Euclidean Geometry:

Conics Sections

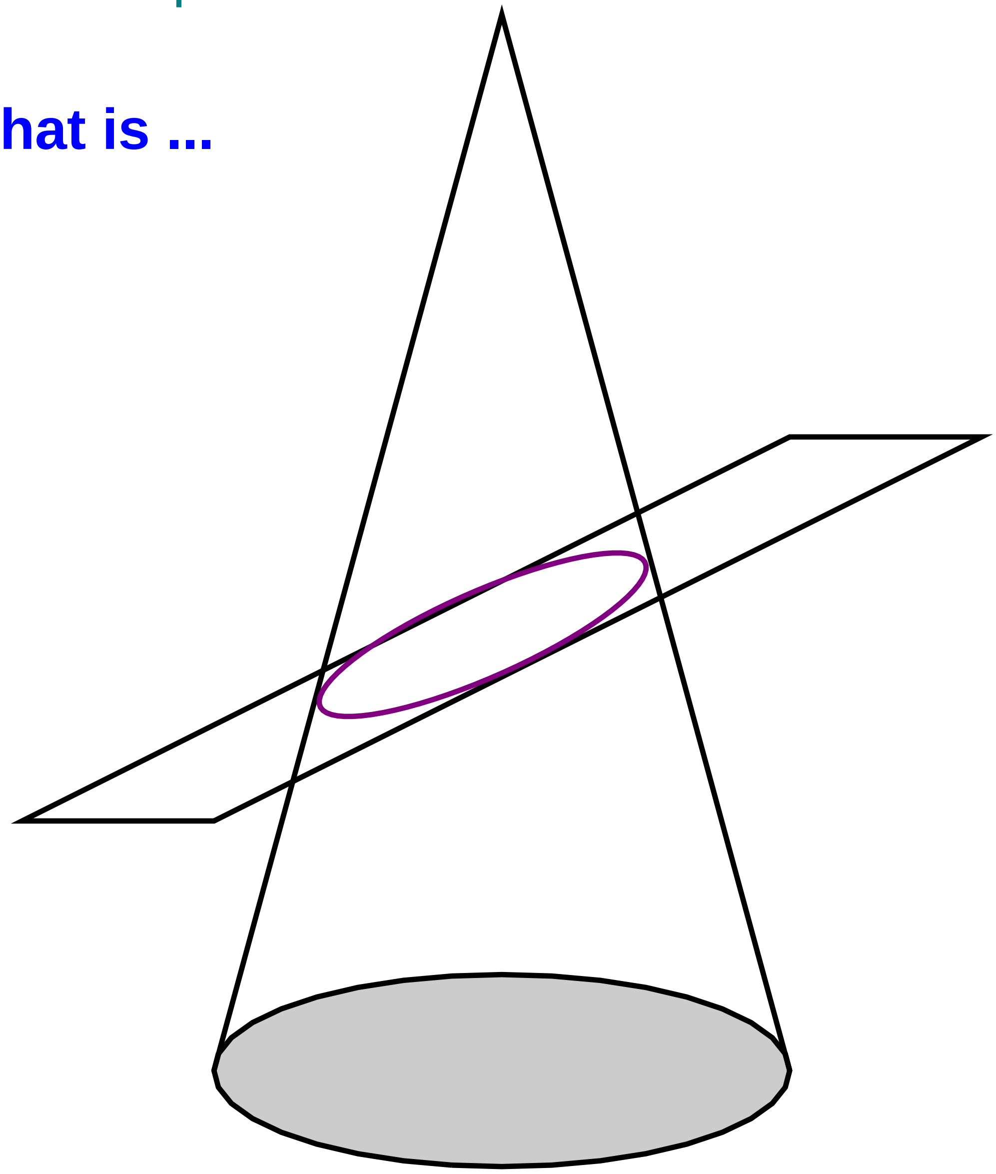
Designed by

Helen Yuen

Theorem.

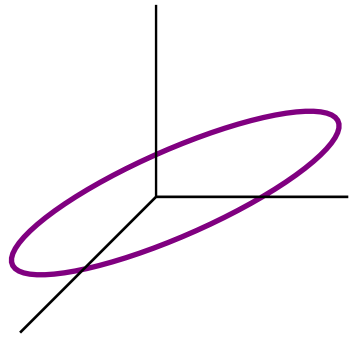
If a cone is cut with a plane which does not intersect the cone’s base, the intersection of the cone & plane is an ellipse.

That is ...



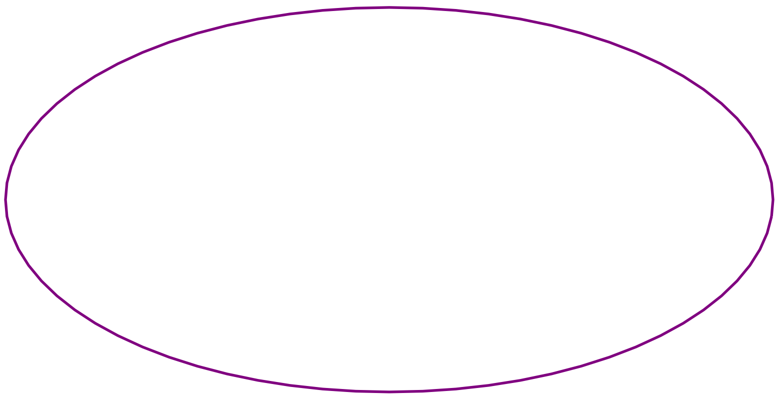
The problem is ...

It is not obvious that this circular shape is indeed an ellipse, because it is not obvious that it has symmetry.
The shape is not parallel to any axis.



Therefore,

we must first define ellipse.



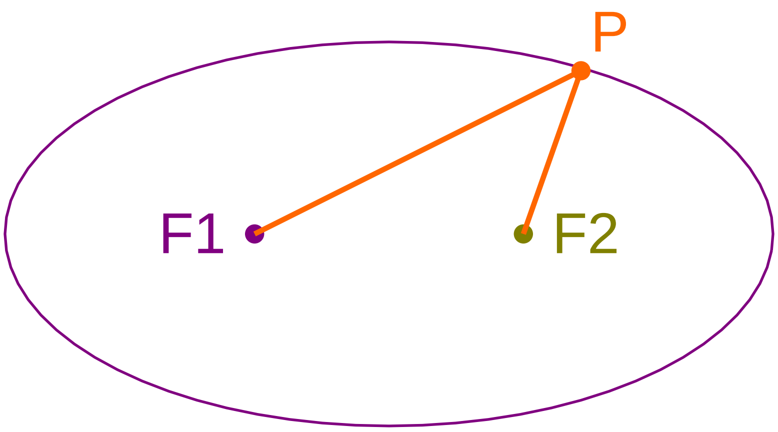
Definition of Ellipse.

Let P be any point on the ellipse.

Let d(x,y) be the distance between

point x and point y

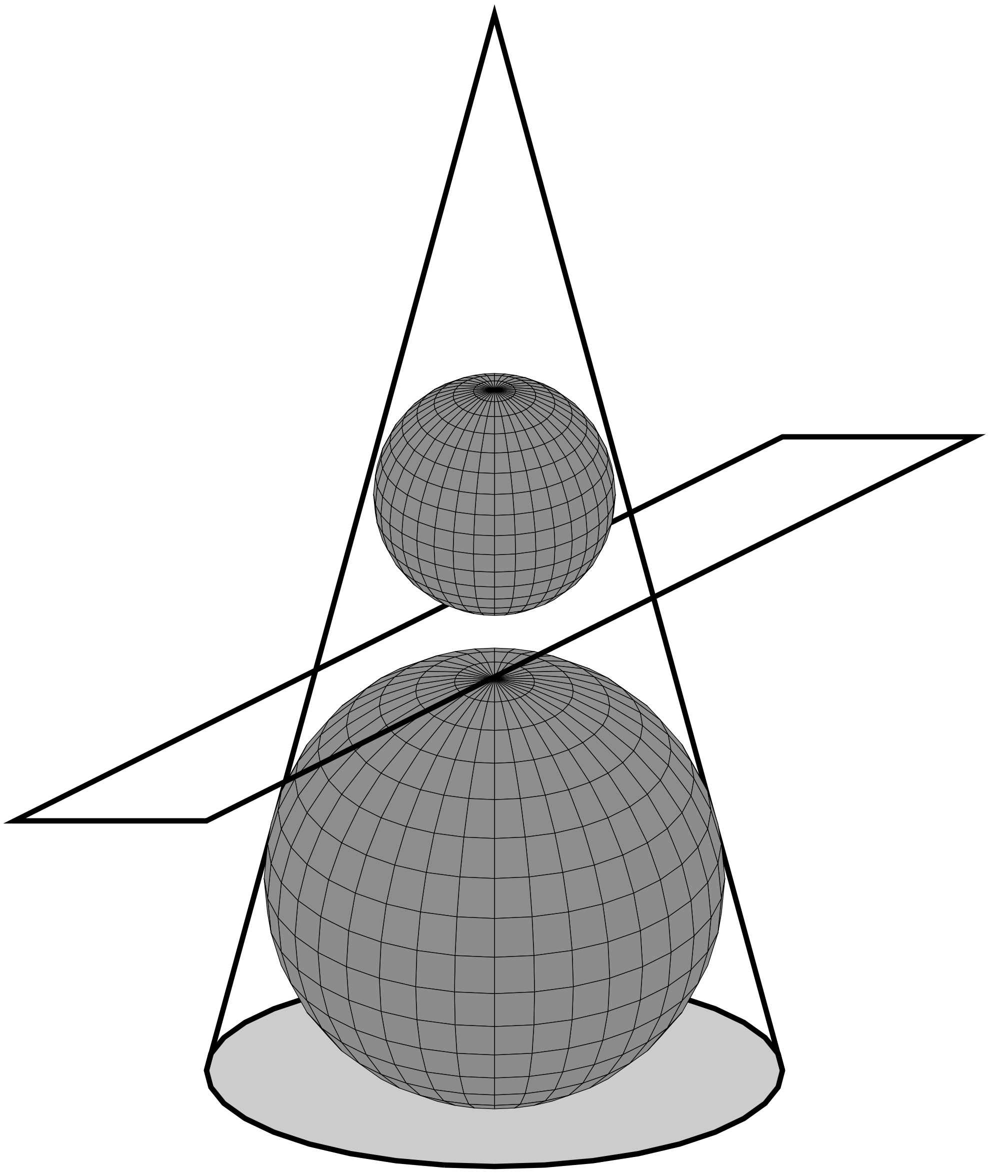
Then d(P,F1) + d(P,F2) = constant #



We’ll now introduce **Dandelin spheres**.

A Dandelin sphere is incribed in the cone.

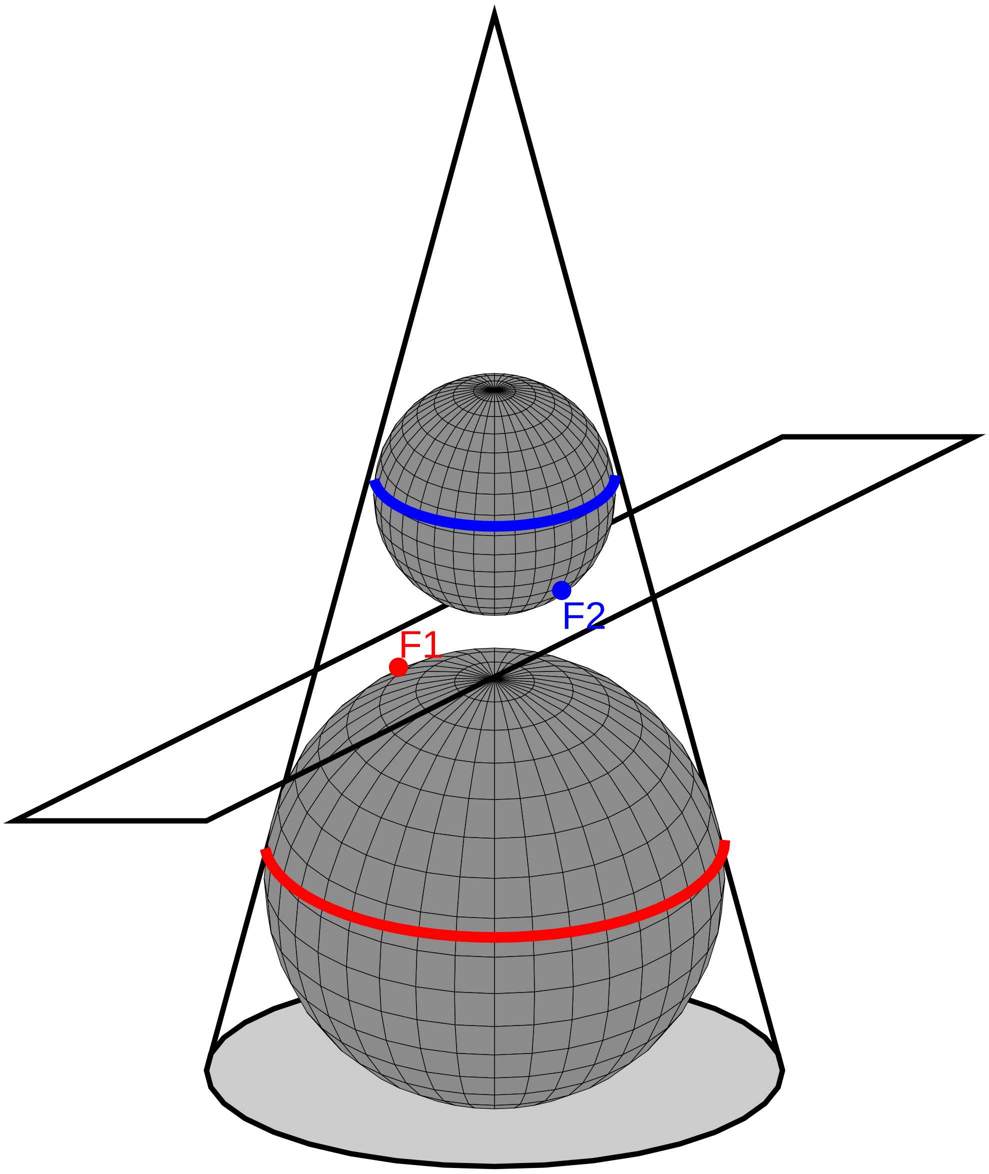
Each Dandelin sphere touches, but does not cross, both the plane and the cone.



That means ...

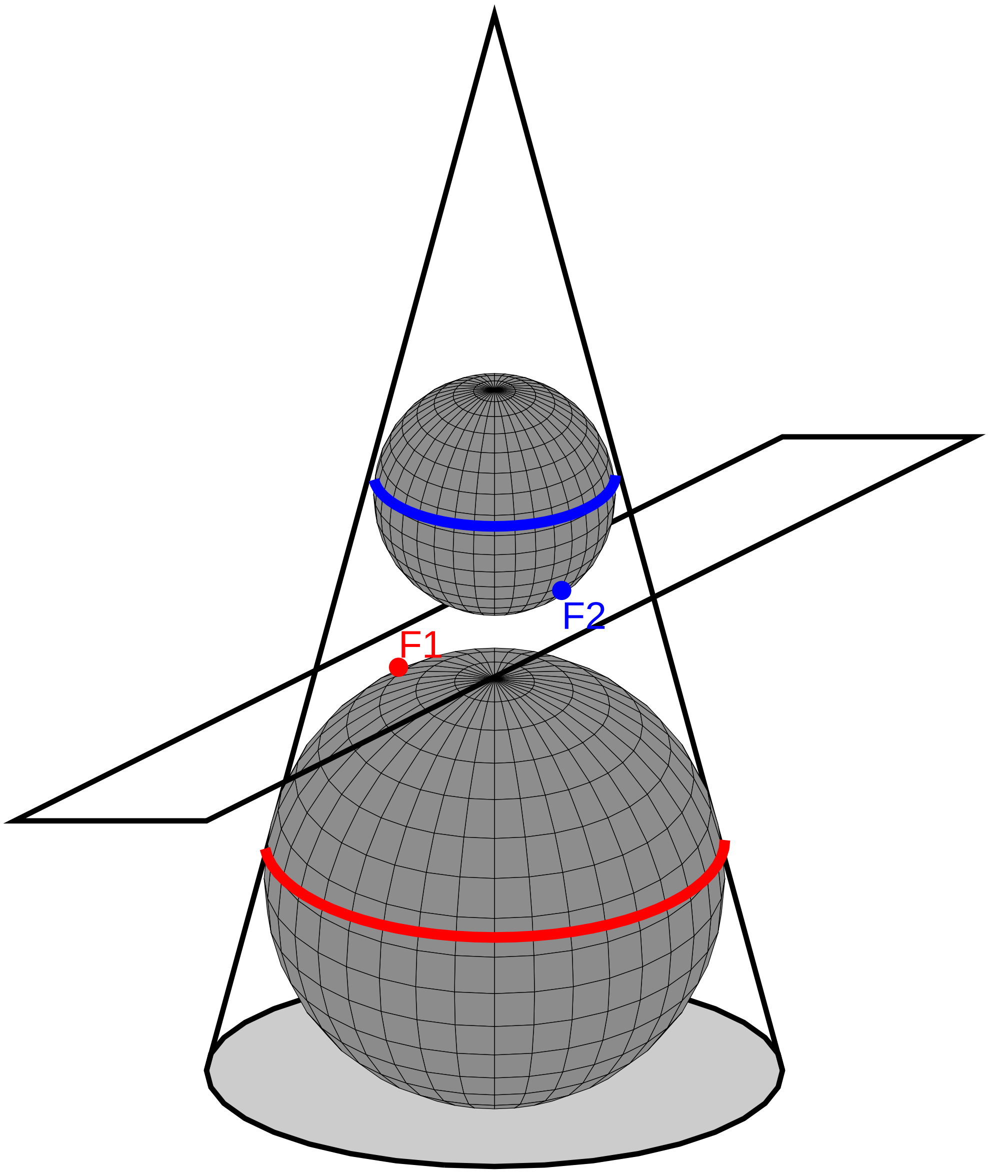
Each sphere touches the cone in a circle,

and touches the plane at a point.



The reason that we use Dandelin sphere ...

is because by Dandelin’s Theorem, each conic section has one Dandelin sphere for each focus.

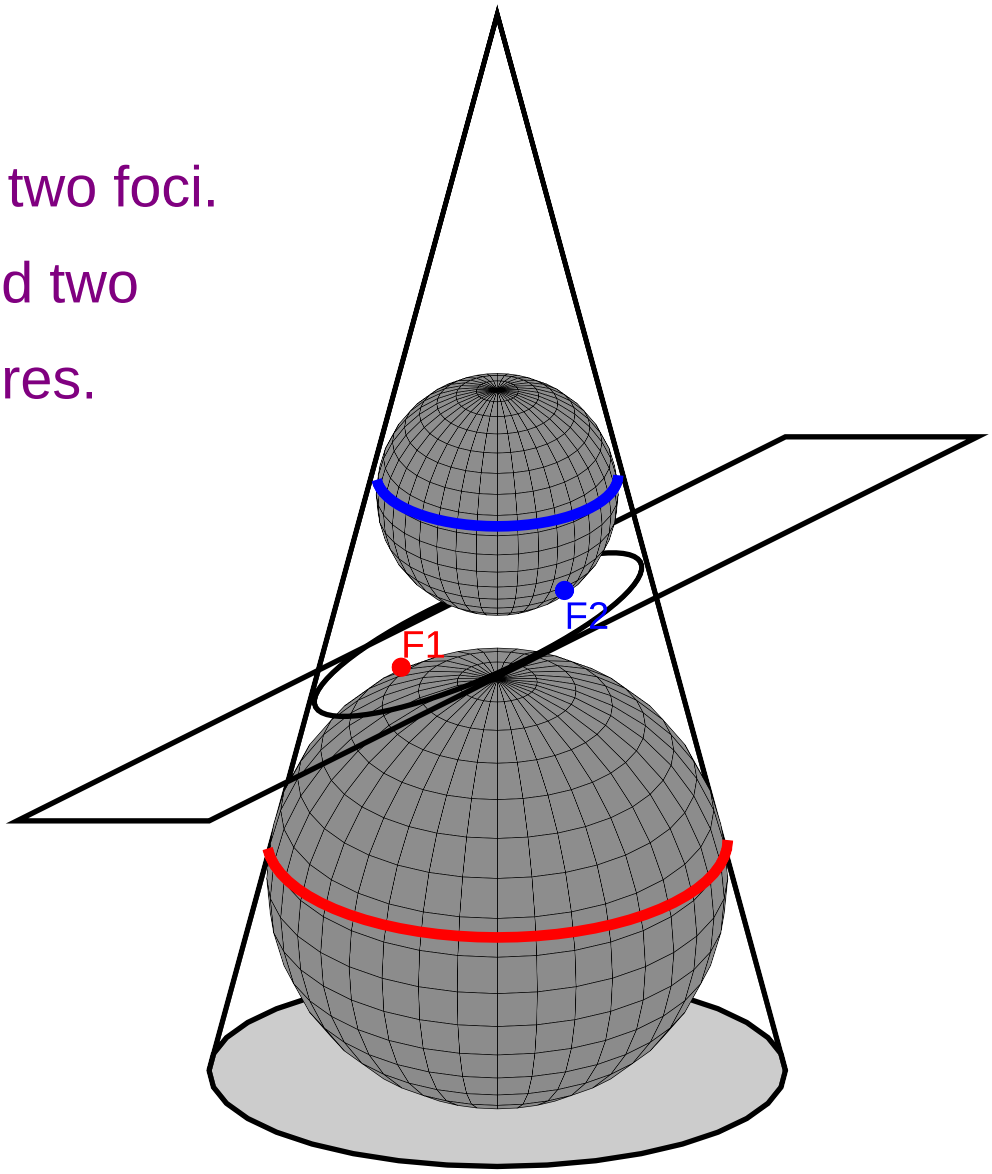


The reason that we use Dandelin sphere ...

is because by Dandelin’s Theorem, each conic section has one Dandelin sphere for each focus.

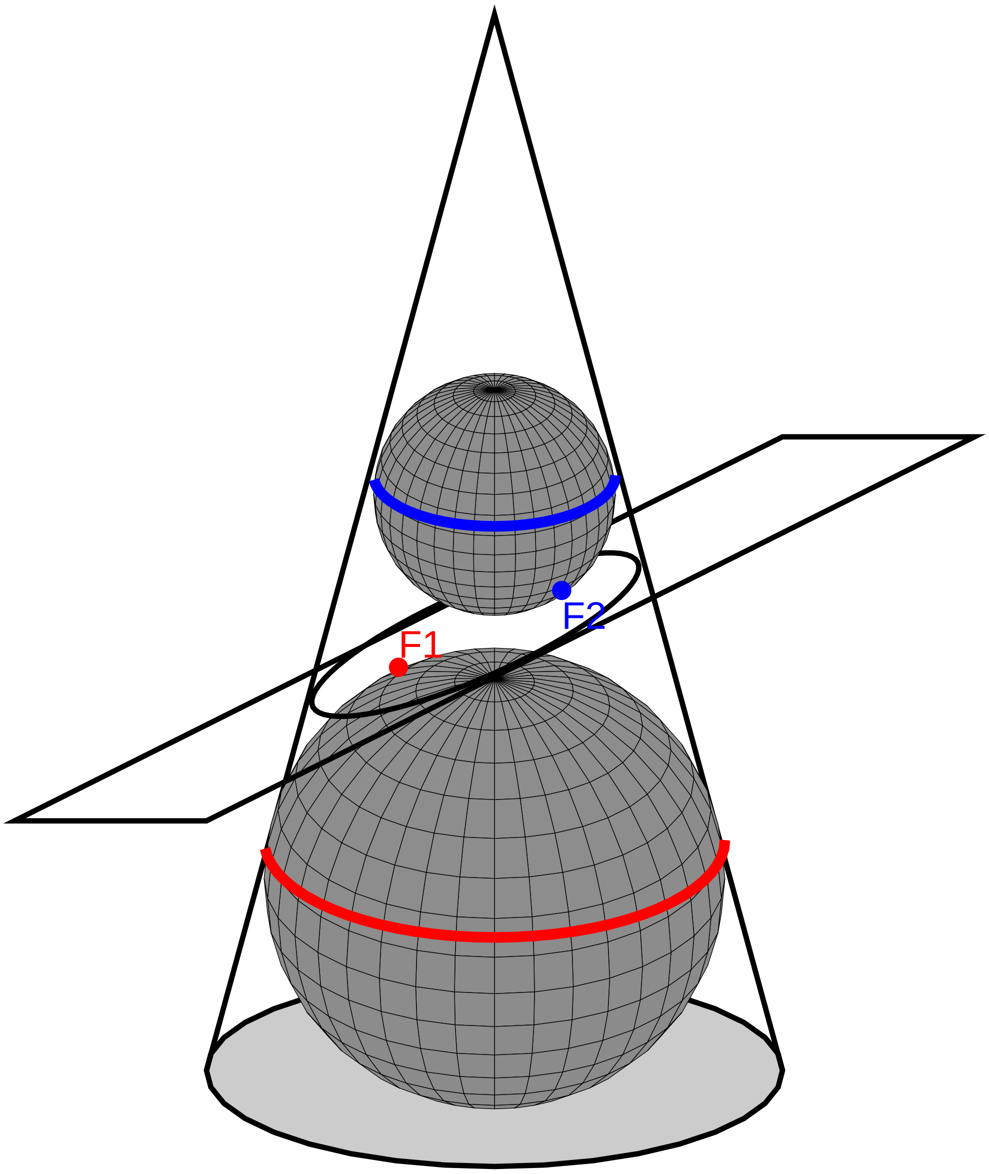
Recall.

An ellipse has two foci. Hence we need two Dandelin spheres.



Proof of the following theorem.

If a cone is cut with a plane which does not intersect the cone's base, the intersection of the cone and the plane is an ellipse.

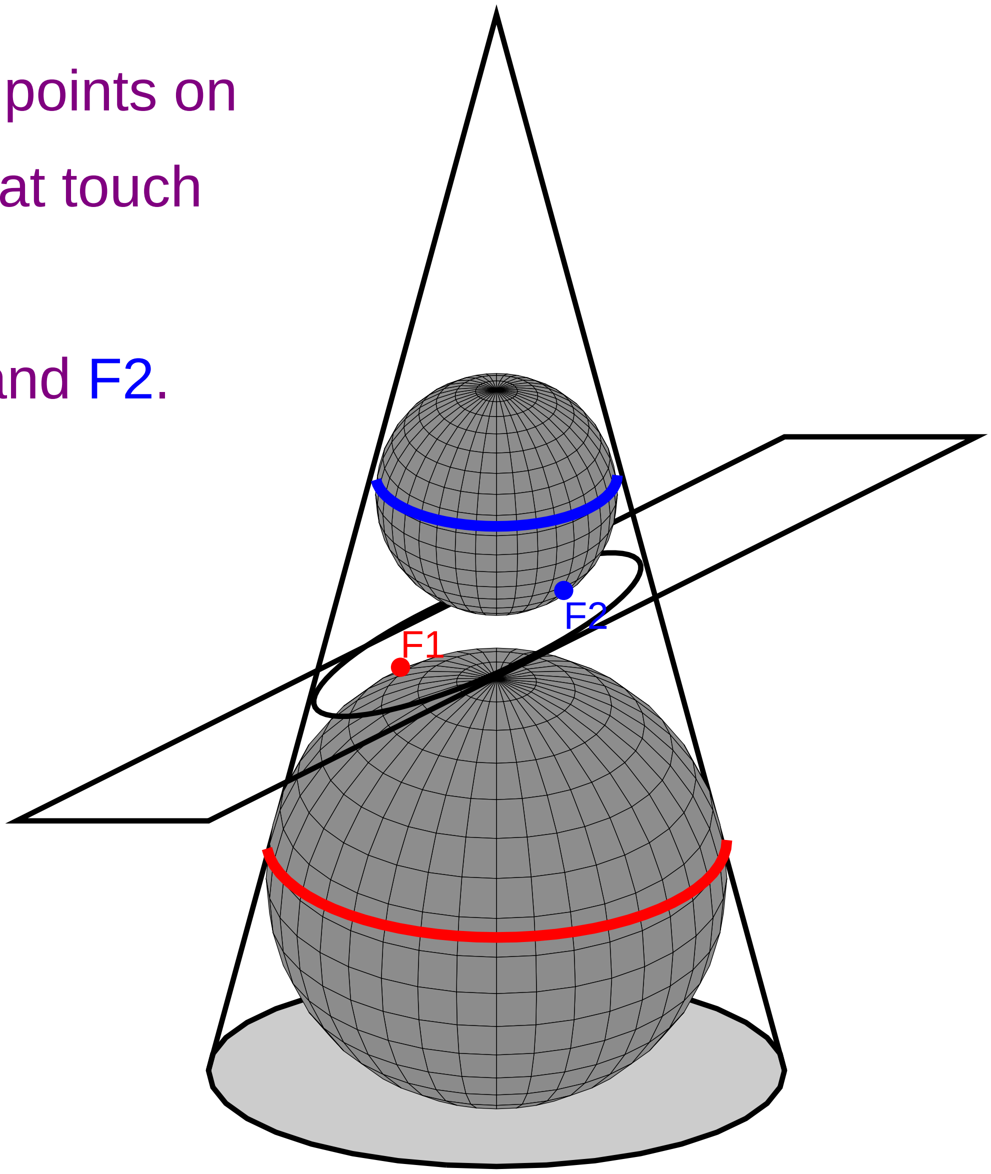


Proof of the following theorem.

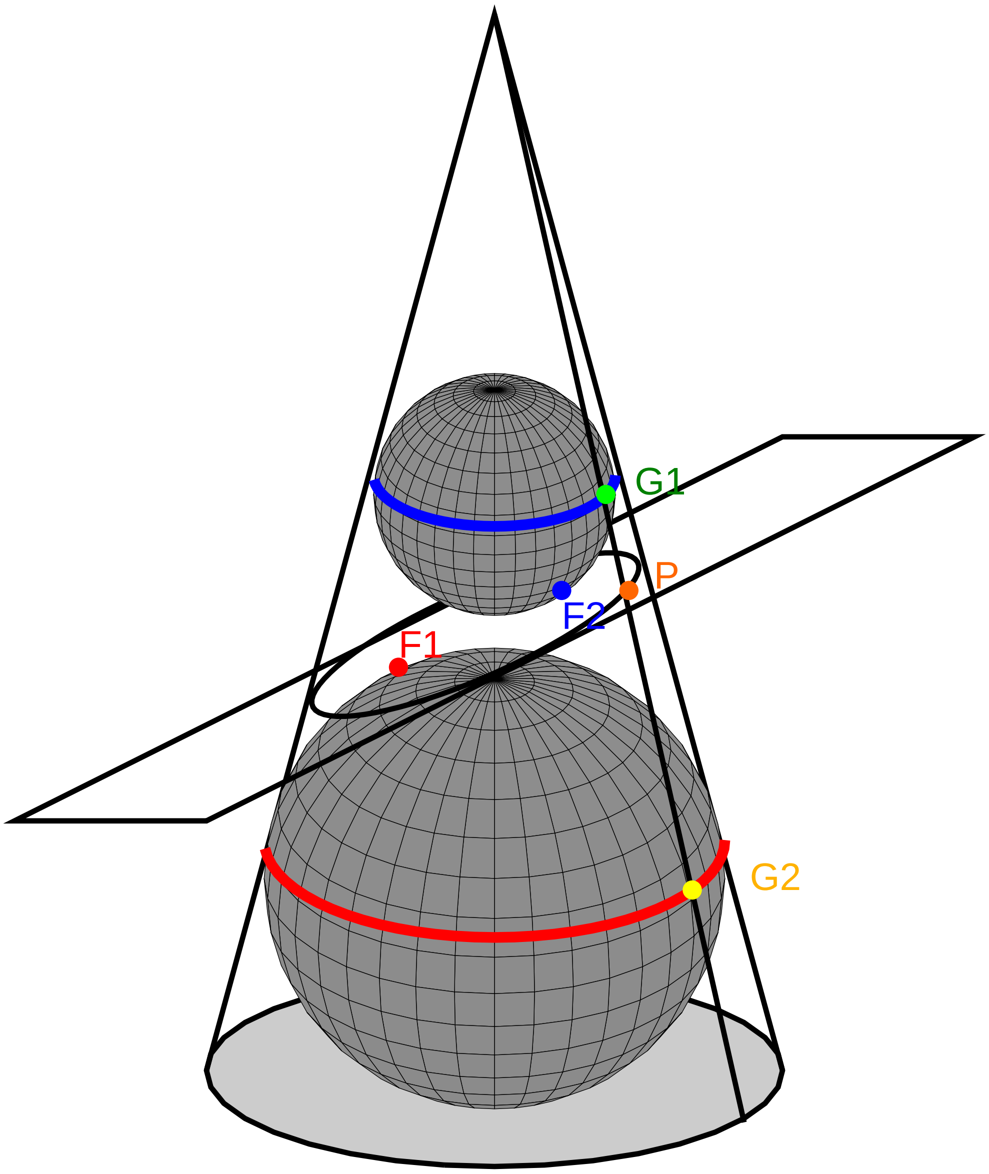
If a cone is cut with a plane which does not intersect the cone's base, the intersection of the cone and the plane is an ellipse.

There are two points on the spheres that touch the plane.

Call them F1 and F2.

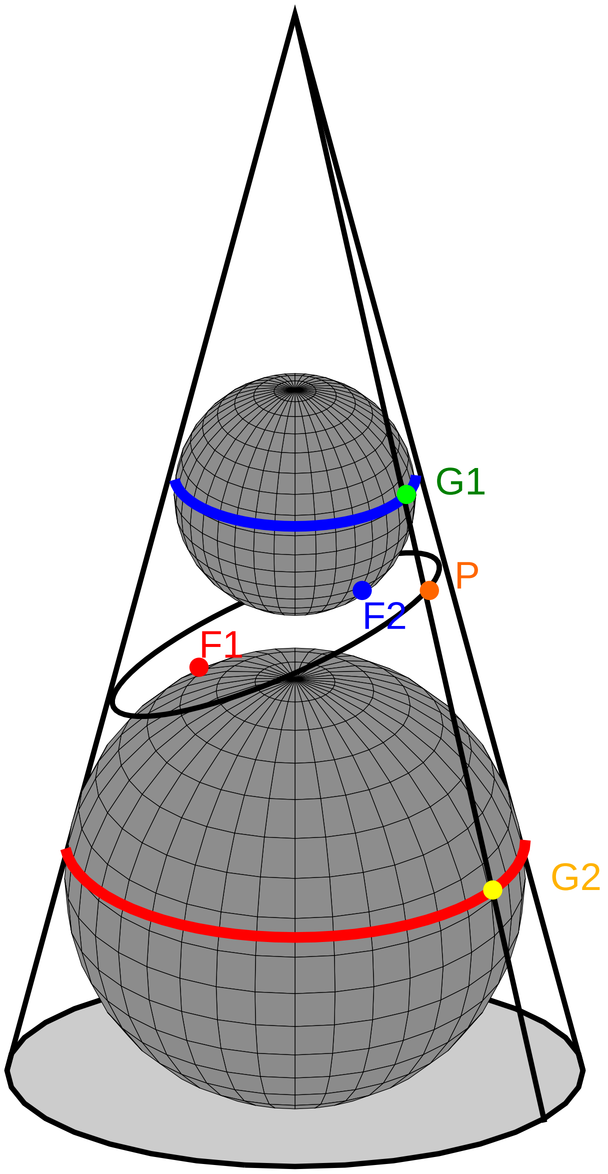


Let P be a typical point on the ellipse.
A line passing through P and the vertex of the cone intersects the two circles at points G1 and G2.



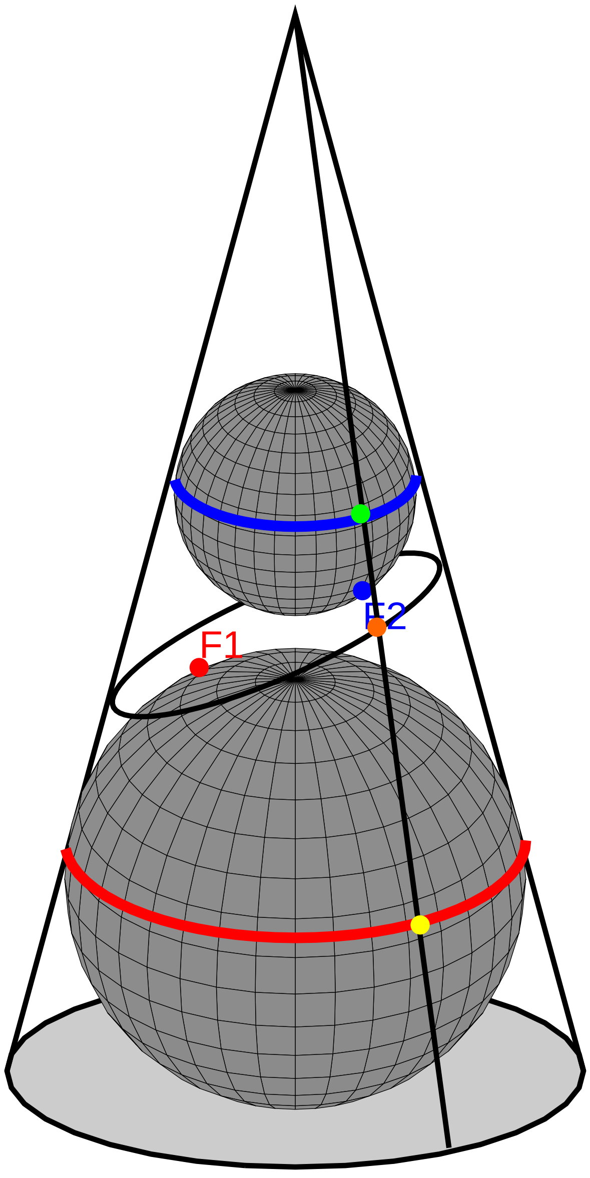
As P moves along the ellipse,

G1 and G2 move along the two circles.



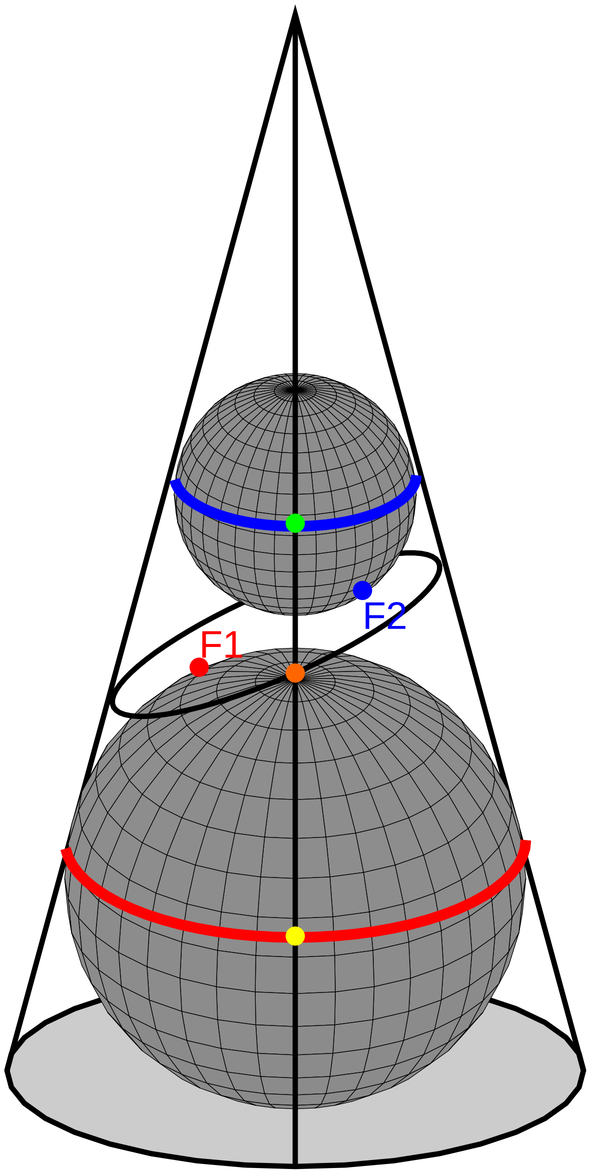
As P moves along the ellispe,

G1 and G2 move along the two circles.



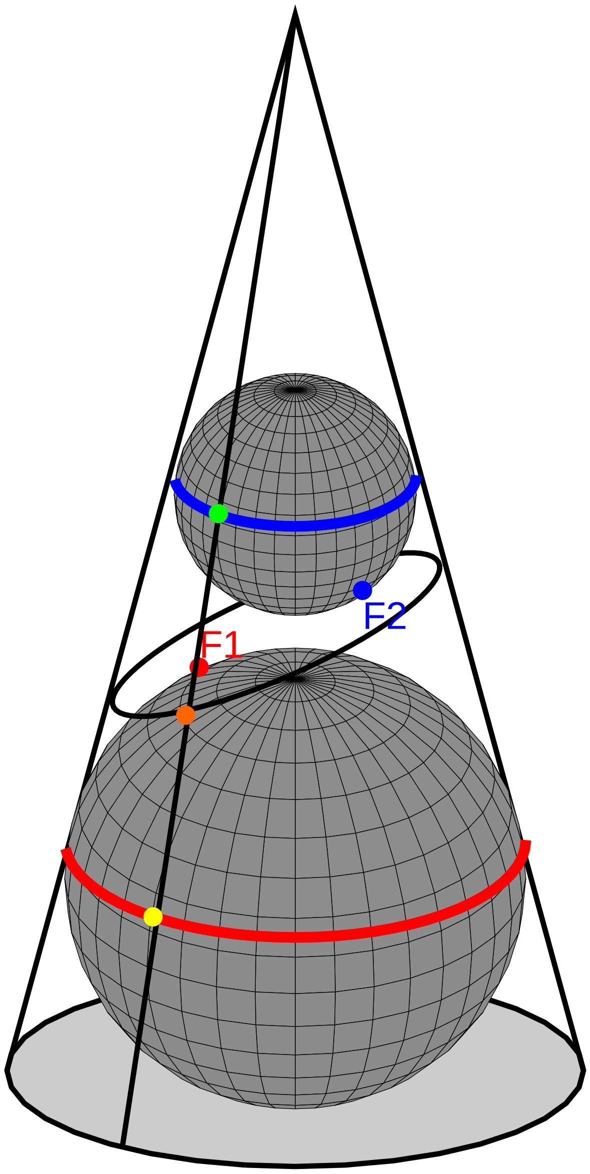
As P moves along the ellispe,

G1 and G2 move along the two circles.



As P moves along the ellispe,

G1 and G2 move along the two circles.



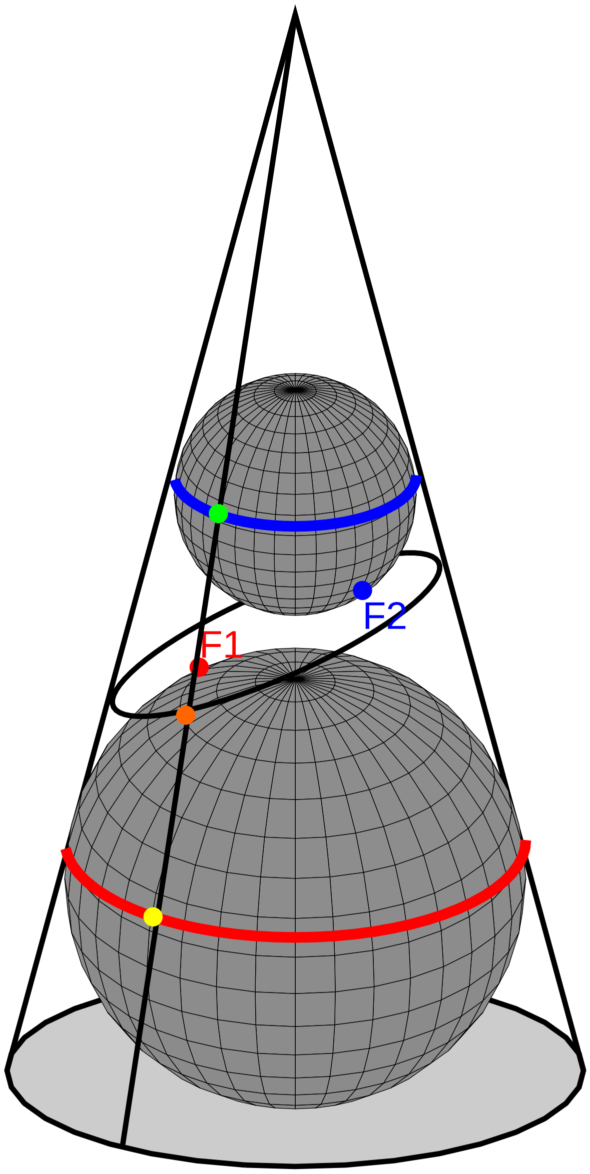
Notice that

d(G1,G2) = constant #

And recall that P is a typical point on the ellipse.

Therefore, the points G1, P, and G2

lies on a straight line.



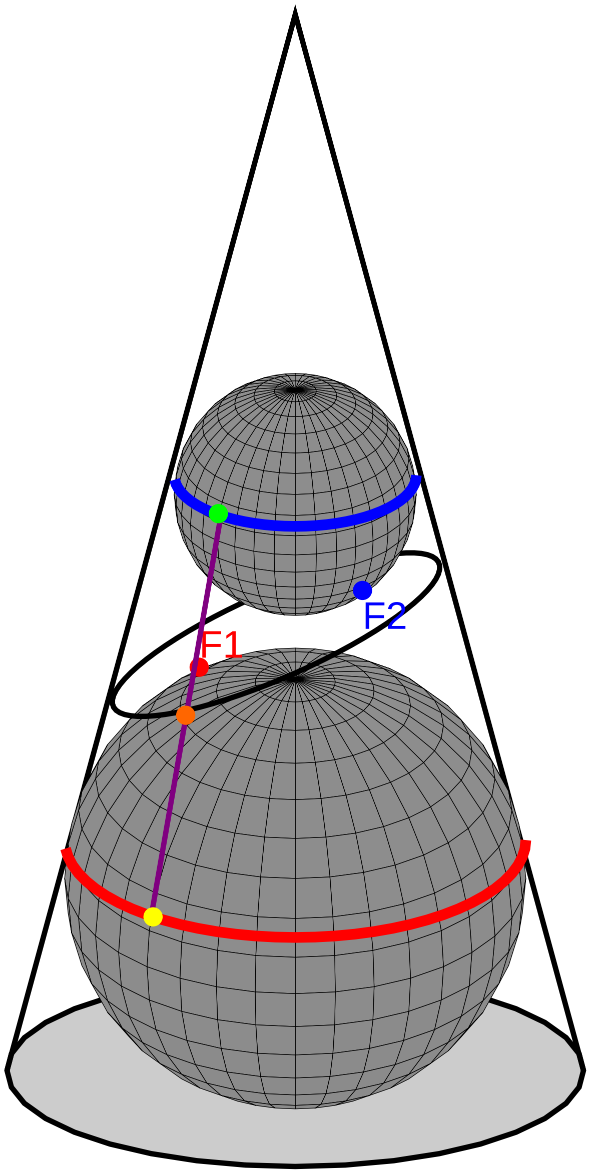
Notice that

d(G1,G2) = constant #

And recall that P is a typical point on the ellipse.

Therefore, the points G1, P, and G2

lies on a straight line.



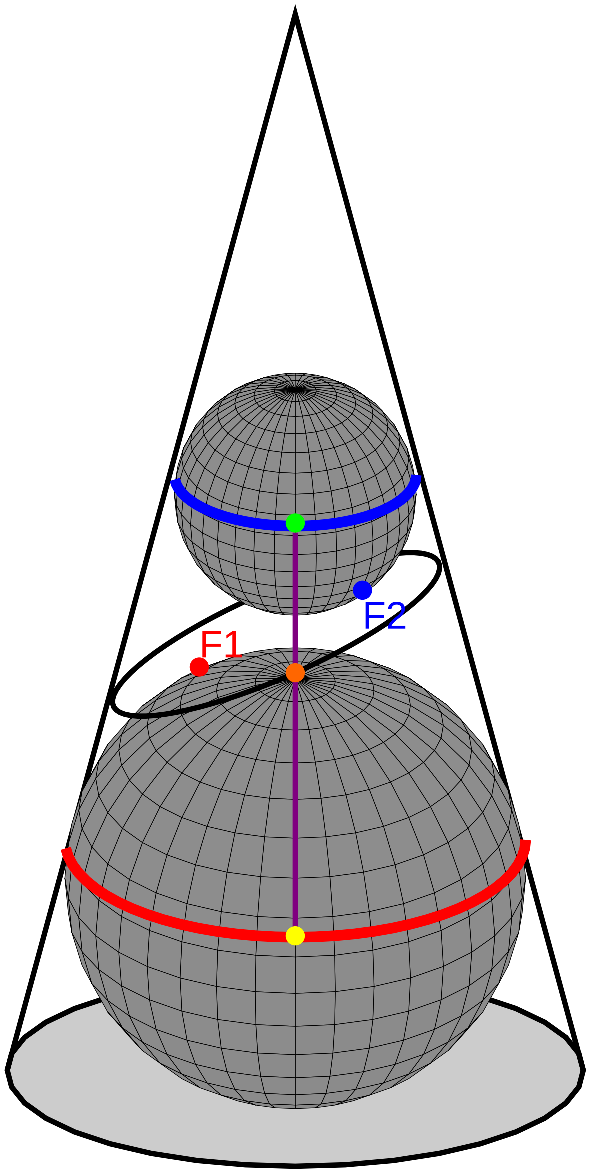
Notice that

d(G1,G2) = constant #

And recall that P is a typical point on the ellipse.

Therefore, the points G1, P, and G2

lies on a straight line.



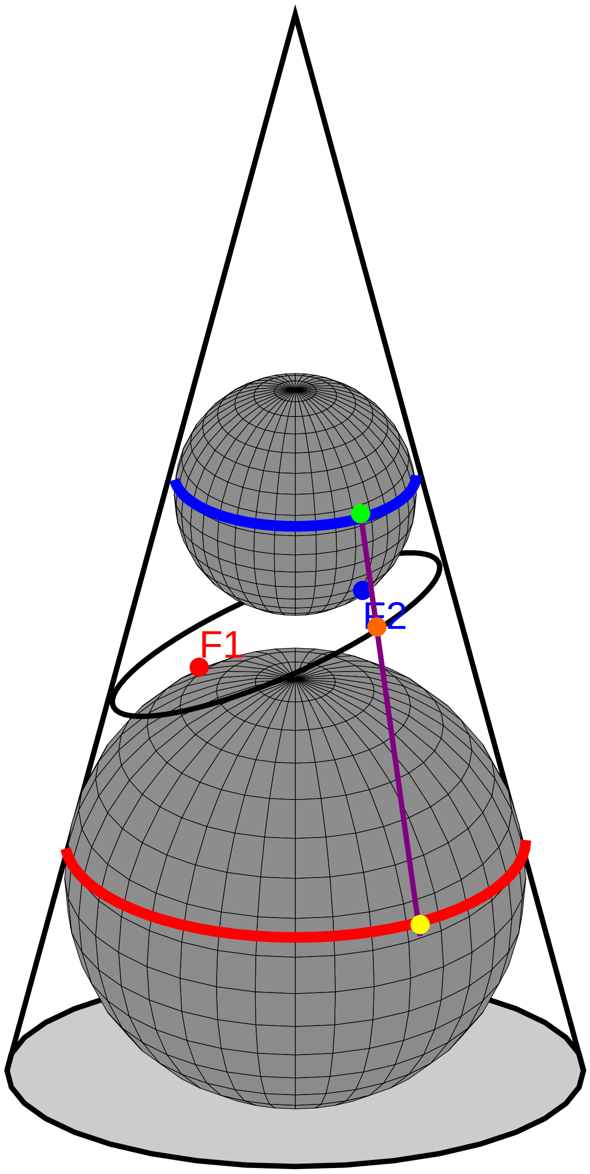
Notice that

d(G1,G2) = constant #

And recall that P is a typical point on the ellipse.

Therefore, the points G1, P, and G2

lies on a straight line.



Notice that

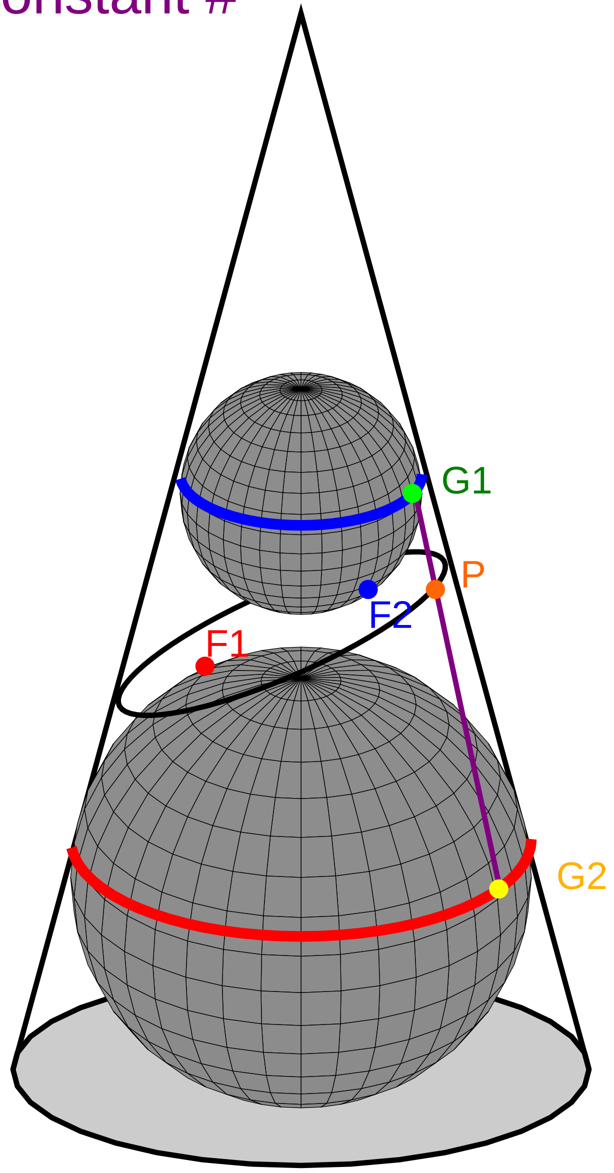
d(G1,G2) = constant #

Furthermore,

d(G1,G2) = d(G1,P) + d(G2,P)

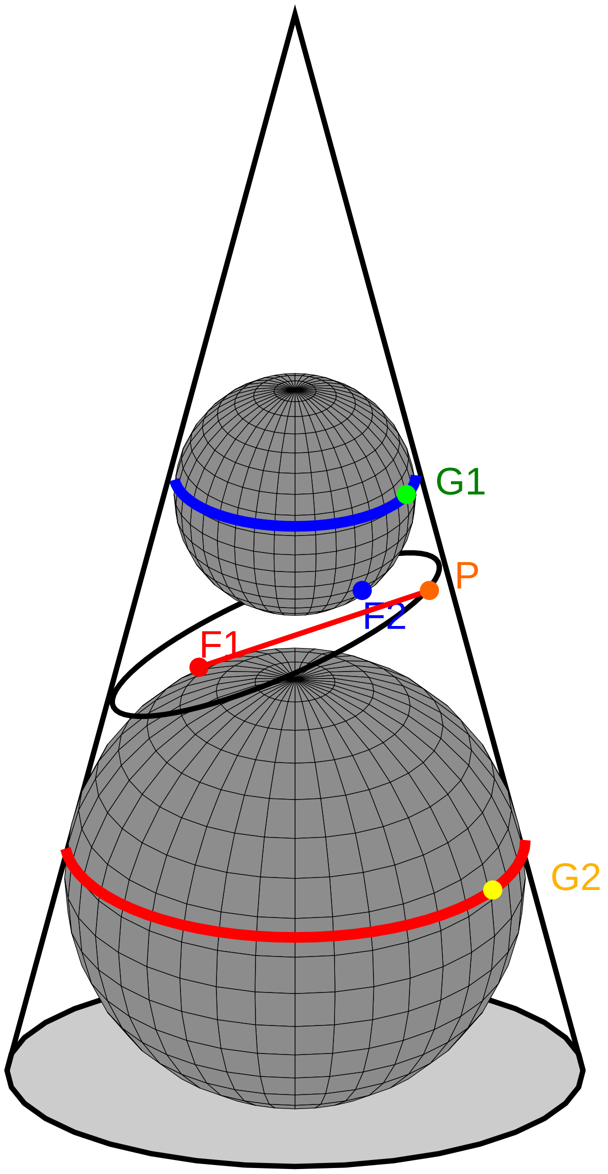
And therefore,

d(G1,P) + d(G2,P) = constant #



We can draw a line from the point F1 to the point P.

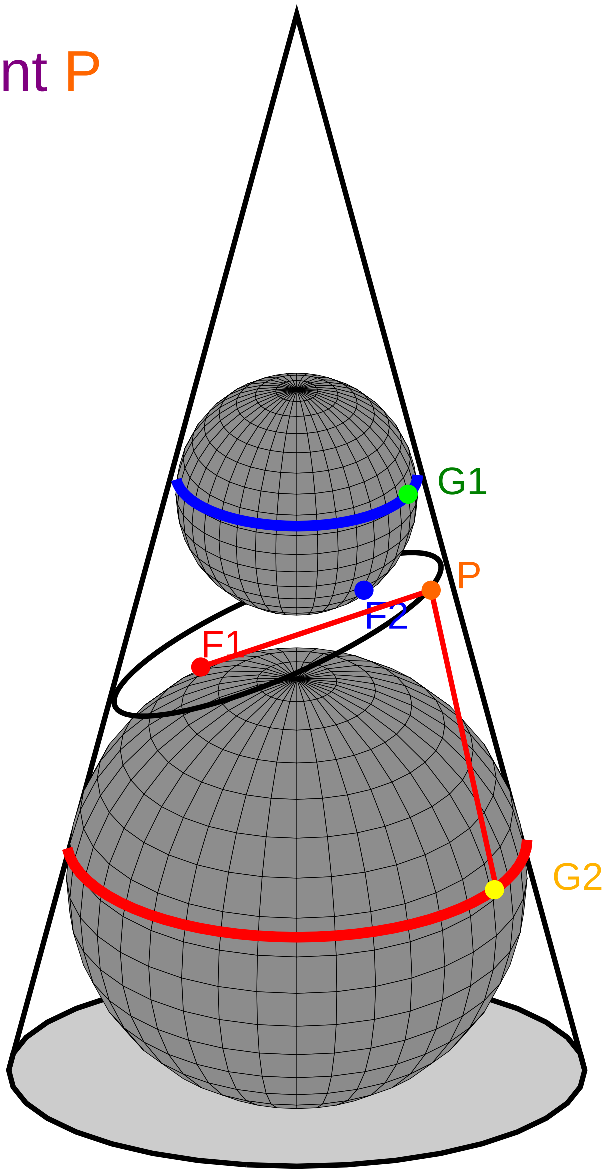
This line is tangent to the bottom large sphere.



We can draw a line from the point F1 to the point P.

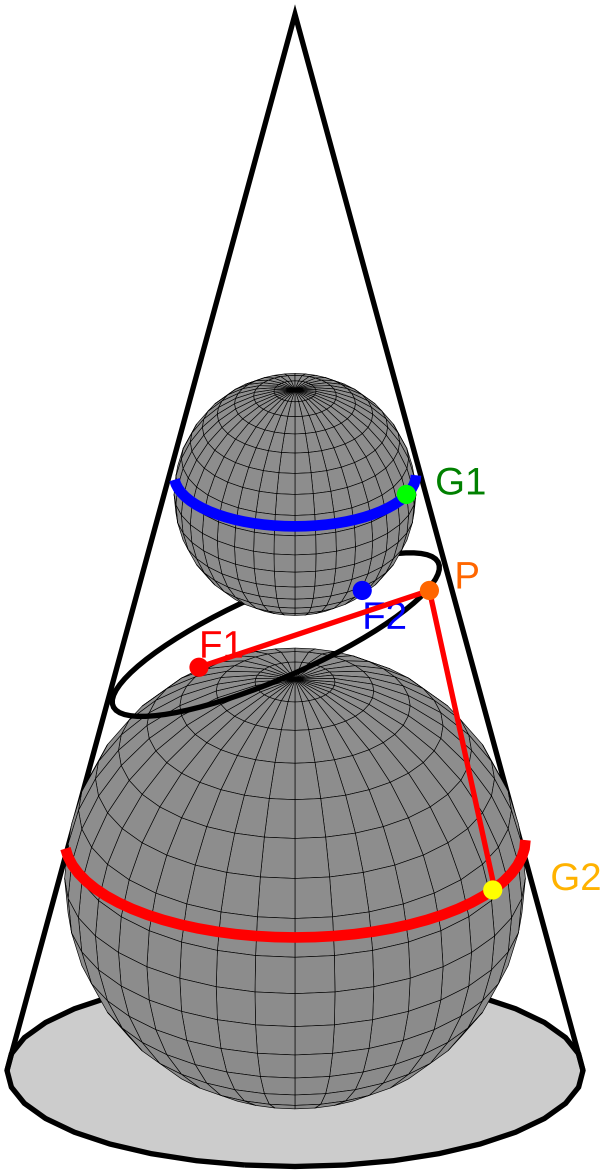
This line is tangent to the bottom large sphere.

Notice that the line from the point G2 to the point P is also tangent to the *same* sphere.



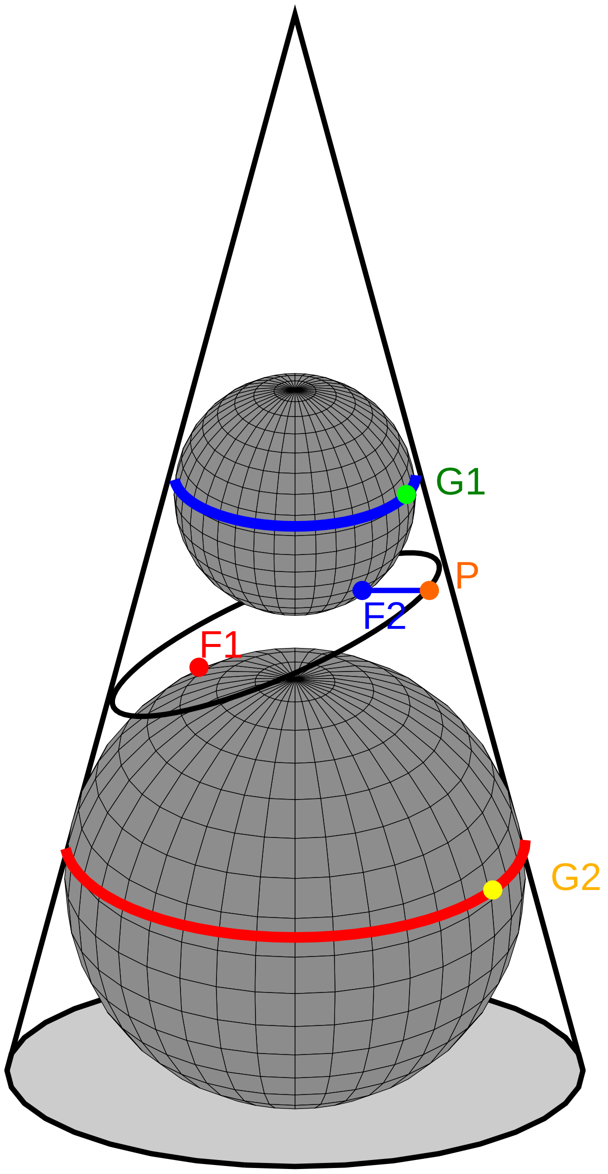
Since both F1 and G2 are tangent to the *same* sphere, and the two red lines intersect at point P, the following formula holds:

$$d(G2,P) = d(F1,P)$$



We can also draw a line from the point F2 to the point P.

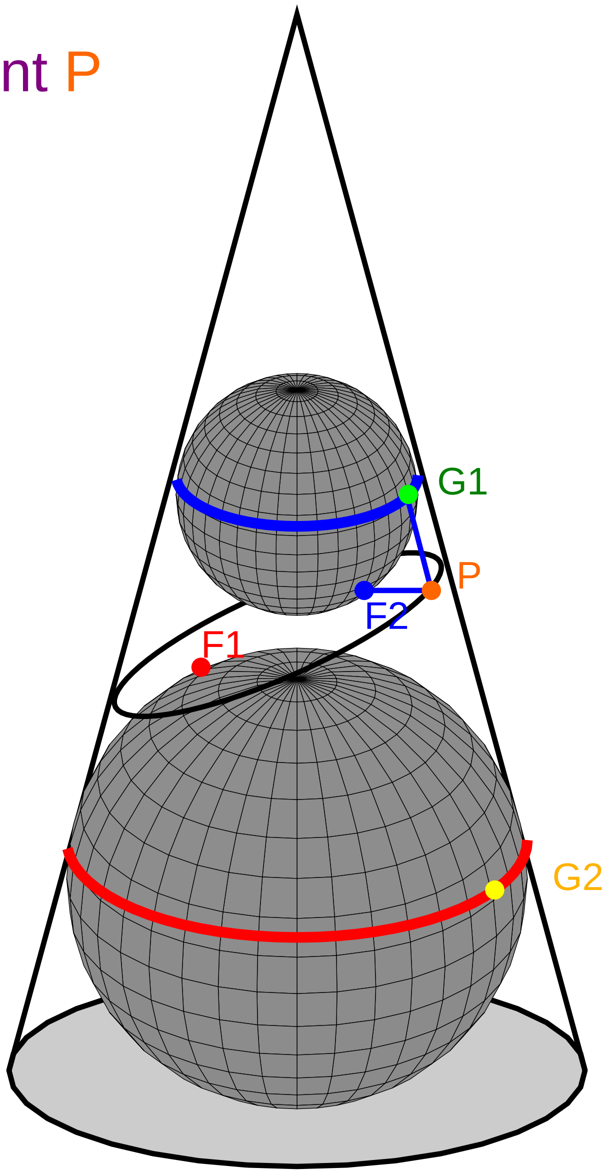
This line is tangent to the top small sphere.



We can also draw a line from the point F2 to the point P.

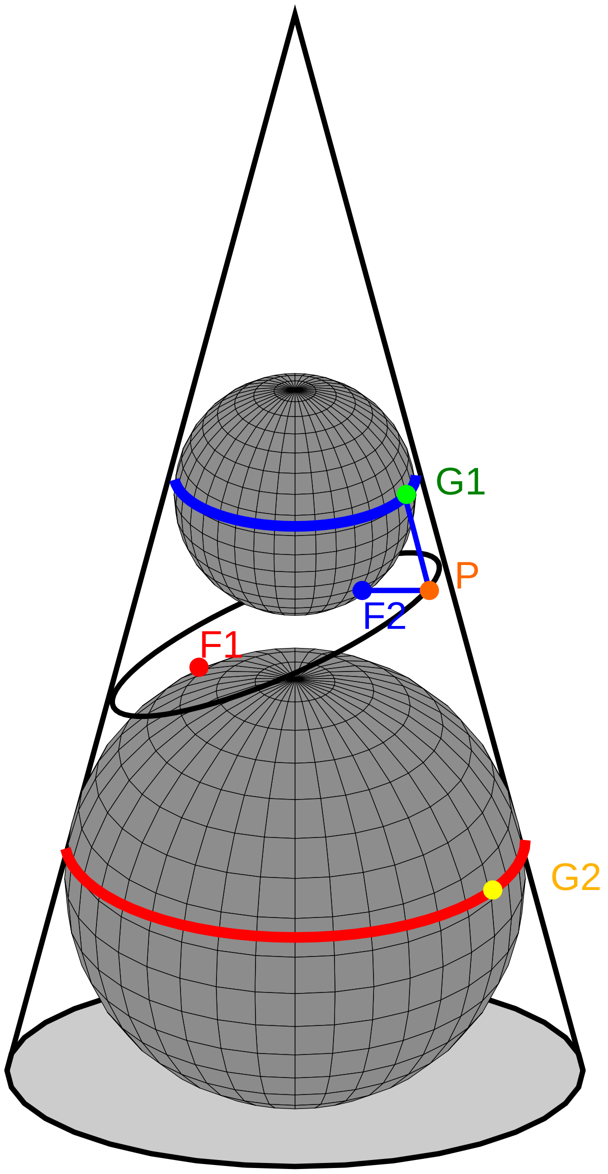
This line is tangent to the top small sphere.

Notice that the line from the point G1 to the point P is also tangent to the *same* sphere.



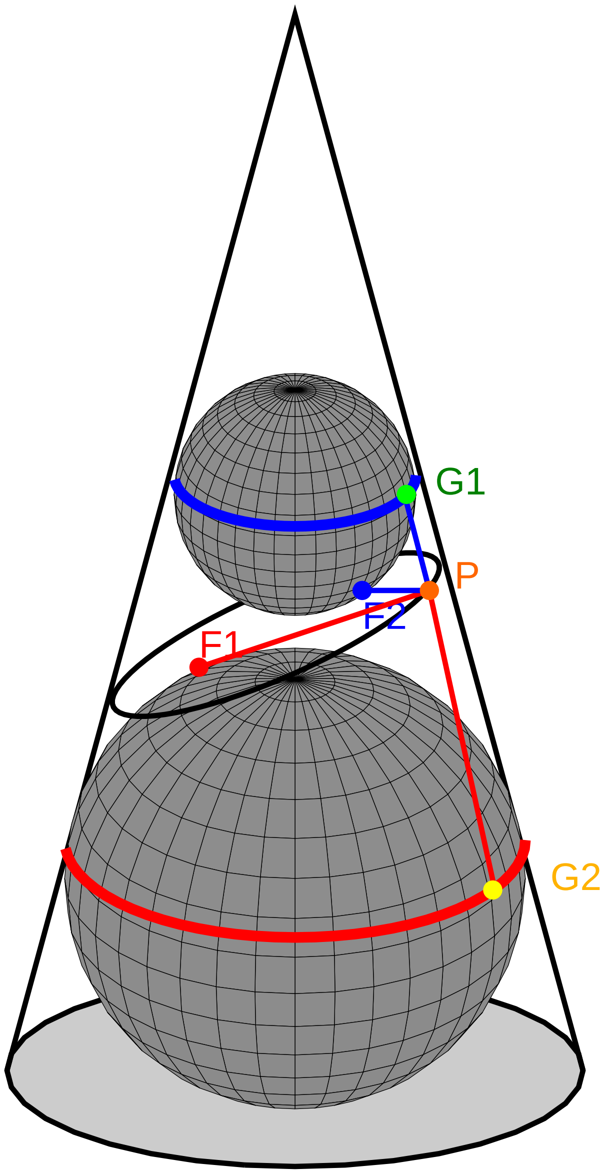
For the same reason as above, the following formula also holds:

$$d(G1,P) = d(F2,P)$$



In summary, $d(G2,P) = d(F1,P)$

$d(G1,P) = d(F2,P)$



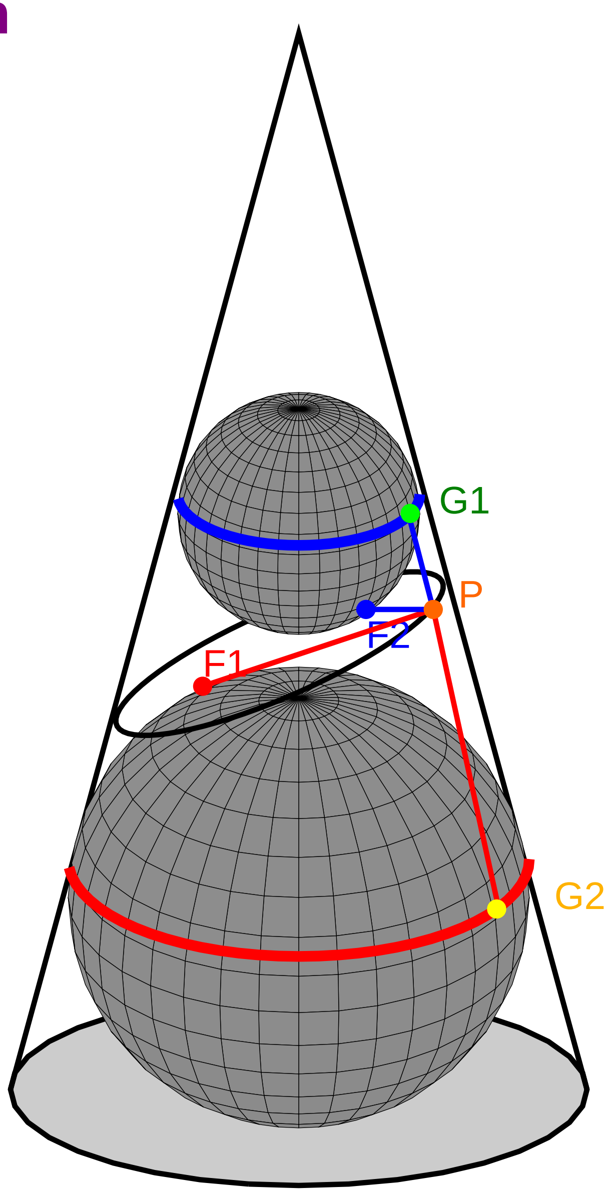
In summary, $d(G2,P) = d(F1,P)$

$d(G1,P) = d(F2,P)$

Since $d(G1,P) + d(G2,P) = \text{constant}$ #

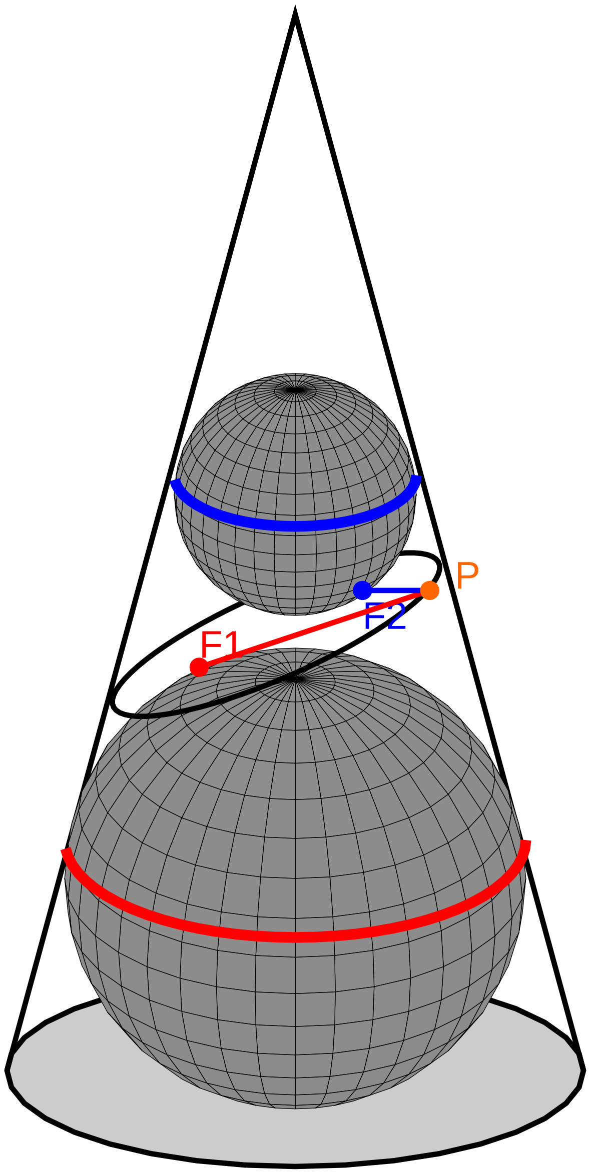
Therefore $d(F1,P) + d(F2,P) = \text{constant}$ #

which is the **definition** of ellipse.



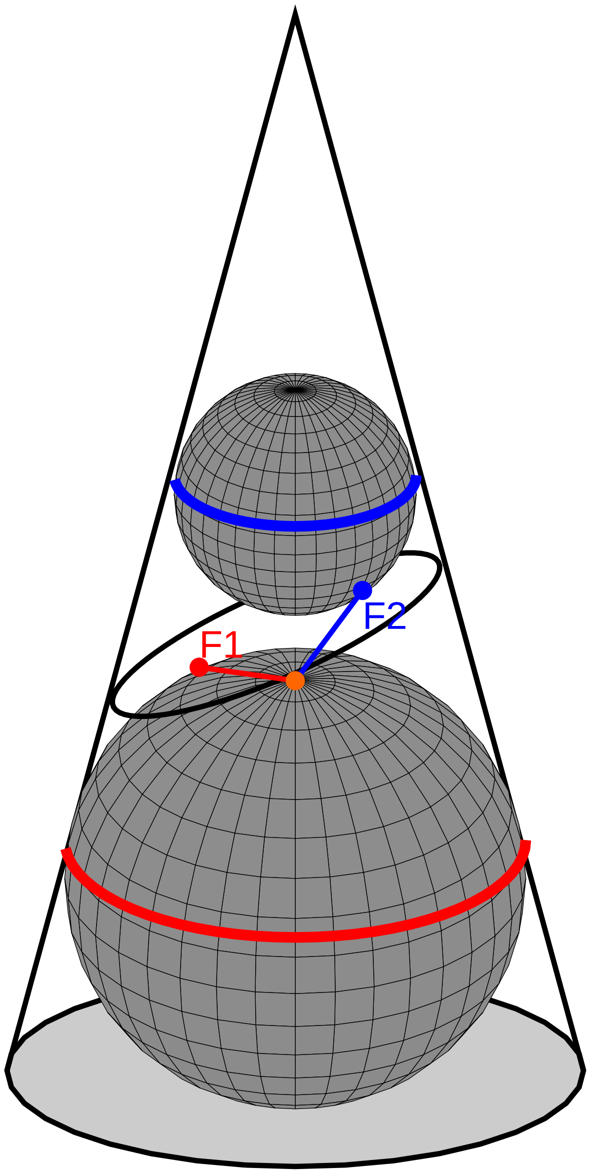
Therefore, the intersection of a plane with a cone is indeed an **ellipse**.

End of Proof ~



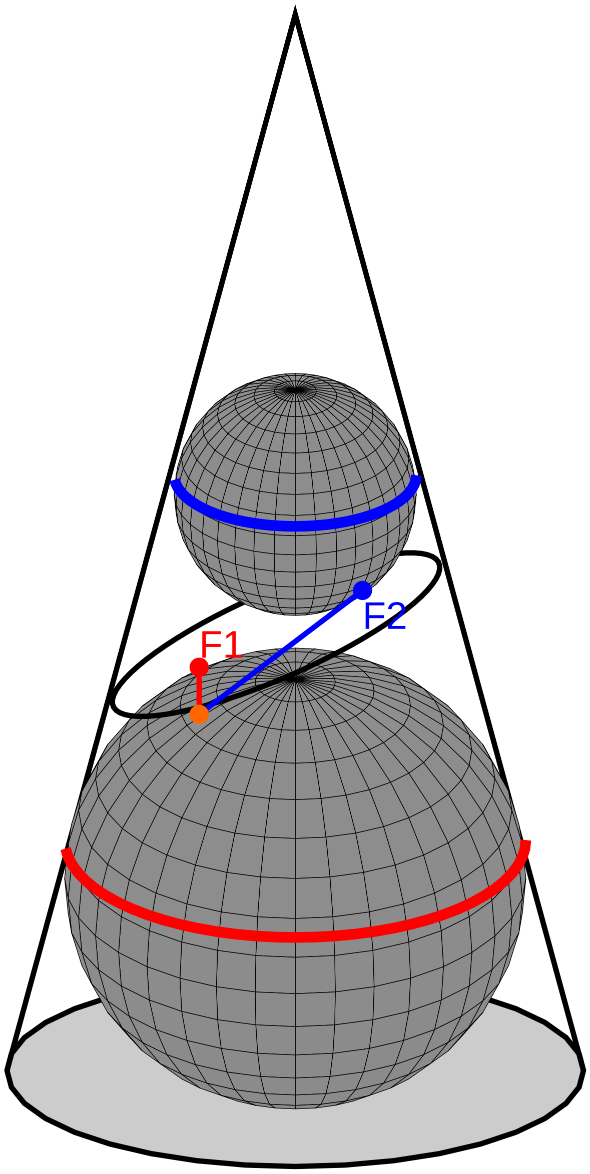
Therefore, the intersection of a plane with a cone is indeed an **ellipse**.

End of Proof ~



Therefore, the intersection of a plane with a cone is indeed an **ellipse**.

End of Proof ~



References

References

http://www.andrews.edu/~calkins/math/webtexts
/geom09.htm

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

Designed by

Helen Yuen

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

Designed by

Helen Yuen

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

Designed by

Helen Yuen

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

Designed by

Helen Yuen

References

http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

http://en.wikipedia.org/wiki/Conic-section

http://en.wikipedia.org/wiki/Dandelin_spheres

http://en.wikipedia.org/wiki/Ellipse

http://www.maths.gla.ac.uk/~wws/cabripages/cone.html

Designed by

Helen Yuen