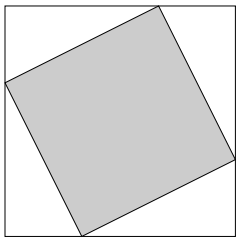
## Mathematics 308—February 5, 1996

## Pythagoras' Theorem

Here is a selection of proofs of Pythagoras' Theorem, from the hundreds known.

### Big squares



Simple algebra asserts that

$$(a+b)^2 = a^2 + 2ab + b^2$$

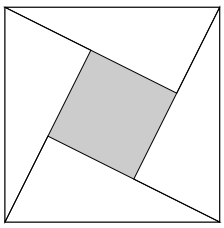
while the picture asserts that

$$\begin{aligned}(a+b)^2 &= c^2 + 4 \cdot \frac{ab}{2} \\ &= c^2 + 2ab\end{aligned}$$

from which we see that

$$c^2 = a^2 + b^2 \ .$$

### Little squares

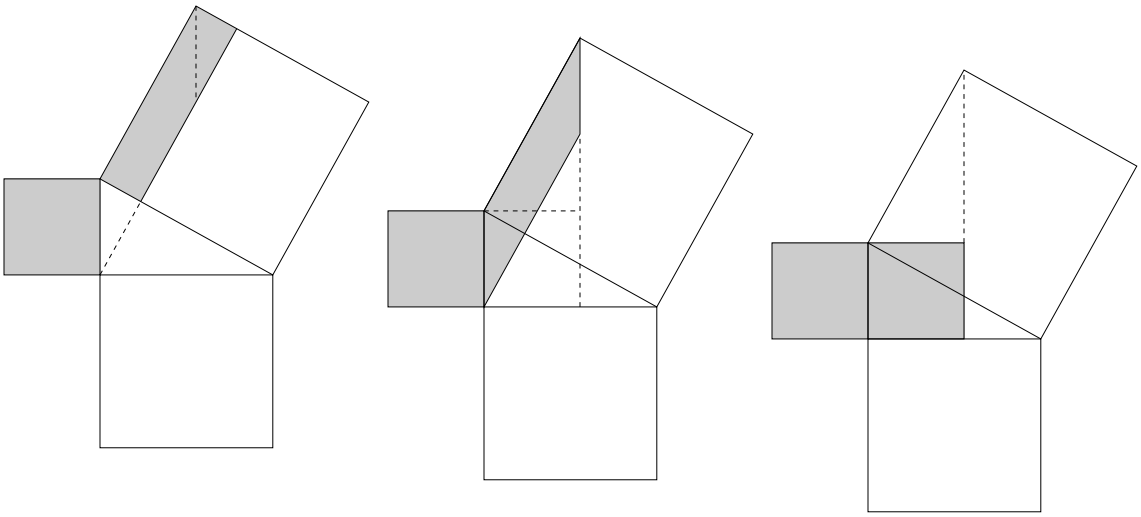


Here

$$\begin{aligned}c^2 &= 4 \cdot \frac{ab}{2} + (a-b)^2 \\ &= 2ab + a^2 + b^2 - 2ab \\ &= a^2 + b^2 \ .\end{aligned}$$

### My favorite

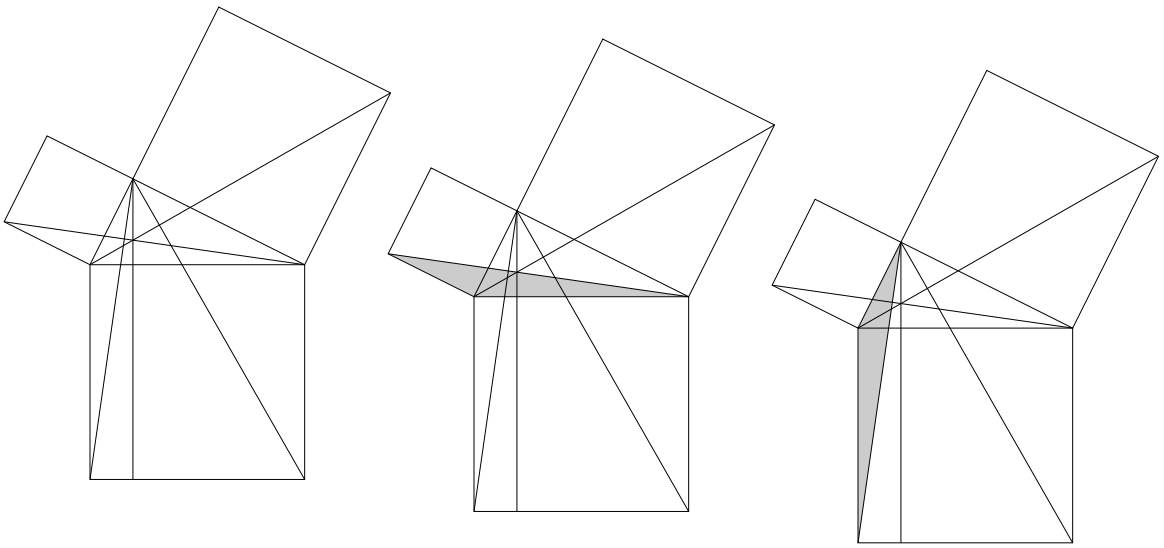
The algebraic proofs are simple, but I prefer a proof which gives a direct decomposition of $c^2$ into geometric components $a^2$ and $b^2$.



The proof dissects the large square into two parts, and asserts that they have area $a^2$ and $b^2$. The basic idea is that shears preserve areas. A sequence of shears produces a rectangle. In order to conclude the proof it must be shown that this rectangle is in fact a square. This depends on the claim that the triangle just above the square is congruent to the original. This follows from the *angle-side-angle* Theorem.

### Euclid's

This is the one given by Euclid in the proof of Theorem I.47 of *The elements*.



It is essentially the same as the last proof, but uses triangles instead. In other words, it is proving rather that

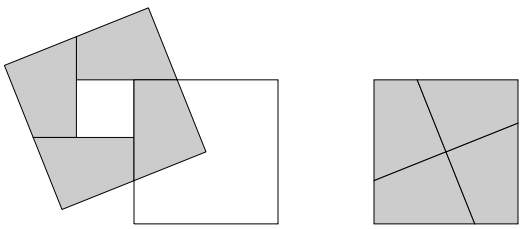
$$\frac{c^2}{2} = \frac{a^2}{2} + \frac{b^2}{2}$$

The main point is that the two gray triangles are congruent, this time by *side-angle-side*. Equivalently, you can see more intuitively that one can be rotated into the other.

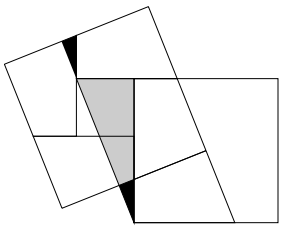
### The most cryptic

This is one a colleague of mine pointed out to me. I have been told that it is due to Sam Dewdney, a man who invented a myriad mathematical games around the end of the last century.

Let $c$ be the side of the larger square, $a$ the side of the one inside it, and $b$ that of the detached one. The picture shows immediately that the area of the largest square decomposes into the areas of the other two, so that it is transparently clear that $c^2 = a^2 + b^2$. What it is not so clear is that $a$, $b$, and $c$ are the sides of a right triangle!



It is clear, however, how to draw a right triangle with sides $a$ and $b$. It remains to see that the length of its hypthenuse is $c$.



This follows from the congruence of the two black triangles.

Another, perhaps simpler, way to locate the triangle is like this: