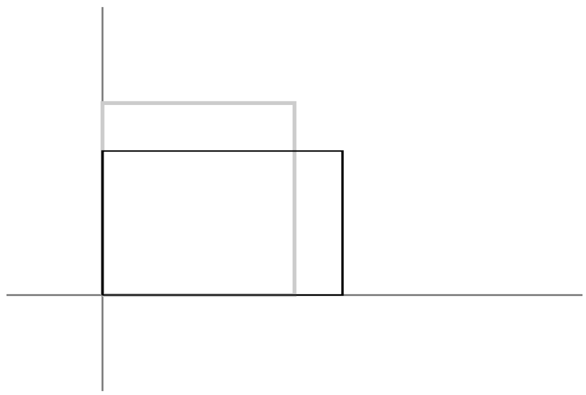
Mathematics 308—Fall 1996

More about scale changes

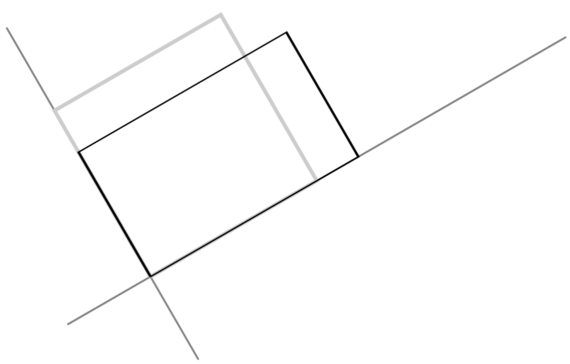
I shall do in this short note what I have already done for shears.

1. Scale changing along general axes

A scale change with respect to the x and y axes has this effect:



For scale changes with respect to arbitrary axes the situation is almost identical to what it was for shears.



Suppose we want to know how to apply a scale change of a and b along perpendicular vectors u and v. Let θ be the angle of u with respect to the x-axis. Then we can obtain our scale change by a combination of three simpler transformations: (1) we rotate by $-\theta$; (2) we change scale by a and b along the x and y axes; (3) we rotate back by θ. The matrix we get is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} .$$

The vector $(\cos\theta, \sin\theta)$ is scaled by a factor of a in this transformation, the vector $(-\sin\theta, \cos\theta)$ scaled by b. They are taken into multiples of themselves by the transformation, and are its **eigenvectors**. It is easy to see that the matrix we get above is symetric, and the theory of eigenvalues and eigenvectors for symmetric matrices asserts taht every symmetric matrix is the matrix of some scale change with respect to perpendicular axes.