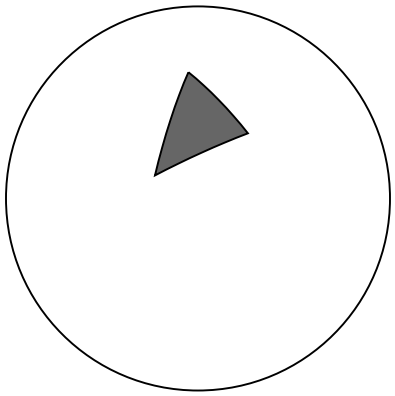
Mathematics 308—Fall 1996

Area on a sphere

These notes are concerned with two results about area on spheres. The first calculates the area of a triangle on the sphere in terms of the sum of its internal angles. The second uses this to show why the angles around a vertex on a convex polyhedron sum to less than 2π.

1. The area of spherical triangles

Suppose we are given a spherical triangle with interior angles α, β, γ. A few examples will suggest that the sum $\alpha + \beta + \gamma$ is always greater than π, which is what it would be if the triangle were in a plane.



In other words, if we call the **defect** of the triangle the quantity

$$D = \alpha + \beta + \gamma - \pi$$

then we expect that *D is greater than 0 for a spherical triangle*. There is in fact a simple interpretation of D which explains this phenomenon. Suppose that the sphere we are looking at has radius 1.

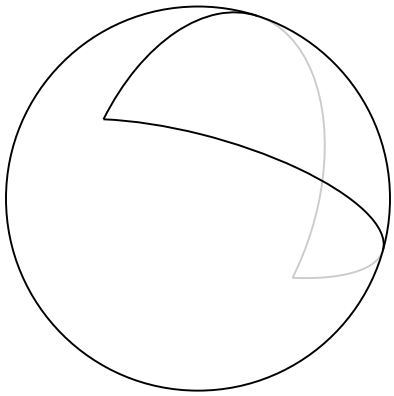
- *The defect of a spherical triangle on the unit sphere is equal to its area.*

This is a simple fact, and the proof is simple, too. The starting point, and in fact the only really tricky item, is that the area of the whole sphere is 4π. This step usually involves a limit process—i.e. calculus—and is in fact much more complicated than the rest of the argument.

The next step is to deduce from this a simple formula for the area of a sector, which is the region on the sphere made up of two great half-circles meeting in two antipodal points—for example the region on the Earth bounded by two longitudes and the two poles. The shape of a sector is completely determined by the interior angle at one end. If this angle is α, then the ratio of the area of the sector to that of the whole sphere is the same as the ratio of α to 2π. In other words, the area of a sector with interior angle α is

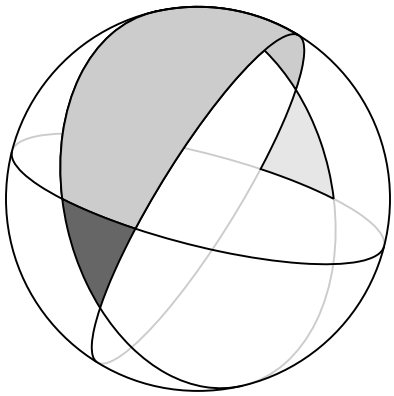
$$4\pi \frac{\alpha}{2\pi} = 2\alpha \ .$$

For example, a hemisphere is a sector with angle π, and has area 2π.



Let the triangle be Δ. Each corner of Δ is the interior angle of a sector, which is the union of Δ and a second region. Say the sector for angle α is Δ together with A, etc. Then as far as areas are concerned

$$A + \Delta = 2\alpha$$
$$B + \Delta = 2\beta$$
$$C + \Delta = 2\gamma$$



On the sphere, every region Ω has an antipodal region Ω'. This region has exactly the same shape as the original, and in particular the same area. The whole sphere is partitioned into the 8 regions

$$A,\ B,\ C,\ \Delta,\ A',\ B',\ C',\ \Delta'$$

so that in terms of areas

$$A + B + C + \Delta + A' + B' + C' + \Delta' = 4\pi$$

which since $A = A'$ etc means that

$$A + B + C + \Delta = 2\pi\ .$$

This is not quite obvious from the definition of this region, but it can be seen from the fact that Δ, A, B, and C' partition a hemisphere.

If we substitute the equations $A = 2\alpha - \Delta$ we see that

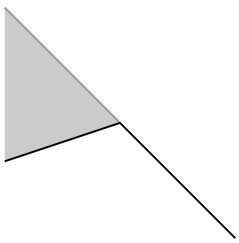
$$(2\alpha - \Delta) + (2\beta - \Delta) + (2\gamma - \Delta) + \Delta = 2(\alpha + \beta + \gamma - \Delta) = 2\pi$$

so that

$$\alpha + \beta + \gamma - \Delta = \pi$$
$$\alpha + \beta + \gamma - \pi = \Delta$$

which is what we wanted to prove.

The exterior angle at a vertex whose interior angle is α is $\pi - \alpha$, or the angle through which one turns in going clockwise around that vertex. It is also the angle through which we turn the normals to the sides when we go around the vertex. On the sphere, each side coming into a vertex lies on a spherical arc. The direction of traversal along the side determines an axis perpendicular to that arc according to the right-hand rule. The exterior angle is also the angle through which the axis turns as we pass from one side of the vertex to the other. We shall use this observation in the next section.



In the plane, when we travel around a triangle the total distance we turn is the sum of the exterior angles we pass and also 2π. In a spherical triangle the sum of the exterior angles is

$$(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 3\pi - (\alpha + \beta + \gamma) = 2\pi - \Delta \ .$$

This formulation remains valid for any simple polygon contained in a hemisphere. This can be seen by chopping the polygon up into triangles.

Exercise 1.1. *What is the correct formula if the sphere has radius R?*

Exercise 1.2. *The reasoning above assumes implicitly that the inside of Δ does not intersect its antipodal region Δ'. Why is that? Exhibit such a region and modify the proof to prove the theorem in that case also.*

2. The defect at the vertex of a convex polyhedron

Suppose that P is the vertex of a convex polyhedron and that touching it are faces $F_1, F_2, \ldots, F_n$ with angles α_i at P.

- *The sum of the α_i is less than 2π.*

This will follow from the result in the previous section! The defect

$$2\pi - \sum \alpha_i$$

will be interpreted as an area on the sphere.

Let ν_i be the unit normal to the face F_i, pointing out from it. The vector ν_i can be considered as a point on the unit sphere. Join each ν_i to ν_{i+1} by an arc on the sphere, with $\nu_{n+1} = \nu_1$. The angle between ν_i and ν_{i+1} is also the spherical distance between them, and the distance one turns the normal ν_i through in moving from F_i around onto F_{i+1}. The axis one makes this turn around is the edge leading out from P separating F_i from F_{i+1}. Note that this edge lies on both faces, and is by definition perpendicular to each of the normal vectors.

If Π is a plane through P with all of the convex polyhedron on one side, the sequence of points ν_i on the sphere will span a polygon contained in the hemisphere cut off by the plane parallel to Π through the origin.

Therefore we can apply the area formula from the previous section: the sum of the exterior angles at the vertices ν_i is less than 2π, and the defect is the area of the spherical polygon spanned by the ν_i. In order to conclude we need to know that the exterior angle at ν_i is the same as the angle α_i. But this follows from the observation that this angle is the angle between the axes defining the sides coming into that vertex.