Happy End Problem

by Fu-Jung Yang

Suppose there are four points positioned anywhere on a flat surface, as long as no points are co-linear. When all the points are joined, do they form a convex quadrilateral?

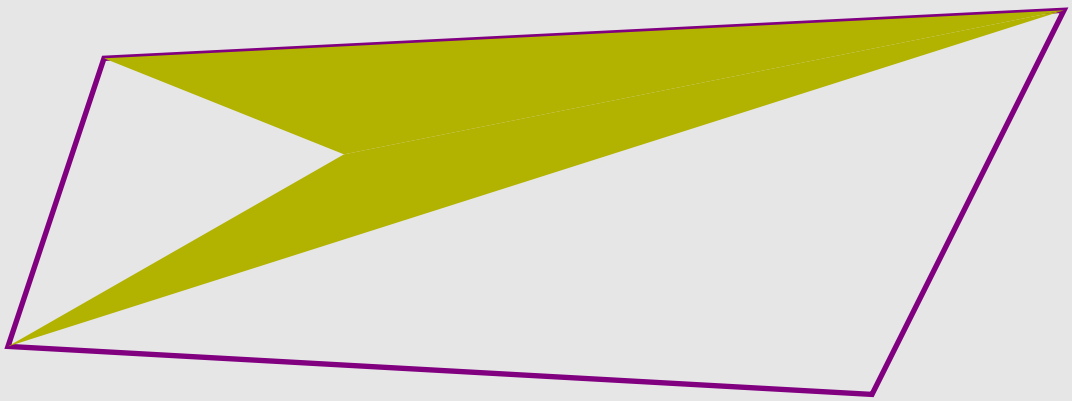
Before I give out the answer,

I would like to explain the term co-linear and convex.

The term co-linear means that three or more of the points form a straight line.

These three green dots shown above are co-linear.

The term convex means that the figure has no indentation.



The purple box is a convex quadrilateral, but the yellow quadrilateral is not because of the indentation.

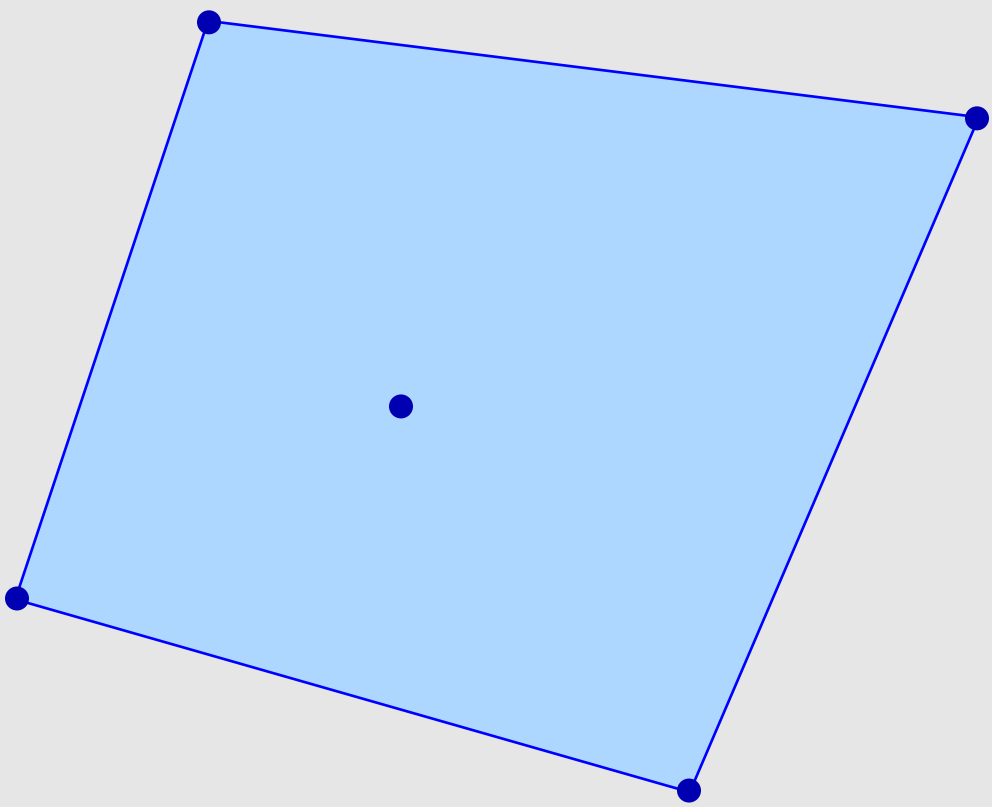
Clearly, the answer is no.

Then, how about using five points?

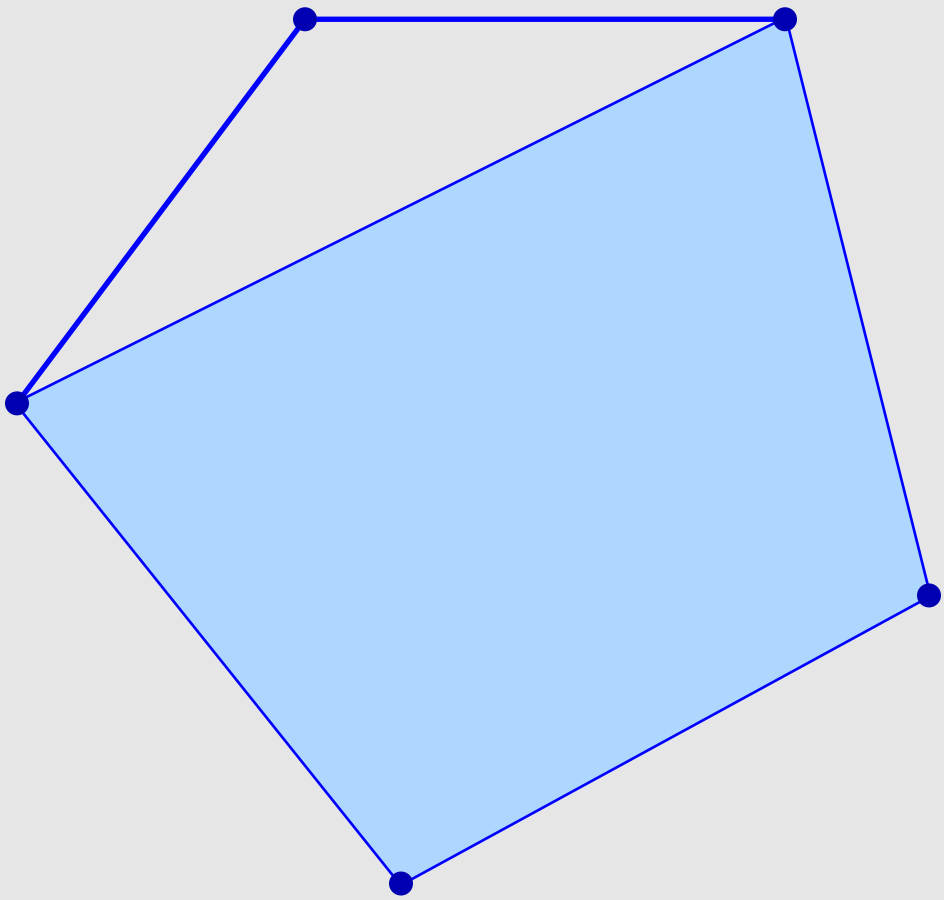
I claim that the answer is positive.

There are three different ways in which a convex polygon encloses all five points.

The simplest case occurs when a convex polygon results from joinging four points to create a quadrilateral, which fences in the remaining point and automatically satisfies the requirement.



In the second case, if the convex polygon is a pentagon incorporating all five points, then any four of these points can be connected to form a quadrilateral. For example, the shaded area in the picture below.



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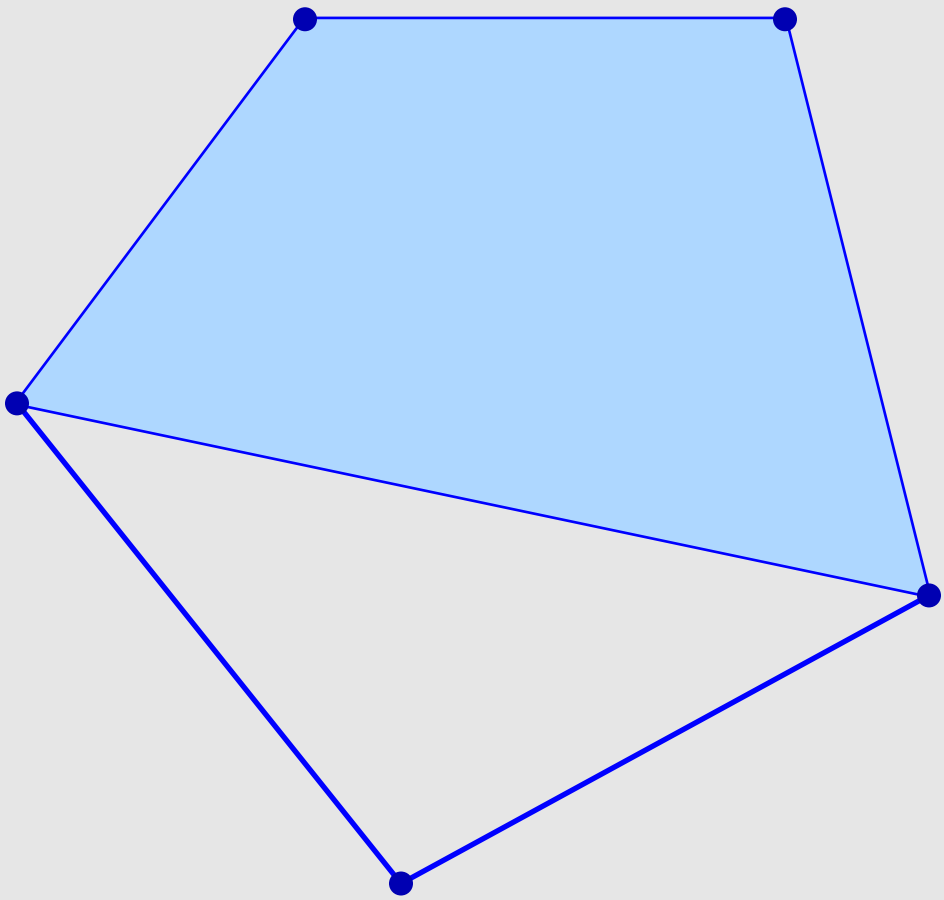

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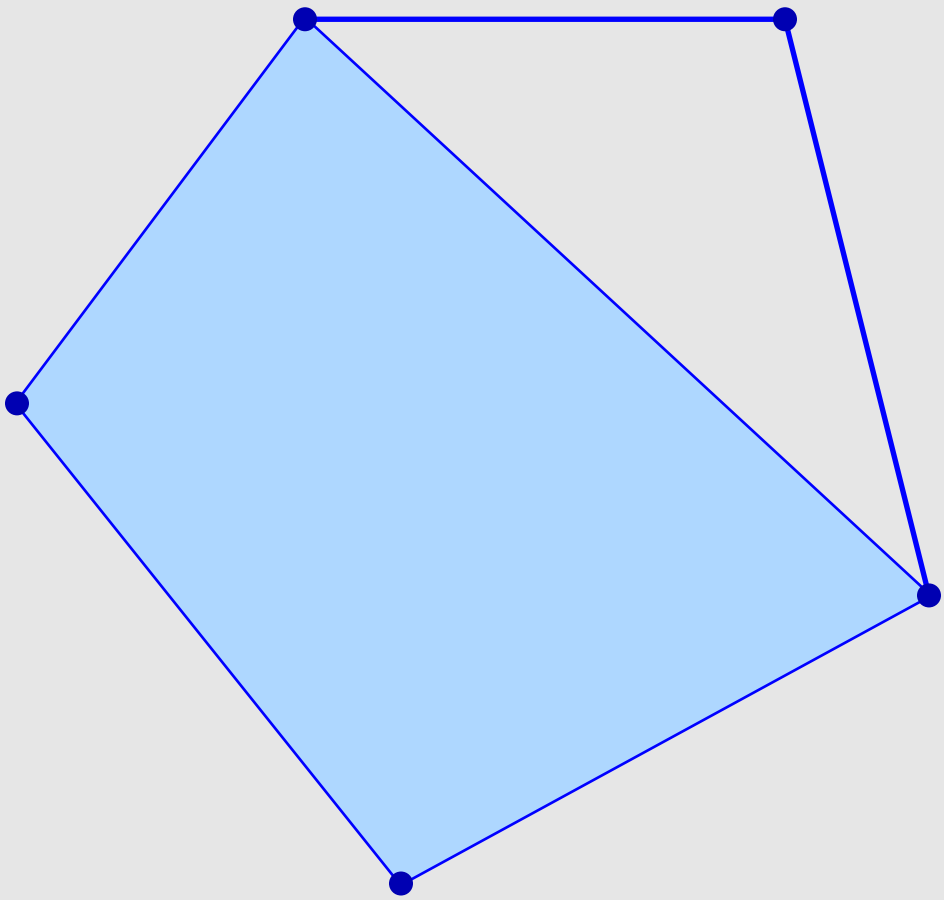

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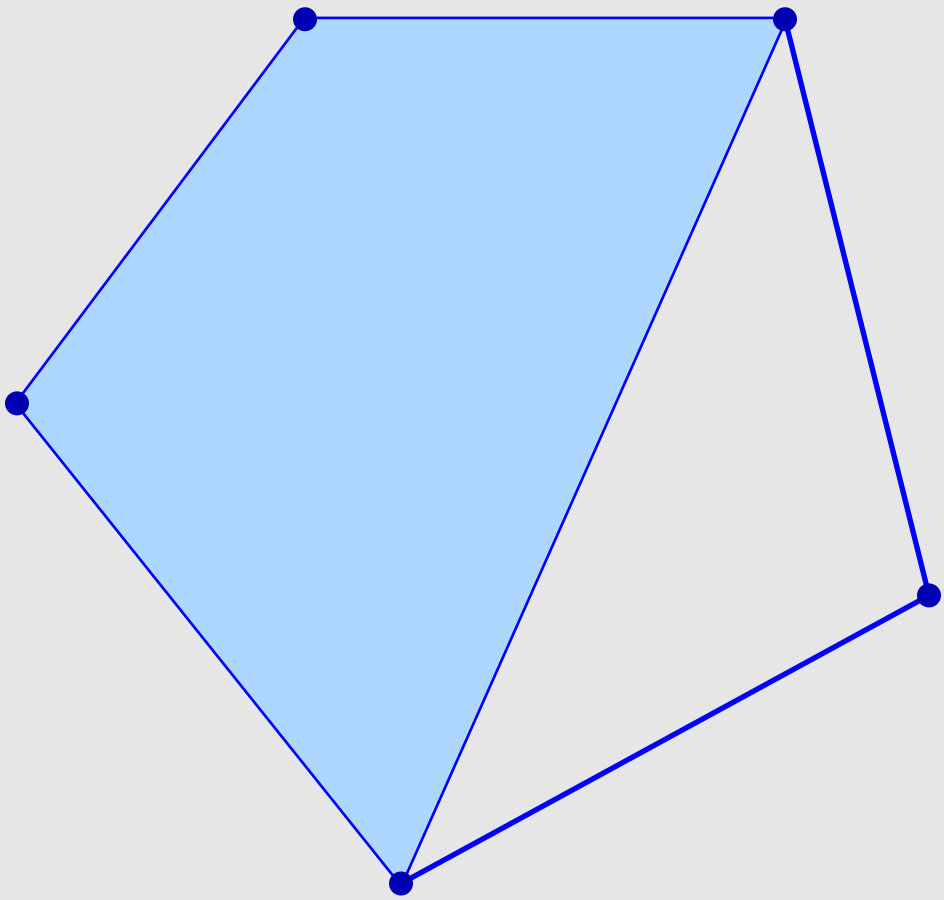

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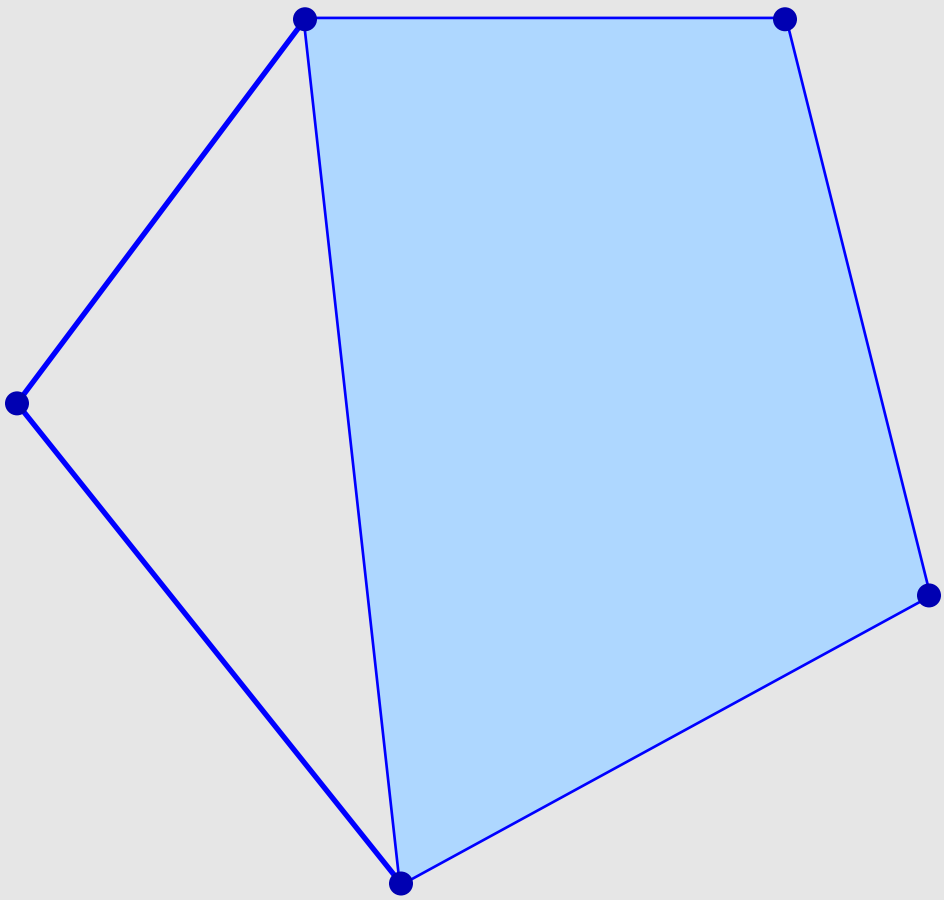
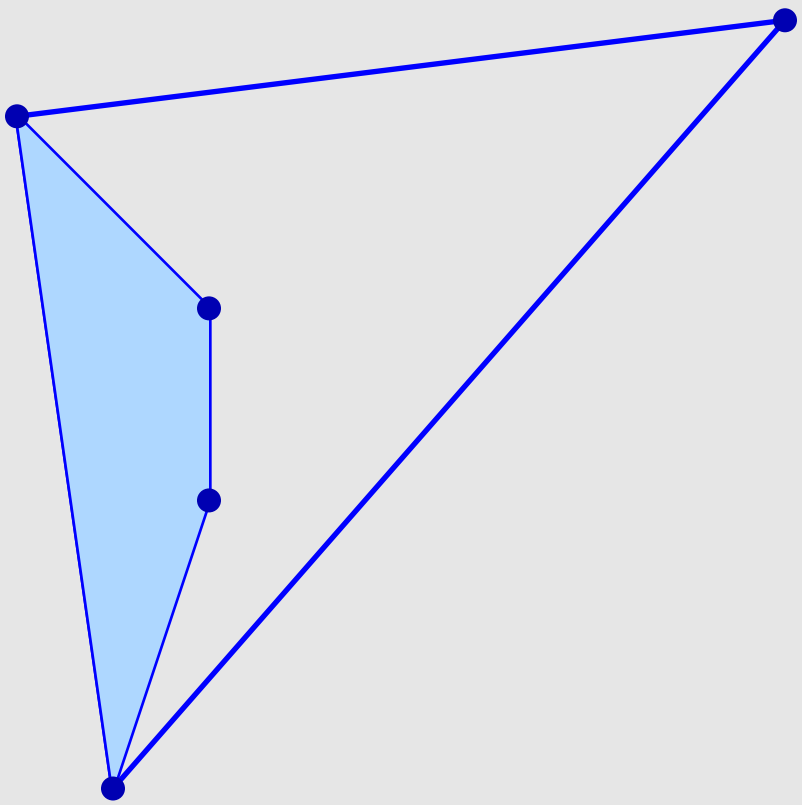


Finally, if the convex polygon includes just three points to create a triangle, the two points left inside the triangle define a line that splits the triangle so that two of the triangle's points are one side of the line. These two points plus the two interior points automatically form a convex quadrilateral.

For example, the shaded area in the picture below.

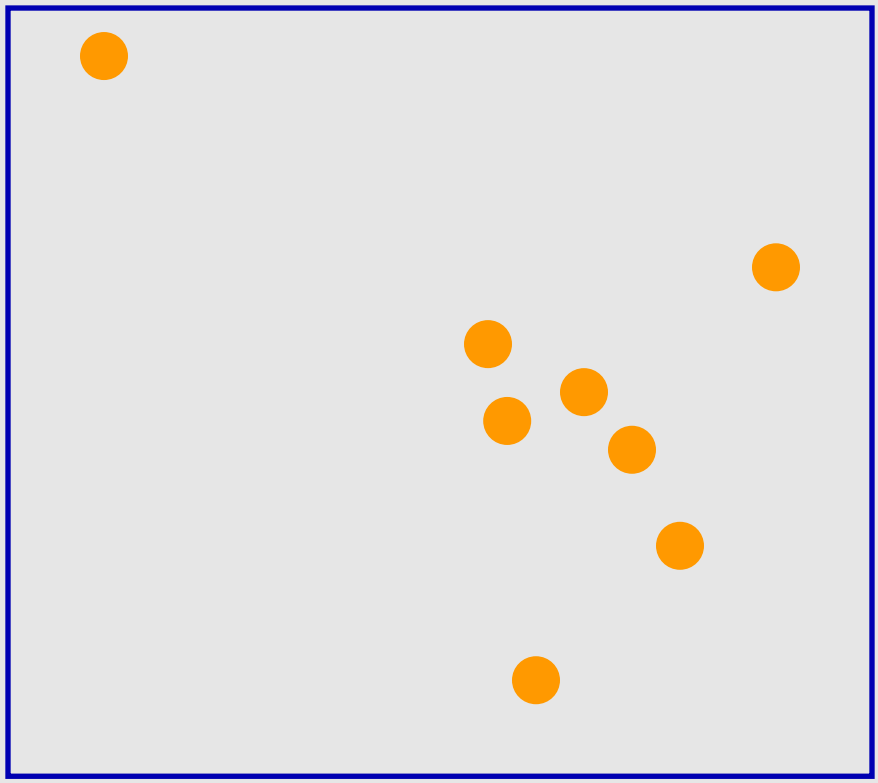


Therefore, convex quadrilaterals are not gauranteed for any arrangements of four points on a flat surface.

However, a convex quadrilateral has to be present in any given set of five points lying in a plane, with no three points in a line.

In general, among points randomly scattered across a flat surface, you can always find a set of points that forms a particular convex polygon, if there are enough points.

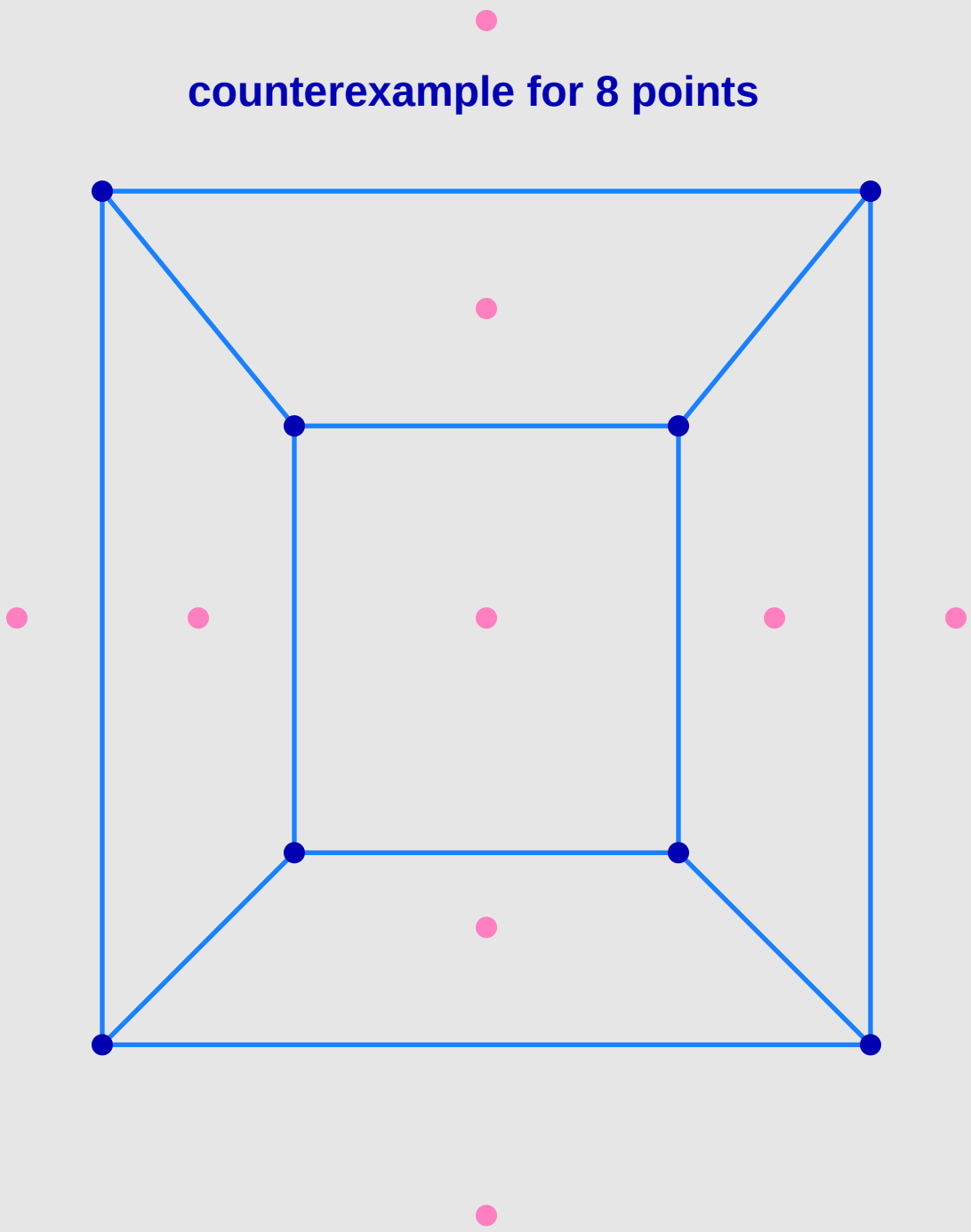
Now, we are going to study another example.
You can always find a covex pentagon in a set of nine point but not among just eight points.



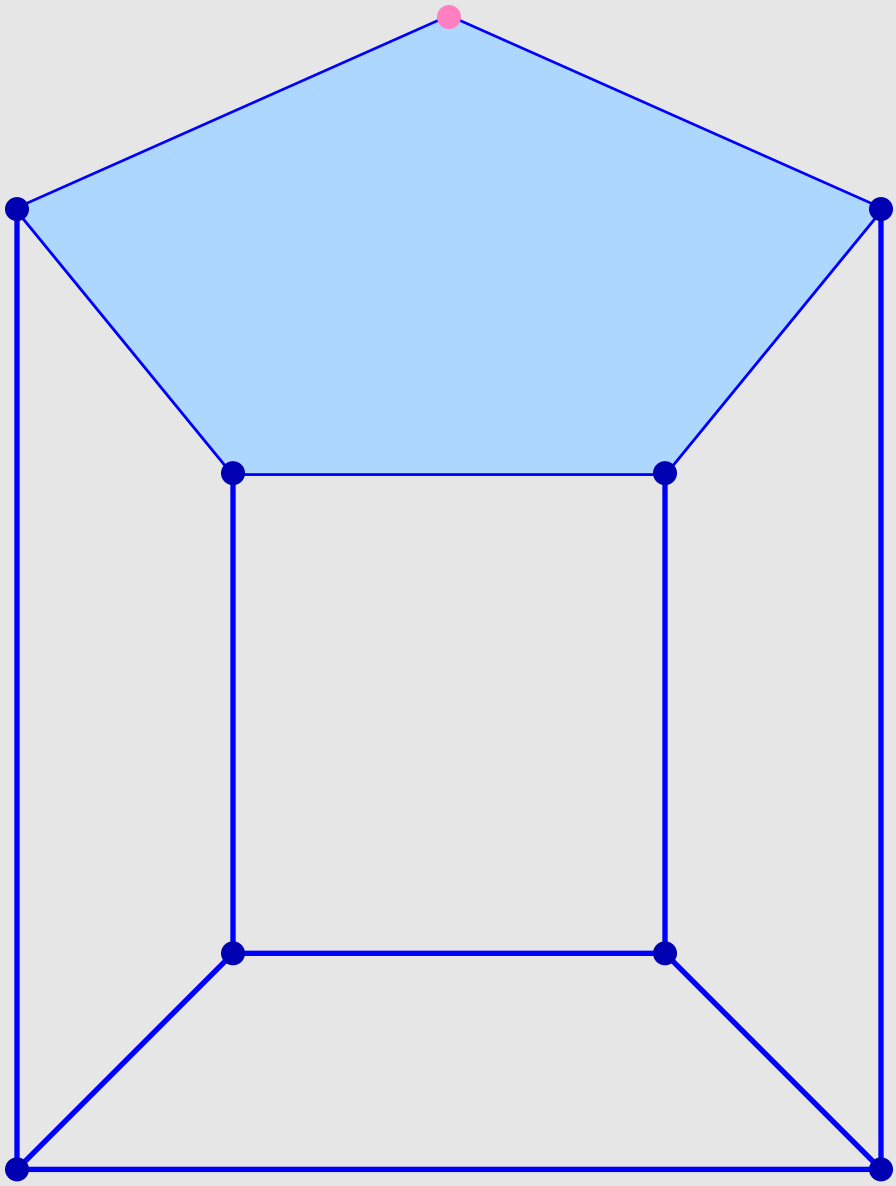
No pentagons are possible for the arrangement of eight points shown in the figure above, so in order to make a convex pentagon, we need more than eight points.

E. Makai proved that we need nine points to make a convex pentagon after demonstrating that a counterexample could be found for eight points.

counterexample for 8 points



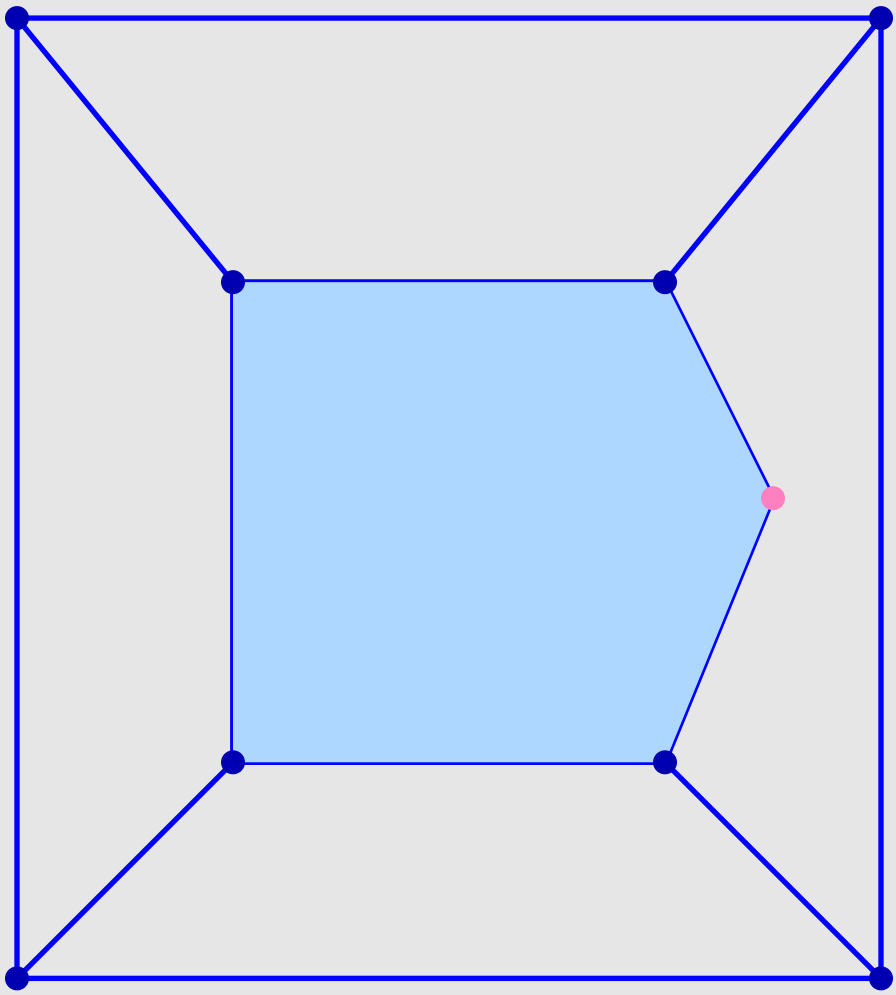
No convex pentagon can be formed by the set of eight blue dots. However, if we add a point from one of the regions marked by the pink dots, we would have a convex pentagon.



For example, the area shaded in blue is a convex pentagon.

It contains four blue and one pink points.

No more than two of the points are co-linear.



Here is another example, the area shaded in blue is a convex pentagon.

Again, it contains four blue and one pink points, and no more than two of the points are co-linear.

If we express everything that we have seen so far mathematically, the problem asks:

For any integer n greater than or equal to 3, determine the smallest positive integer N(n) such that any set of at least N(n) points in general position in the plane, i.e. no three of the points are on a line, contains n points that are the vertices of a convex n-gon.

Szekeres and Erdos proposed that if the number of points that lie in a plane is $1+2^{n-2}$, you can always select n points so that they form a convex polygon with n sides. Thus, for a quadrilateral, n is 4, and the number of points required will be $1+2^{4-2}=1+2^{2}=5$. For a convex pentagon, n is 5, and the number of points required will be 9.

Interestingly, no one has yet proved the conjecture concerning the precise number of points required to guarantee the presence of a convex polygon.

The problem is solved only for the values n=3, 4, and 5. No one has even demonstrated that any set of 17 points in the plane always contains six points that are the vertices of a convex hexagon.

Furthermore, the parameter space becomes so large that searching for counterexample at random even for the case n=6 takes an extremely long time.

Therefore, the general problem remains open.

Erdos and Szeheres (1935) showed that N(n) always exists and derived the bound

$$2^{n-2}+1 =< N(n) =< \binom{2n-4}{n-2}+1,$$

For n>=4, by Chung and Graham (1998), this has since been reduced to N(n)=<G(n) for

$$G(n)=\binom{2n-4}{n-2},$$

by Kleitman and Pachter (1998), N(n)=<H(n) for

$$H(n)=\binom{2n-4}{n-2}+7-2n,$$

by Toth and Valtr (1998), N(n)=<M(n) for

$$M(n)=\binom{2n-5}{n-2}+2$$

Before I finish showing my project, I would like to tell you the story behind the name of "happy end problem".

Erdos dubbed this geometrical problem the "happy end" problem. Probabily, the ending he had in mind was the subsequent marriage of Szekeres and Klein, who now live in Sydney, Australia.

References:

http://mathworld.wolfram.com/HappyEndProblem.html.

http://www.sciencenews.org/

http://www.ams.org/journal-getitem?pii=S0273-0979-00-00877-6

"knots"by Maggie Ying from Dr.Casselman's class

Thank you for your time!

Have a nice day!