Mathematics 308—February 5, 1996

Animating linear transformations

Finding the exponential of a matrix

If X is a matrix then its exponential is defined by the infinite converging series

$$e^X = I + X = \frac{X^2}{2} + \frac{X^3}{3!} + \cdots$$

This series is used directly to calculate a few simple examples, and to produce two rules. The examples and the rules can then be used to calculate e^X for any X, after a certain amount of calculation.

Example. Let

$$X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Then according to the series development

$$e^X = \begin{bmatrix} 1 + a + a^2/2 + a^3/3! + \cdots & 0 \\ 0 & 1 + b + b^2/2 + b^3/3! + \cdots \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Example. If

$$X = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

then

$$e^X = I + X = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

since $X^2 = 0$ and consequently $X^n = 0$ for $n \geq 2$.

Example. If

$$X = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

then

$$\begin{aligned} X &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \\ X^2 &= \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix} \\ X^3 &= \begin{bmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{bmatrix} \\ X^4 &= \begin{bmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{bmatrix} \\ X^5 &= \begin{bmatrix} 0 & -\omega^5 \\ \omega^5 & 0 \end{bmatrix} \end{aligned}$$

and the calculations then cycle through every fourth power. Hence

$$\begin{aligned} e^X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} + \begin{bmatrix} -\omega^2/2 & 0 \\ 0 & -\omega^2/2 \end{bmatrix} + \begin{bmatrix} 0 & \omega^3/3! \\ -\omega^3/3! & 0 \end{bmatrix} + \begin{bmatrix} \omega^4/4! & 0 \\ 0 & \omega^4/4! \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 - \omega^2/2 + \omega^4/4! - \cdots & -\omega + \omega^3/3! - \omega^5/5! + \cdots \\ \omega - \omega^3/3! + \omega^5/5! - \cdots & 1 - \omega^2/2 + \omega^4/4! - \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \end{aligned}$$

The first of the two rules we need is a generalization of the familiar rule taht $e^{x+y} = e^x e^y$ when x and y are numbers.

$$\begin{aligned} e^{X+Y} &= I + (X+Y) + \frac{(X+Y)^2}{2} + \text{terms of degree} \geq 3 \\ &= I + (X+Y) + \frac{X^2 + XY + YX + Y^2}{2} + \text{terms of degree} \geq 3 \\ e^X e^Y &= \left(I + X + \frac{X^2}{2} + \cdots\right)\left(\frac{I + Y + Y^2}{2 + \cdots}\right) \\ &= I + (X+Y) + \frac{X^2 + 2XY + Y^2}{2} + \cdots \end{aligned}$$

and we see that the usual rule must be modified:

$$e^{X+Y} = e^X e^Y$$

under the assumption that $XY = YX$.

The second rule concerns similar matrices. I recall that a matrix X is **diagonalizable** if we can find another matrix S such that $S^{-1}XS$ is a diagonal matrix Y. Since we know how to find e^Y, you may wonder if we can use the diagonalization to find e^X. The second rules allows this:

$$e^{S^{-1}XS} = S^{-1} e^X S$$

The prototypes

The examples and the rules can be used to extend our list of examples.

Example. Let

$$X = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

Then

$$X = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + 0b00 = A + B$$

These two components commute with each other ($AB = BA$), so by the first rule

$$e^X = e^A e^B = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^a & e^a b \\ 0 & e^a \end{bmatrix}$$

Example. Let

$$X = \begin{bmatrix} a & -b \\ a & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = A + B$$

and again

$$E^X = e^A e^B = \begin{bmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{bmatrix}$$

How to find the exponential of any matrix

If X is a 2×2 matrix, then calculating e^X falls into one of three cases.

(1) It has real eigenvalues, and is diagonalizable.

(2) It has real eigenvalues, but is not diagonalizable.

(3) It has complex eigenvalues. In this case it is diagonalizable, but the diagonalization involves complex numbers, which we wish to avoid.

In each case we can find a matrix V such that $V^{-1}XV = Y$ has a simple form, and we can calculate according to the third rule for exponentials

$$e^X = V e^Y V^{-1}$$

We must distinguish the three cases. I recall that the characteristic polynomial is

$$\lambda^2 + \mathrm{trace}(X)\,\lambda + \det(X)$$

where

$$\mathrm{trace}(X) = \tau = a + d, \quad \det(X) = \delta = ad - bc\ .$$

Its roots are

$$\lambda = \frac{\tau}{2} \pm \frac{\sqrt{\tau^2 - 4\delta}}{2}$$

(1a) The matrix X is diagonalizable, with real distinct eigenvalues. In this case

$$\tau^2 - 4\delta > 0$$

We make up a matrix V whose columns are eigenvectors. Then

$$\begin{aligned} V^{-1}XV &= D \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ X &= VDV^{-1} \\ e^X &= V e^D V^{-1} \end{aligned}$$

If the matrix has only a single eigenvalue, then there are two possibilities. It may be equal to a scalar multiple of the identity, in which case we can calculate e^X directly. This is case (1b). Otherwise we get case (2).

(2) The matrix X has two equal eigenvalues, and is not diagonalizable. In this case

$$\tau^2 - 4\delta = 0$$

and X is not a scalar multiple of I, and in not even diagonal. Here we let V be a 2×2 matrix whose first column is an eigenvector and whose second column is any vector which is not a multiple of the first column. Then

$$V^{-1}XV$$

will have the form

$$E = \begin{bmatrix} \lambda & x \\ 0 & \lambda \end{bmatrix}$$

for some x which falls out of the calculation, and

$$e^X = V e^E V^{-1}$$

where E is one of the prototypes.

(3) The matrix X is diagonalizable, but with complex eigenvalues and eigenvectors.

$$\tau^2 - 4\delta < 0$$

Here we can diagonalize by means of complex matrices, but we can also do something with only real matrices. Let $\lambda = a + ib$ be an eigenvalue. Then we can find an eigenvector for it, say $v = (x, y)$ where x and y are complex numbers. Let V be a matrix whose first column is v and whose second column is its **conjugate** $\overline{v}$. Then

$$V^{-1} X V = \Lambda$$

where

$$\Lambda = \begin{bmatrix} a + ib & 0 \\ 0 & a - ib \end{bmatrix}$$

For example, if

$$X = X_0 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is the prototype then

$$V_0 = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

and

$$V_0^{-1} X_0 V_0 = \Lambda$$

From the two equations

$$\begin{aligned} V^{-1} X V &= \Lambda \\ V_0^{-1} X_0 V_0 &= \Lambda \end{aligned}$$

we get

$$(V\, V_0^{-1})^{-1} X (V\, V_0^{-1}) = X_0$$

and

$$e^X = U e^{X_0} U^{-1}$$

where

$$U = V\, V_0^{-1} \; .$$

The matrix U is in fact real. This is because

$$U = \begin{bmatrix} x & \overline{x} \\ y & \overline{y} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} = \begin{bmatrix} x & \overline{x} \\ y & \overline{y} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{bmatrix} = \begin{bmatrix} (x + \overline{x})/2 & (\overline{x} - x)/2i \\ (y + \overline{y})/2 & (\overline{y} - y)/2i \end{bmatrix}$$

Its first column is therefore made up of the real components of v, its second column the negative imaginary components.

$$U = \begin{bmatrix} \mathrm{real}\,(x) & -\mathrm{imaginary}\,(x) \\ \mathrm{real}\,(y) & -\mathrm{imaginary}\,(y) \end{bmatrix}$$

where if $z = p + iq$ then

$$\mathrm{real}\,(z) = p, \quad \mathrm{imaginary}\,(z) = q \; .$$

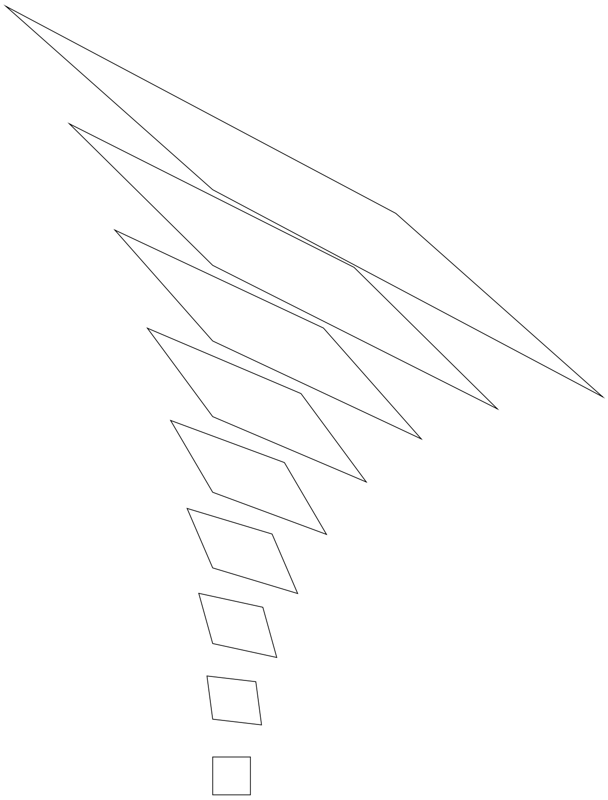
The animation

Once we know how to calculate exponentials, we can animate a linear transformation in the following way. First of all, we are not to be given the final linear transformation, but the **generator** of the motion, say X. Then the matrices e^{tX} are interpolations between I and e^X. So we apply a succession of transformations e^{tX} for t equal to 0, Δt, $2\Delta t$, etc. up until $t = 1$. We calculate the successive exponentials according to the rule

$$e^0 = I, \quad e^{(n+1)\Delta t} = e^{\Delta t} e^{n\Delta t}$$

For example here is what you get with generator

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$



The PostScript routines you are given

The file `eigs.inc` contains these routines:

`E`	defines $e = 2.718281828$
`PI`	defines $\pi = 3.141592654$
`det`	returns determinant of a 2×2 matrix on stack
`trace`	returns trace
`matrixmul`	A, B on stack replaced by AB
`matrixvectormul`	A, v replaced by Av
`eigentype`	returns `Real`, `Unipotent`, or `Complex`
`real-eigensolve`	returns U, $e = [\lambda_0 \ \ \lambda_1]$ such that $U^{-1}AU$ is diagonal
`unipotent-eigensolve`	returns U, λ
`complex-eigensolve`	returns U, $[\mathrm{real}(\lambda) \ \ \mathrm{imaginary}(\lambda)]$