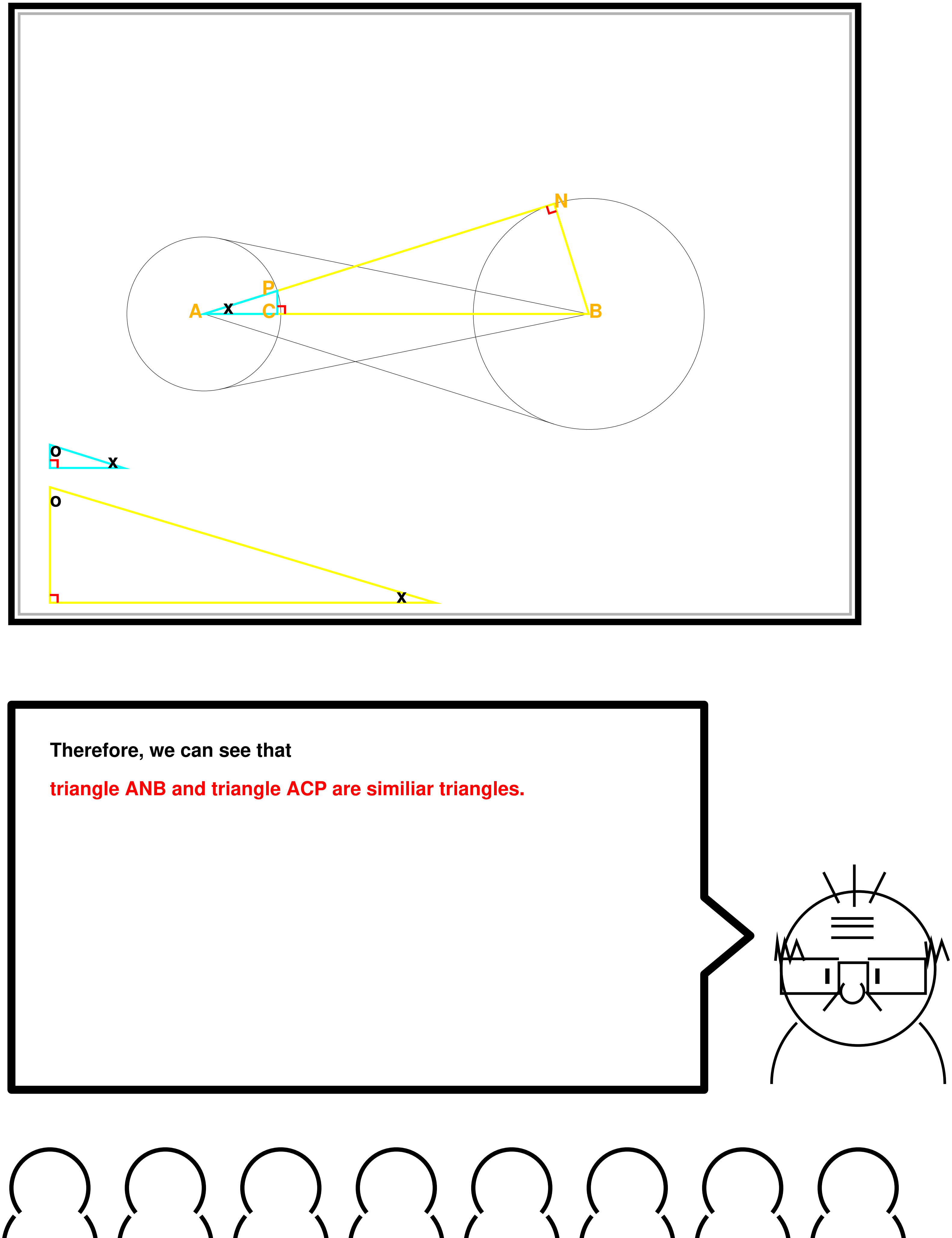

A
P
C
Q
R
D
S
B
Therefore, since line PQ and RS are perpendicular to line AB,
line PQ and RS are perpendicular to PR and QS.
This claims that:
PQRS is a rectangle.
!
!
!
!

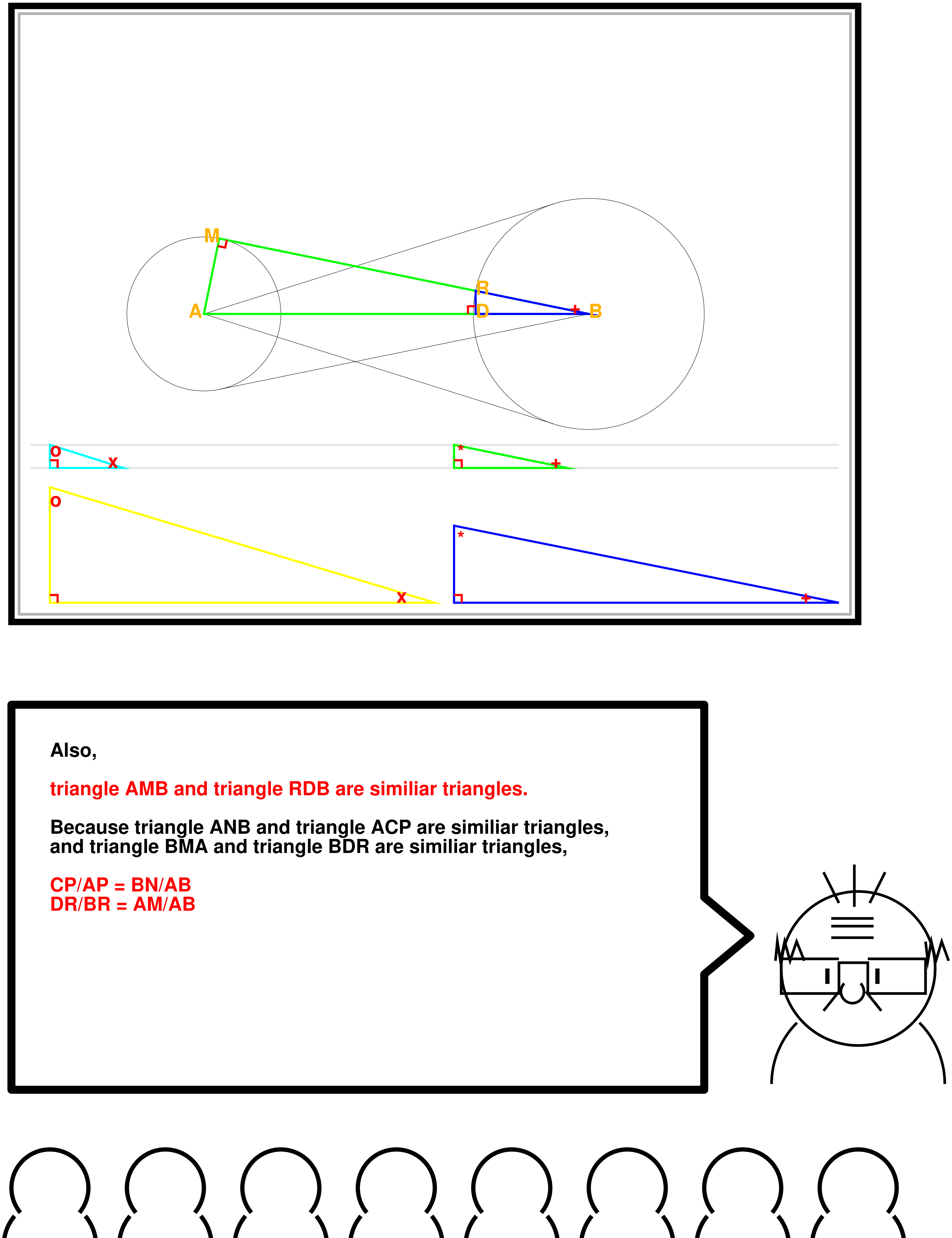

P
R
Q
S
Proof 1: Q.E.D.
By the property of rectangle,
line PQ = line RS
Q.E.D.


Second Proof: By NRich Program, University of Cambridge.
P
R
A
C
D
B
Q
S
Connet point A and B. C then is the midpoint of line PQ, and D is the midpoint of line RS. Therefore,
triange ACP and triangle BRS are right triangles.

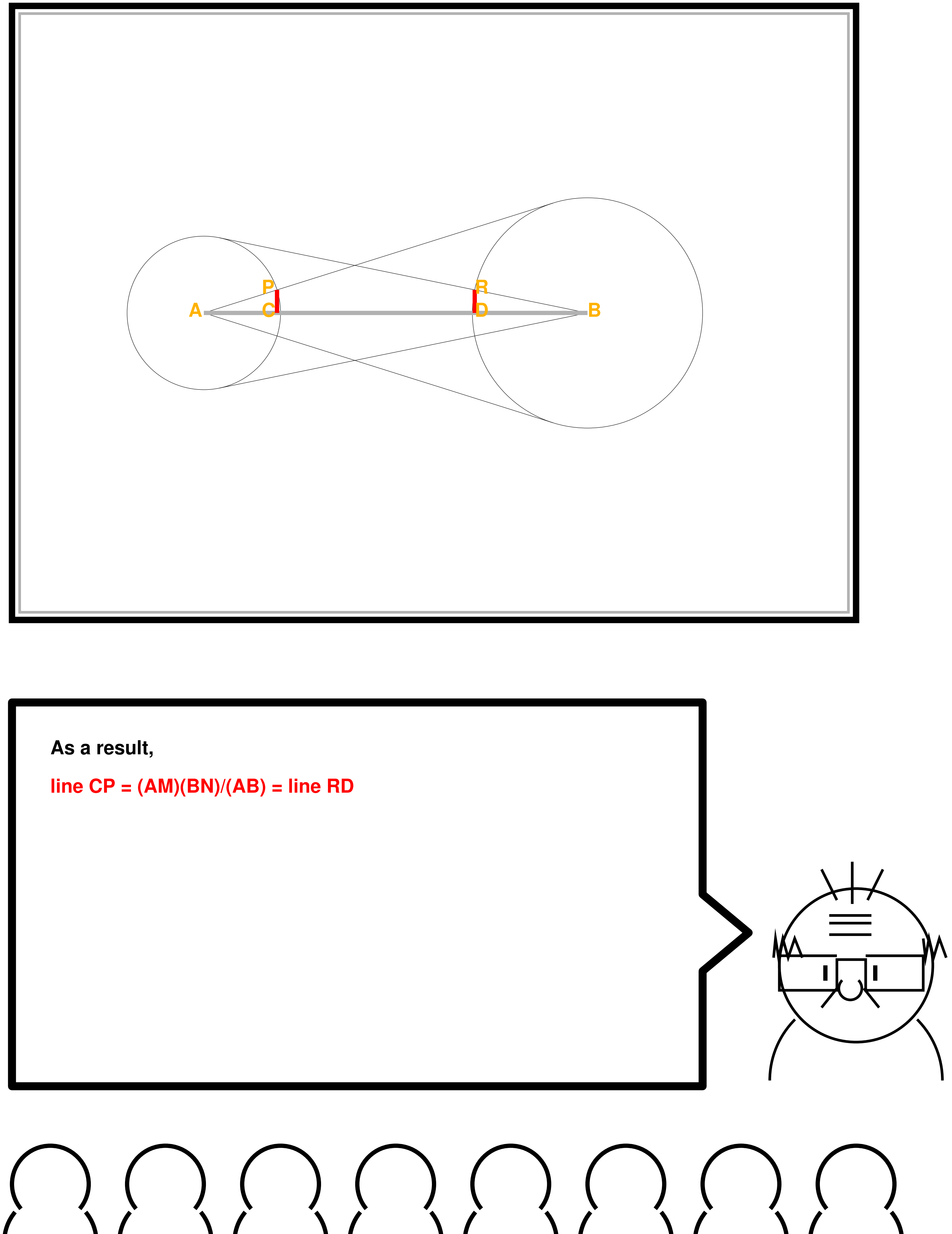

M
N
A
B
On the other hand, since line BM is tangent the left circle, and AN is tangent to the right circle,
angle AMB and angle ANB are also right angles.

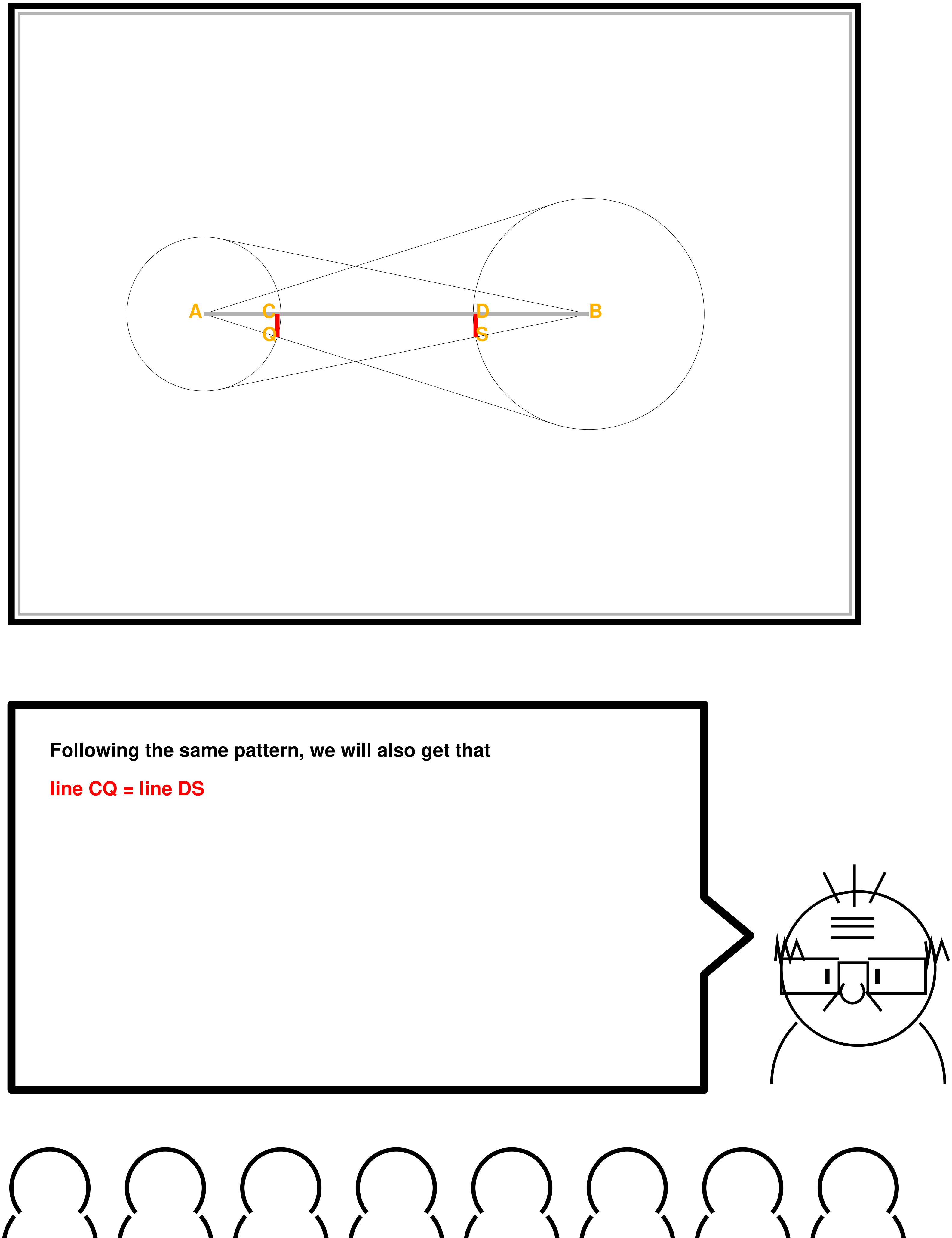

M
N
P
R
A
C
D
Q
S
Also, we can see that:
angle NAB and PAC are equal;
angle MBA and angle RBD are equal.

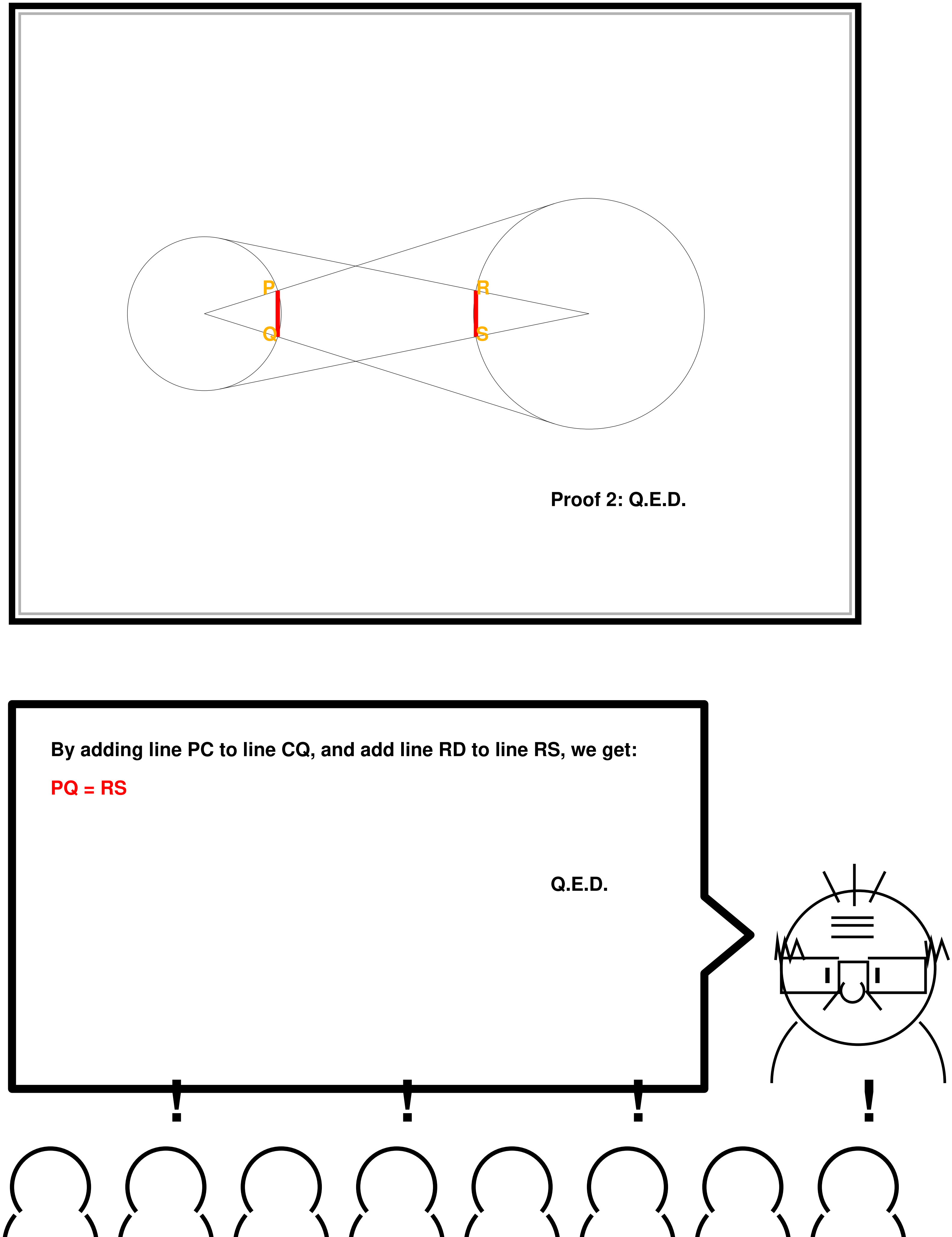

N
P
A
x
C
B
o
x
o
x
Therefore, we can see that
triangle ANB and triangle ACP are similiar triangles.



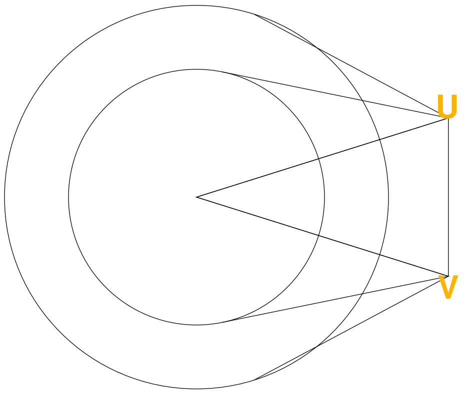
M
R
A
D
B
o
x
*
+
Also,
triangle AMB and triangle RDB are similiar triangles.
Because triangle ANB and triangle ACP are similiar triangles, and triangle BMA and triangle BDR are similiar triangles,
CP/AP = BN/AB
DR/BR = AM/AB


P
R
A
C
D
B
As a result,
line CP = (AM)(BN)/(AB) = line RD


A
C
Q
D
S
B
Following the same pattern, we will also get that
line CQ = line DS

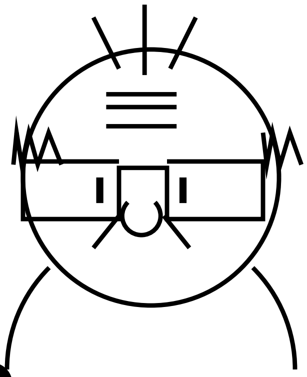

P
R
Q
S
Proof 2: Q.E.D.
By adding line PC to line CQ, and add line RD to line RS, we get:
PQ = RS
Q.E.D.
!
!
!
!

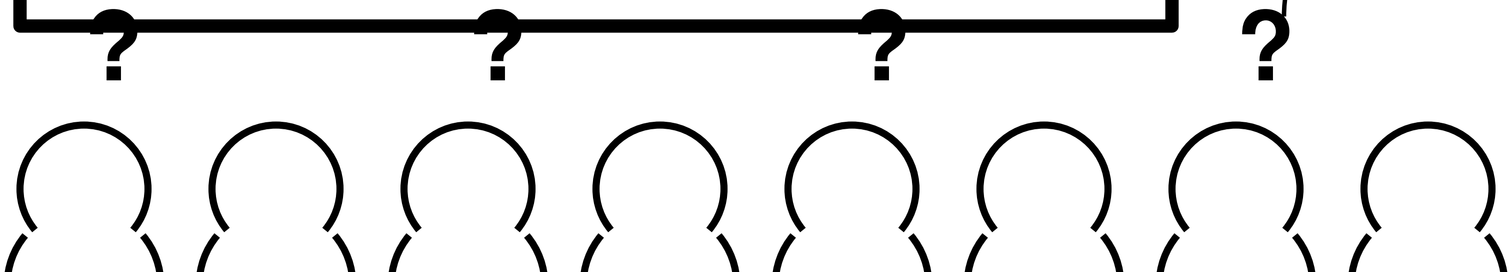
Third Proof: By E. J. Barbeau, M. S. Klamkin, and W.O.J. Moser.

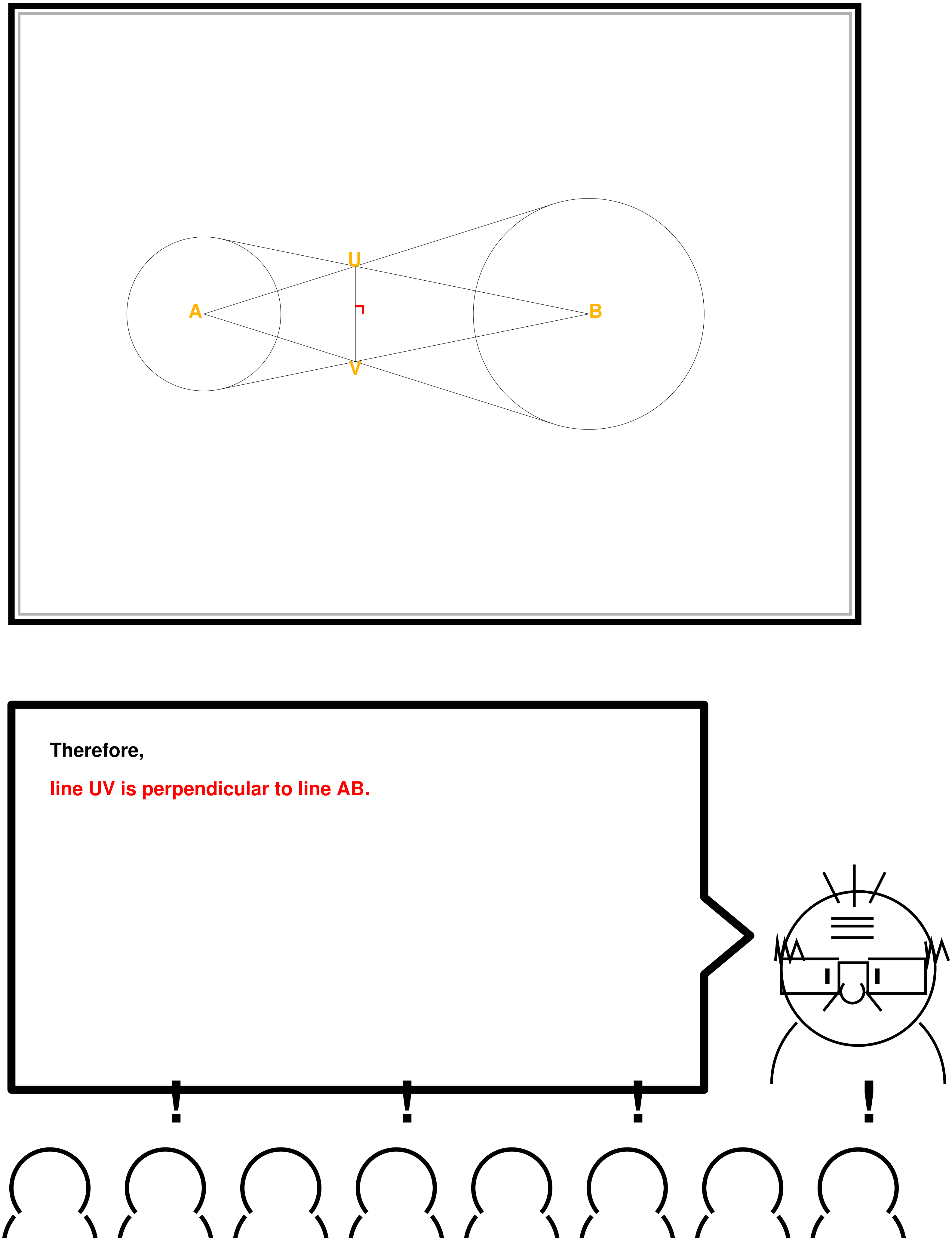


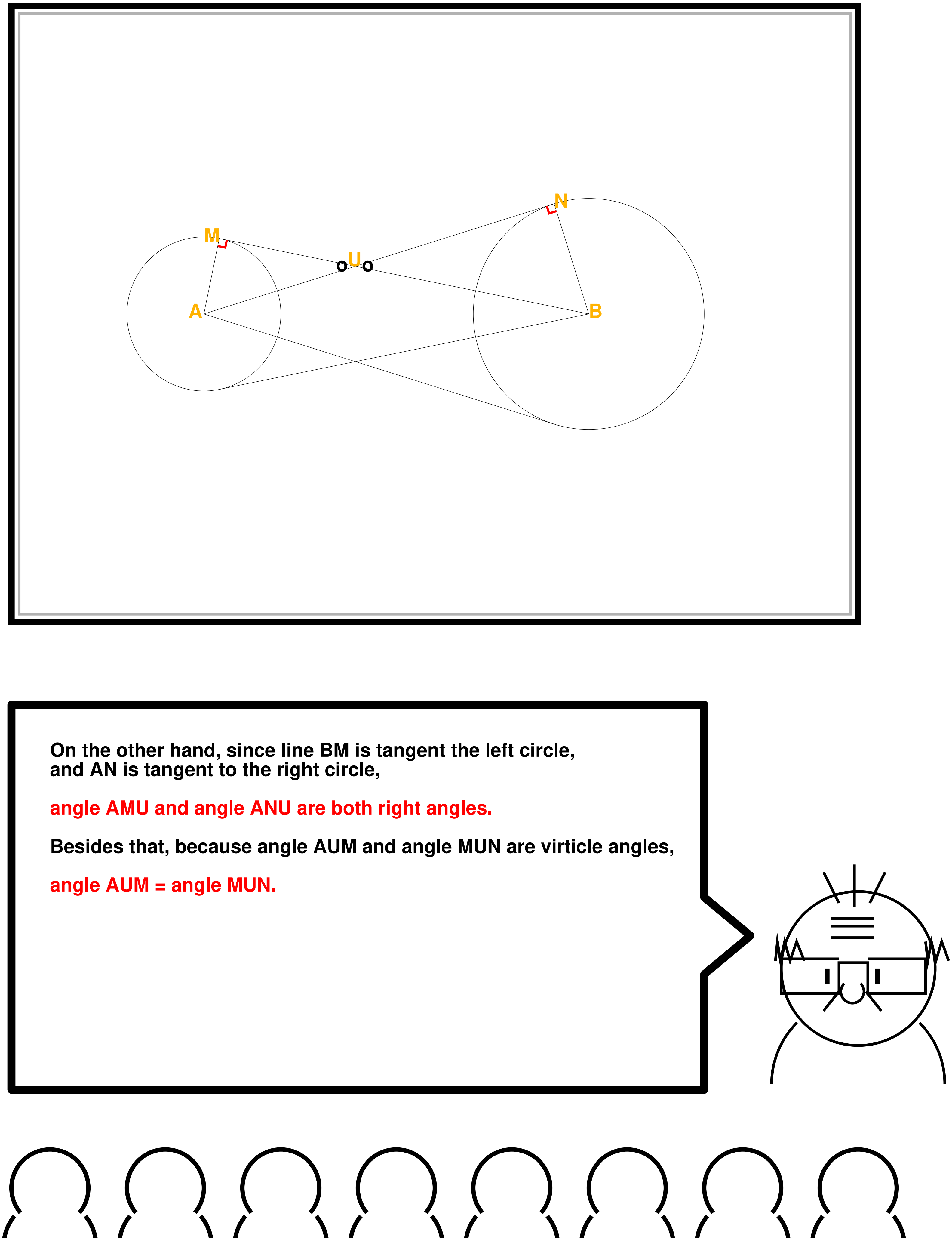
By folding the left circle and the right circle, we see that

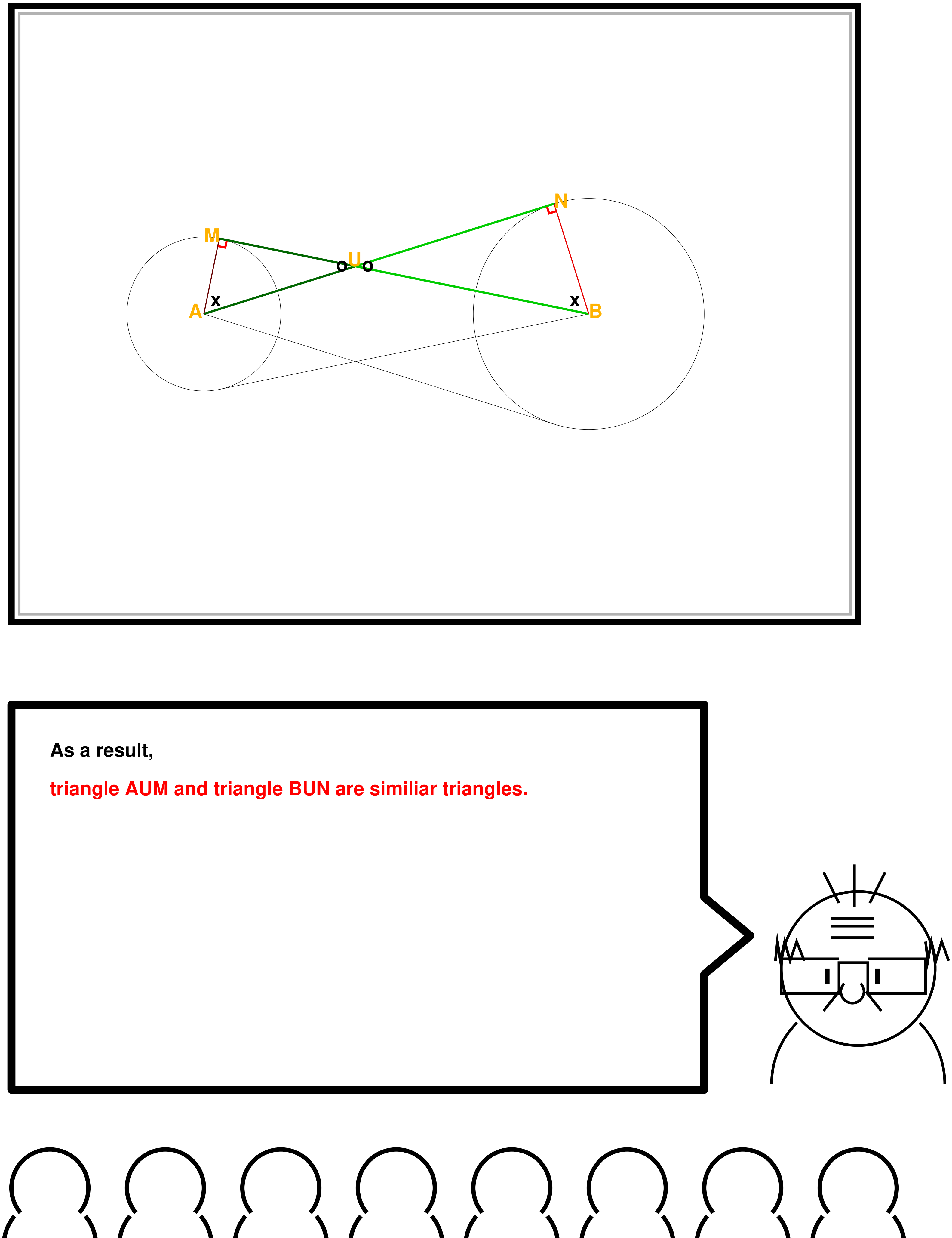
the left circle and the right circle is symmetric on line UV.



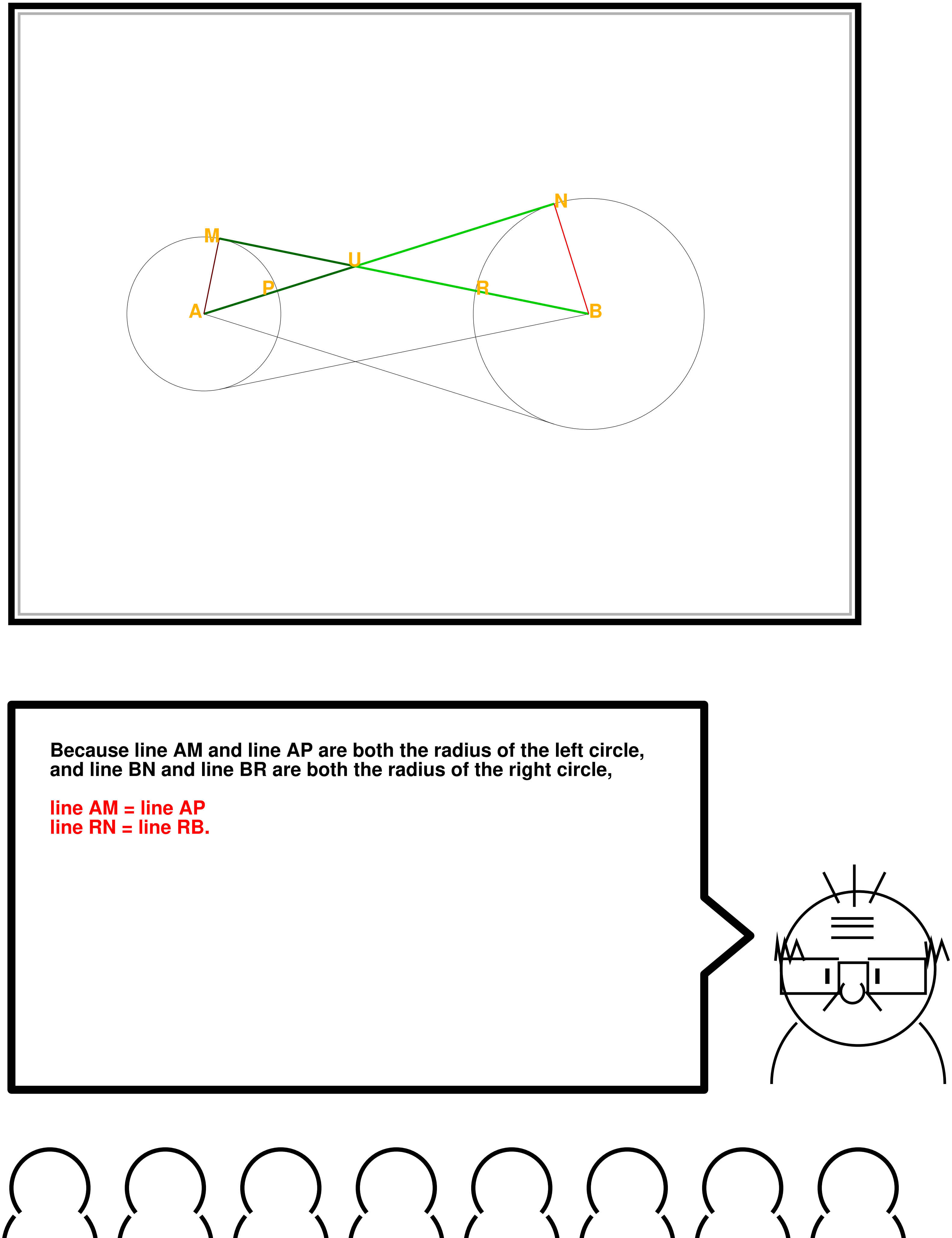



U
A
B
V
Therefore,
line UV is perpendicular to line AB.
!
!
!
!

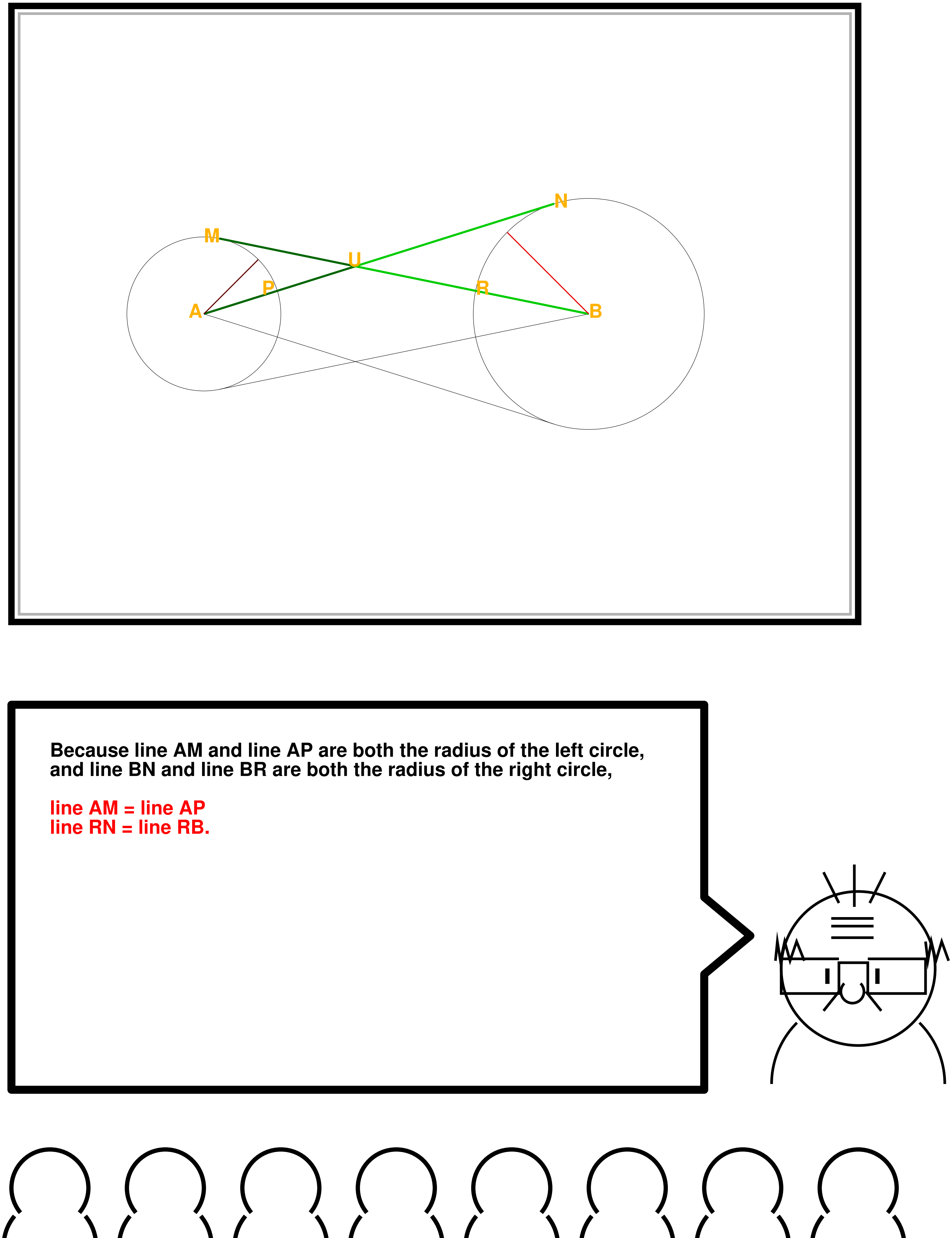

M
U
N
A
B
On the other hand, since line BM is tangent the left circle,
and AN is tangent to the right circle,
angle AMU and angle ANU are both right angles.
Besides that, because angle AUM and angle MUN are virticle angles,
angle AUM = angle MUN.

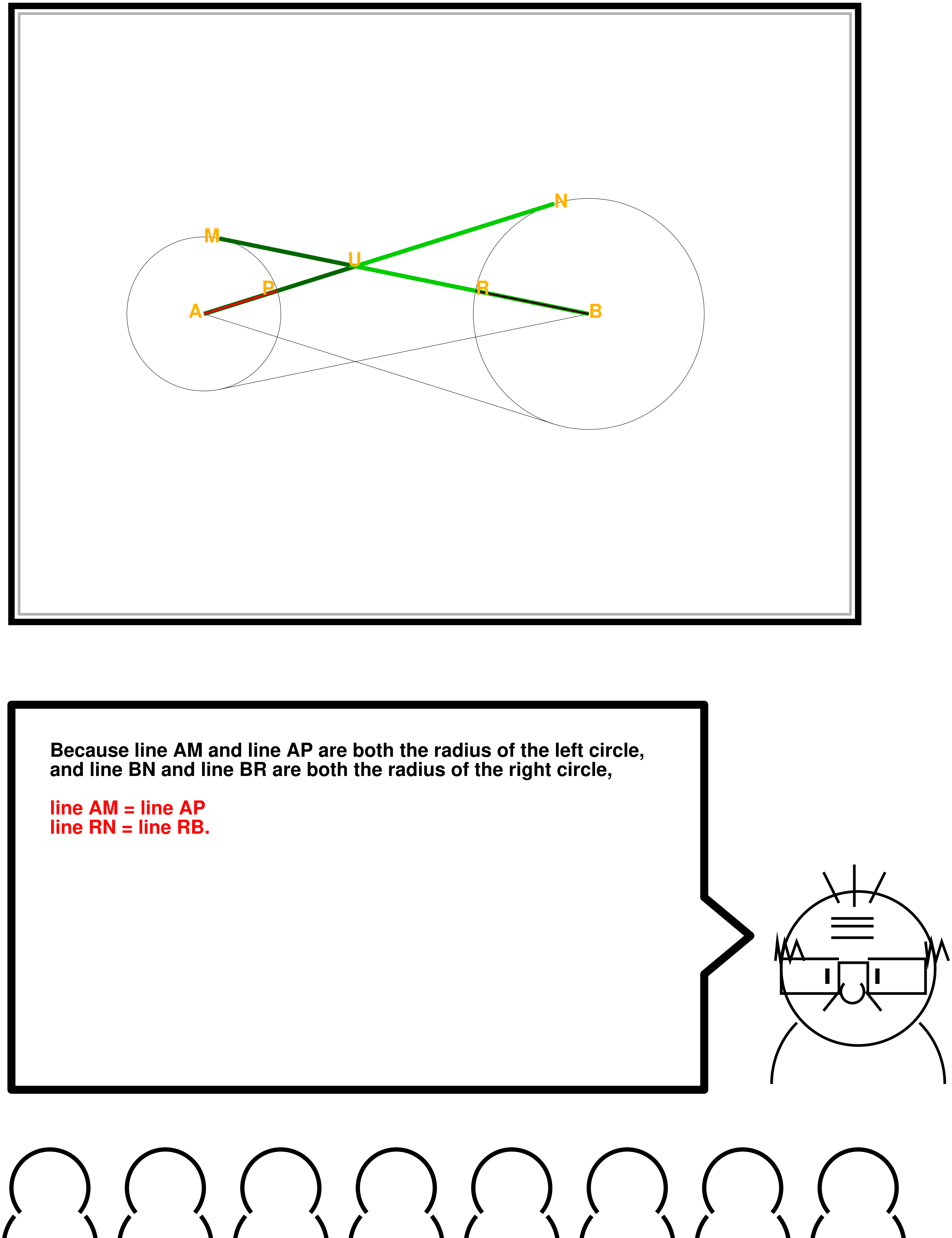


N
M
U
o o
x
A
x
B
As a result,
triangle AUM and triangle BUN are similiar triangles.

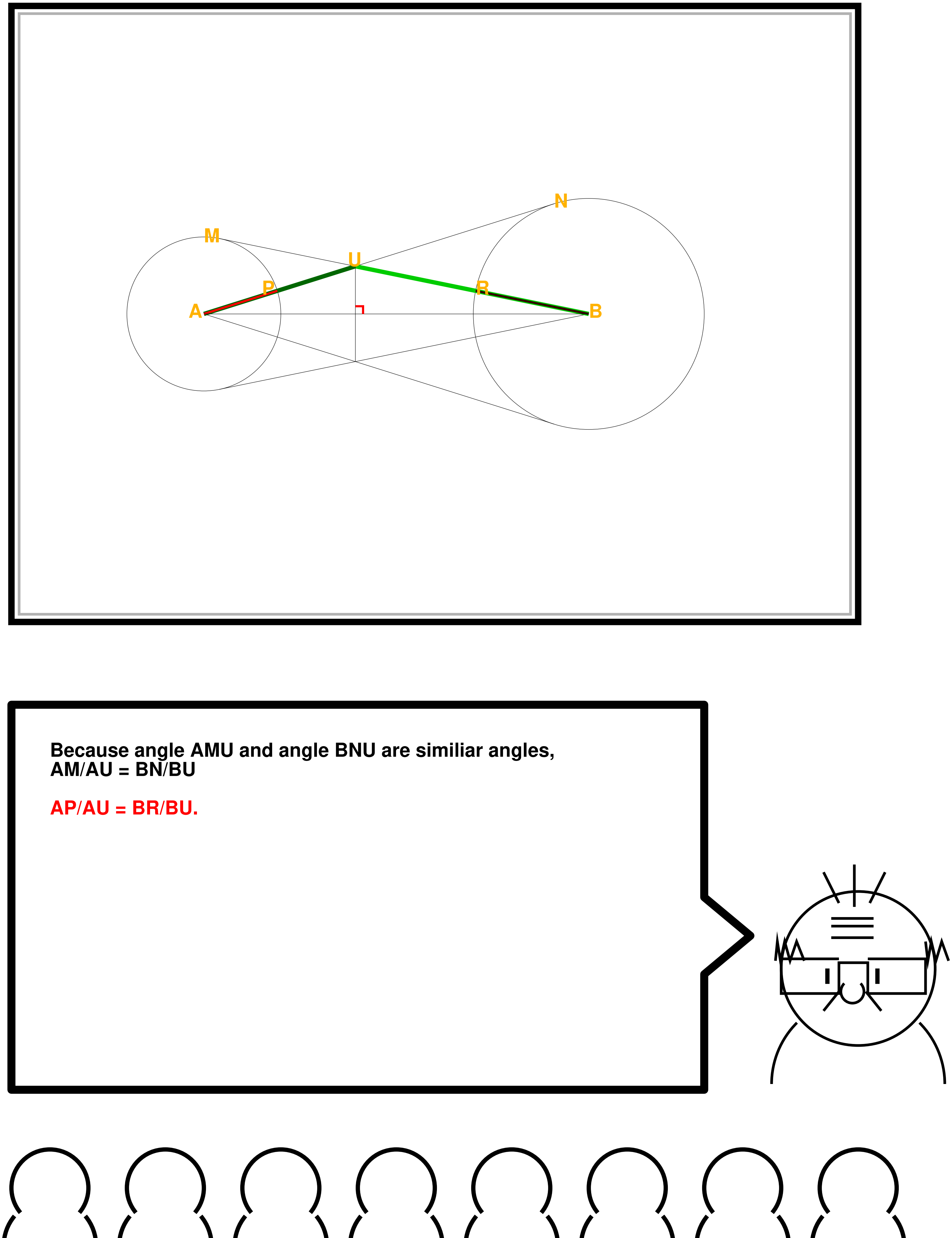


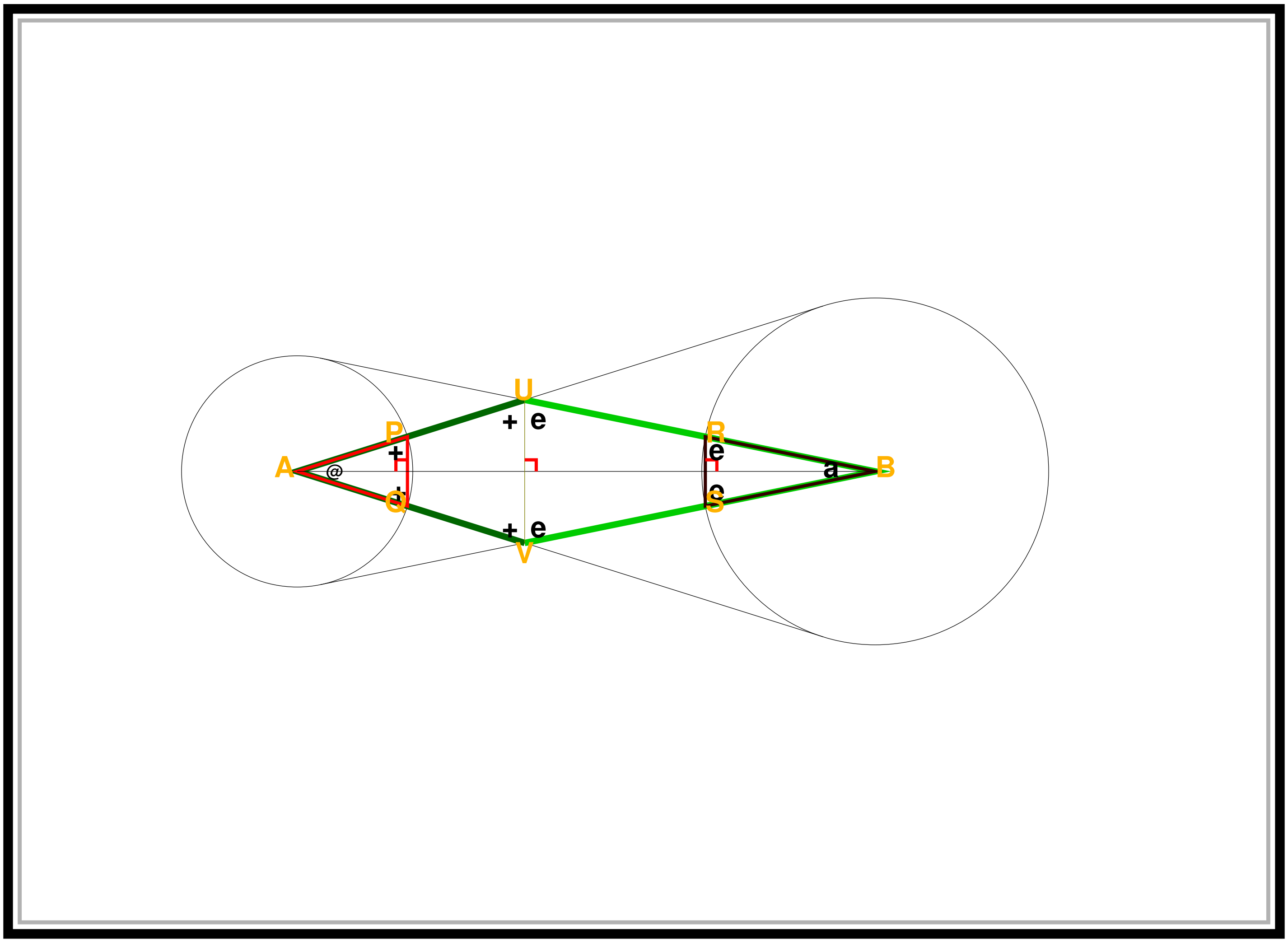
M
N
U
P
R
A
B
Because line AM and line AP are both the radius of the left circle,
and line BN and line BR are both the radius of the right circle,
line AM = line AP
line RN = line RB.


M
N
U
P
R
A
B
Because line AM and line AP are both the radius of the left circle,
and line BN and line BR are both the radius of the right circle,
line AM = line AP
line RN = line RB.



N
M
U
P
R
A
B
Because line AM and line AP are both the radius of the left circle,
and line BN and line BR are both the radius of the right circle,
line AM = line AP
line RN = line RB.


N
M
U
P
R
A
B
Because angle AMU and angle BNU are similiar angles,
AM/AU = BN/BU
AP/AU = BR/BU.

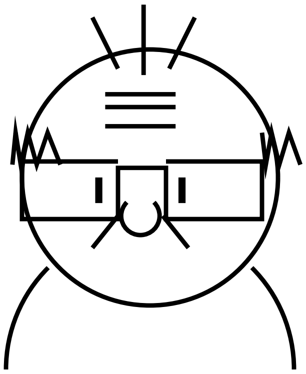


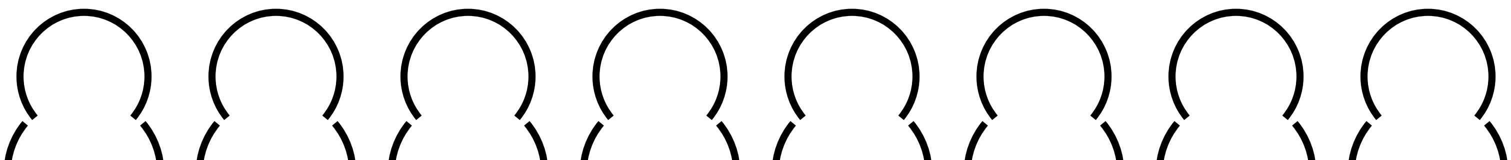
Then we get angle APC and angle BRD. Since the graph is symmetric on line AB,
angle AQC = angle APC, and angle ASD = angle ARD.
Also, because angle APC and angle AUV are corresponding angles, and angle BRD and angle BUV are corresponding angles,

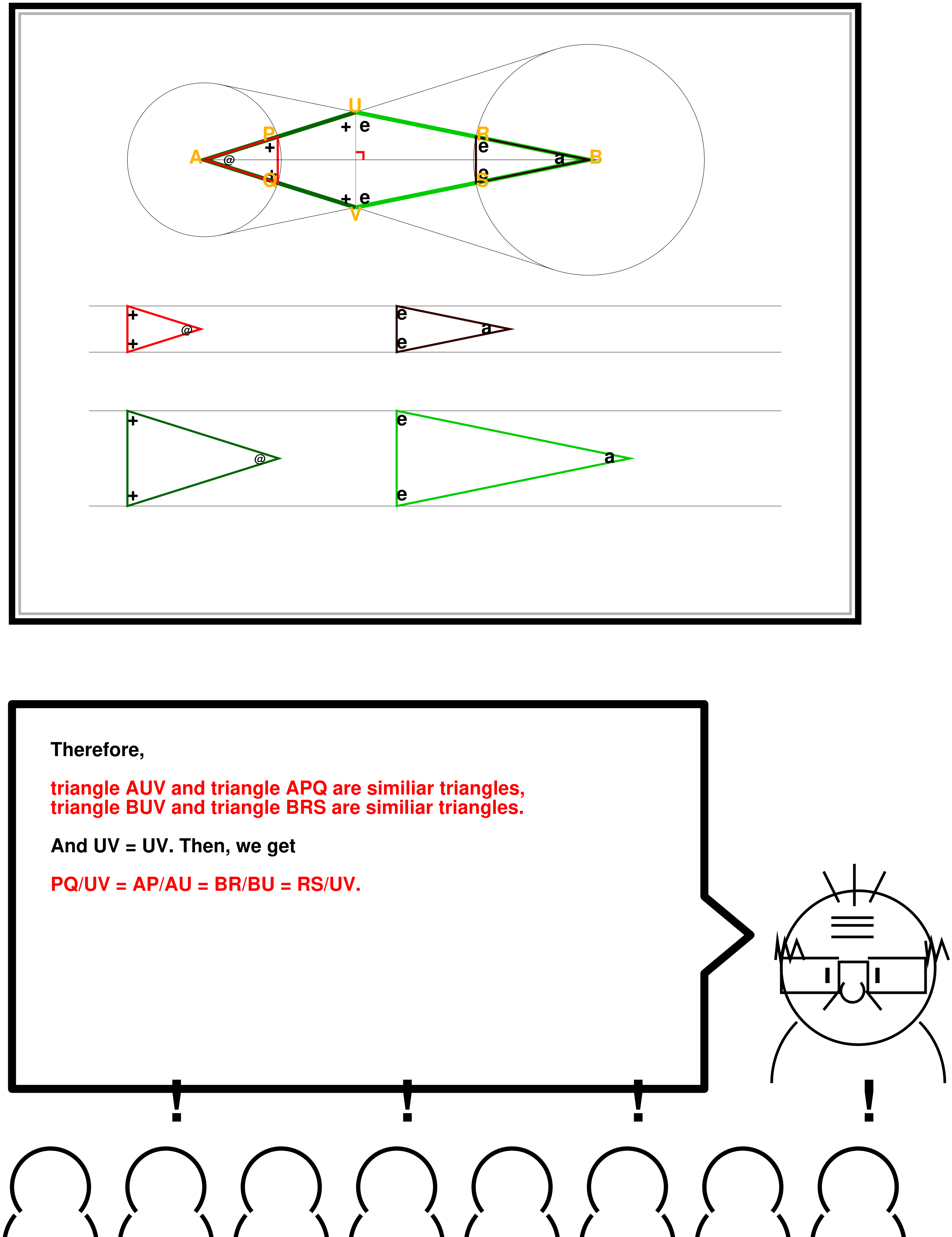
angle AUV = angle APC = angle AQC = angle AVU;
angle BRD = angle BUV = angle BSD = angle BVU.

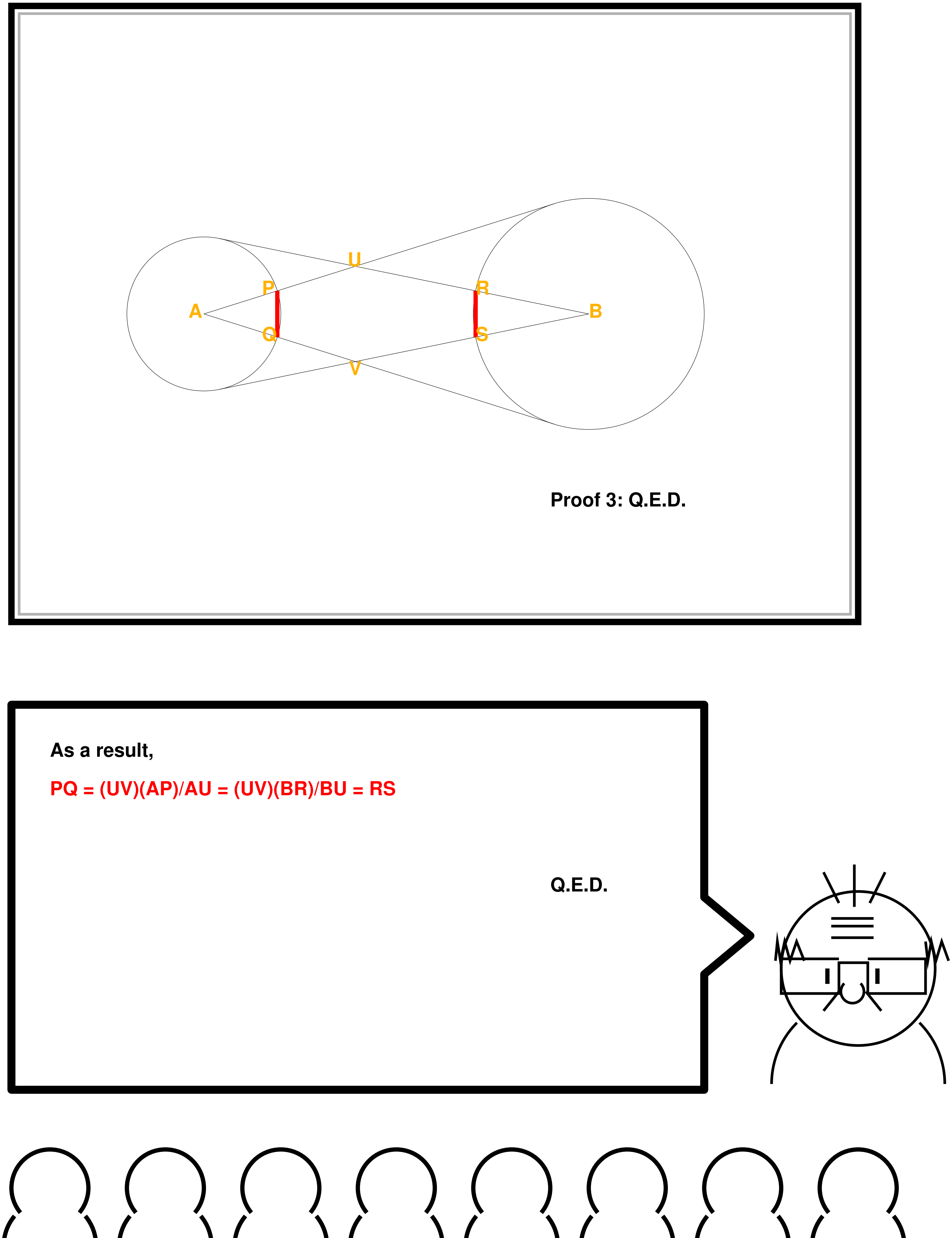
Also, because angle PAQ and angle UAV are the same angle, and angle RBS and angle UBV are the same angle,

angle PAQ = angle UAV
angle UBV = angle RBS.






U
P
A
Q
R
B
S
V
Therefore,
triangle AUV and triangle APQ are similiar triangles,
triangle BUV and triangle BRS are similiar triangles.
And UV = UV. Then, we get
PQ/UV = AP/AU = BR/BU = RS/UV.
!
!
!
!


U
P
R
A
B
Q
S
V
Proof 3: Q.E.D.
As a result,
PQ = (UV)(AP)/AU = (UV)(BR)/BU = RS
Q.E.D.

References:

Proof #1: Antonio Gutierrez

Proof #2: NRich Program, University of Cambridge

Proof #3: E. J. Barbeau, M. S. Klamkin, W. O. J. Moser, Problem 185

Source website:
http://www.cut-the-knot.org/Curriculum/Geometry/Eyeball.shtml#Explanation