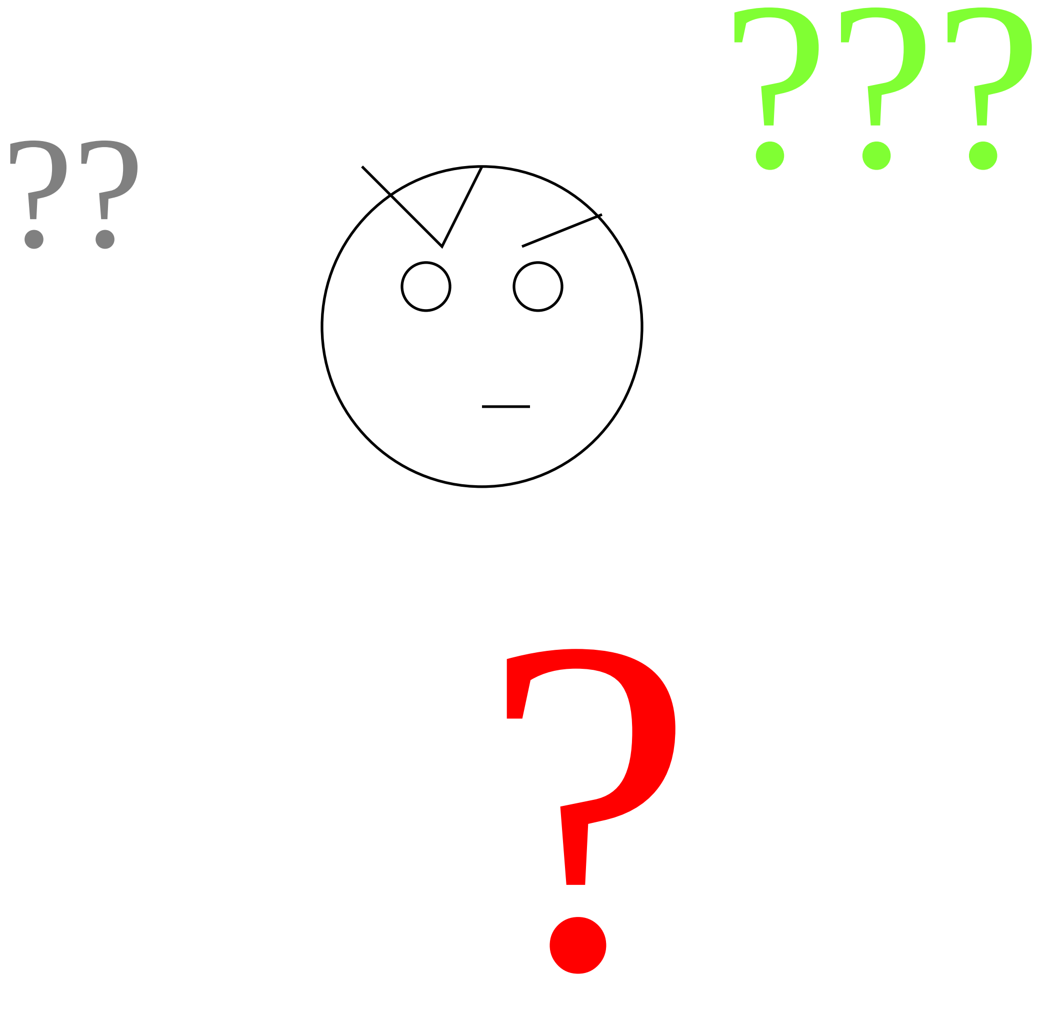
Monge's Three Circle Theorem

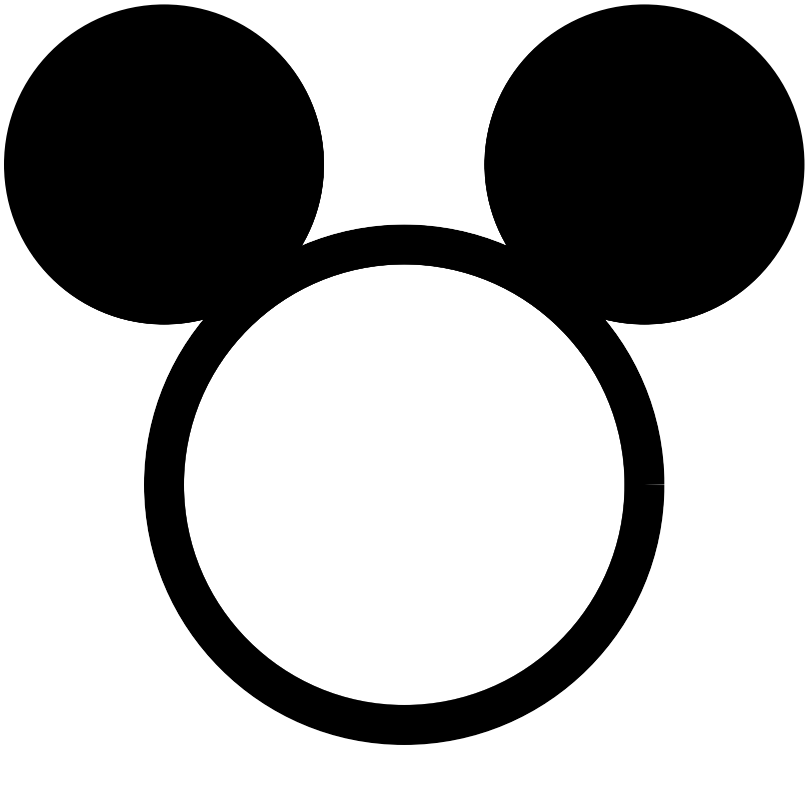
By Sandy Chang

What is Monge’s Three Circle Theorem

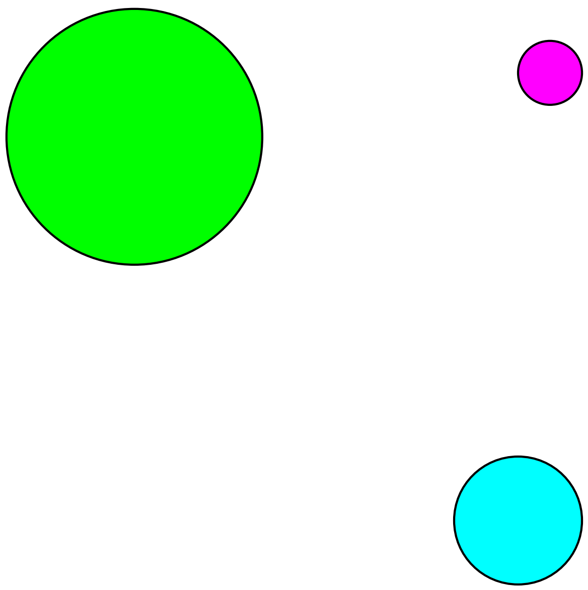


Monge's

Three Circle

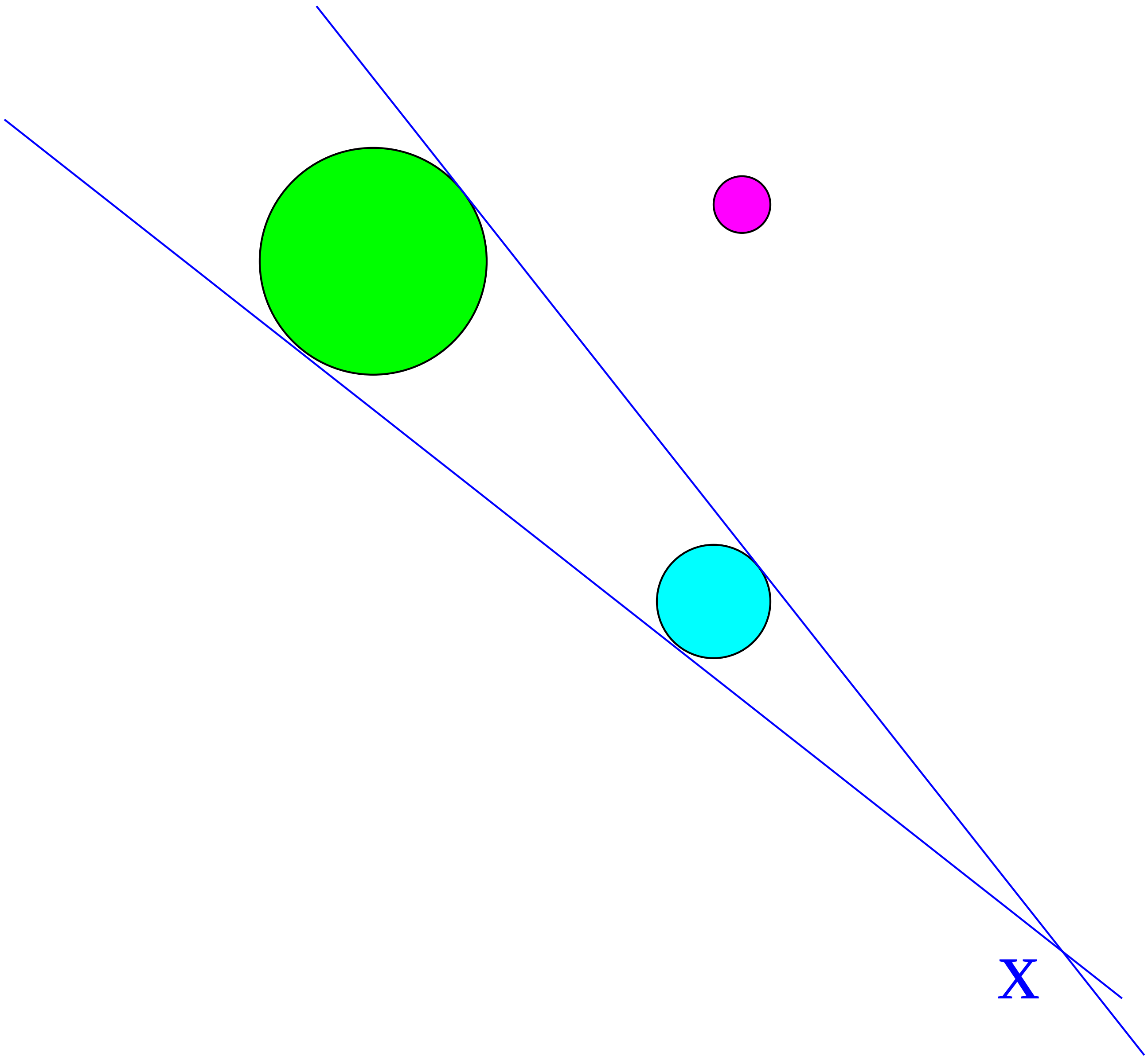


Draw a green, a blue, and a pink circle



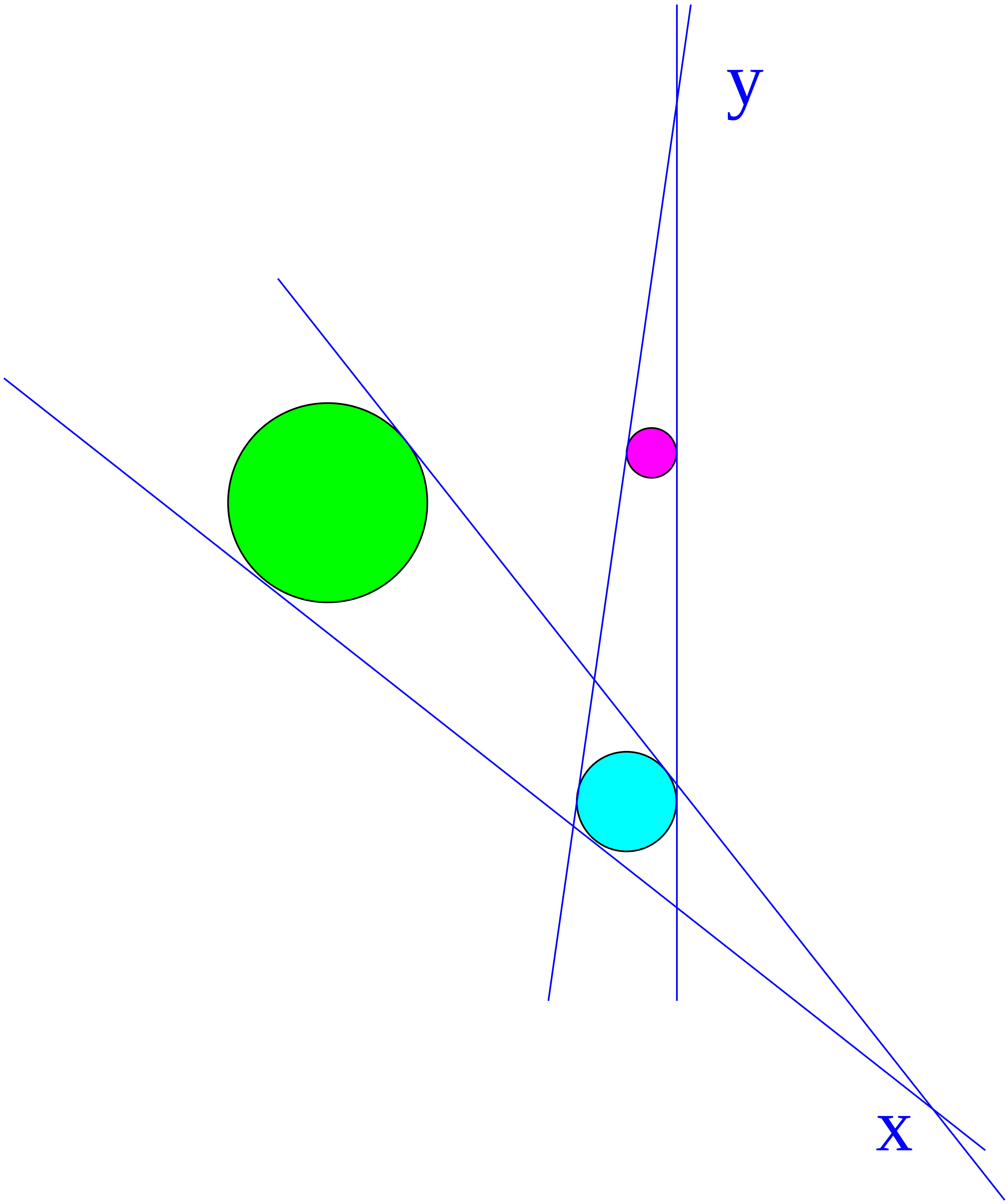
The common external tangents of

the green and blue circle meet at x



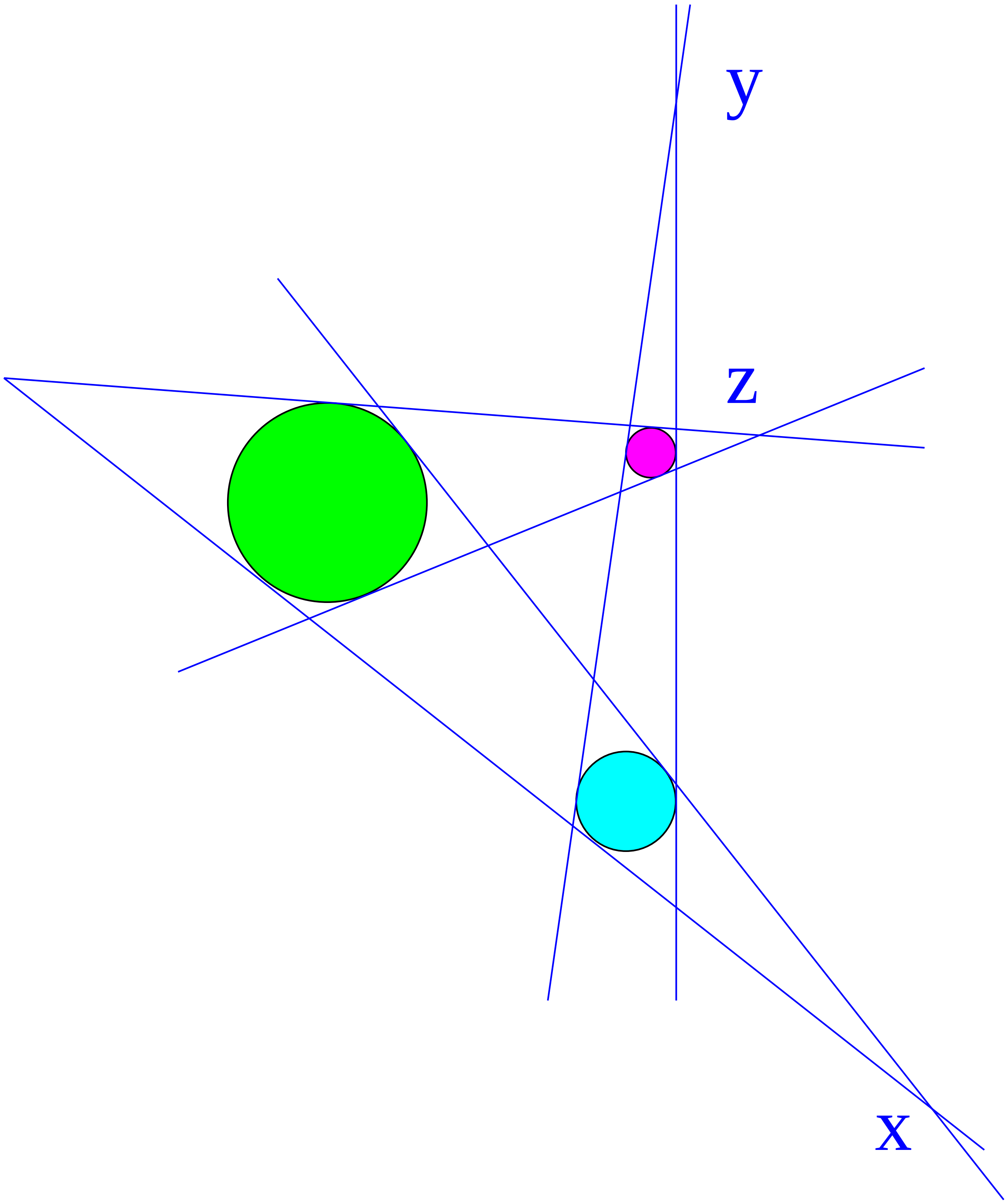
The common external tangents of

the blue and pink circle meet at y

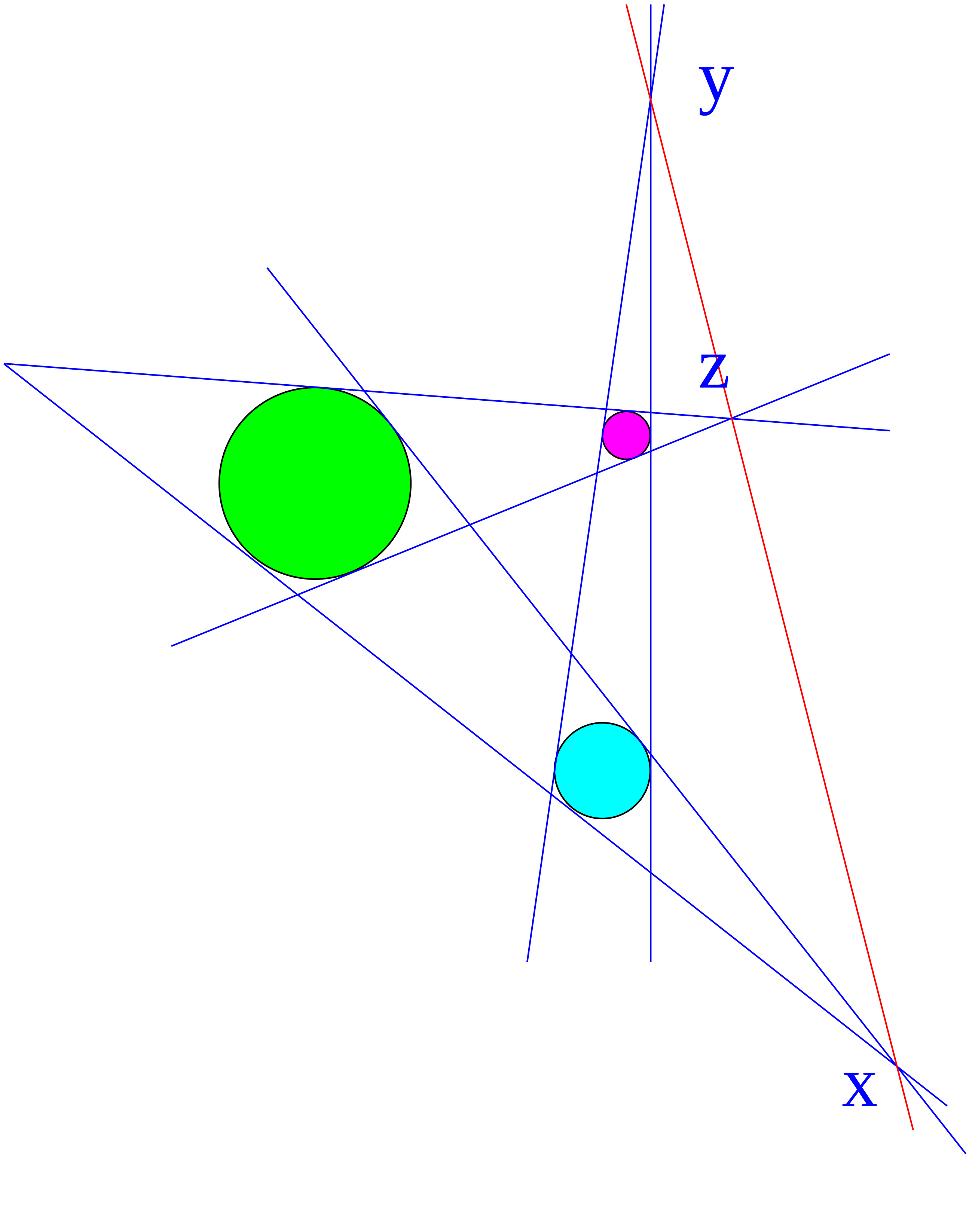


The common external tangents of

the green and pink circle meet at z



y, x, z are collinear



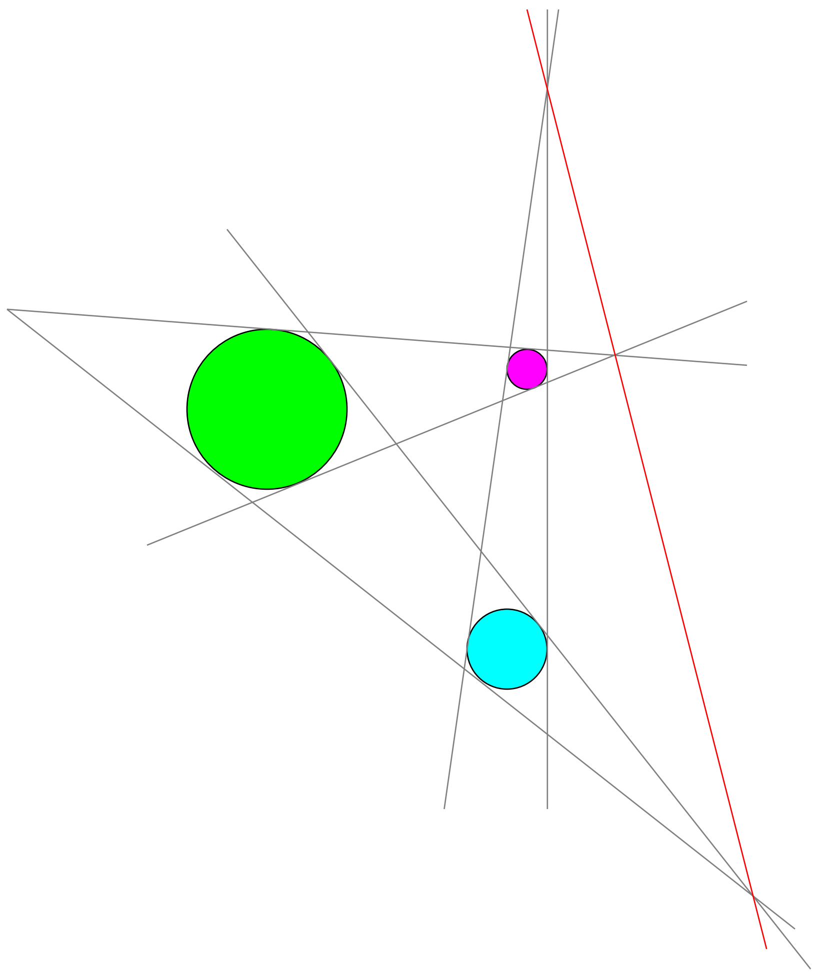
Monge's Circle Theorem

Let there be 3 circles that do

not overlap each other. Then, the

3 intersection points of the common

external tangents lie on a straight line.



Using Desargues' theorem, I will prove this theorem in two ways. First, let me explain Desargues' theorem.

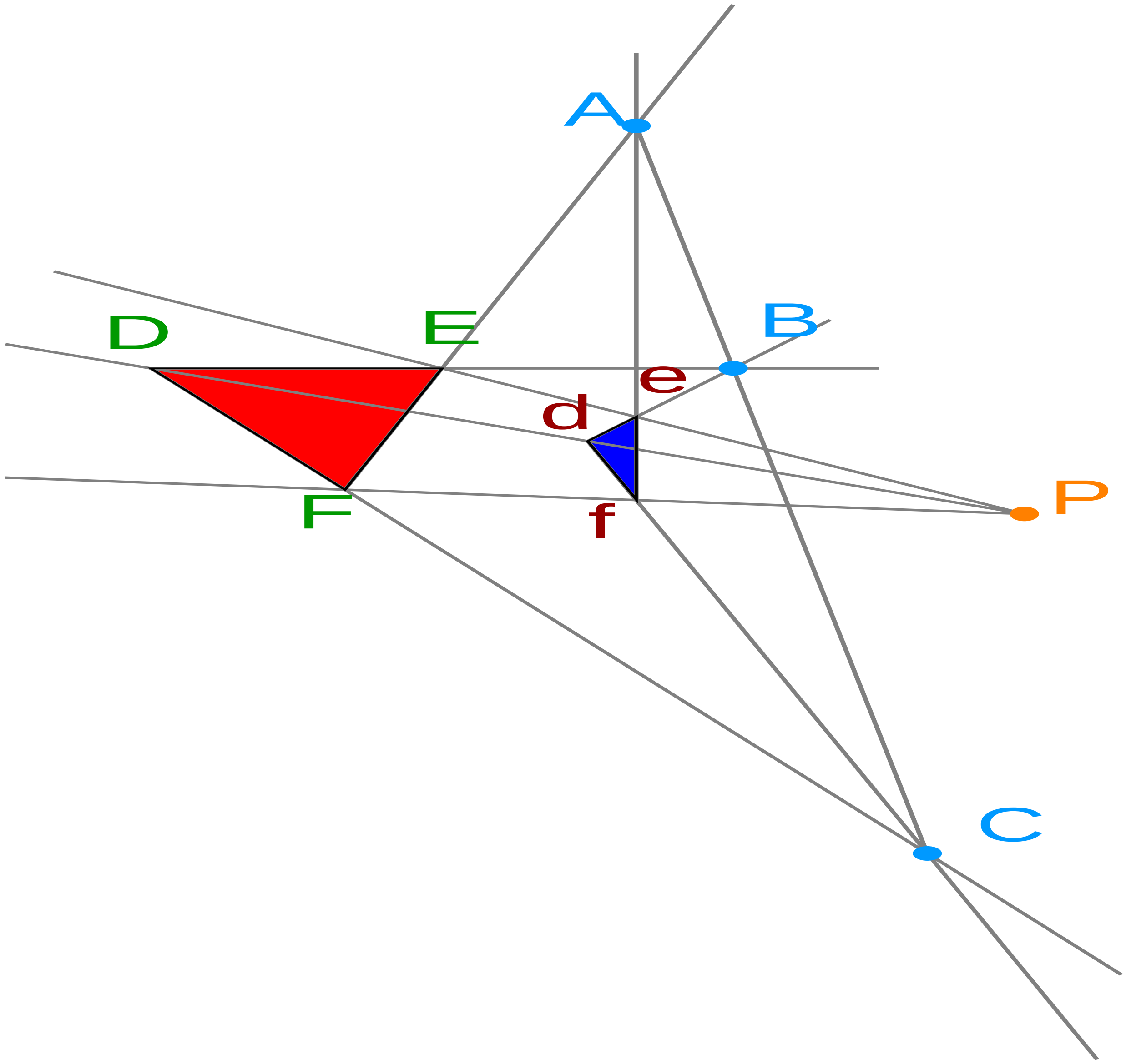
Desargues' Theorem

For two triangles DEF and def, if the joinning corresponding vertices all meet at P,

(that is, line Dd, line Ee, and line Ff all meet at P)

then A, B, C are collinear.

where A is the joint point of the extended line EF and ef, B is the joint point of DE and de, and C is the joint point of DF and df.

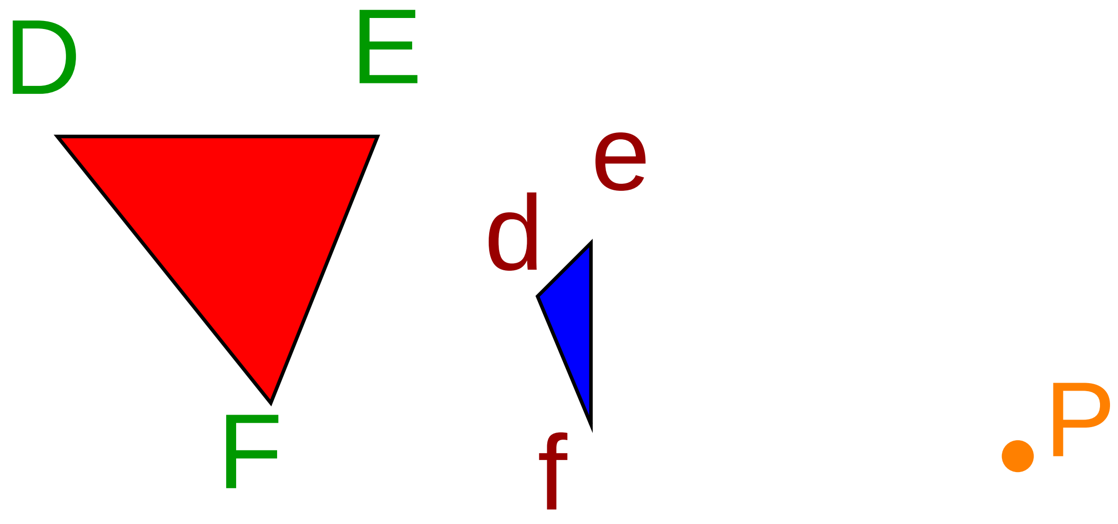


Lets prove

Desargues' Theorem

If points Dd, Ee, Ff are concurrent,

then they all pass through point P.



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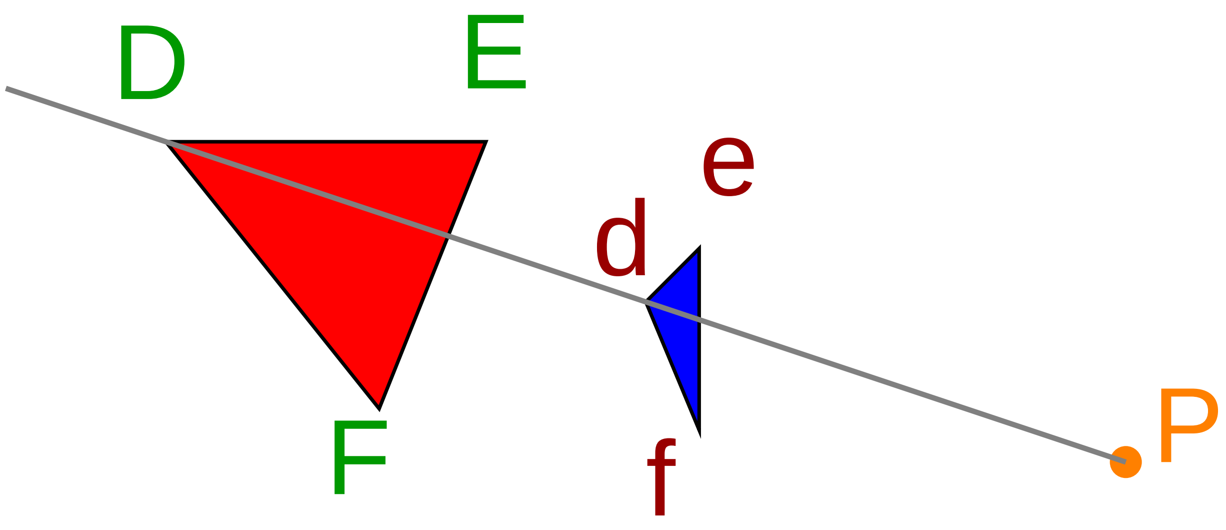

If points Dd, Ee, Ff are concurrent,

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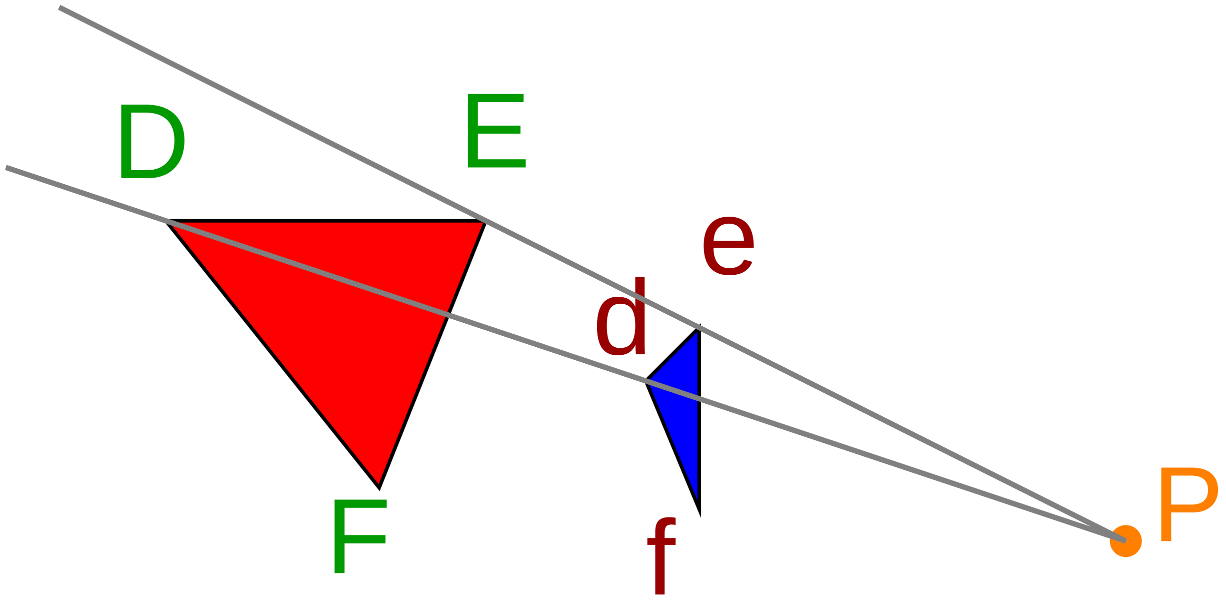

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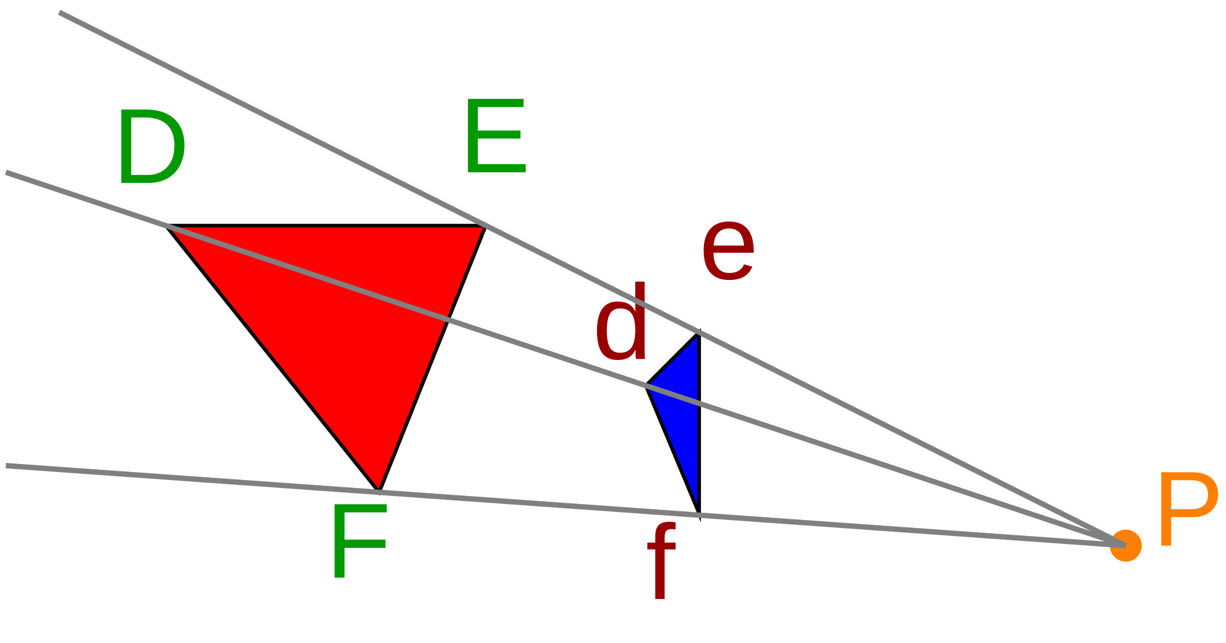
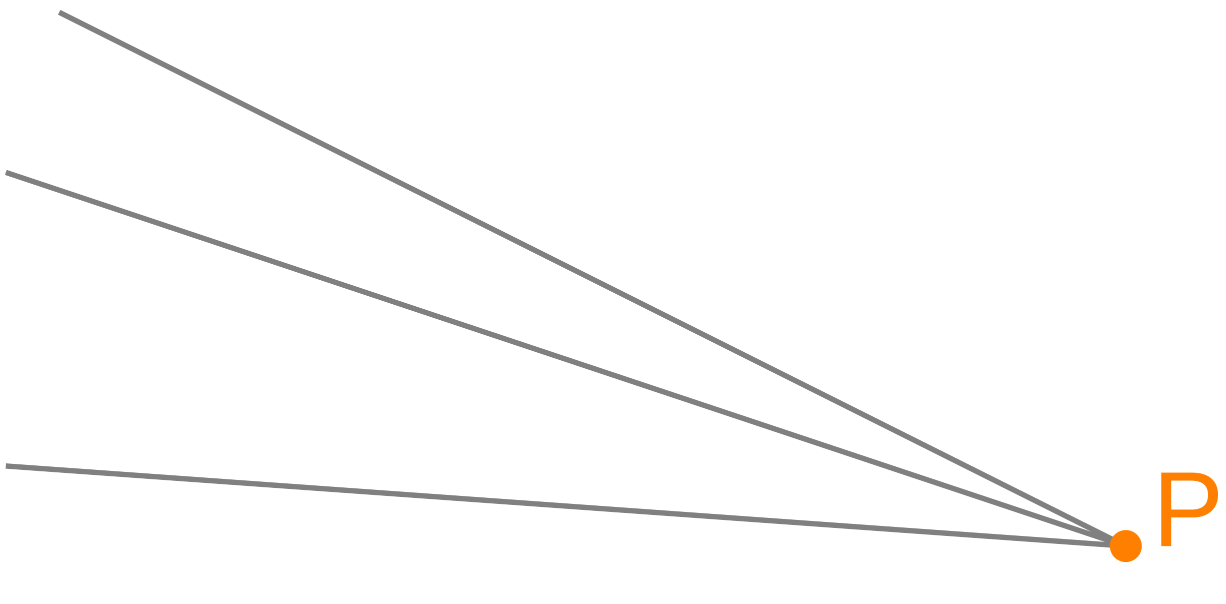


Alternatively, we can start with

point P and 3 rays from P,

we will get the same result.



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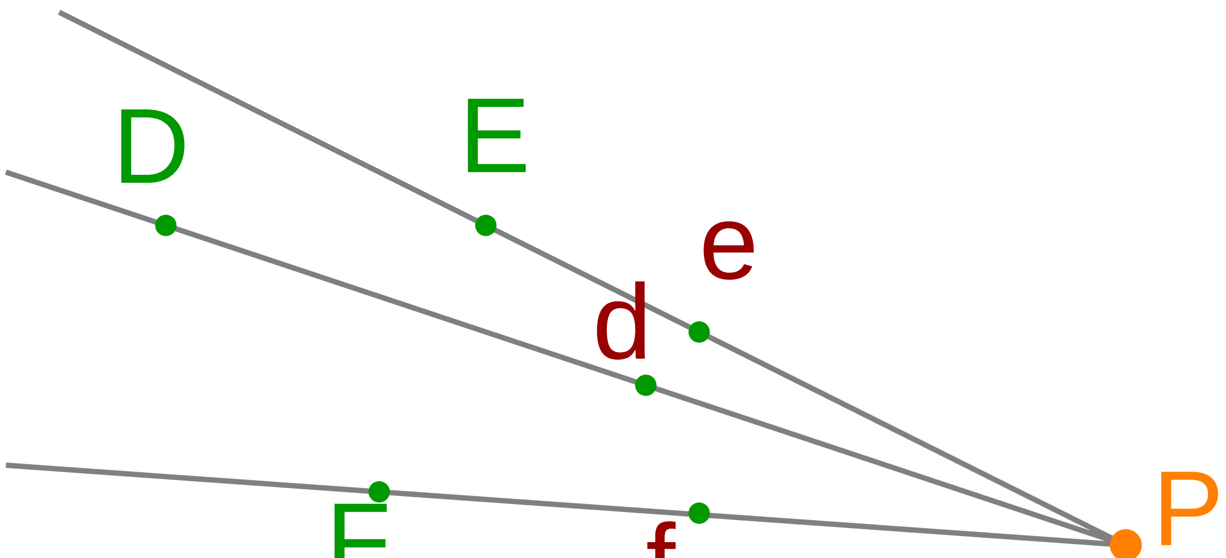

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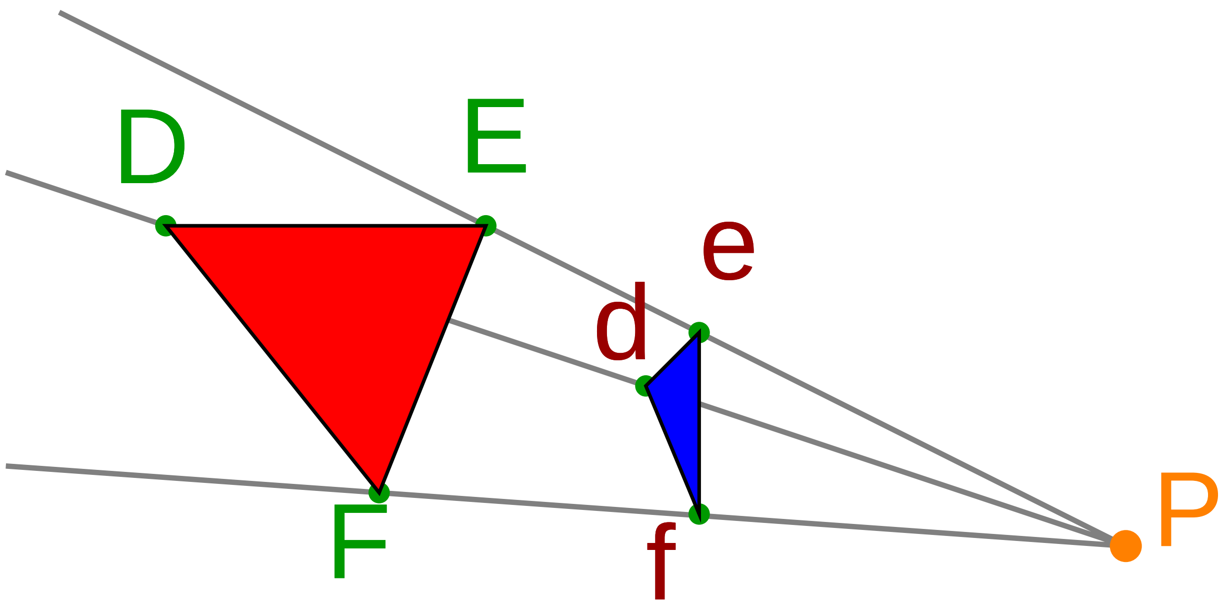
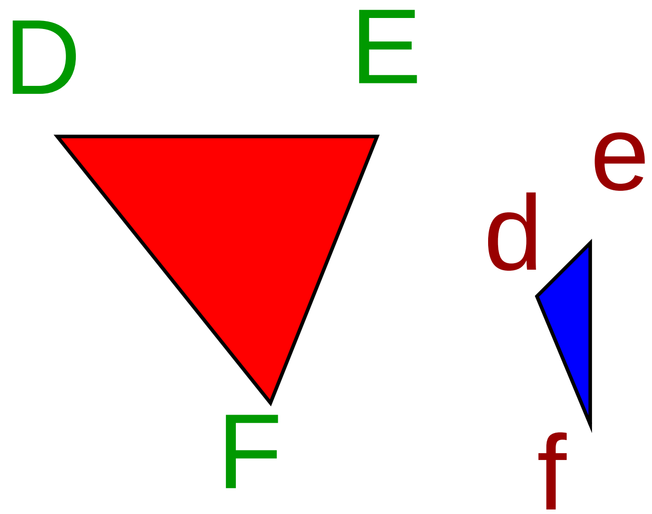
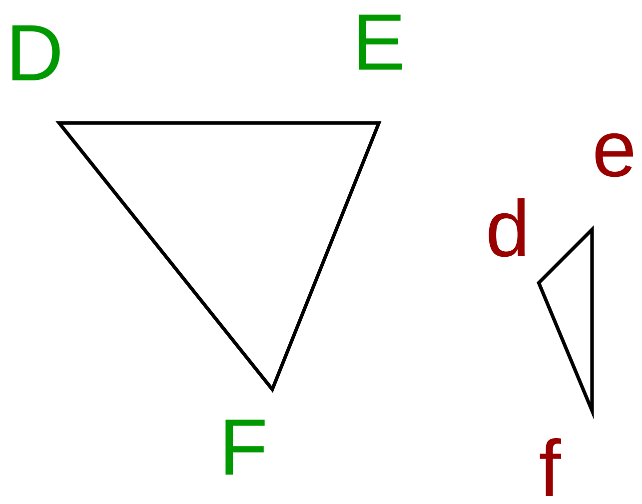


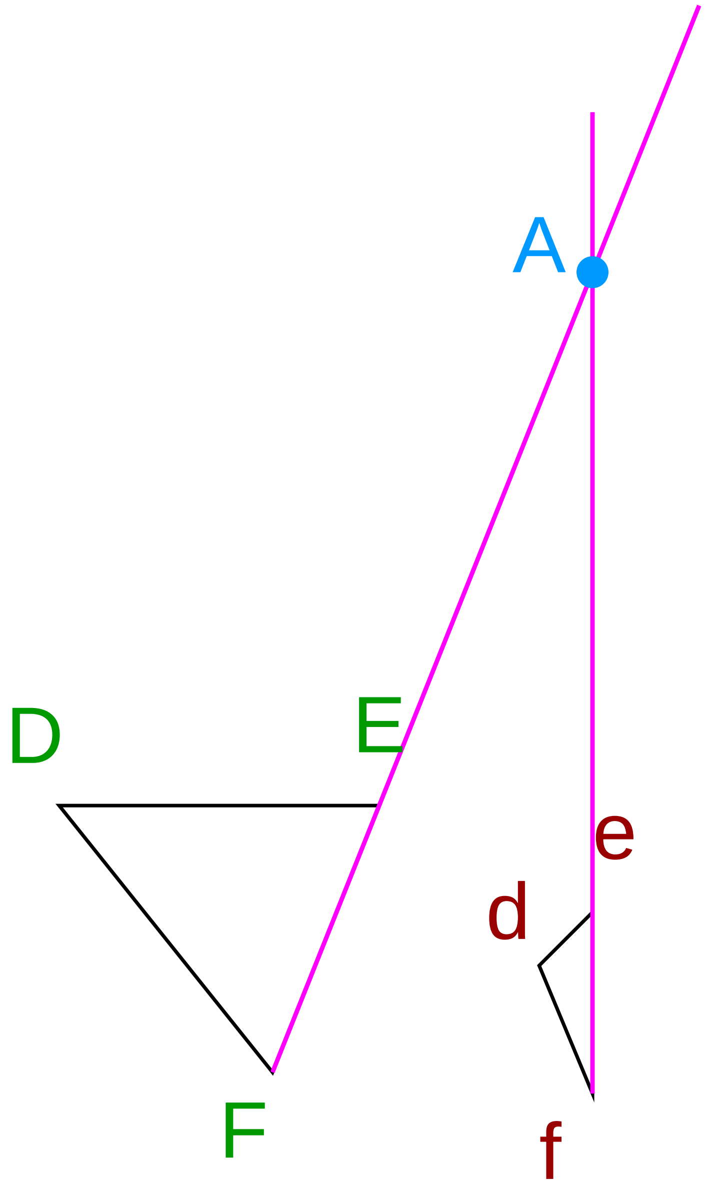
Lets show the existence of

points A, B, and C. Again, we

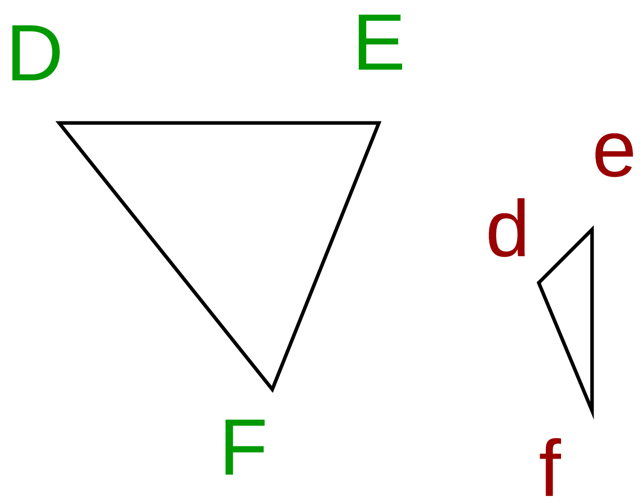
start with the two triangles

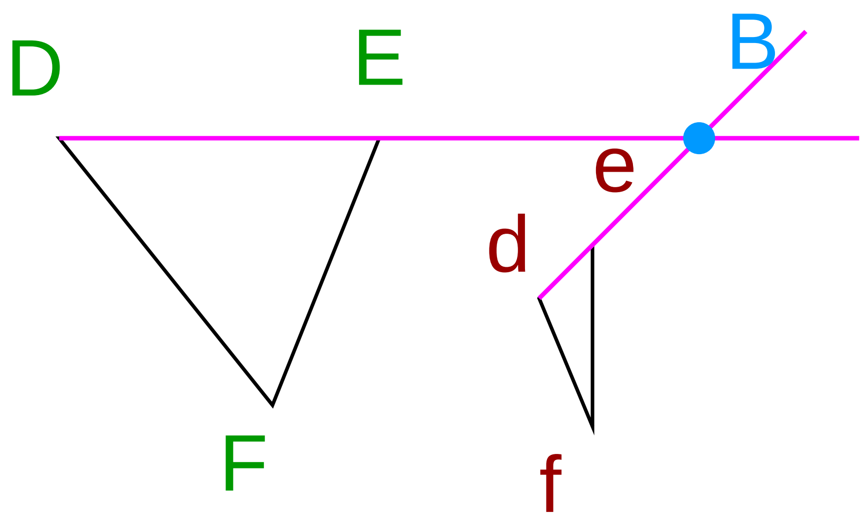


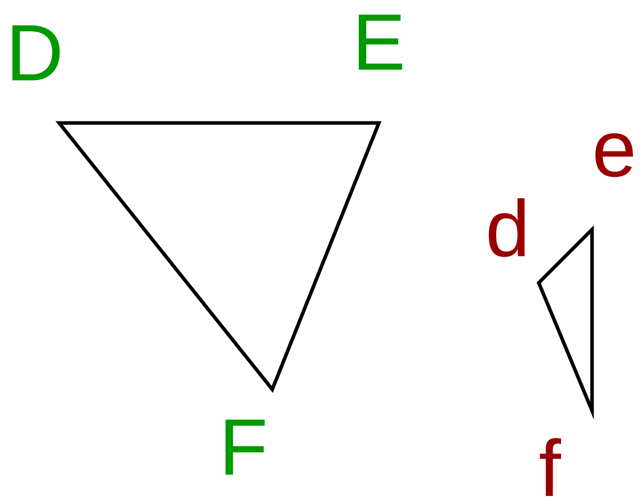

D
E
F
d
e
f

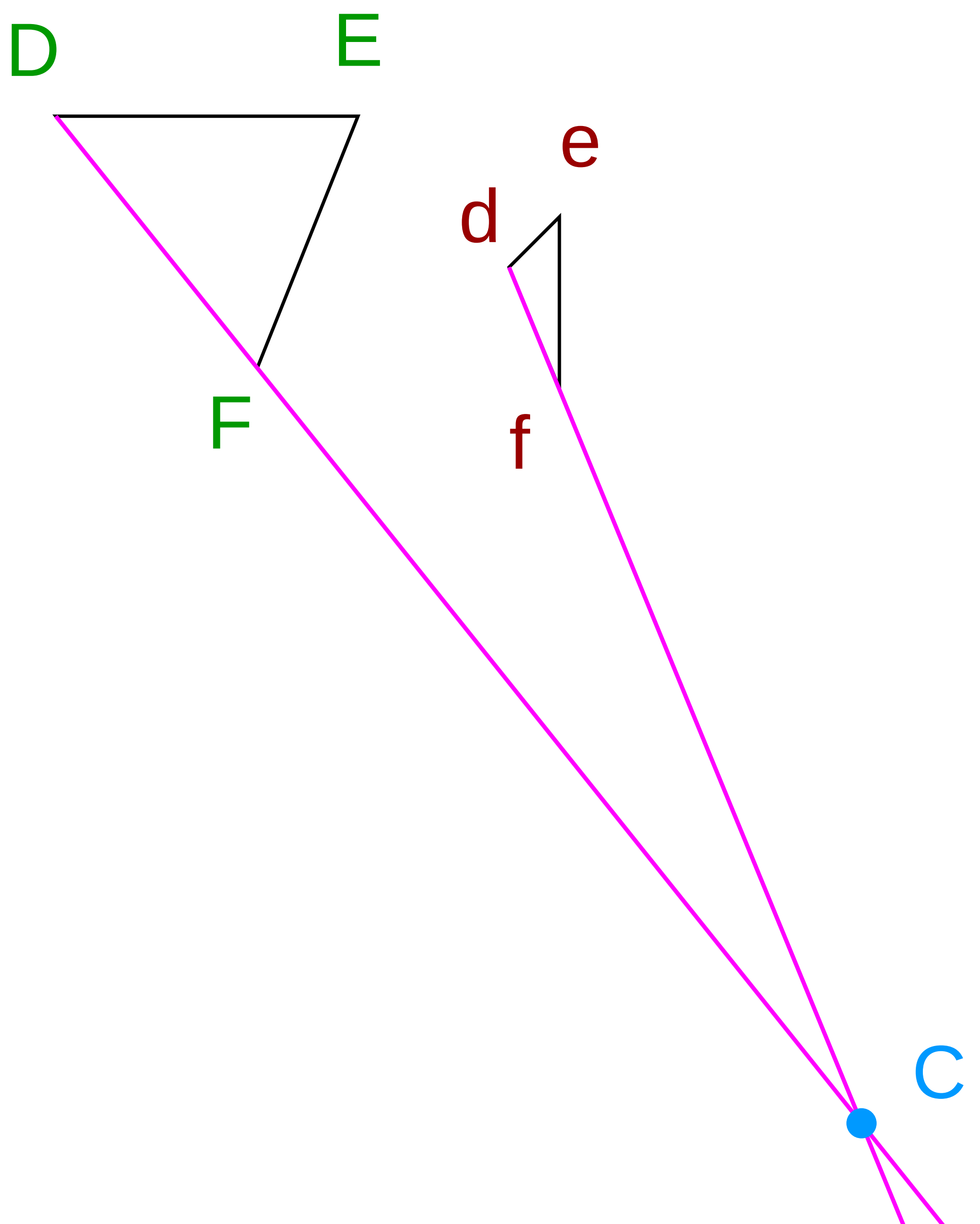


A
D
E
e
d
F
f

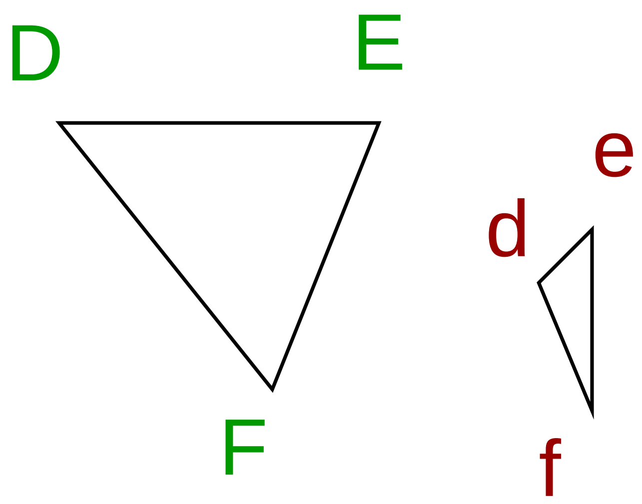

D
E
F
e
d
f

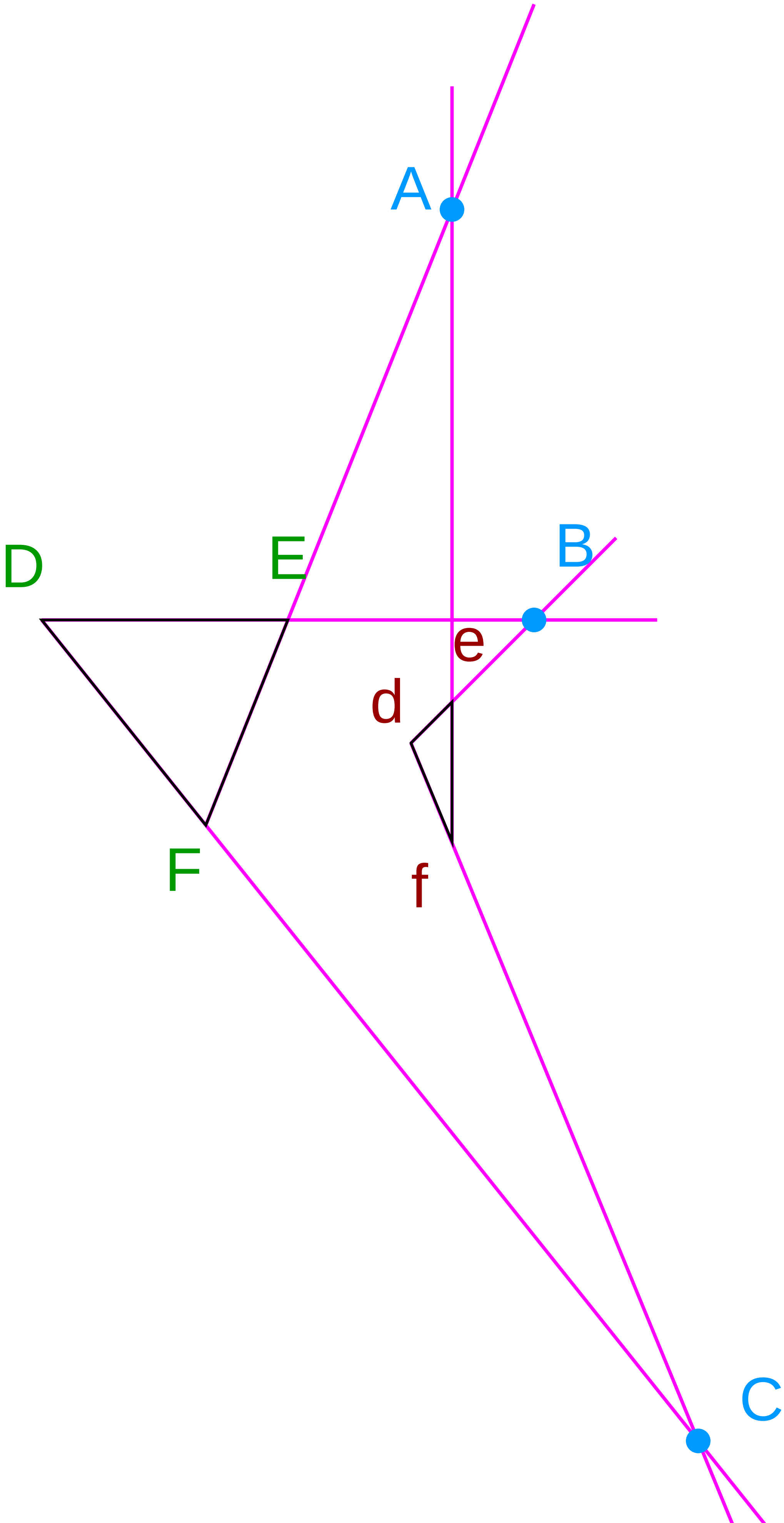

D
E
B
e
d
F
f


D
E
F
d
e
f

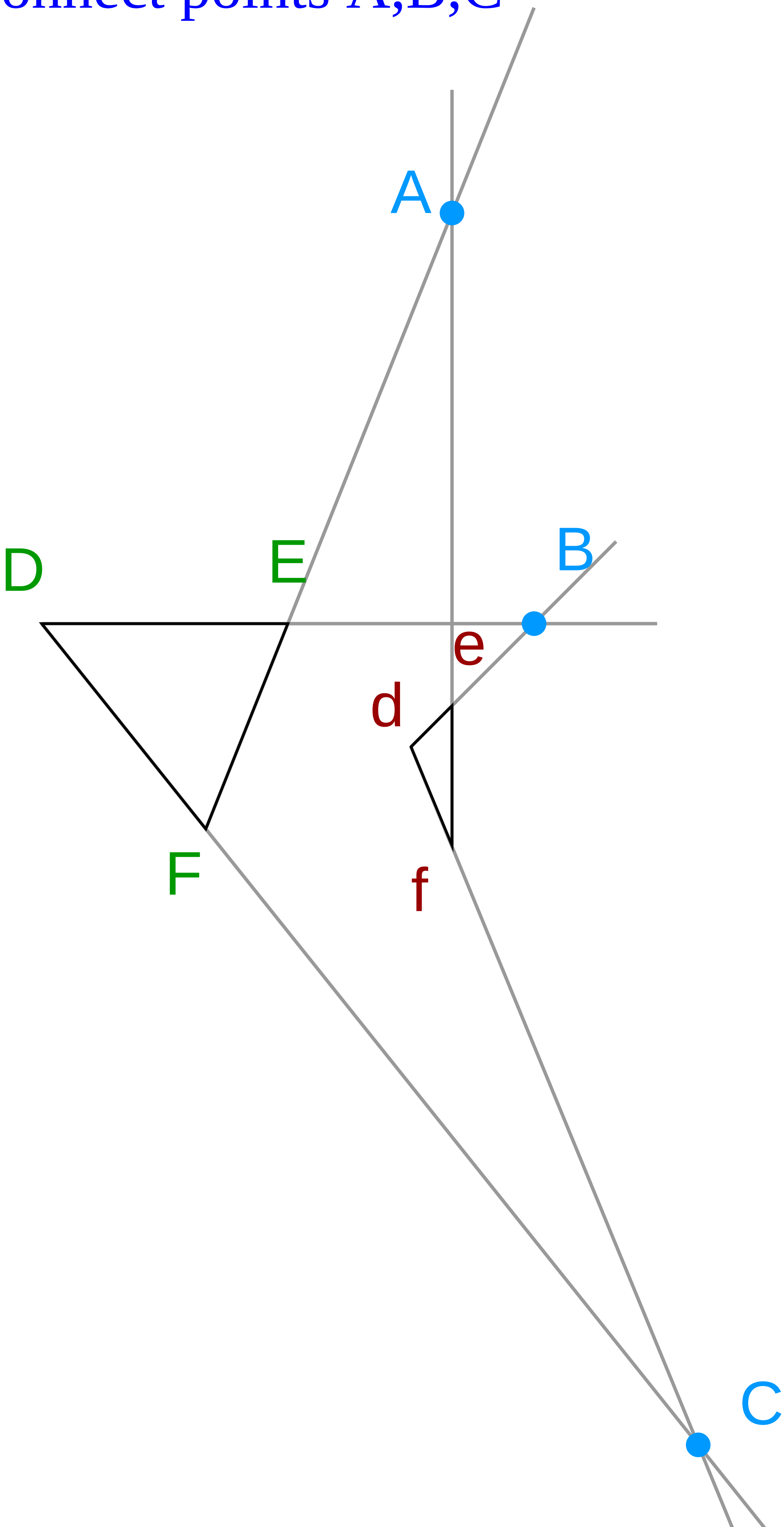


D
E
e
d
F
f
C

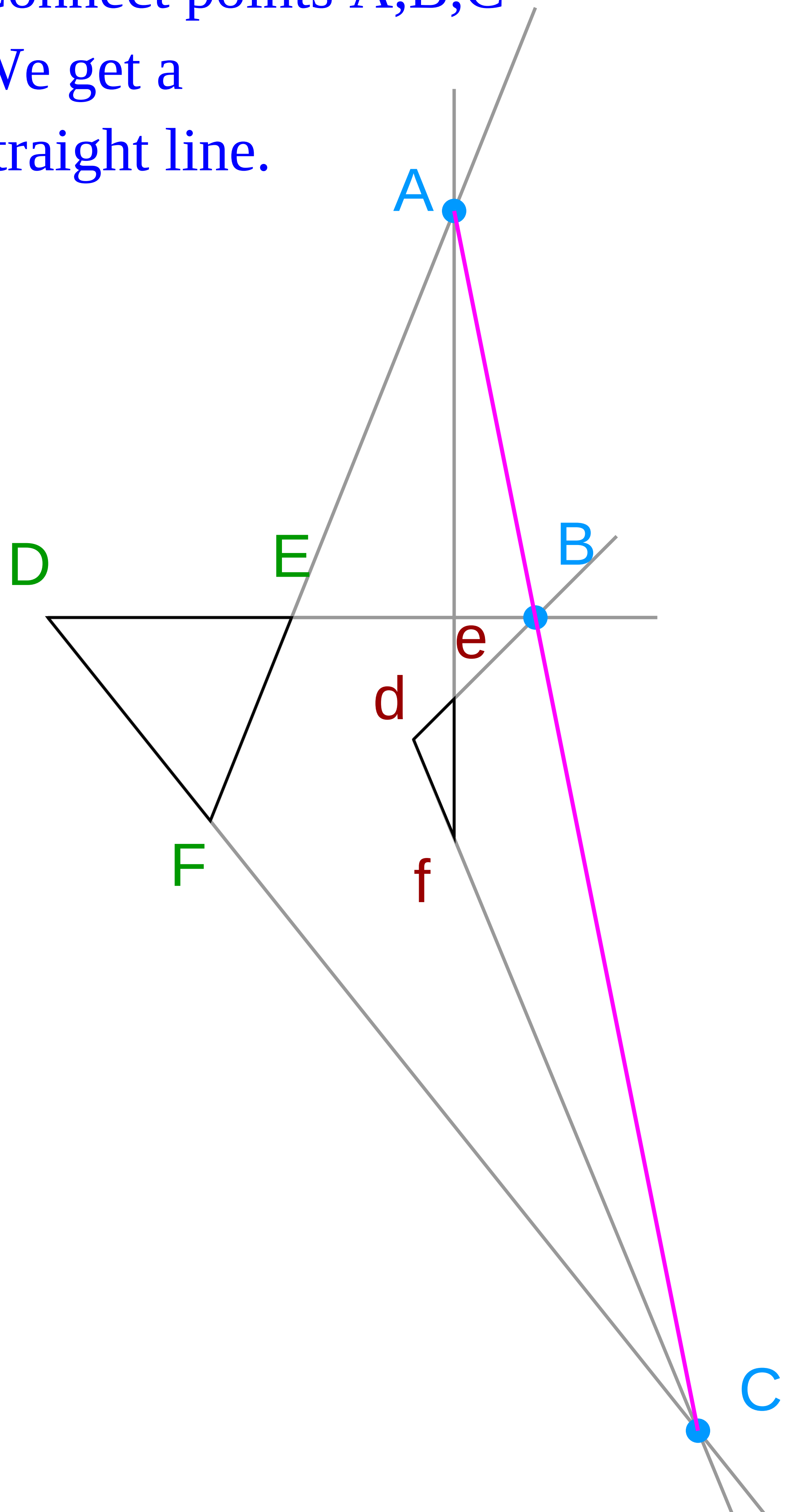

D
E
F
e
d
f

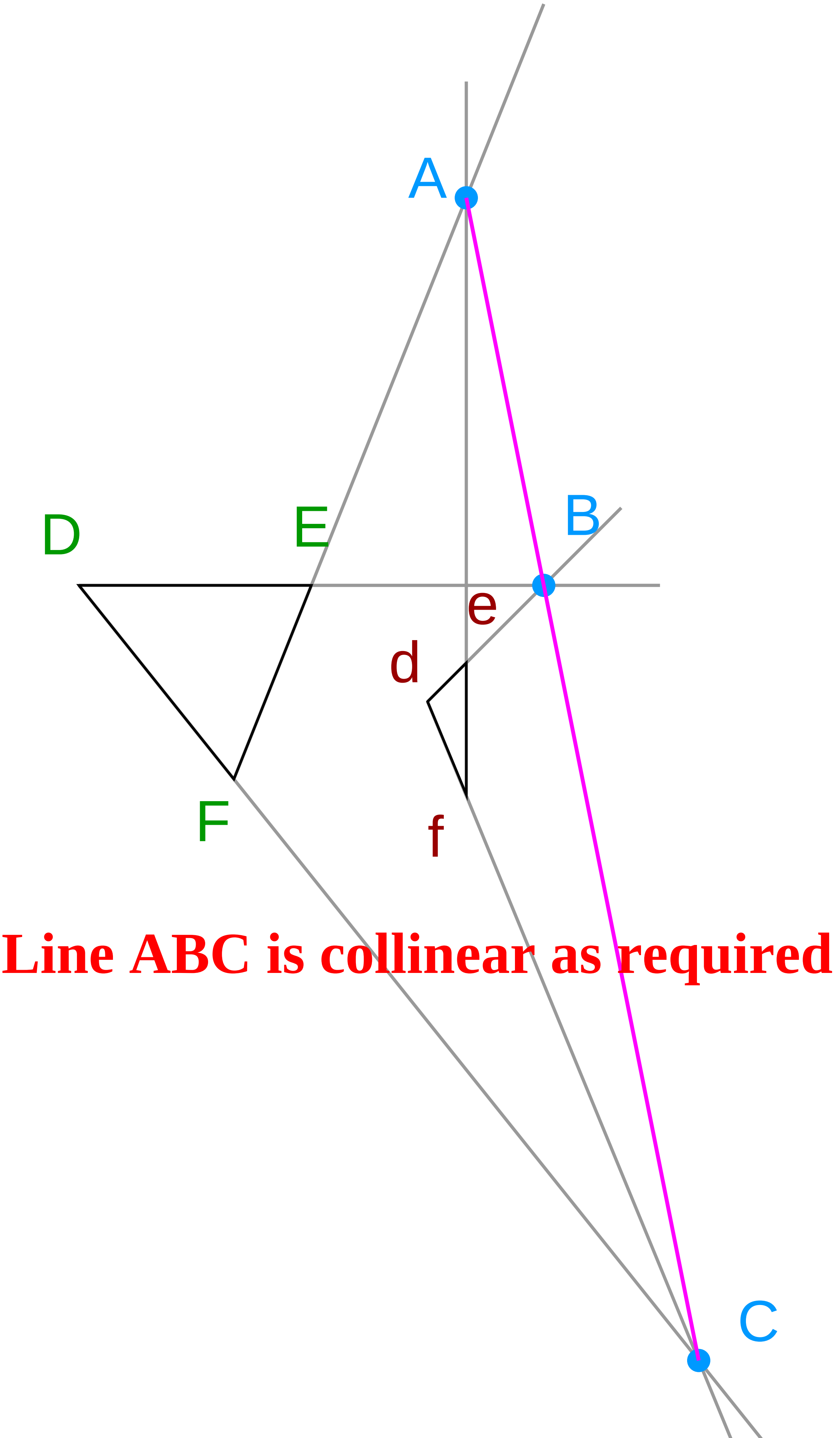

A
B
C
D
E
F
d
e
f

Connect points A,B,C

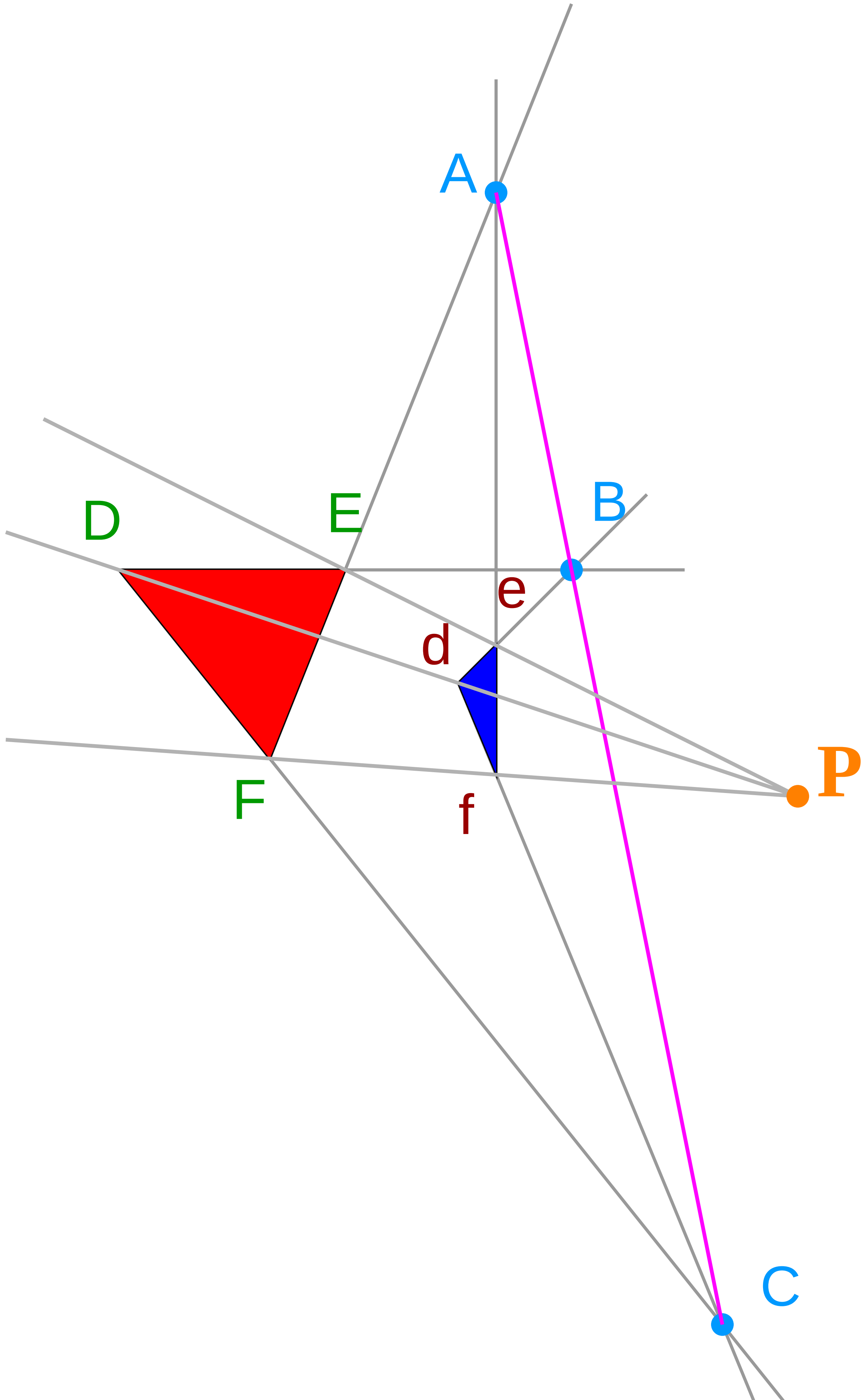


Connect points A,B,C
We get a
straight line.




A
B
C
D
E
F
d
e
f
Line ABC is collinear as required

"Desargues' Theorem"

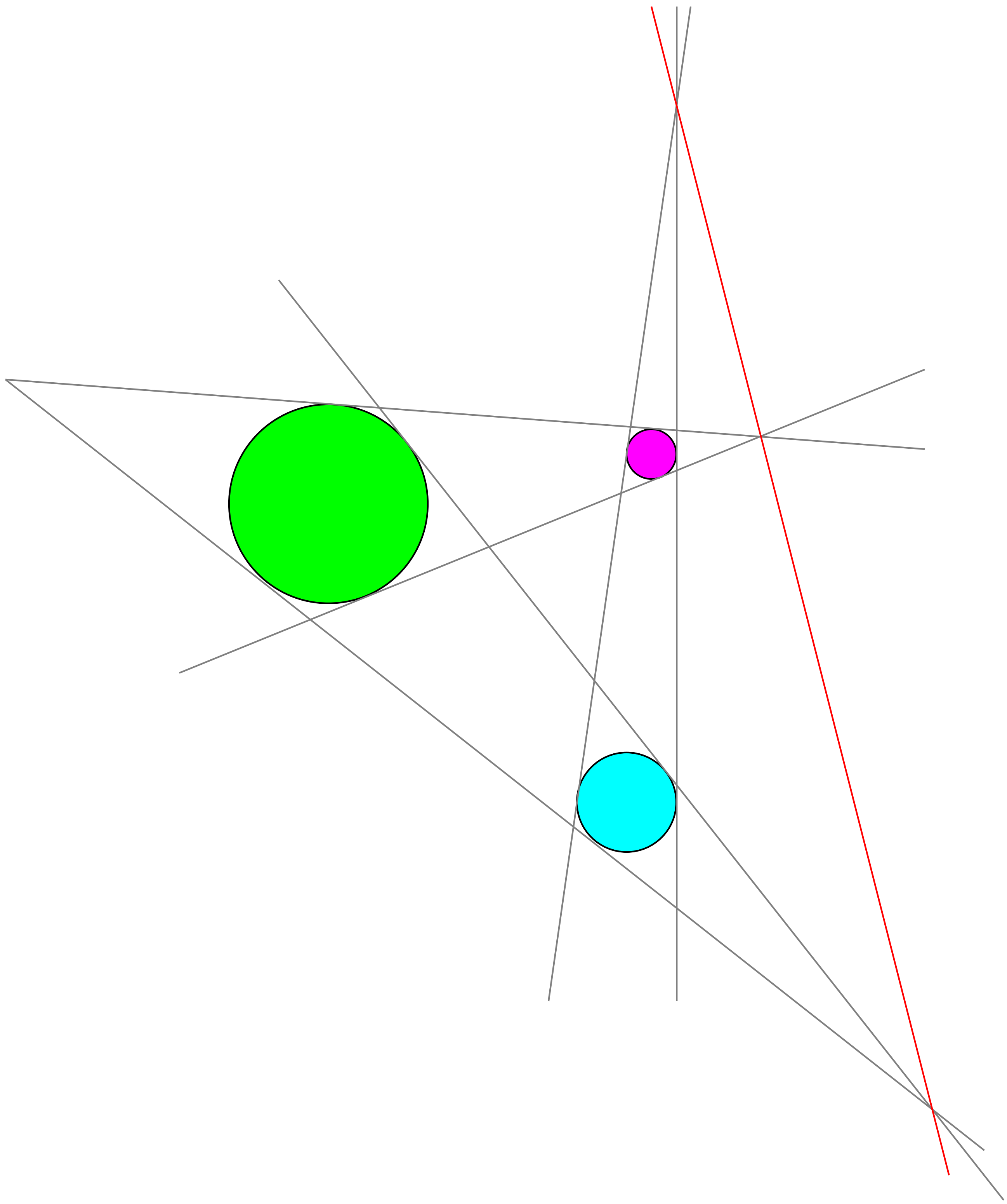


Now, we can use

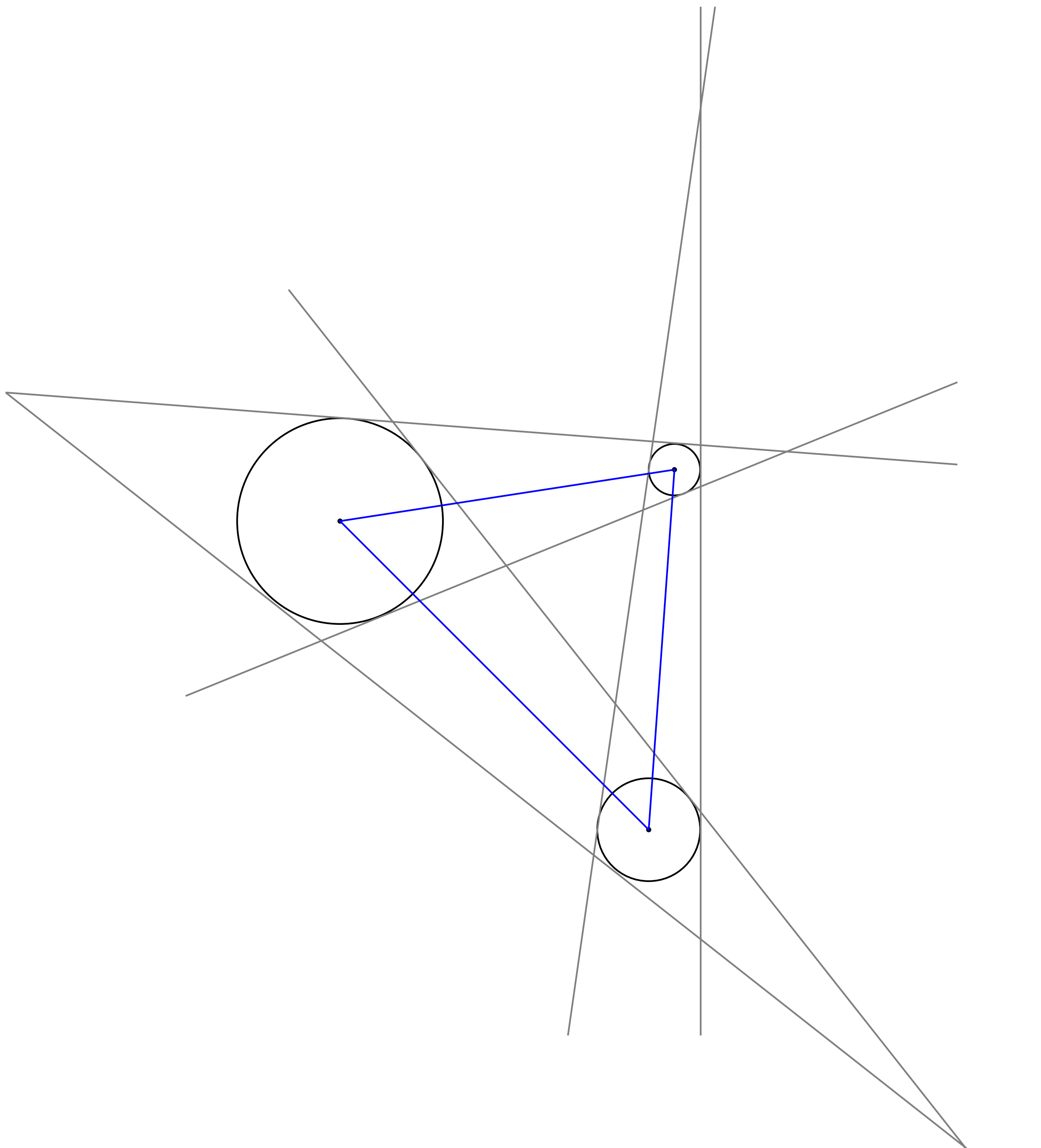
Desargues to prove

Monge’s Circle Theorem

Proof 1 of Monge’s Circle Theorem

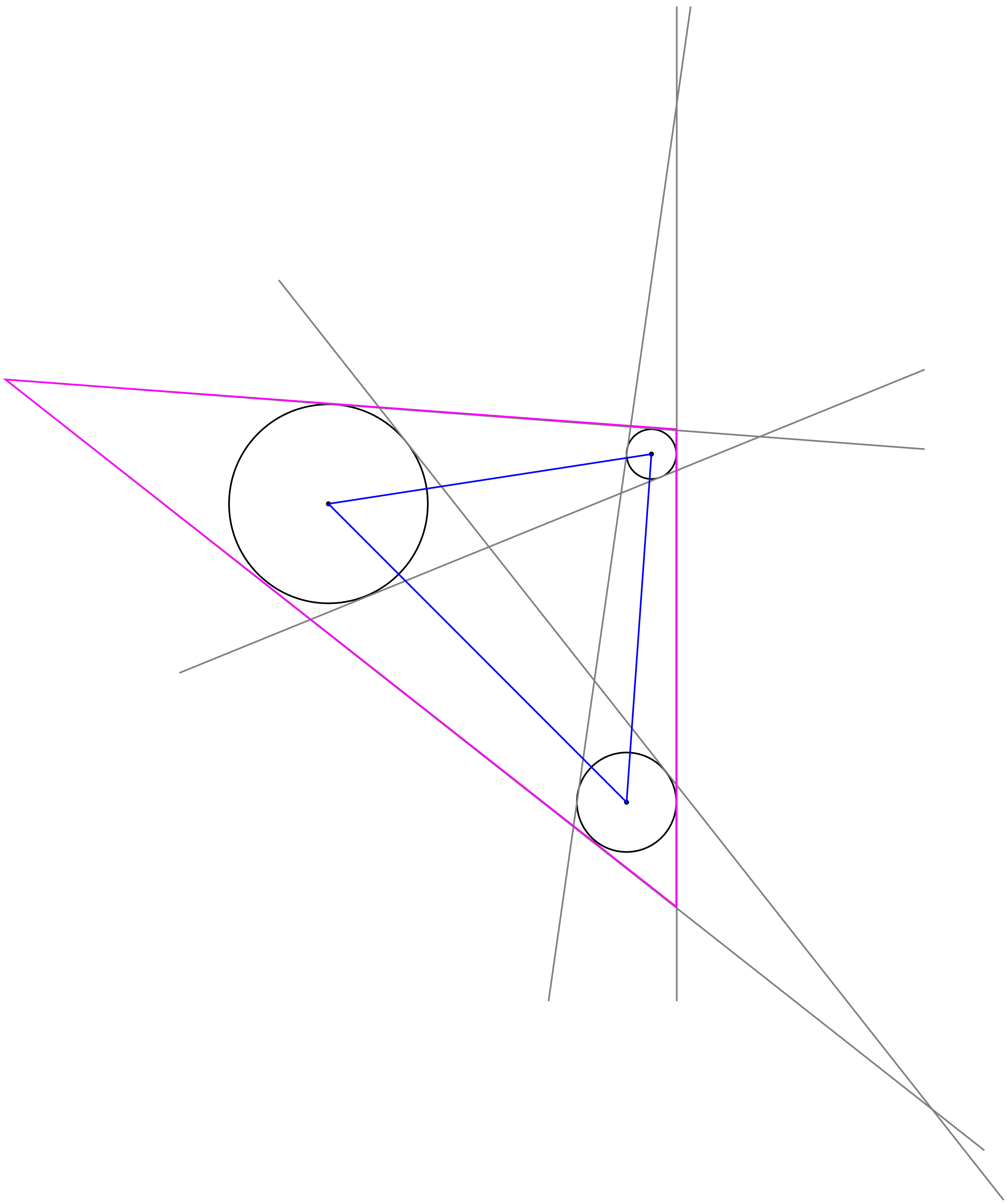


1. Connect all 3 centers

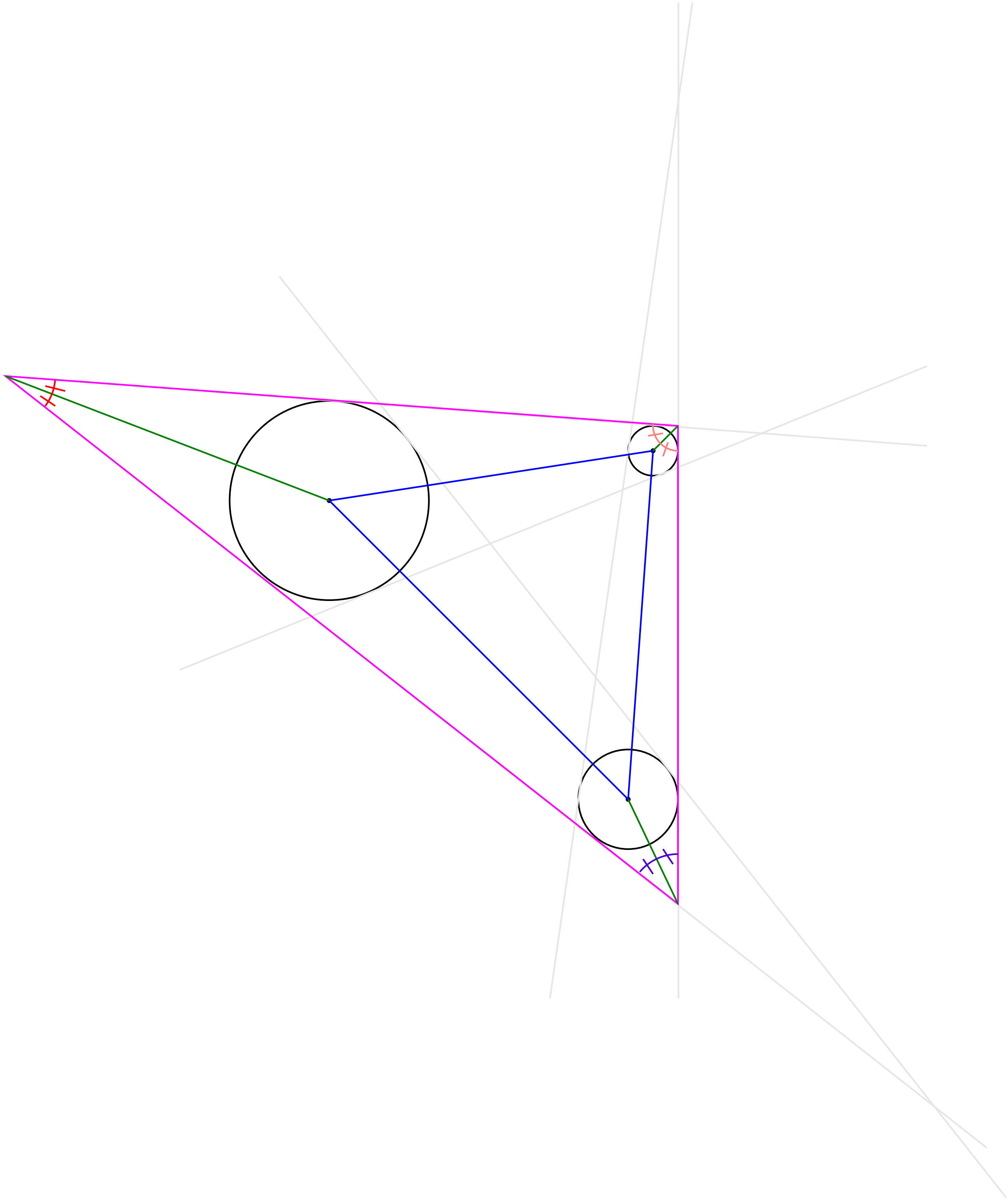


1. Connect all 3 centers

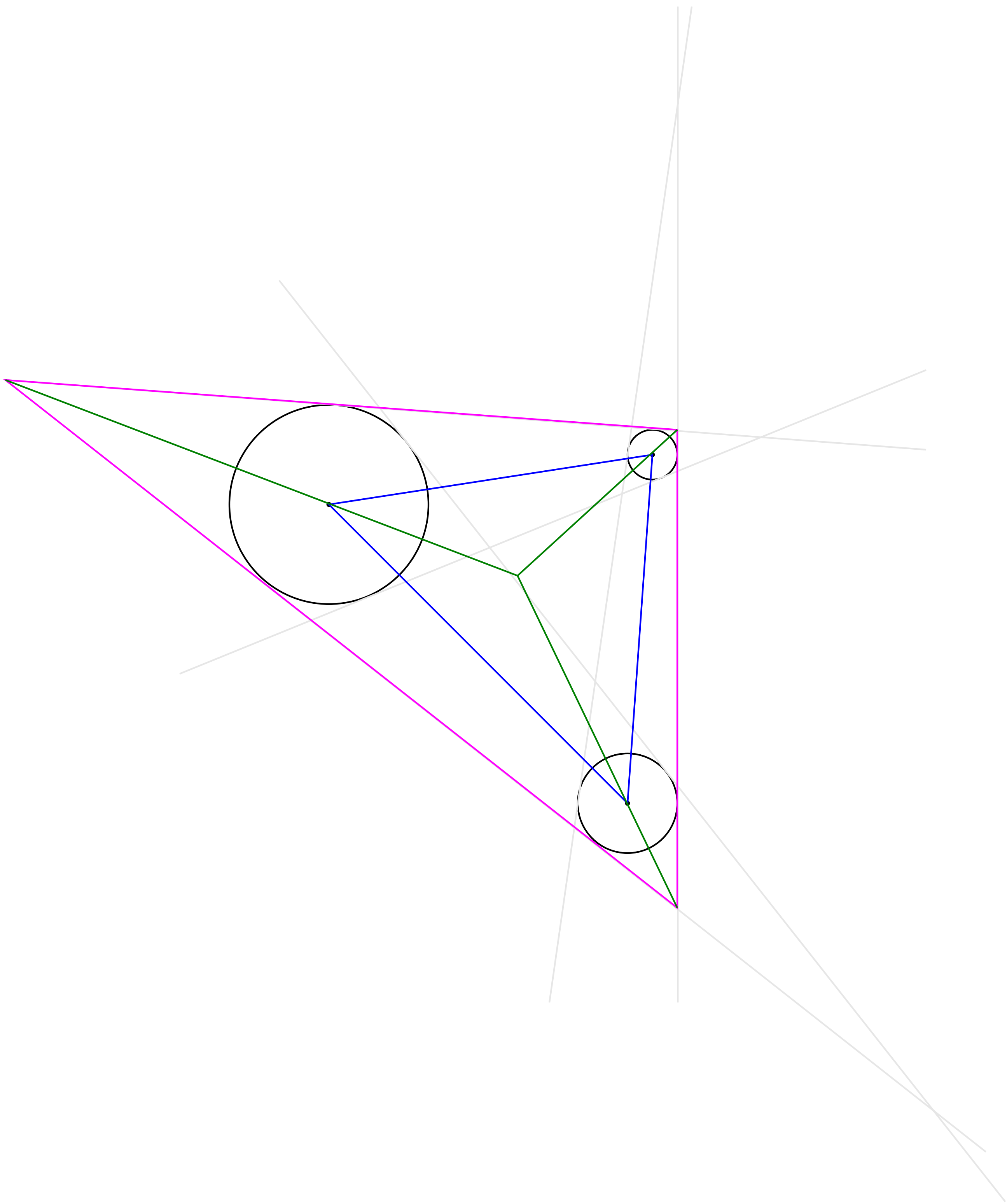
2. Connect the external tangent



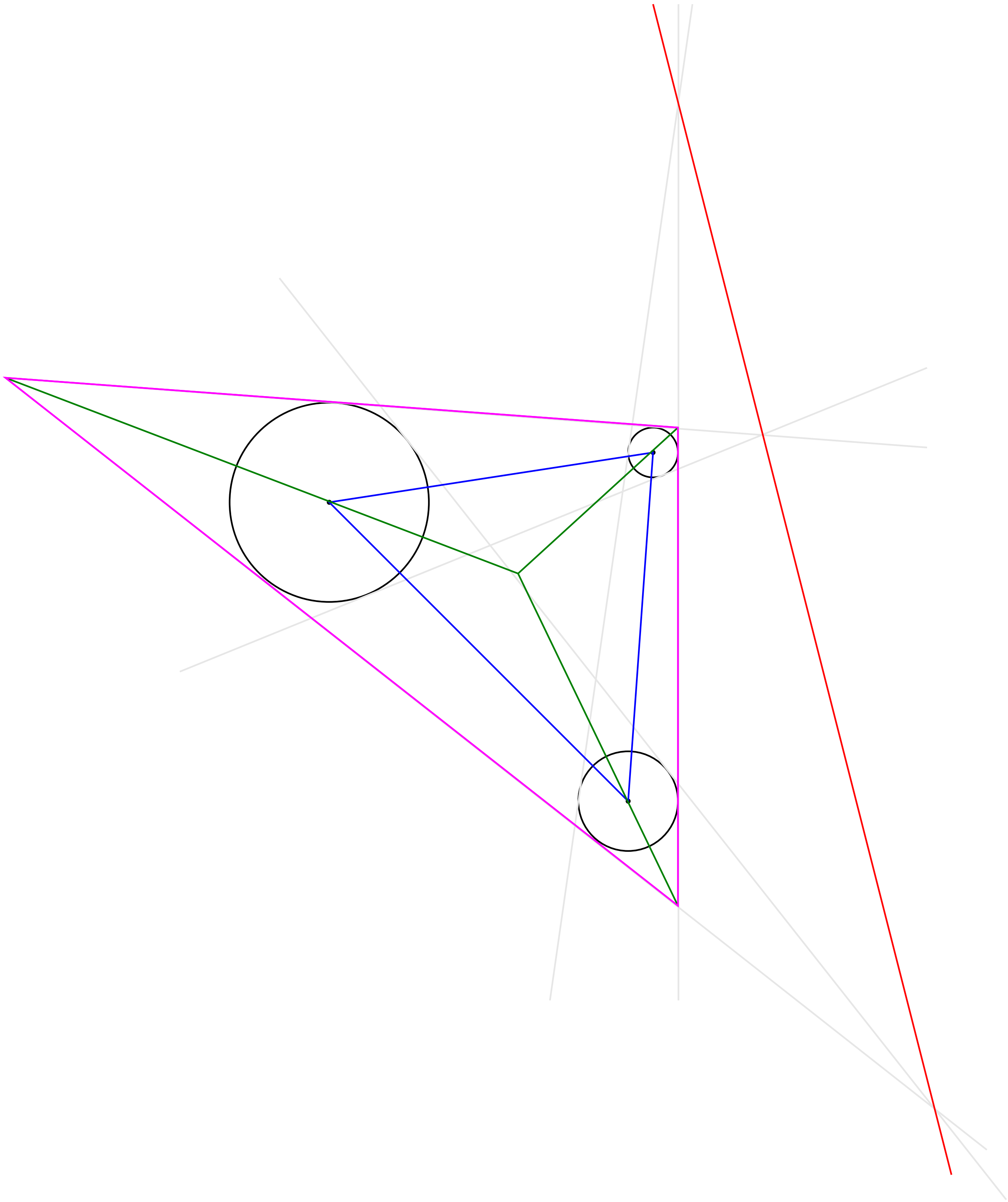
The green lines bisect the angle



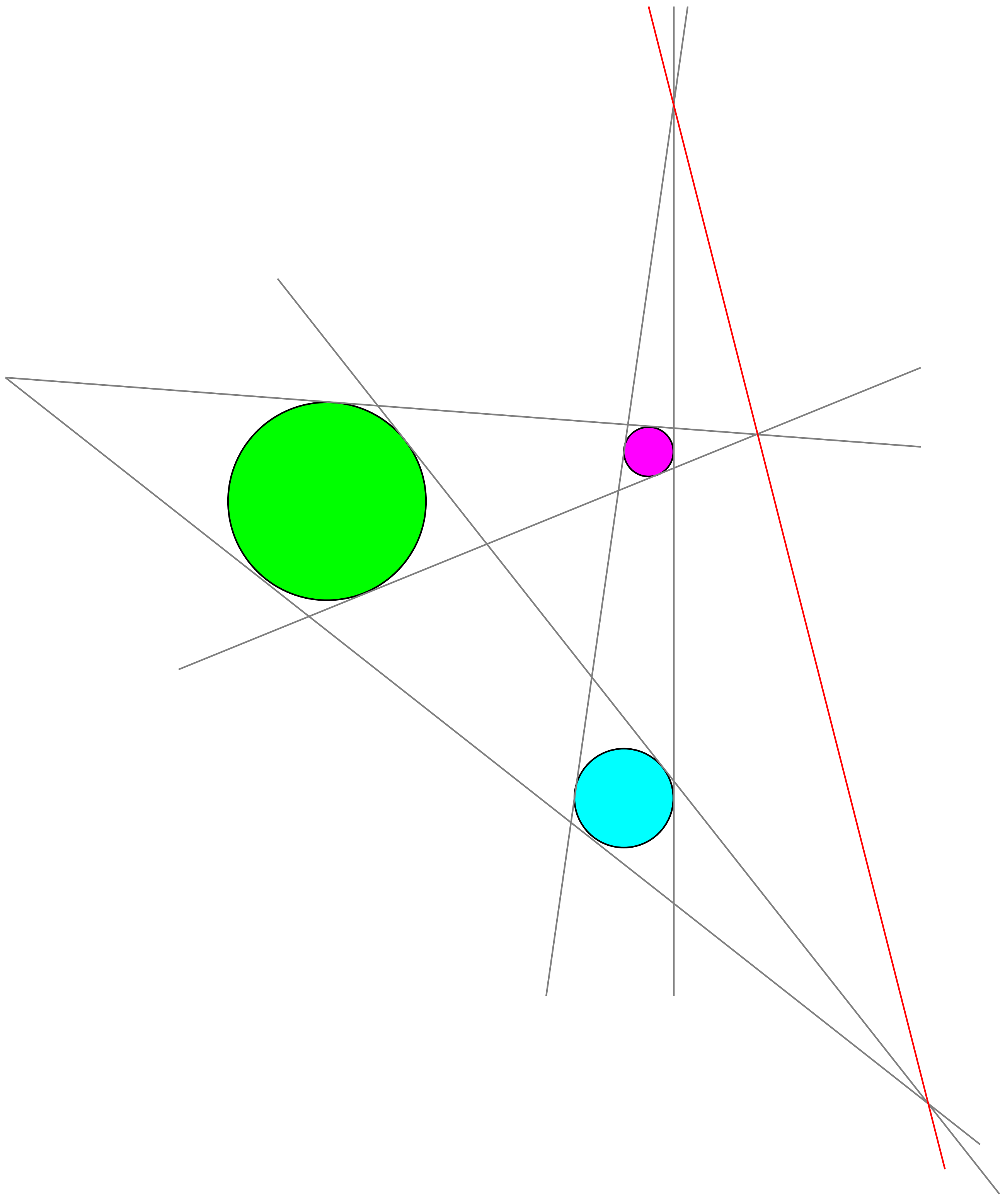
3 green lines merge at one point



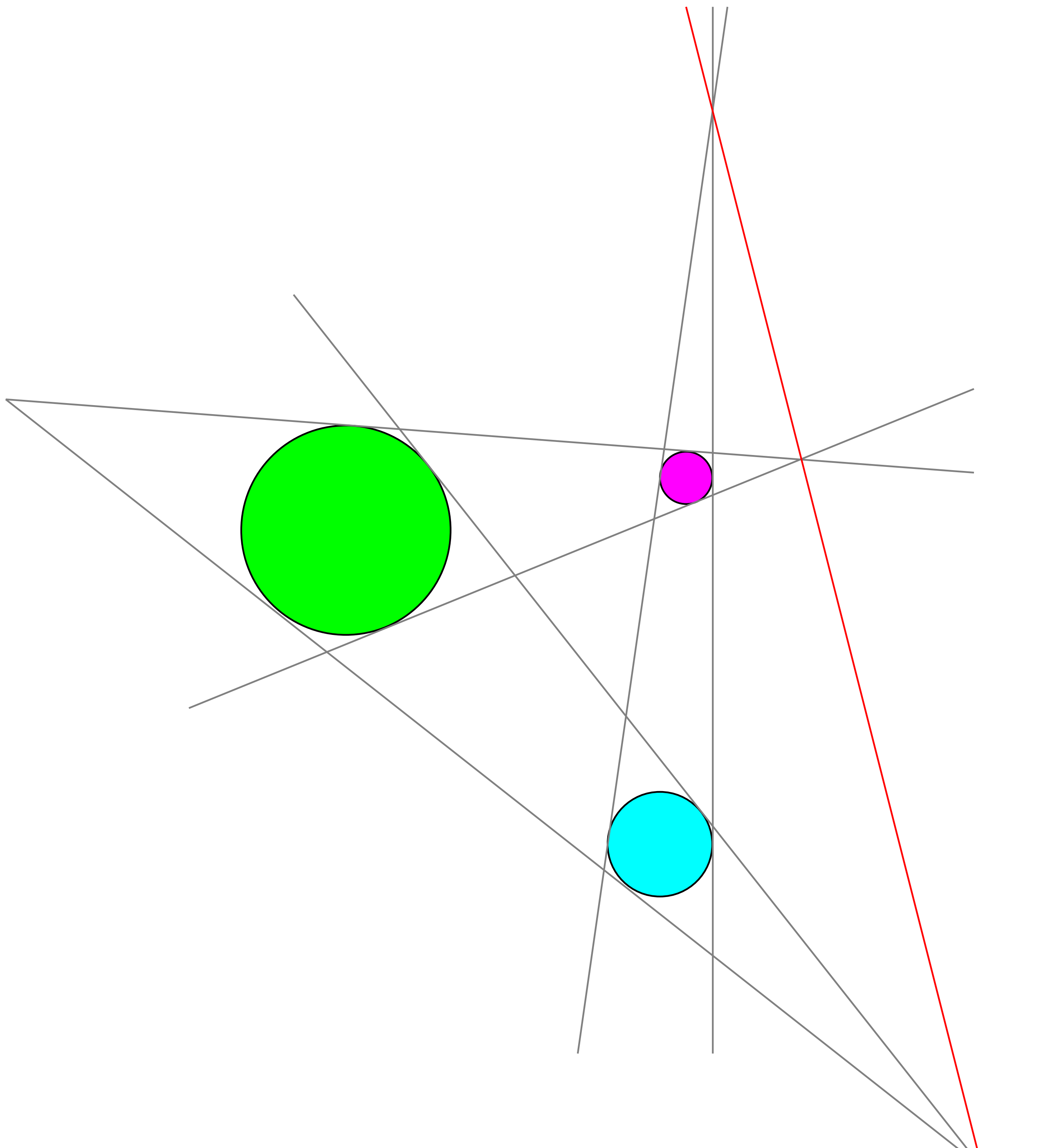
Since the two triangles are perspective from a point; by Desargues' theorem, they are also perspective from a line.



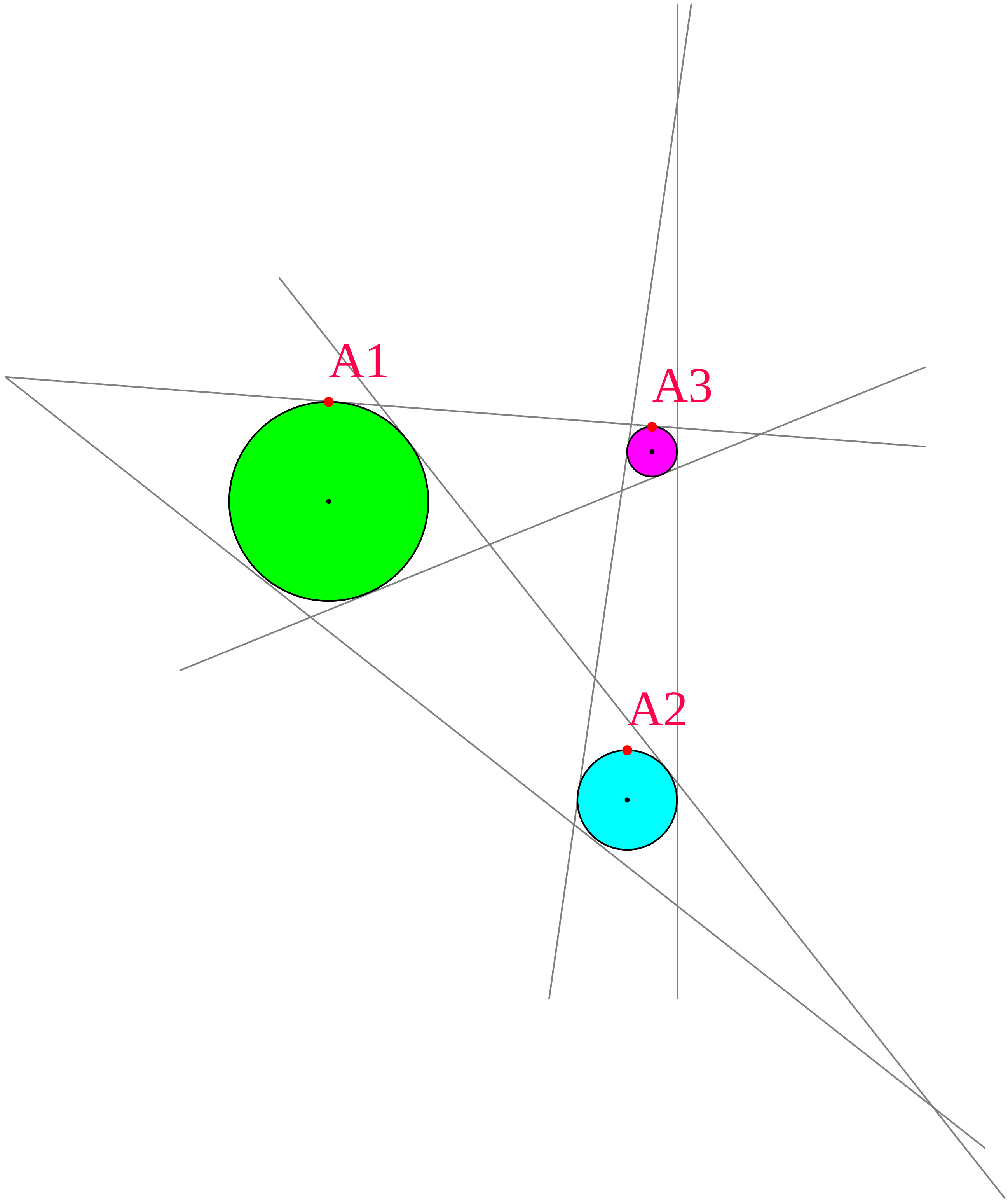
So, they lie on the same line
as required by Monge's Circle Theorem



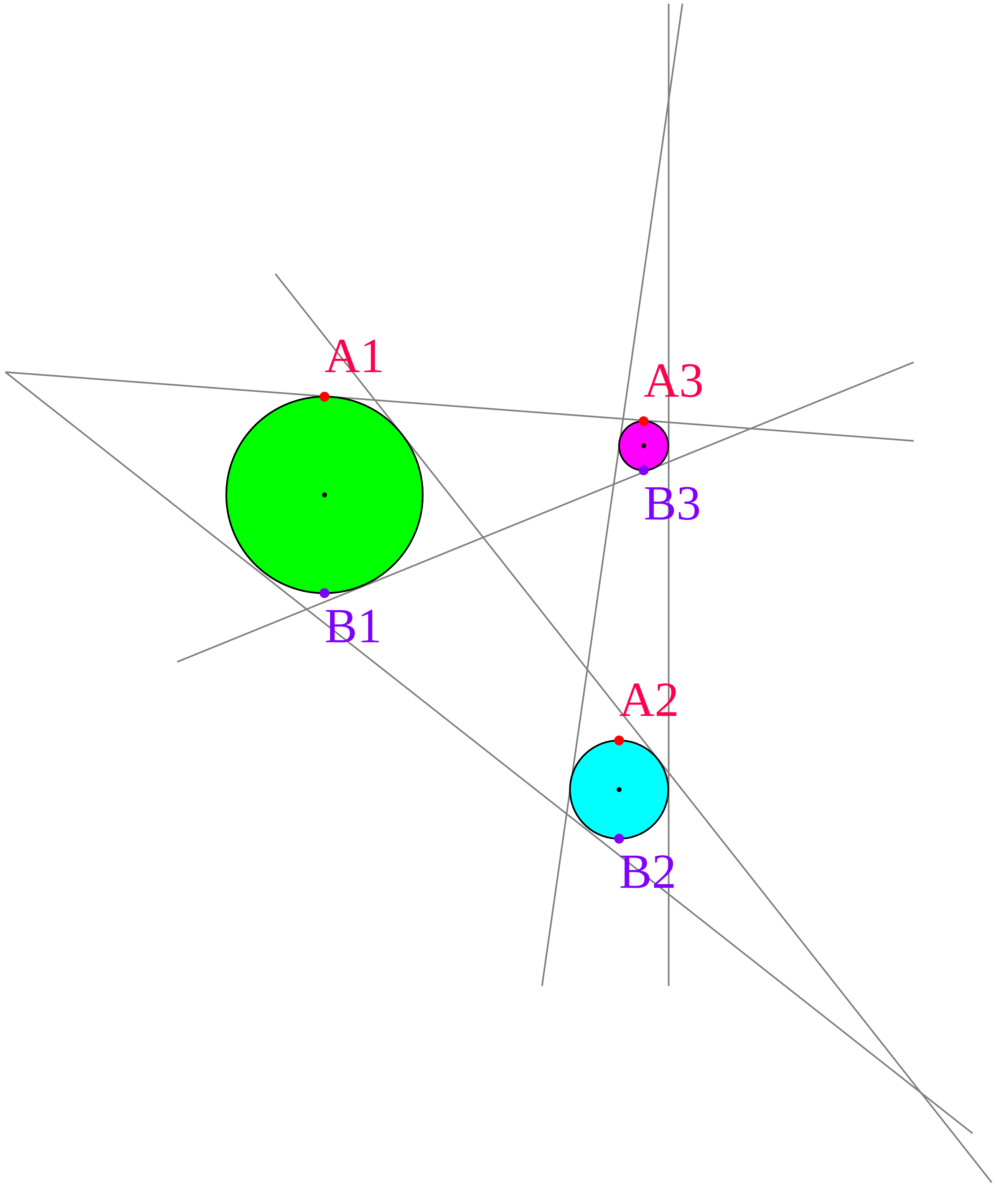
Proof 2 of Monge’s Circle Theorem



Pick a point on the green circle to be A1
Pick a point on the blue circle to be A2
Pick a point on the pink circle to be A3
where A2 and A3 correspond to A1.

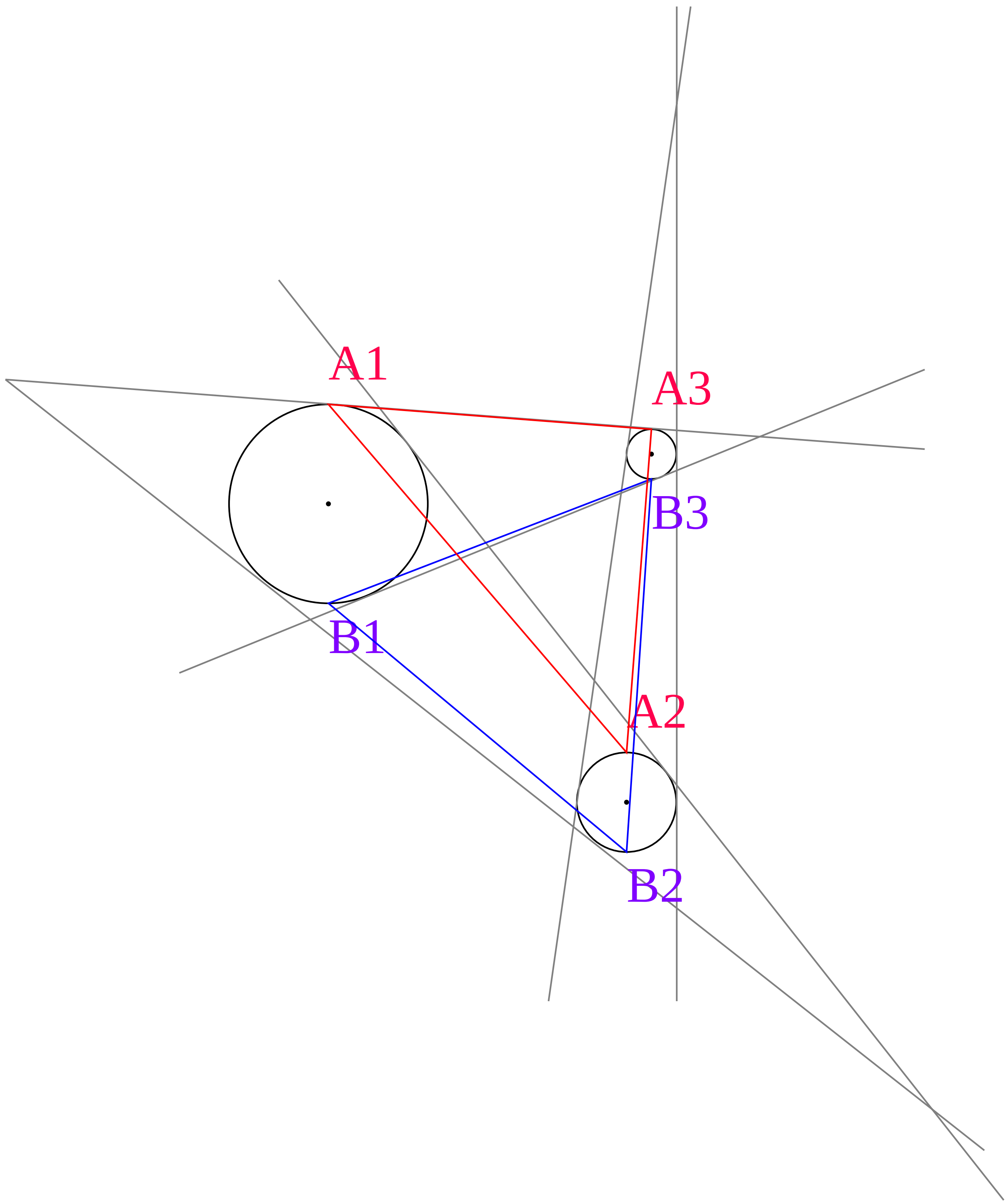


Set the point opposite of A1 to be B1
Set the point opposite of A2 to be B2
Set the point opposite of A3 to be B3
where B2 and B3 correspond to B1,

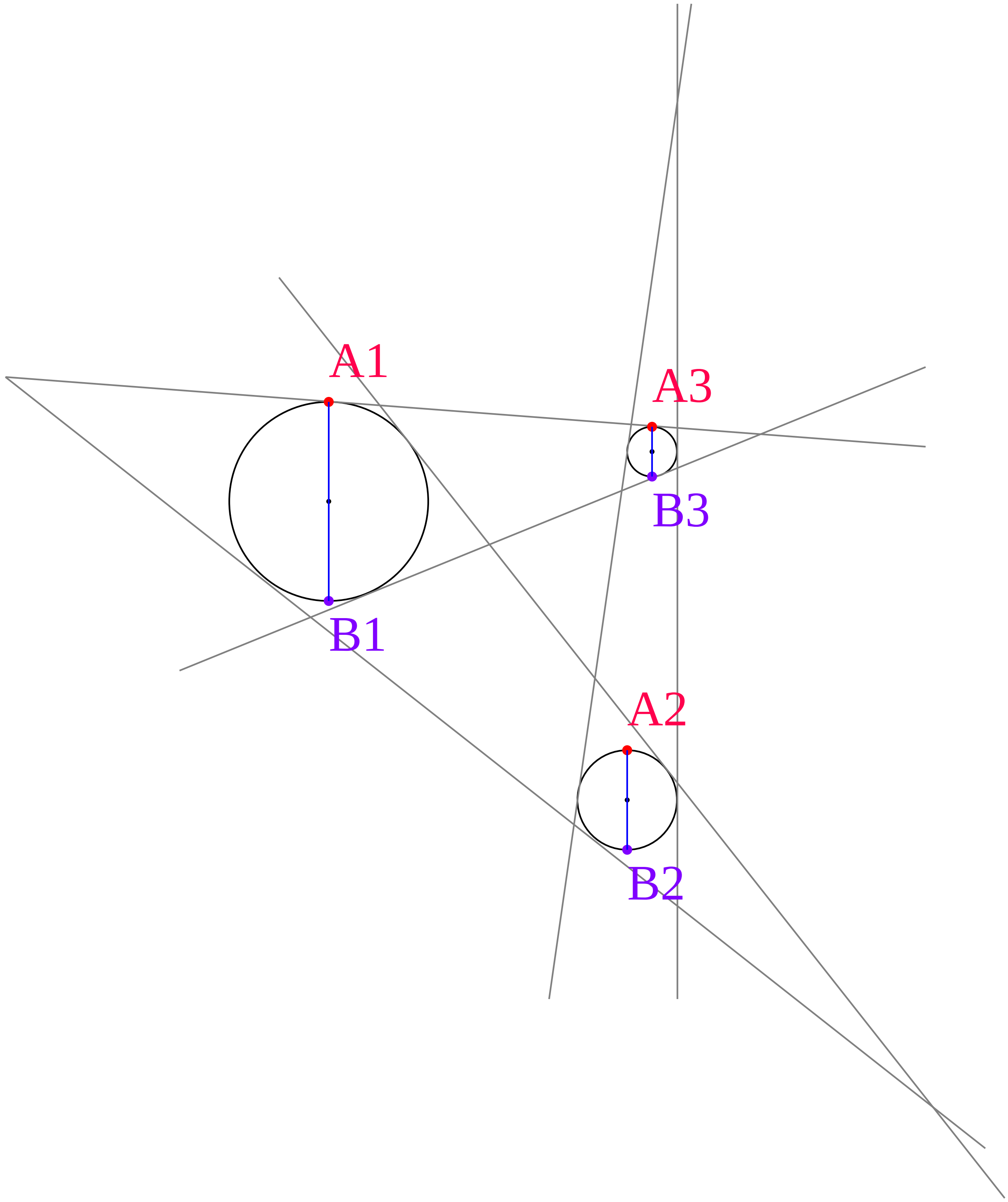


A1, A2, and A3 form a triangle.

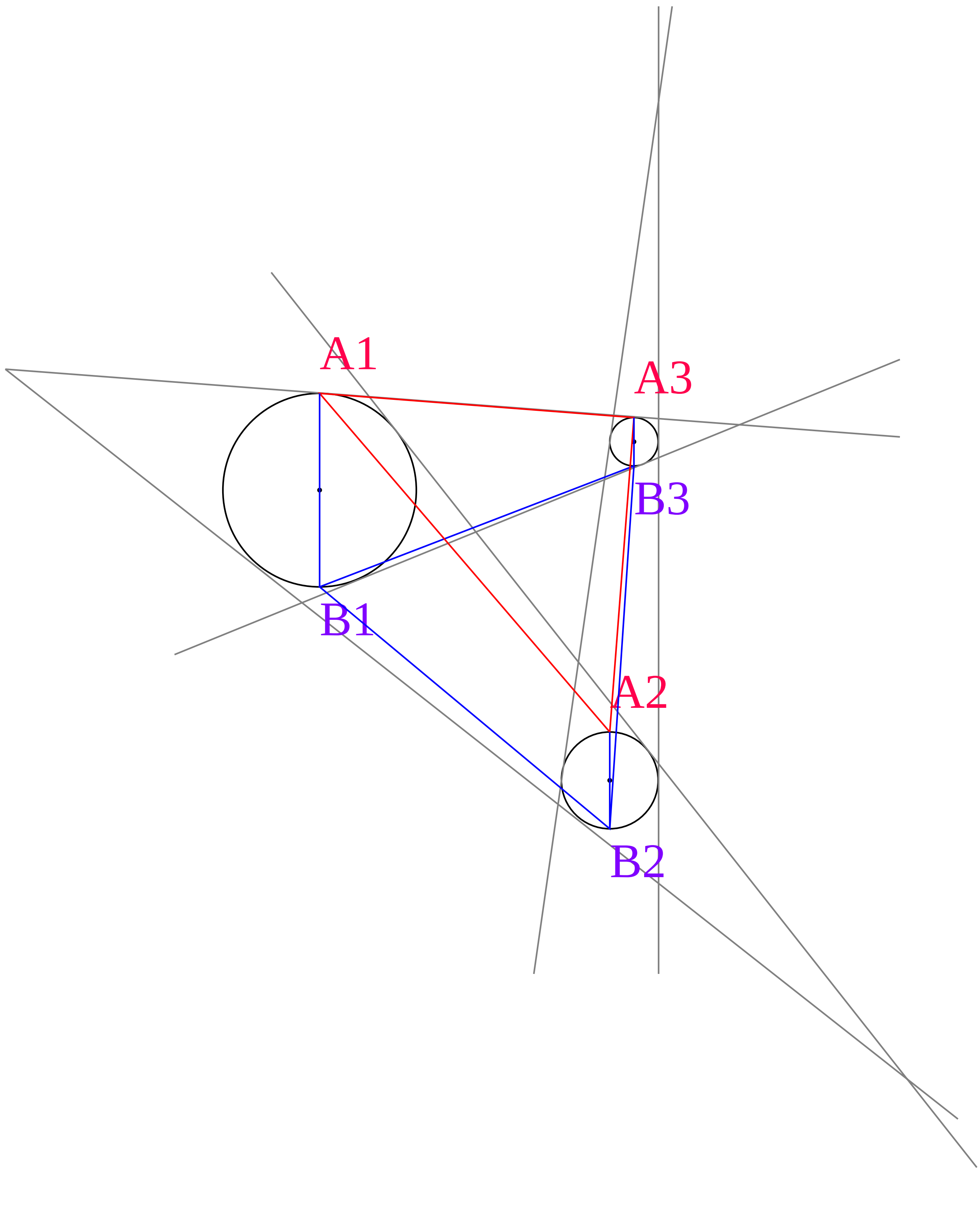
B1, B2, and B3 also form a triangle.



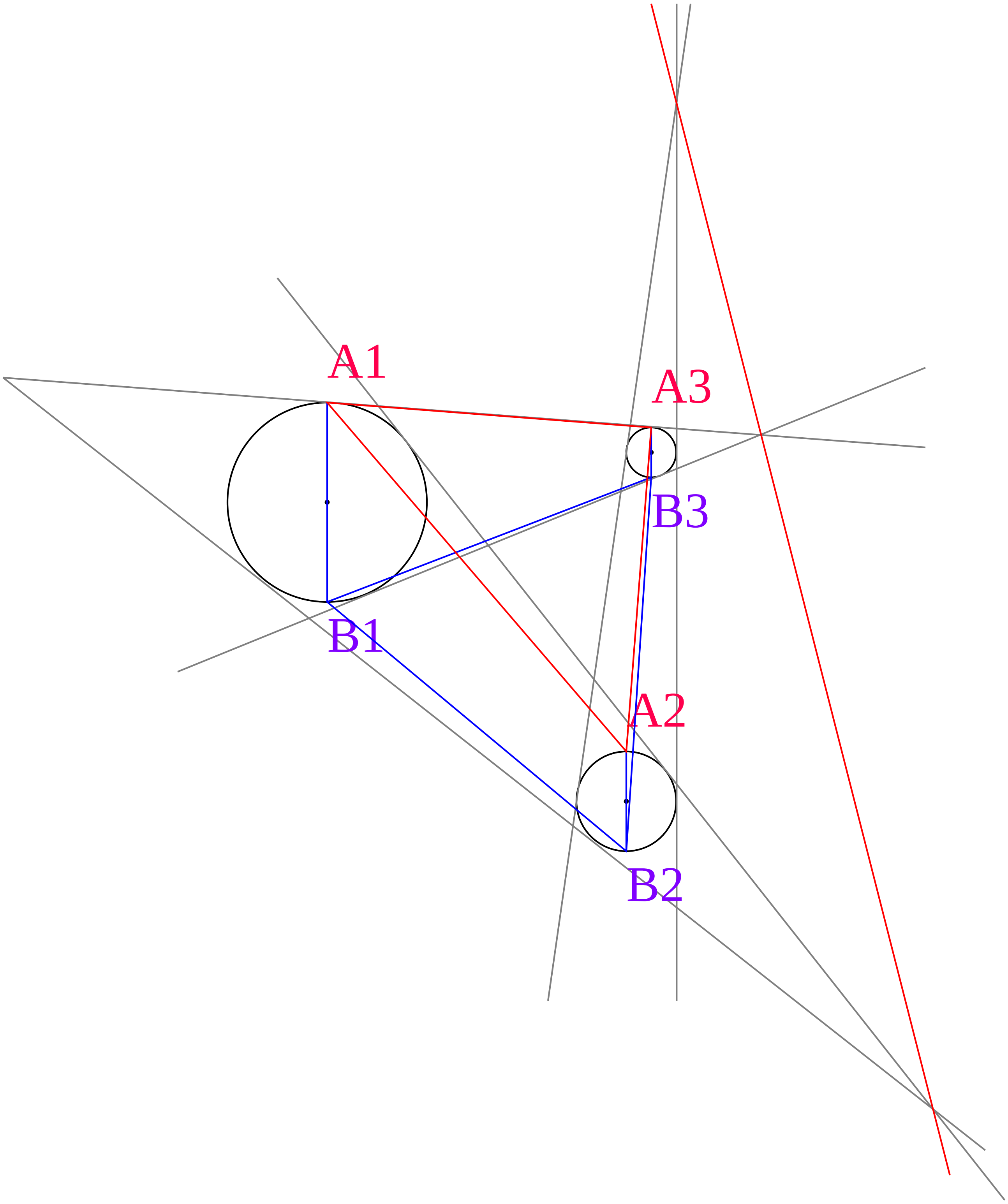
Now, connect A1,A2,A3, and B1,B2,B3 to the center of the corresponding circles.



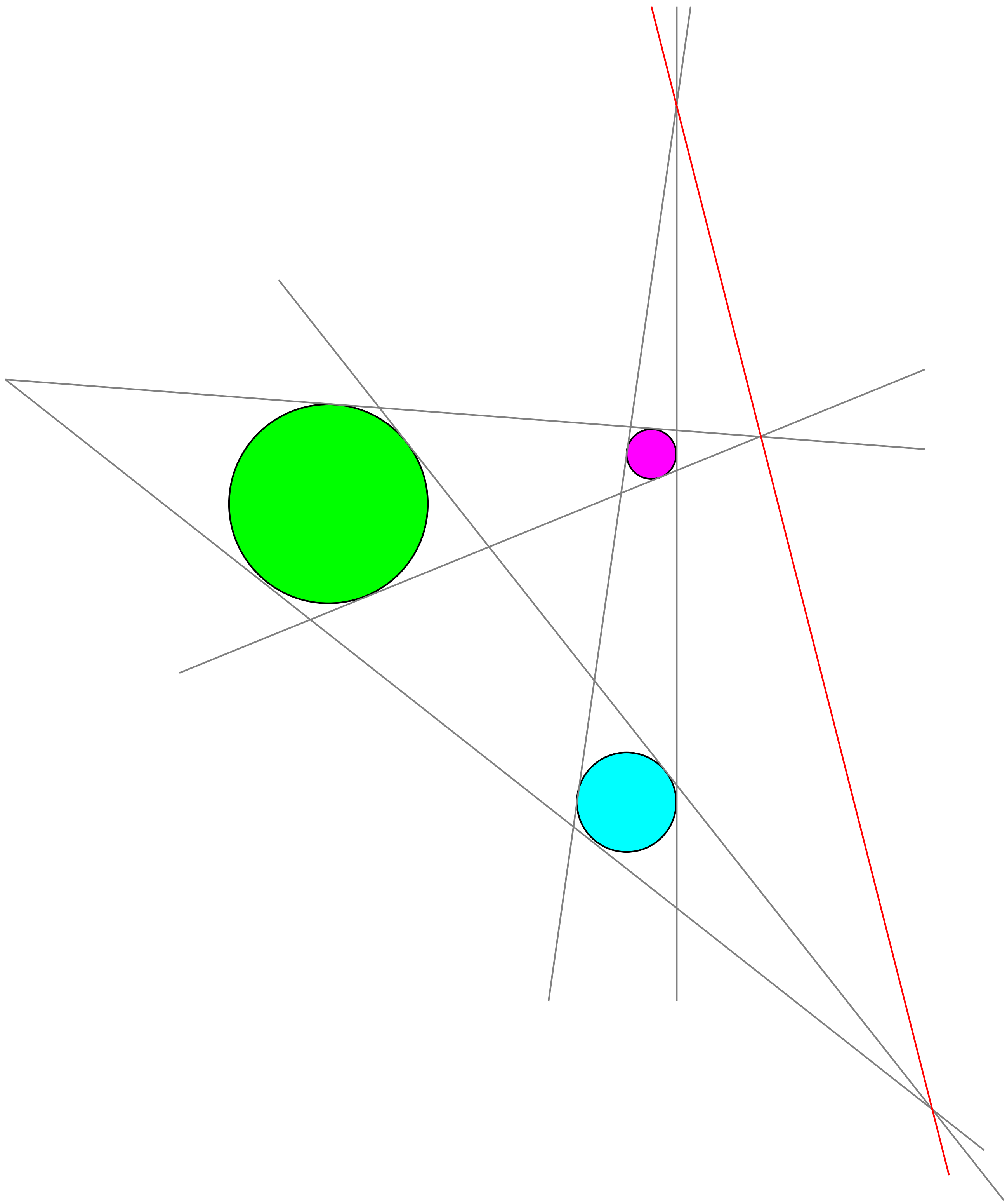
It is clear that A1B1||A2B2||A3B3



So, again, by Desargues' theorem,
Since the two triangles are perspective from a point, they are also perspective from a line.



I have proved that they lie on the same line as required by Monge's Circle Theorem



Reference

http://www.cut-the-knot.org/Curriculum/Geometry/MongeTheorem.shtml

http://www.cut-the-knot.org/Curriculum/Geometry/Desargues.shtml

http://users.ox.ac.uk/~jrlucas/maths/desarapp.pdf