Mathematics 308—February 15, 1996

Introduction to three dimensional geometry

With this section we begin looking at three dimensional geometry. There will be two main topics: (1) the intrinsic geometry of three dimension al objects and (2) how to draw them — which is to say render them in only two dimensions. We will begin by discussing linear transformations in three dimensions, and the first ones we'll look at are the ones which are geometrically the simplest. But as a preliminary, I will discuss dot products.

Dot products

Let be the $3D$ vector

$$u = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \ .$$

Then

$$v = x\mathbf{i} + y\mathbf{j}$$

is its orthogonal projection onto the (x, y) plane. By Pythagoras' theorem we know that $||v||^2 = x^2 + y^2$, but also that $||u||^2 = ||v||^2 + z^2$ which gives us all in all

$$||u||^2 = x^2 + y^2 + z^2 \ .$$

This is also

$$u \bullet u$$

where for any two vectors we define their dot product according to the formula

$$u \bullet v = x_u x_v + y_u y_v + z_u z_v \ .$$

In addition to the rule

$$||u||^2 = u \bullet u$$

the dot product has some simple algebraic properties that are immediate from the definition:

$$\begin{aligned}(u + v) \bullet w &= u \bullet w + v \bullet w \\ (u - v) \bullet w &= u \bullet w - v \bullet w \\ (cu) \bullet v &= c(u \bullet v) \ .\end{aligned}$$

The distance between the heads of two vectors u and v is $||u - v||$, whose square is

$$(u - v) \bullet (u - v) = u \bullet u - 2\, u \bullet v + v \bullet v \ .$$

If u and v are perpendicular, then Pythagoras' theorem asserts that this is just $u \bullet u + v \bullet v$, so that $u \bullet v = 0$. This gives two instances of a general rule: $u \bullet v = 0$ if u and v are perpendicular and $u \bullet v = ||u||\,||v||$ if they are parallel. More generally:

Proposition. *We have*

$$u \bullet v = ||u||\,||v||\, \cos\theta$$

where θ is the angle between u and v.

We have seen the cases $\theta = 0$ and $\theta = \pi/2$. But why is it true in general? It is because of the cosine rule for general triangles. If u and v are vectors, then they form two sides of a triangle. On the one hand, the square of the length of the third side is

$$||u - v||^2 = ||u - v|| \bullet ||u - v|| = ||u||^2 + ||v||^2 - 2u \bullet v$$

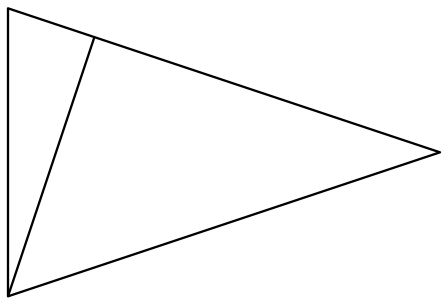
and on the other, by the cosine rule, it is

$$||u||^2 + ||v||^2 - 2||u||\,||v|| \cos\theta$$

so by comparison

$$u \bullet v = ||u||\,||v|| \cos\theta$$

To finish off I shall recall the proof of the cosine rule.



Let the side opposite the origin have length c. By 'dropping' a perpendicular from the origin onto this side we decompose it into two pieces of length, say, x_1 and x_2. Thus

$$c^2 = (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2$$

On the other hand the original angle θ is decomposed into two parts θ_1, θ_2. We know that

$$\cos\theta = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

Finally, let y be the length of the perpendicular. By Pythagoras' Theorem applied to each of the small triangles and trigonometry

$$\begin{aligned} x_1^2 &= a^2 - y^2 \\ x_2^2 &= b^2 - y^2 \\ x_1 &= a\sin\theta_1 \\ y &= a\cos\theta_1 \\ x_2 &= a\sin\theta_2 \\ y &= a\cos\theta_2 \end{aligned}$$

so that

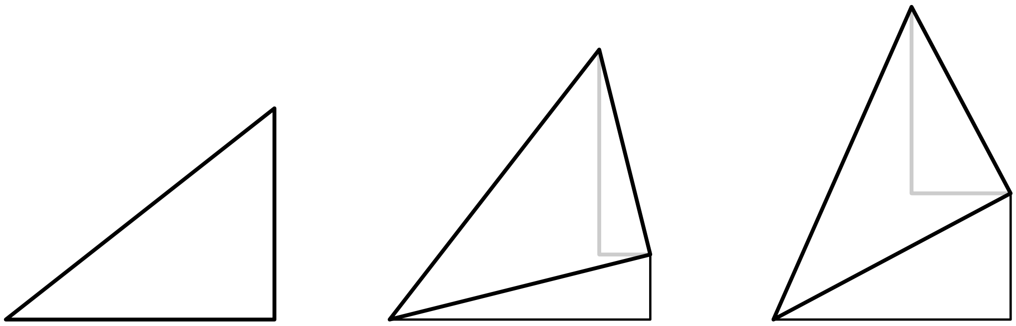
$$\begin{aligned} c^2 &= (a^2 - y^2) + (b^2 - y^2) + 2ab\sin\theta_1\sin\theta_2 \\ &= a^2 + b^2 - 2ab\,\cos\theta_1\cos\theta_2 + 2ab\sin\theta_1\sin\theta_2 \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

Finally, just to be complete, I recall the proof of the formula for the cosine of the sum of two angles.

Theorem. *For any pair of angles α and β*

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

This follows from the slightly more general rule: if we rotate the point (x, y) through θ around the origin, we get a point with coordinates $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. This is formally a consequence of the fact that rotation is a linear transformation. But it should be more directly apparent from this picture (which is actually a diagram illustrating linearity):



Orthogonal transformations

An **orthogonal transformation** in three dimensions is a linear transformation which preserves the lengths of the vectors it moves. Since lengths are calculated in terms of dot products this condition translates to the algebraic condition

$$Tu \bullet Tu = u \bullet u \ .$$

If u and v are vectors then the square of the distance between the points at their heads is

$$\|u - v\|^2 = (u - v) \bullet (u - v) = u \bullet u - 2\, u \bullet v + v \bullet v \ .$$

Because T is linear

$$\|T(u) - T(v)\|^2 = \|T(u - v)\|^2 = \|u - v\|^2 \ .$$

Therefore T preserves the distances between all points in space and

$$Tu \bullet Tv = u \bullet v$$

as well. In other words:

Proposition. *A linear transformation is orthogonal if and only if it preserves dot products.*

The relation between dot products and angles implies that an orthogonal transformation preserves all angles, too.

At any rate, it is immediate from the definition that

Proposition. *The combination of two orthogonal linear transformations is again orthogonal.*

How is orthogonality determined from the matrix of a linear transformation?

Proposition. *A linear transformation is orthogonal if and only if its matrix satisfies these two conditions: (1) its columns are vectors with length* 1*; (2) any two distinct columns are perpendicular.*

That these conditions are necessary follows immediately from the previous result in view of the fact that the columns of the matrix are by definition the images of the vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$.

Suppose on the other hand that T is a linear transformation whose columns satisfy these conditions. If $v = (x, y, z)$ then

$$v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad Tv = xT(\mathbf{i}) + yT(\mathbf{j}) + zT(\mathbf{k})$$

and

$$\begin{aligned} Tv \bullet Tv &= (xT(\mathbf{i}) + yT(\mathbf{j}) + zT(\mathbf{k})) \bullet (xT(\mathbf{i}) + yT(\mathbf{j}) + zT(\mathbf{k})) \\ &= x^2\, T(\mathbf{i}) \bullet T(\mathbf{i}) + xy\, T(\mathbf{i}) \bullet T(\mathbf{j}) + xz\, T(\mathbf{i}) \bullet T(\mathbf{k}) \\ &\quad + yx\, T(\mathbf{j}) \bullet T(\mathbf{i}) + y^2\, T(\mathbf{j}) \bullet T(\mathbf{j}) + yz\, T(\mathbf{j}) \bullet T(\mathbf{k}) \\ &\quad + zx\, T(\mathbf{k}) \bullet T(\mathbf{i}) + zy\, T(\mathbf{k}) \bullet T(\mathbf{j}) + z^2\, T(\mathbf{k}) \bullet T(\mathbf{k}) \\ &= x^2 + y^2 + z^2 \\ &= v \bullet v\ . \end{aligned}$$

Here is another version of the last part of the argument. If K is the matrix, the conditions above are directly equivalent to the single matrix equation

$$^{t}K\, K = I\ .$$

The dot product of a vector with itself, on the other hand, can be written as a matrix product, using the transpose of the column vector v:

$$v \bullet v = {}^{t}v\, v\ .$$

If K is an orthogonal matrix then

$$\begin{aligned} (Kv) \bullet (Kv) &= {}^{t}(Kv)\,(Kv) \\ &= {}^{t}v\,{}^{t}K\, Kv \\ &= {}^{t}v\, v \end{aligned}$$

Since the inverse of matrix is both a left and right inverse, we have

$$K\,{}^{t}K = I$$

as well, which means that the rows are mutually perpendicular unit vectors also. In fact the conditions on rows are equivalent to the conditions on columns. Since the determinant of a product is the product of determinants

$$\det({}^{t}K)\ \det(K) = \det(I) = 1$$

and since $\det({}^{t}K) = \det(K)$

$$\det(K)^2 = 1, \quad \det(K) = \pm 1\ .$$

An orthogonal transformation is said to be a **rotation** if $\det(K) = 1$.

If u, v, w are three vectors in three dimensions which do not all lie on as plane, their **orientation** is determined according to the **right hand rule** in the following fashion: align your index finger along u and your middle finger along v. If your thumb points in the same general direction as w, the orientation is positive, otherwise negative. A linear transformation is said to preserve orientation if its columns are oriented positively. A linear transformation preserves orientation when its matrix has positive determinant. In other words, the rotations are the orthogonal transformations which preserve orientation. Reflections reverse it.

An **oriented orthogonal frame** in three dimensions is a triple of vectors u, v, w of unit length which are mutually perpendicular and oriented positively. In this terminology, a linear transformation is a rotation precisely when the columns of its matrix make up an oriented orthogonal frame.

Proposition. *The third vector in an oriented orthogonal frame is determined by the first two.*

If u and v are given, then w must be perpendicular to both of them, it lies along the unique line perpendicular to the plane spanned by them. Since it has unit length, there are two choices possible. Orientation picks out one of the two. Explicitly, the third is the cross-product of the first two.

Proposition. *Any of the vectors in an orthogonal triple can be any vector of unit length.*

In other words, if we are given, say, a vector w of unit length, we can always we can always choose u and v to fill the single vector out into an oriented triple.

The previous result amounts to the claim that we need only pick out one of them, say u. We just choose u to be any unit vector in the plane perpendicular to w.

There is something subtle about this assertion, because as we may see later on rigourously, *there is no way to make the choice of u depend continuously on w.* This fact is related to a simple geometric problem. If u and v are vectors of length 1, then their heads lie on the unit sphere. If they do not lie on the same axis, then there will be a unique great-circle arc from one to the other of shortest length. But if they are antipodal points, all great-circle arcs between them have the same length, and there is no good way to single out any of them.

At any rate, this fact explains why a constructive and explicit proof proceeds according to cases: (i) Suppose u lies along the z axis. Then $w = \pm\mathbf{k}$. The triples $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $-\mathbf{i}$, $-\mathbf{j}$, $-\mathbf{k}$ will fill the bill.

(ii) Otherwise, let u be the normalized cross-product of w and $\mathbf{k}$, and let v be the cross-product of w and u.

Geometrically, this amounts to the following construction. Call the **equator** of the unit sphere its intersection with the (x, y)-plane. If w does not lie on the z-axis, then there will exist a unique point e on the equator closest to it, and two points on the equator at right angles to it. These are the unit vectors on the line of intersection of the equator with the plane perpendicular to w. Let u be the one such that w, u, e are positively oriented.

This problem arises also in drawing some simple pictures in three dimensions, as we shall see later on.

We know how to describe geometrically two kinds of orthogonal transformations. (1) Pick an oriented axis, and rotate through some angle θ around that axis, where the direction of positive rotation is chosen according to the **right-hand rule**. This is called an **axial rotation**. (2) Pick a plane, and reflect through that plane. This is called a **planar reflection**.

The first preserves orientation, the second reverses it.

We shall not have any need at the moment to look at reflections further. Our main interest is in rotations.

I first point out that it is very simple to write down the matrix of the rotation of angle θ around the z-axis. It is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The next item we take care of is a practical one.

How can we find the effect of an axial rotation R on a vector?

Let α be a unit vector along the axis. If u is any vector at all, then we can split u into two components, one component u_0 parallel to α and another one $u_\perp$ perpendicular to it. The projection formula for vectors tells us that

$$u_0 = (u \bullet \alpha)\, \alpha$$

since $\|\alpha\| = 1$, and

$$u_\perp = u - u_0$$

Let

$$v_\perp = \alpha \times u_\perp = \alpha \times u$$

so that now $u_\perp$ and $v_\perp$ are vectors of the same size in the plane perpendicular to α. Rotation of u around α is linear, so $Ru = Ru_0 + Ru_\perp$. Of course $Ru_0 = u_0$. What is $Ru_\perp$? Suppose the angle of rotation is θ. Then rotation takes the whole plane perpendicular to α into itself, and since we know what $2D$ rotations do we can write

$$Ru_\perp = (\cos\theta)\, u_\perp + (\sin\theta)\, v_\perp$$

Assembling all this we see that

$$Ru = u_0 + (\cos\theta)\, u_\perp + (\sin\theta)\, v_\perp$$

The second matter we take care of is a theoretical one. The definition of a rotation as an orthogonal linear transformation which preserves orientation is rather abstract. In fact:

Theorem. *Every rotation in three dimensions is an axial rotation.*

The problem is to find the axis. The key ideas will be eigenvalues and eigenvectors, since the axis can be characterized as the vectors taken into themselves by a rotation, or in other words the eigenvectors with eigenvalue 1. In other words, we want to show that every rotation in three dimensions has an eigenvalue equal to 1.

Before I begin, let me discuss the eigenvalues and eigenvectors of $\mathbf{rot}_z(\theta)$. The characteristic polynomial is

$$\lambda^3 - (2c+1)\lambda^2 + (2c+1)\lambda - 1 = (\lambda - 1)(\lambda^2 - 2c + 1)$$

where $c = \cos\theta$, and the eigenvalues are 1, $e^{\pm i\theta}$. The only real eigenvectors lie on the z-axis.

Lemma. *If $u = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is any vector, possibly with complex coordinates, then*

$$u \bullet \overline{u} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \bullet (\overline{x}\mathbf{i} + \overline{y}\mathbf{j} + \overline{z}\mathbf{k}) = |x|^2 + |y|^2 + |z|^2$$

and is zero only if all coordinates are 0.

A matter of calculation.

Lemma. *The eigenvalues of any orthogonal transformation are complex numbers of absolute value* 1.

Say u is an eigenvector. In general it will have complex coordinates. At any rate $Tu = cu$ and then by taking conjugates we see also that $T\overline{u} = \overline{cu}$. But then

$$Tu \bullet T\overline{u} = c\overline{c}\, u \bullet \overline{u} = u \bullet \overline{u}$$

so that $c\overline{c} = 1$, since by definition an eigenvector is not 0.

If T is an orthogonal transformation, its characteristic polynomial is a cubic polynomial, hence has at least one real eigenvalue c. The Lemma says $c^2 = 1$, $c = \pm 1$. If $c = 1$ we are done. If $c = -1$, then consider the other two eigenvalues. They are either more copies of ± 1, or conjugate numbers of magnitude 1. In the second case the determinant of the transformation is -1, a contradiction. In the first case at least one of the remaining eigenvalues must be 1.

Since the eigenvalues of the 2×2 matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

are $e^{\pm i\theta}$, the same argument shows that we can read off the rotation angle from the eigenvalues as well. The only ambiguity is the orientation of the axis or, equivalently, the sign of θ. Each choice just amounts to choosing the orientation of the axis suitably.

If you are given a rotation matrix and asked to find its axis, you just look for eigenvectors for eigenvalue 1. If asked to find the angle of rotation, note that the trace of the matrix is equal to $1 + 2\cos\theta$. If asked to determine how to orient its axis, choose a vector perpendicular to the axis and see what happens to it under the rotation.