Schwarz’s Triangle Problem

(a.k.a. Fagnano’s Problem)

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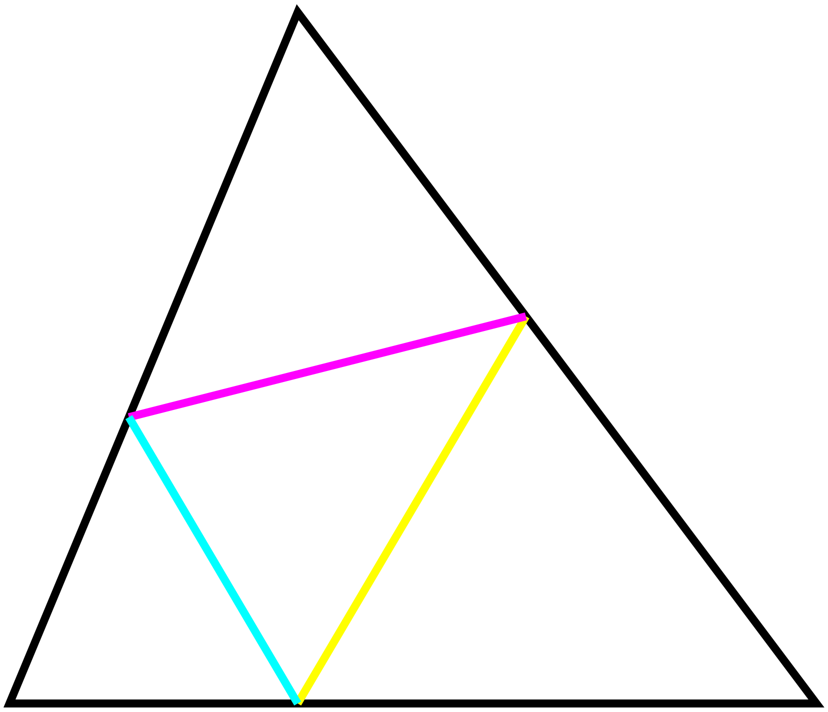
In a given acute-angled triangle, inscribe another triangle with the smallest perimeter.

The History:

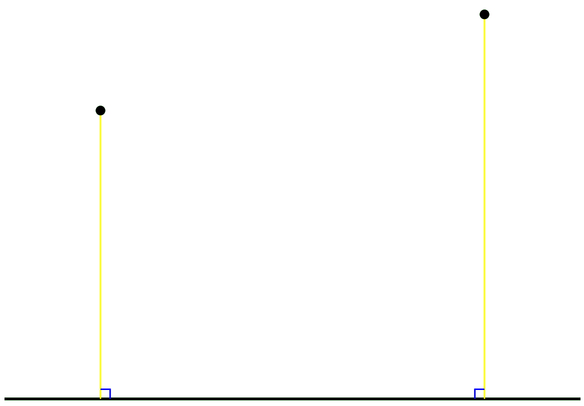
Although this problem was proposed and solved by Fagnano in 1775, His solution used calculus. Since then, several other purely geometric solutions were also discovered. The most famous one of them all was found by Hermann Amandus Schwarz, the same person who brought us the famous Cauchy-Schwarz inequality. This theorem is also called the Schwarz Triangle Problem because the solution only utilizes elementary geometry that can be extended from Euclid’s theorems without much difficulties.



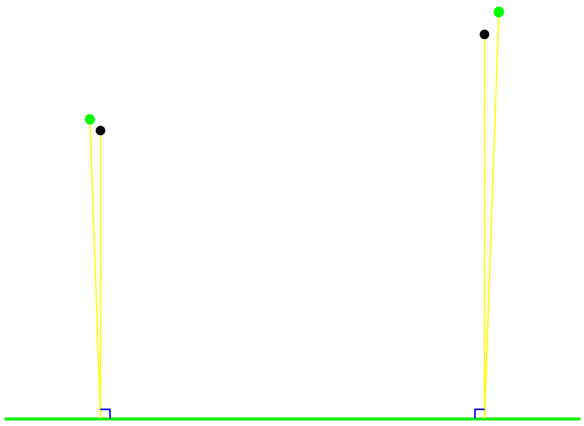
Schwarz used mainly reflection to solve the problem,

so let’s talk about reflection first:

Given a straight line and two points on the same side of the line, find a point on the line with the smallest sum of distances from the point to the other two points.



First, we "reflect" the diagram along the line.



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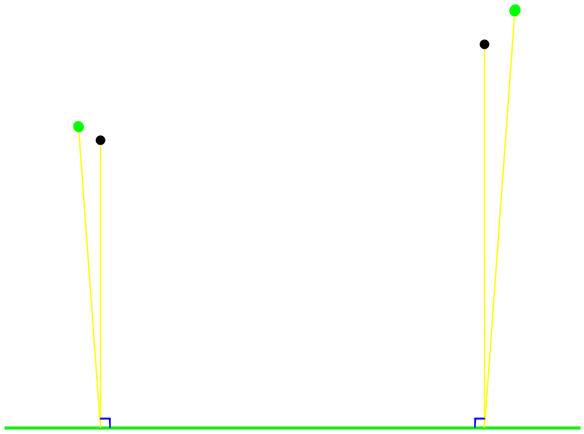

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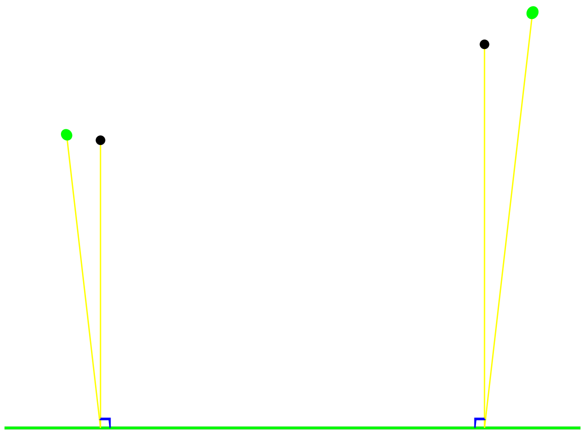

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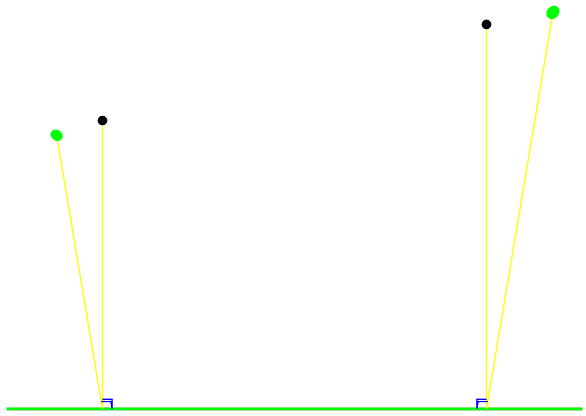

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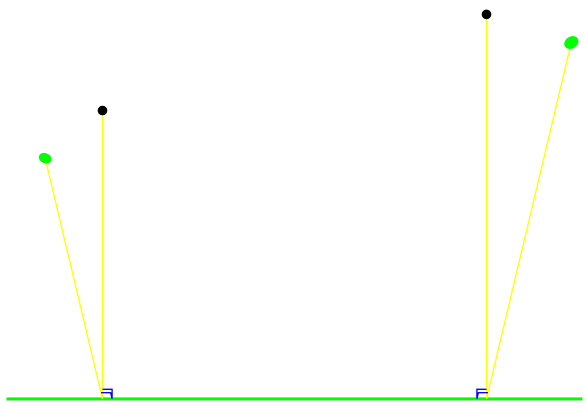

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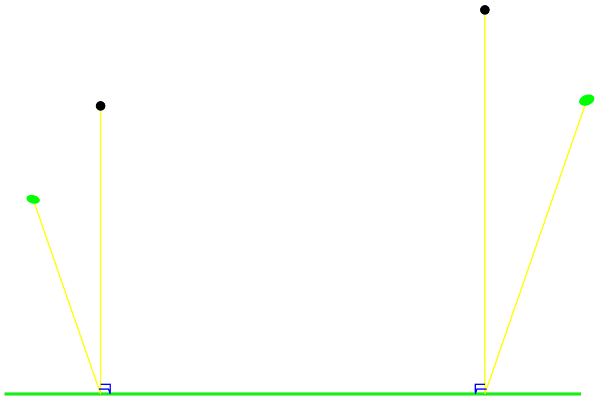

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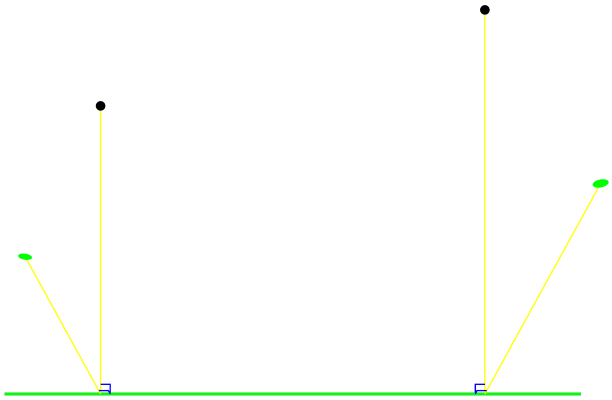

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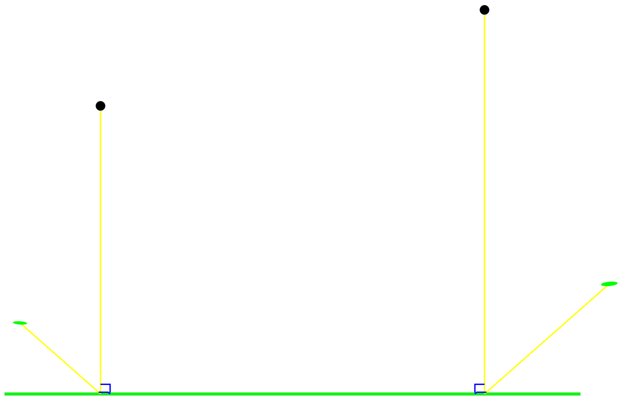

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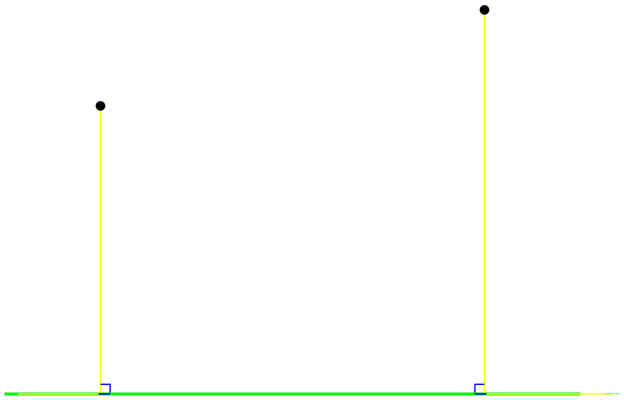

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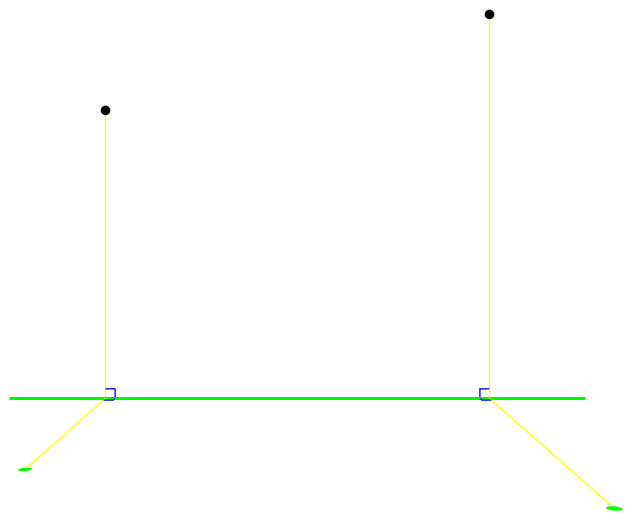

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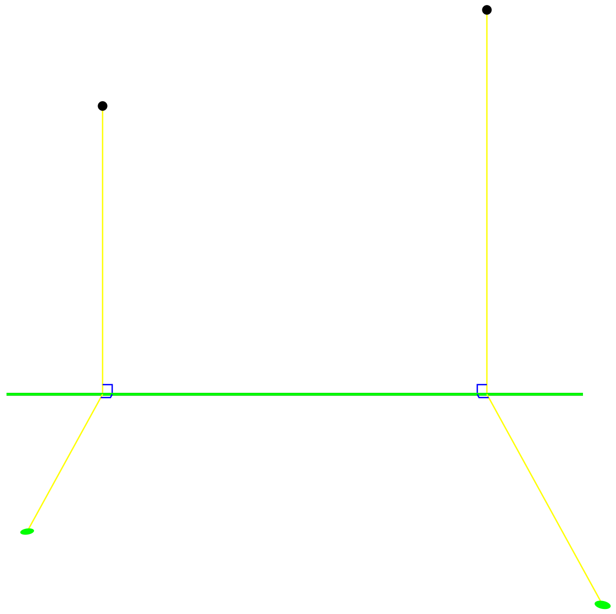

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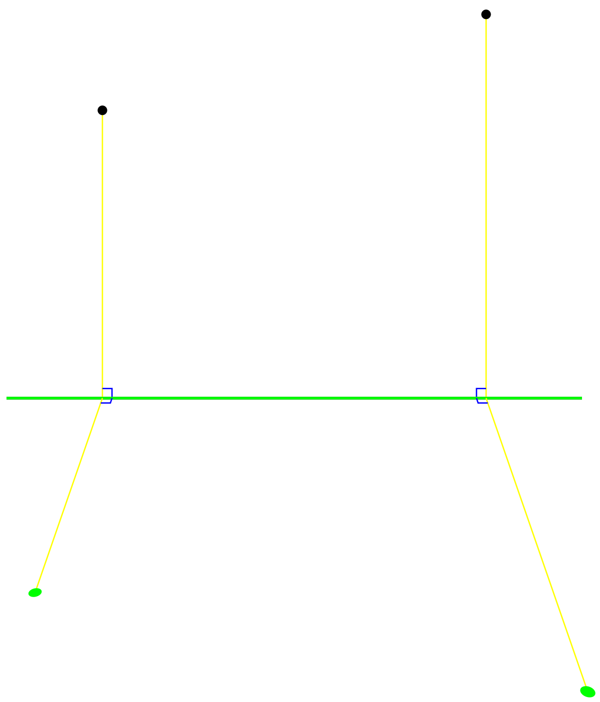

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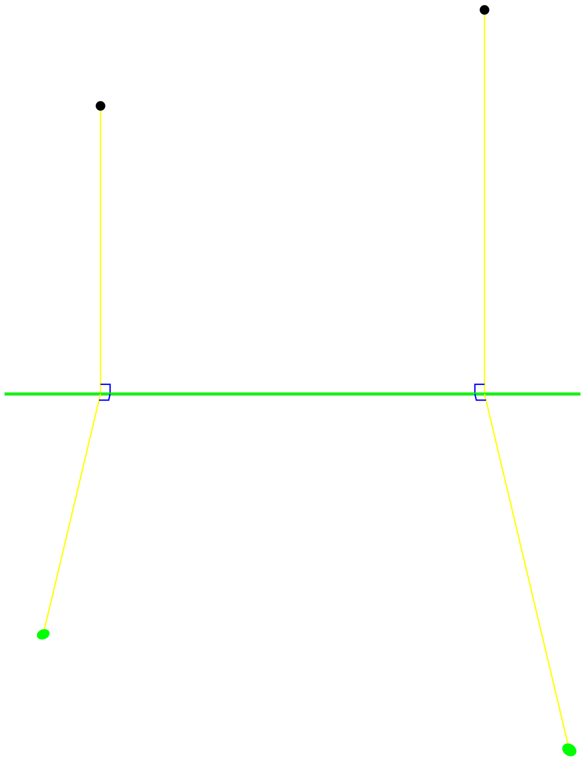

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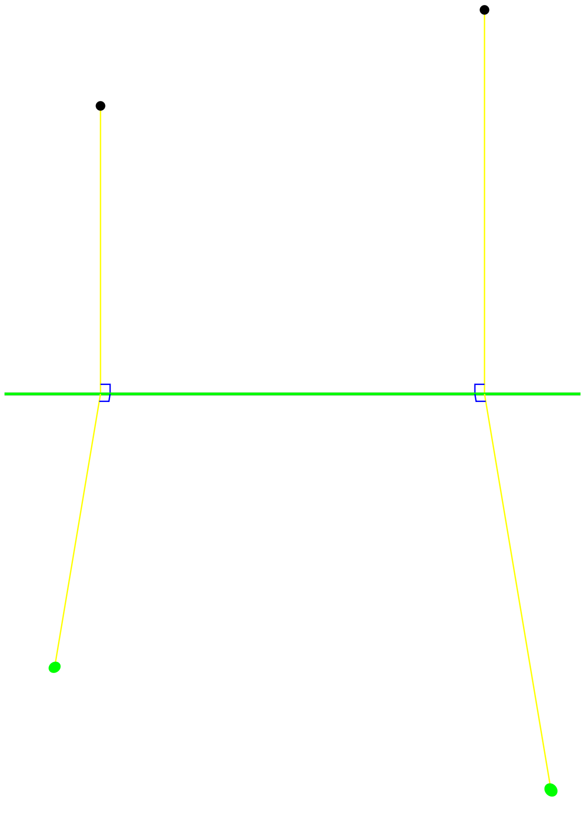

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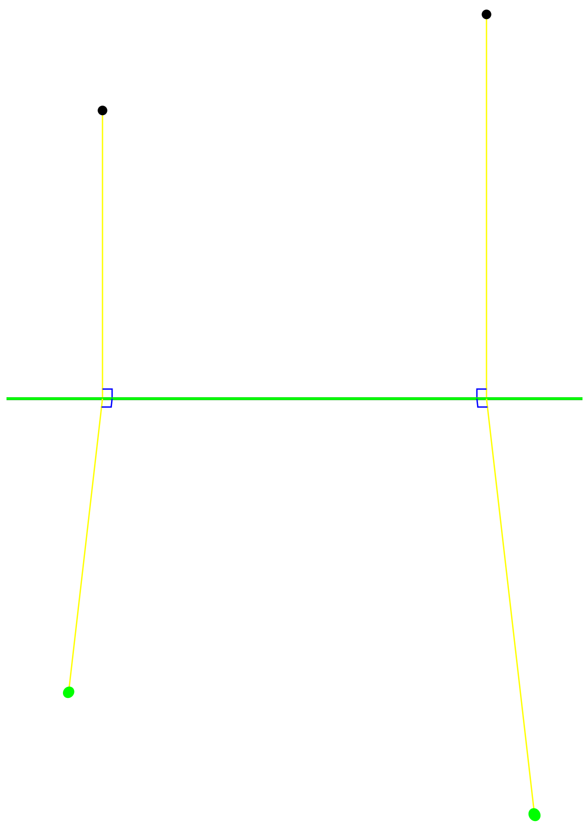

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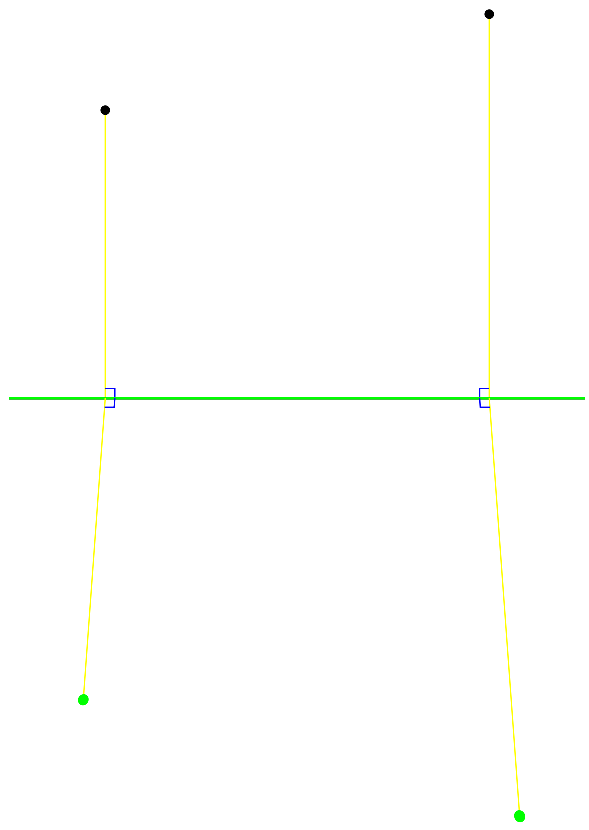

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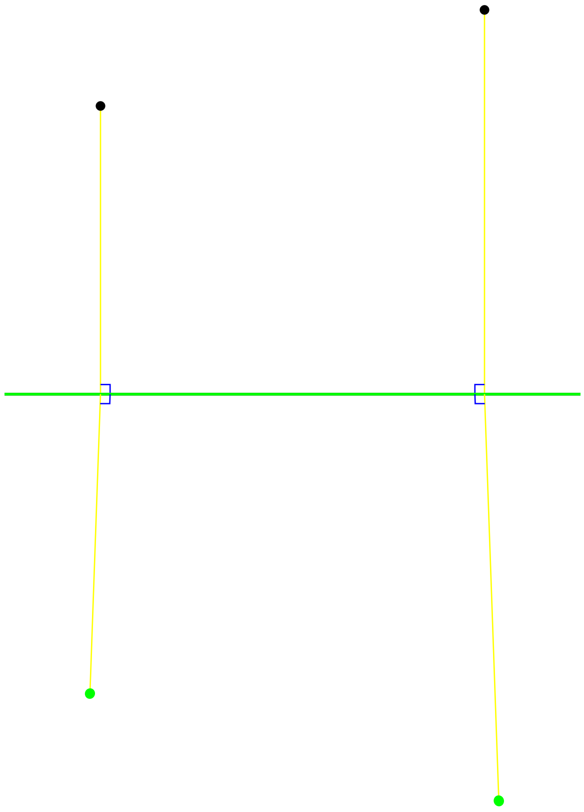

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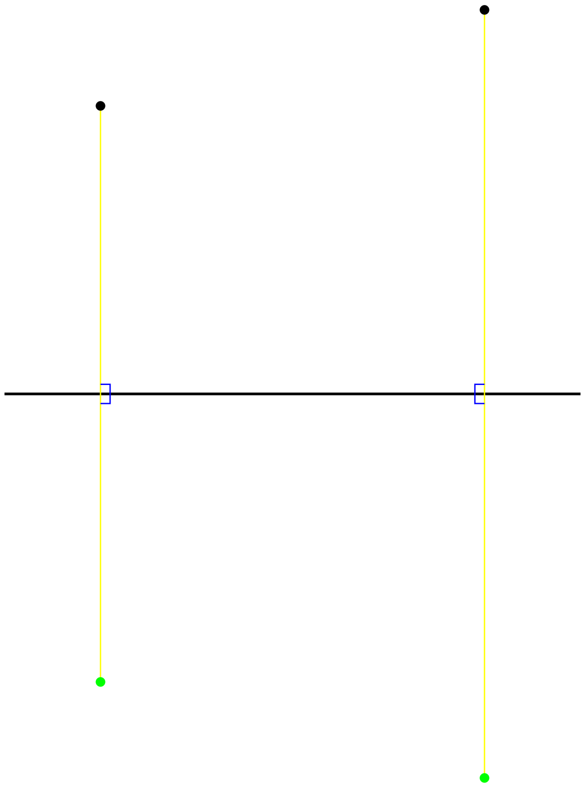
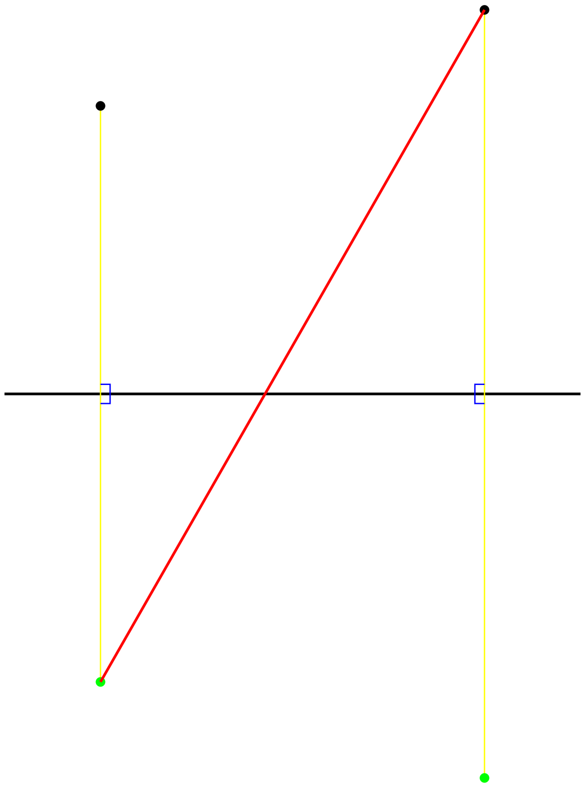


Now Draw a line from one of the original point to the "reflection" of the other point. The intersection of the line and the red line I just drew is the point we want.



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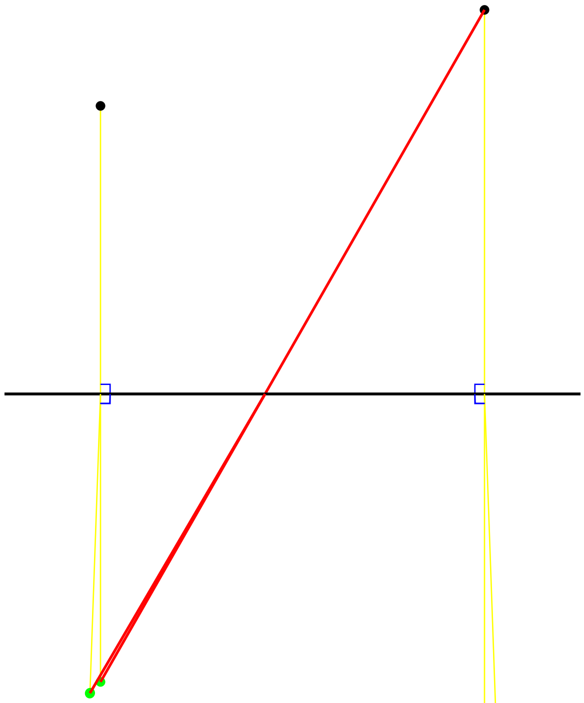

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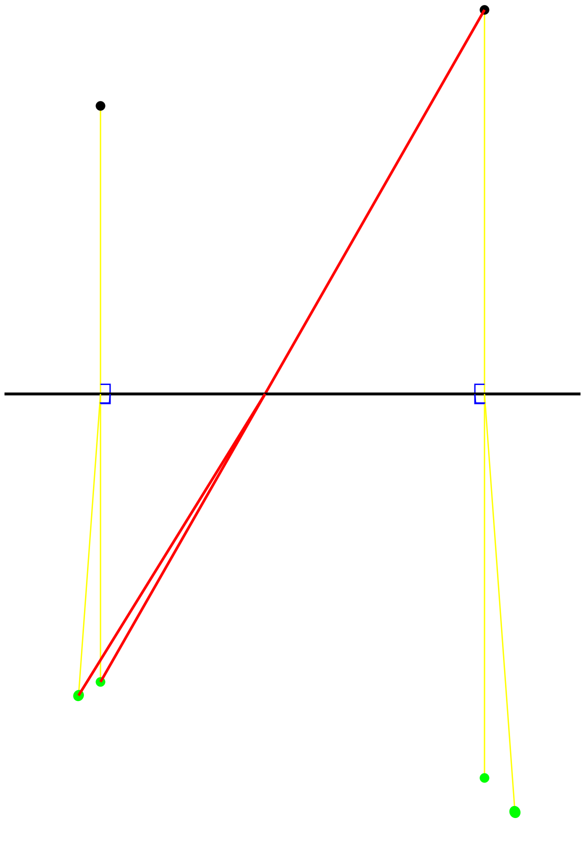

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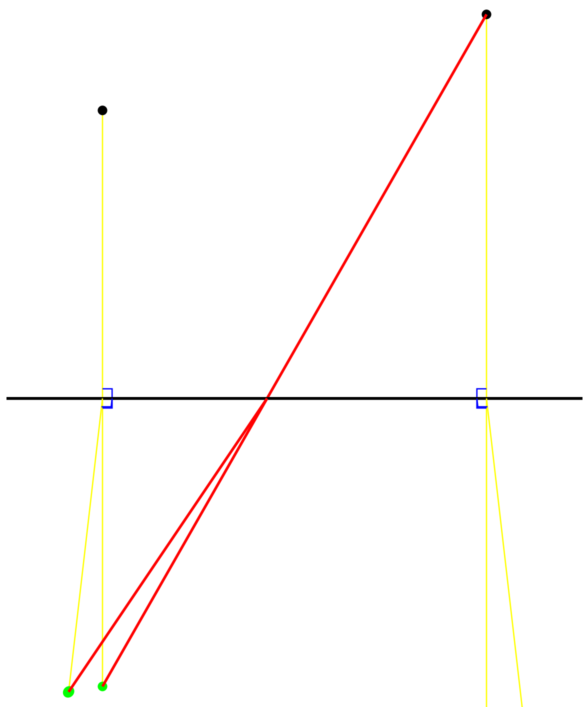

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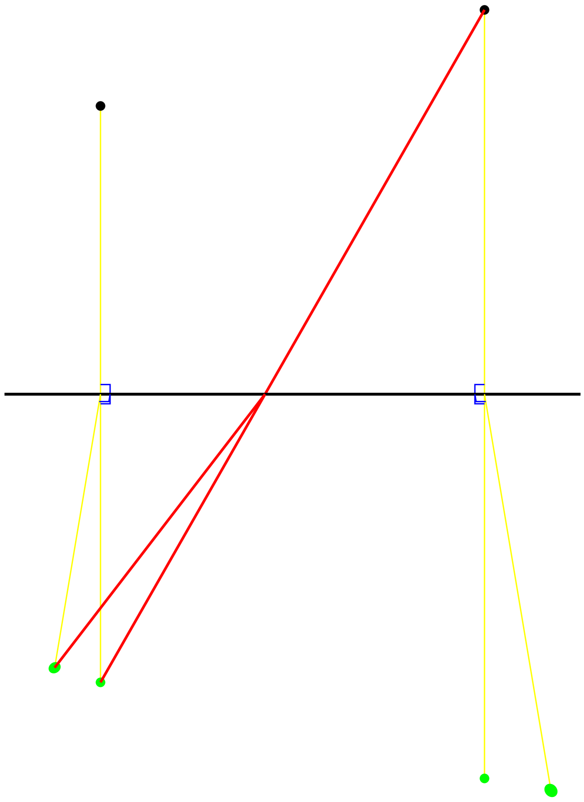

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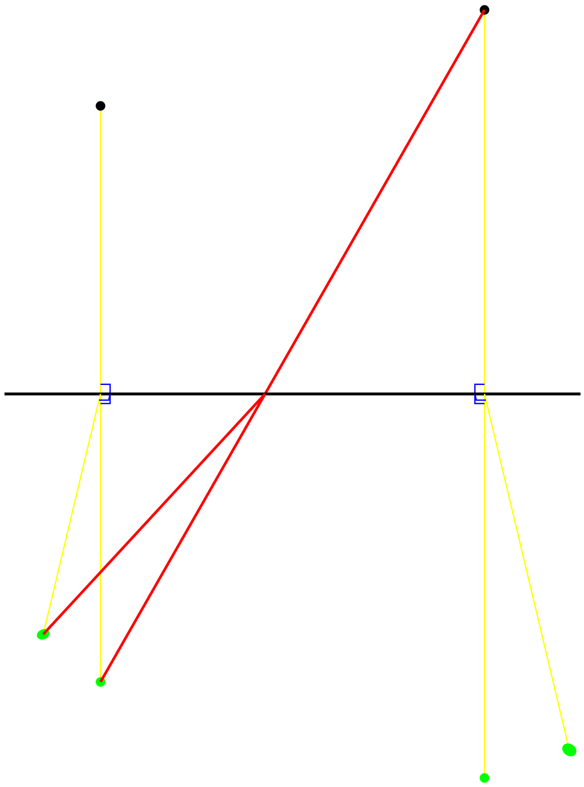

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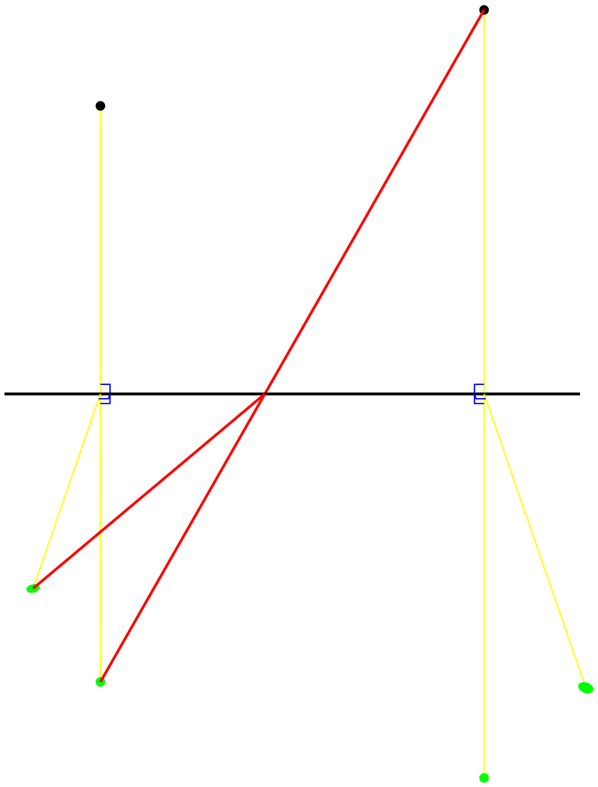

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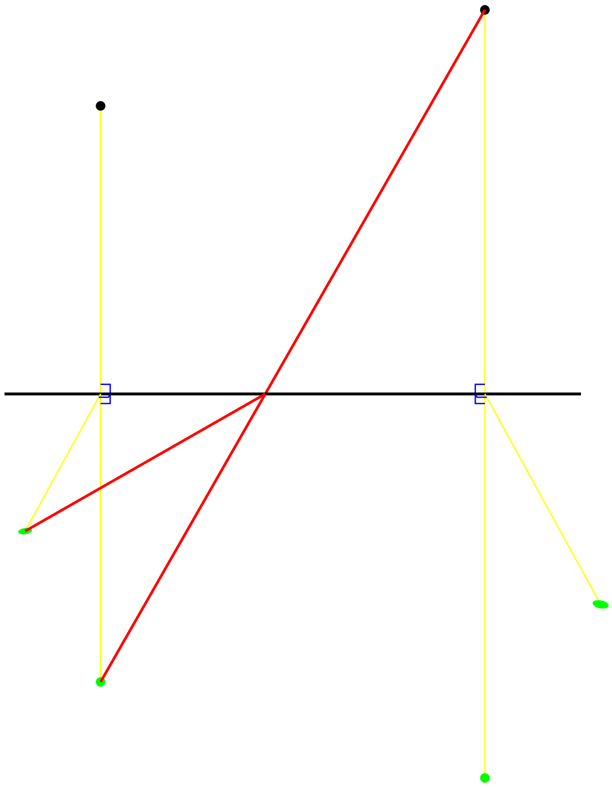

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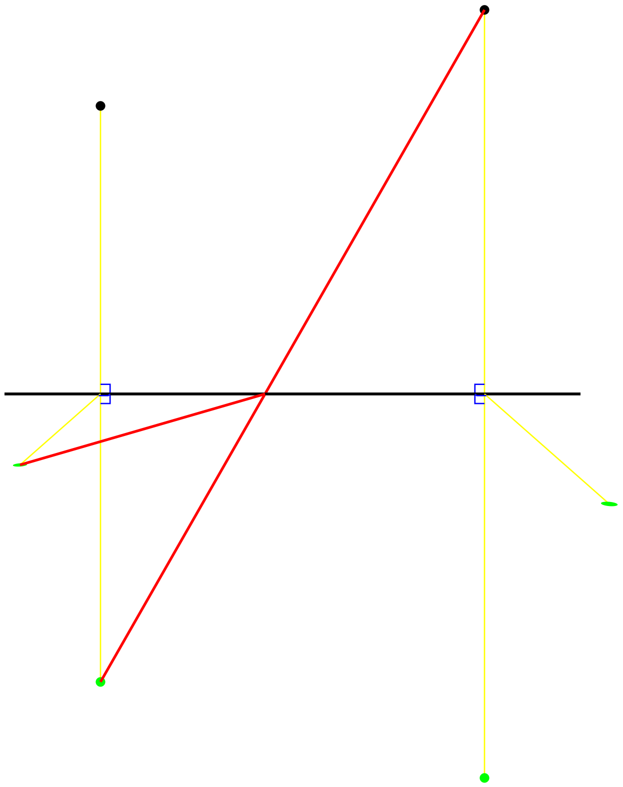

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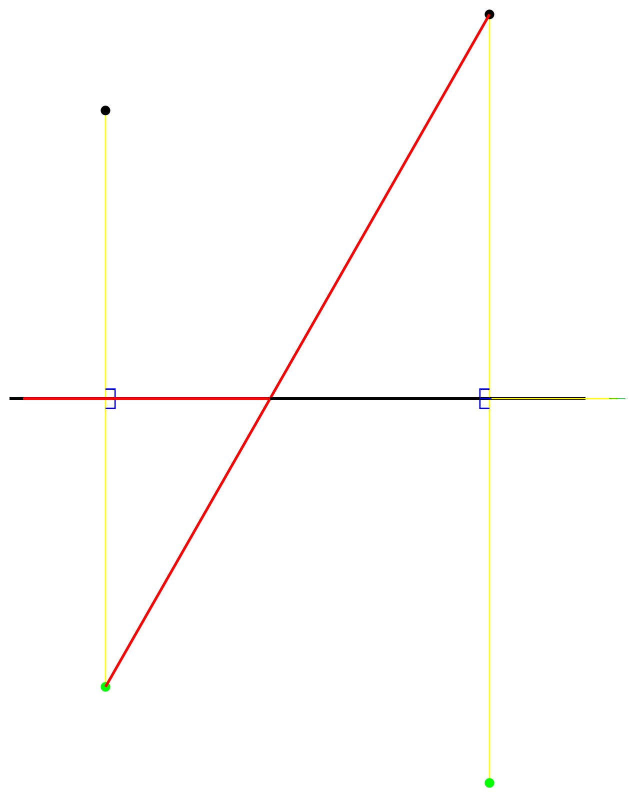

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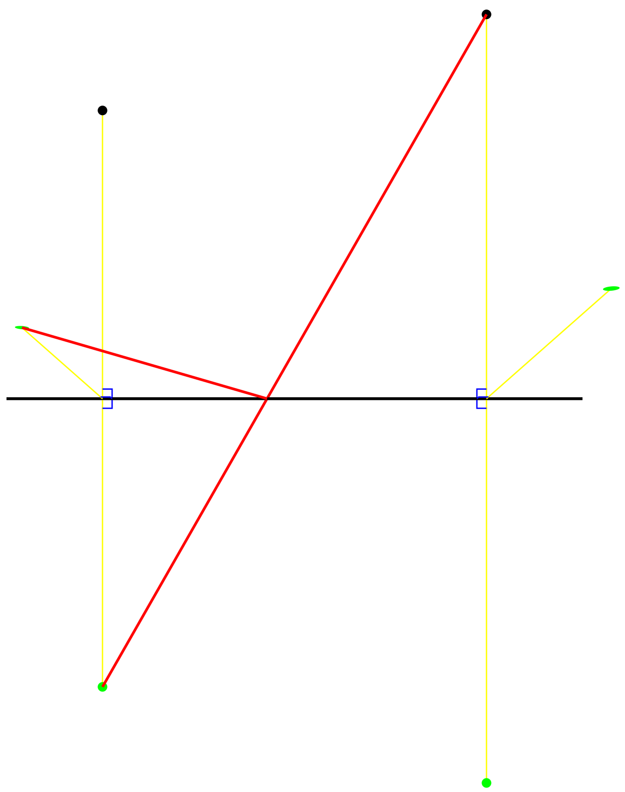

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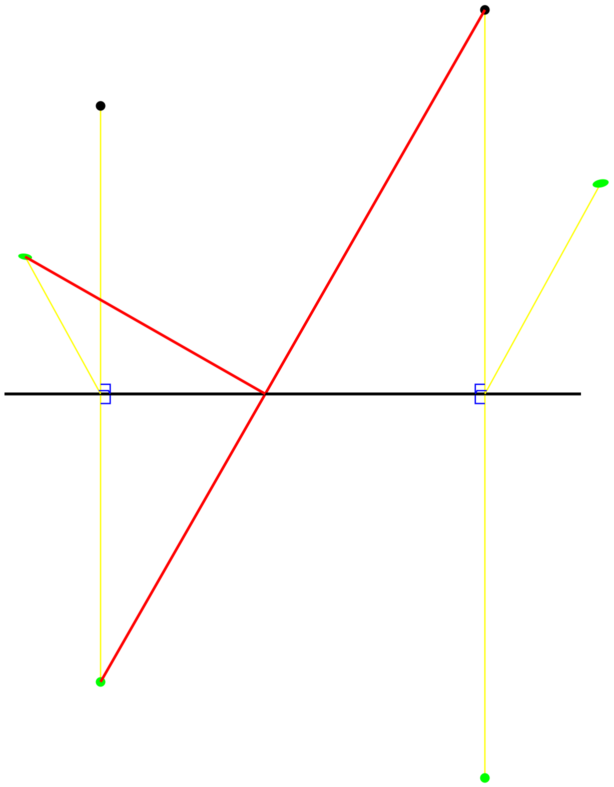

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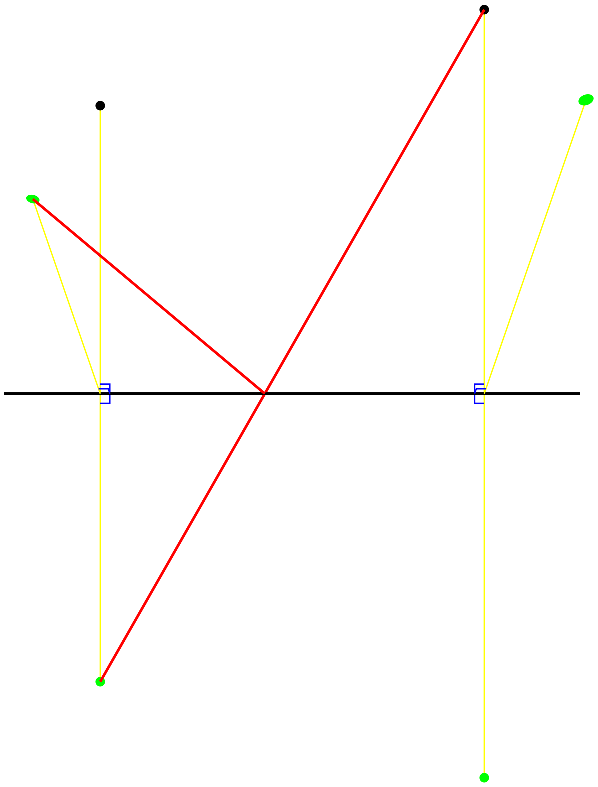

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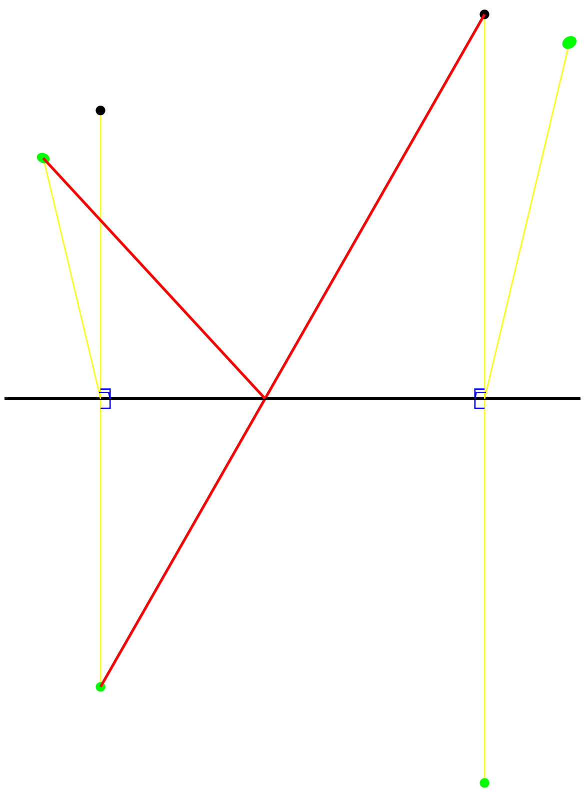

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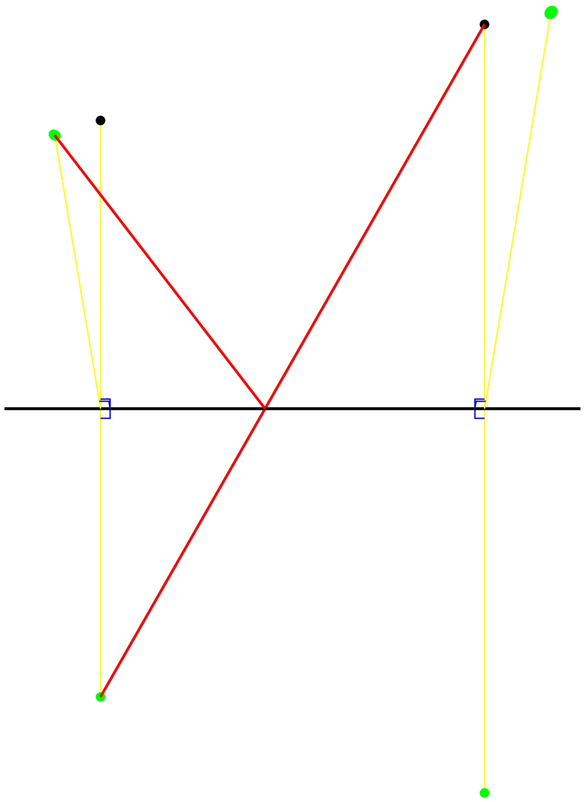

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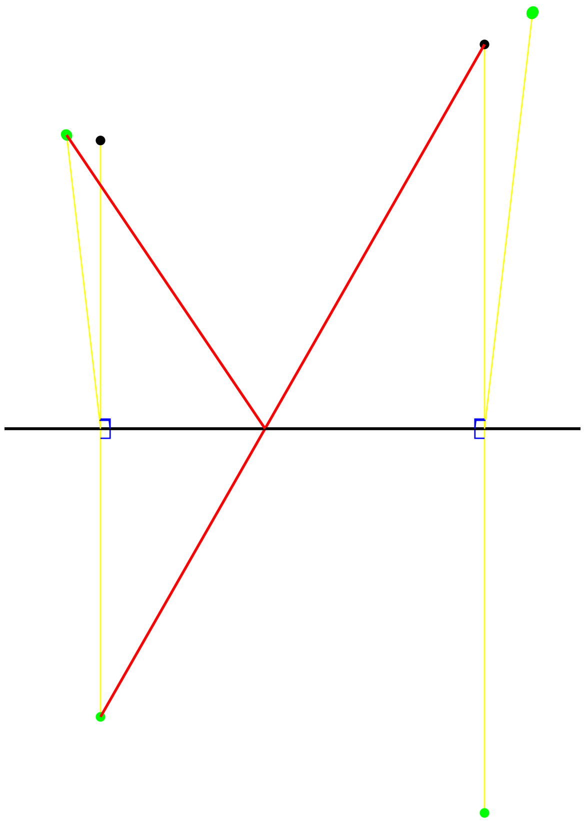

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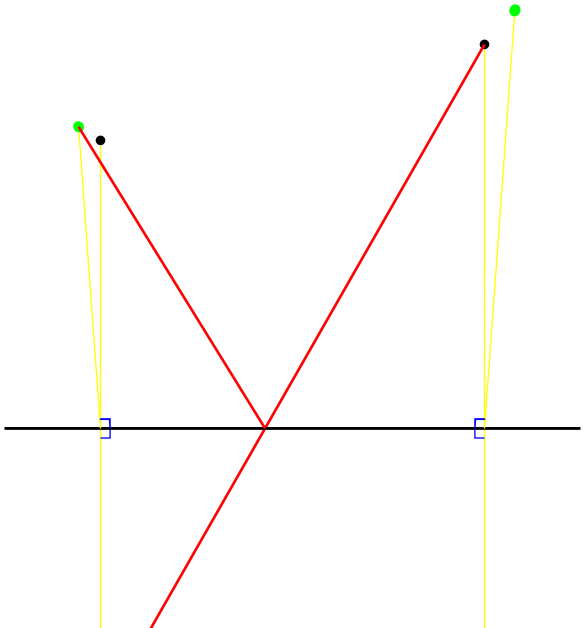

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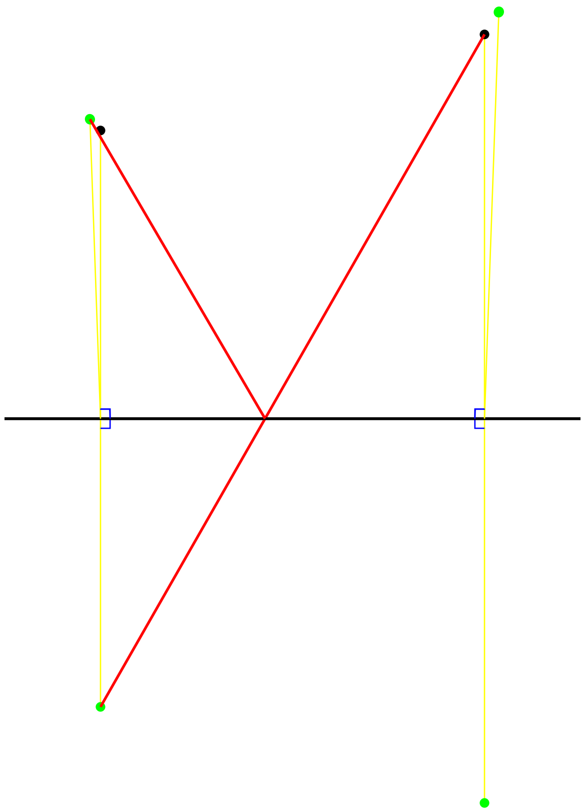

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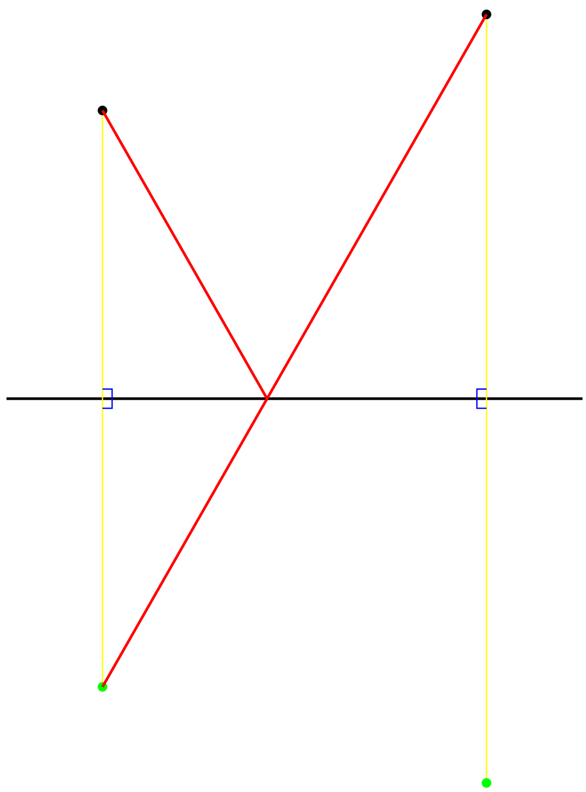
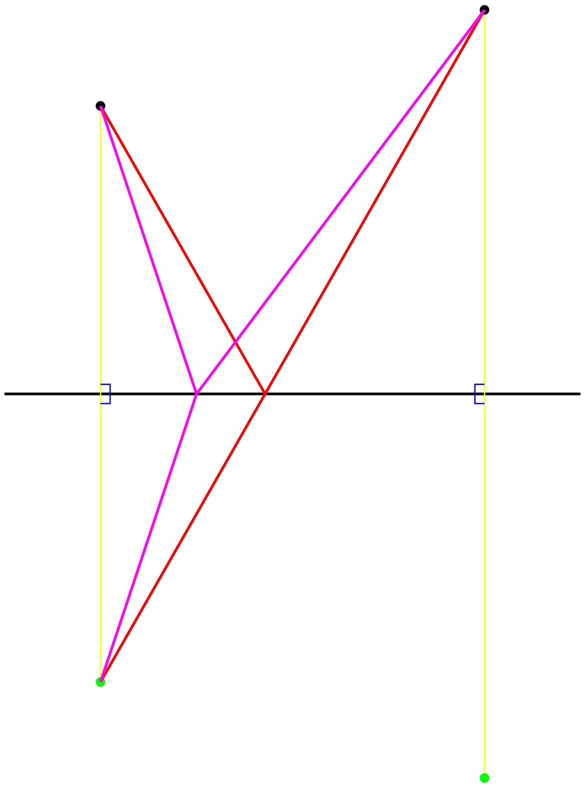
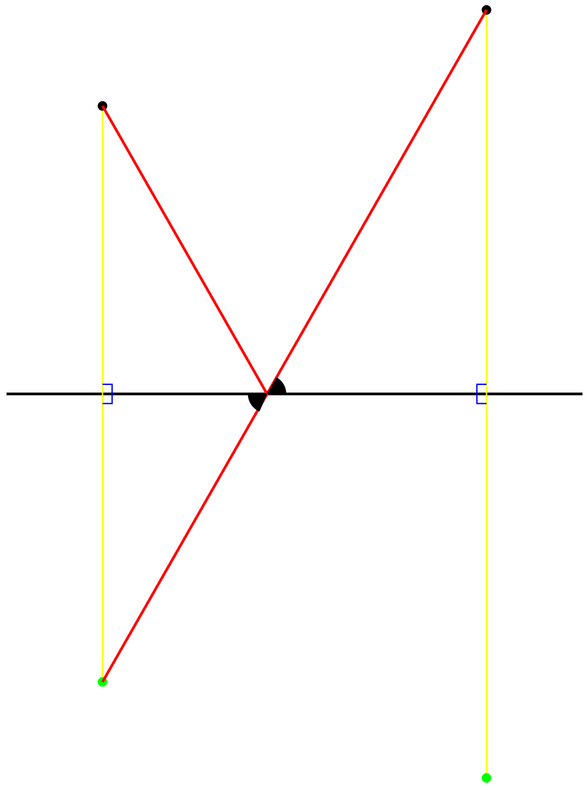


As we can see here, any other points wouldn’t work.
(Triangle Inequality)

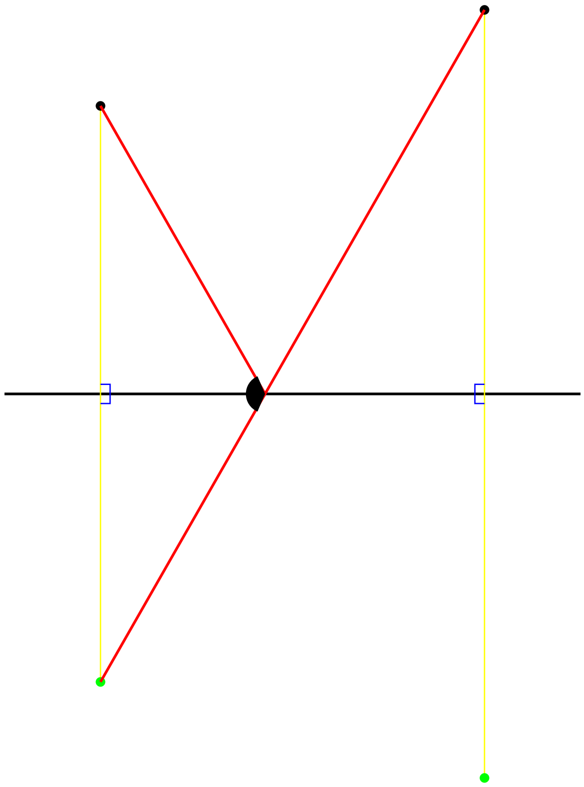


Let's examine the property of said point:

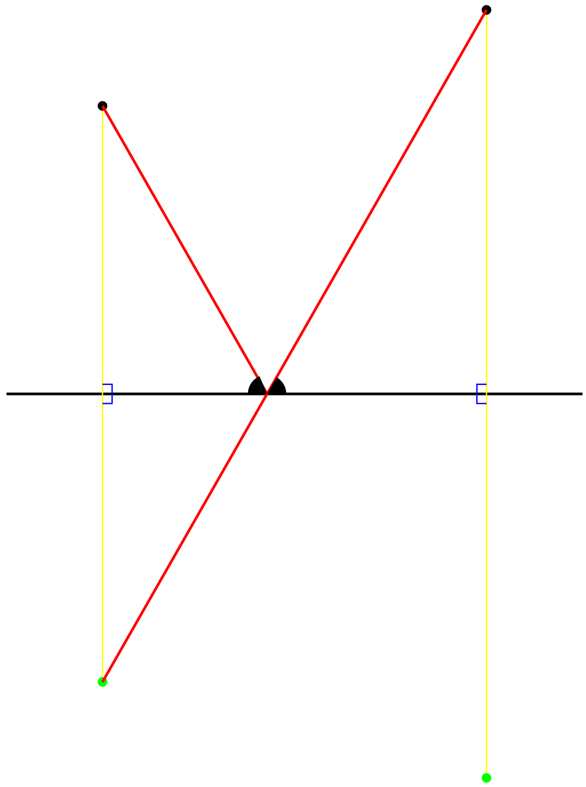
These two angles are equal. (Opposite Angles)

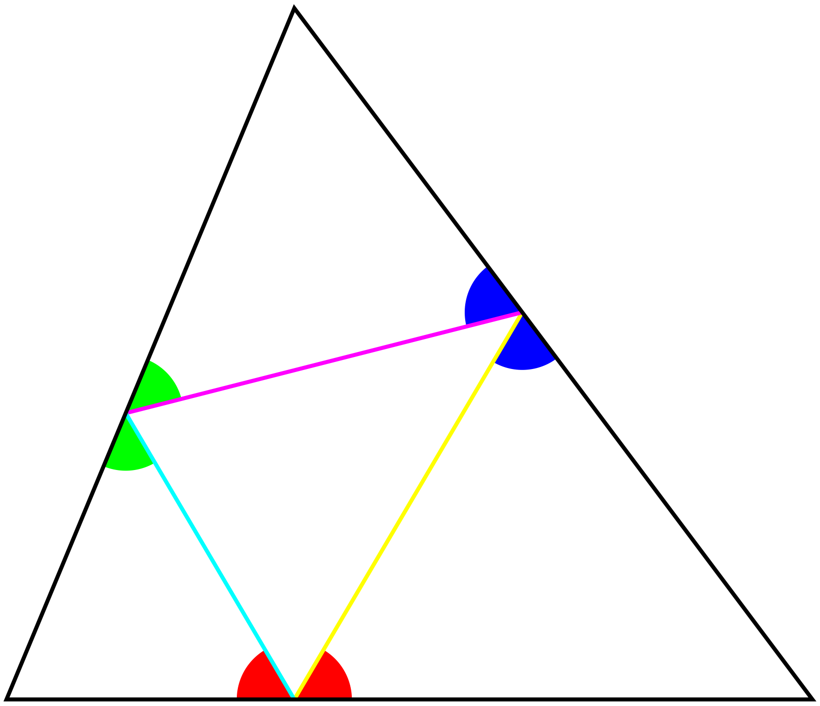


These two angles are equal as well, seeing how one angle is only a refletion of the other.



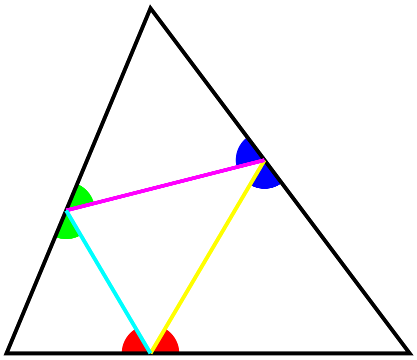
As a result, these two angles are equal.



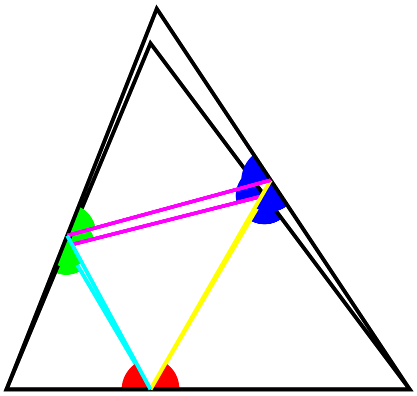


It stands to reason that the inscribed triangle with the smallest perimeter would be one that looks like this, with the angles of the same color equal to each other.

Of course, a proof is required.



As we have seen, reflection is the way to go.
Now "flip" the triangle 5 time.



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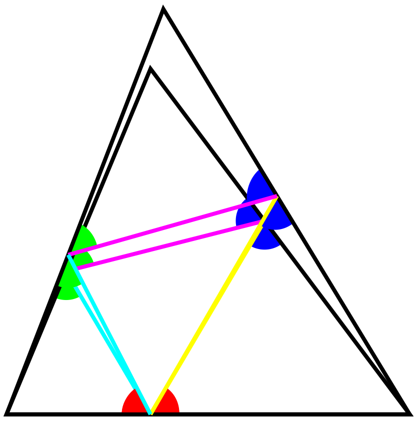

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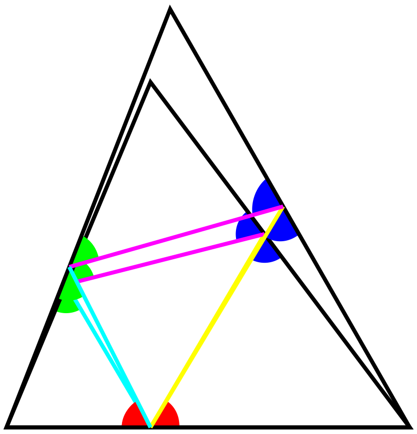

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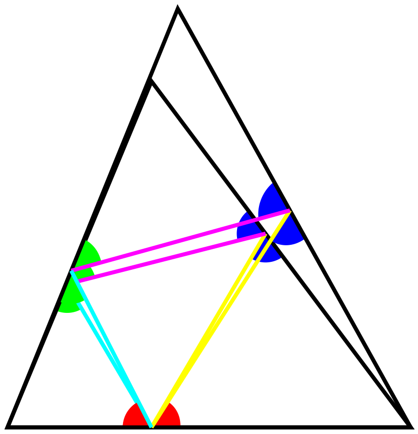

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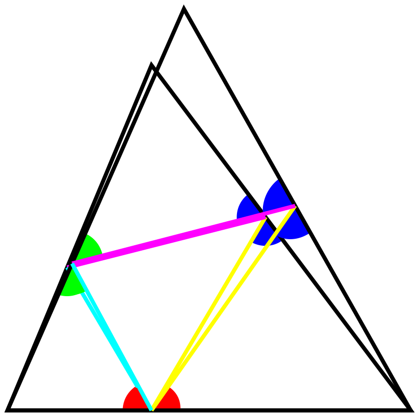

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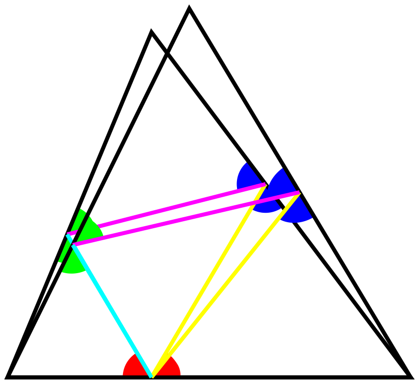

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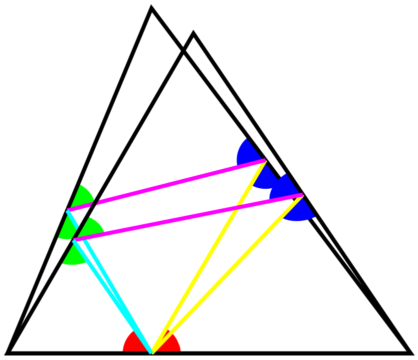

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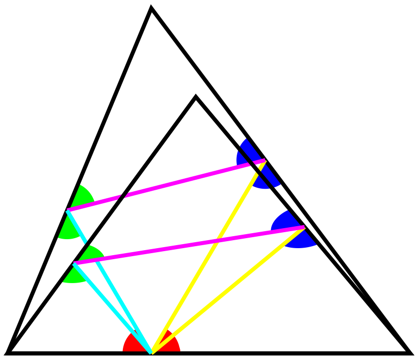

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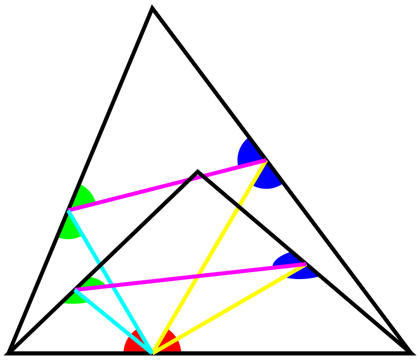

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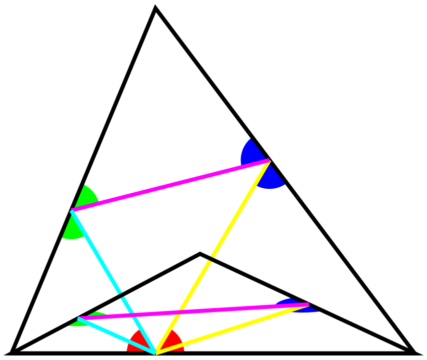

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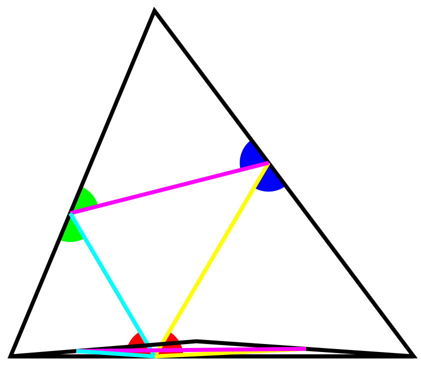

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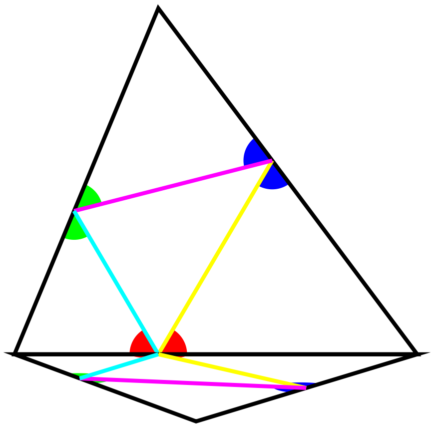

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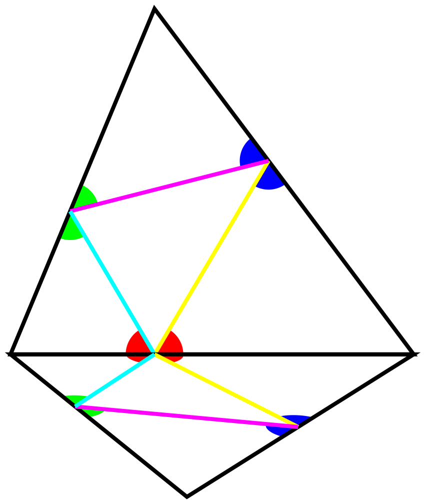

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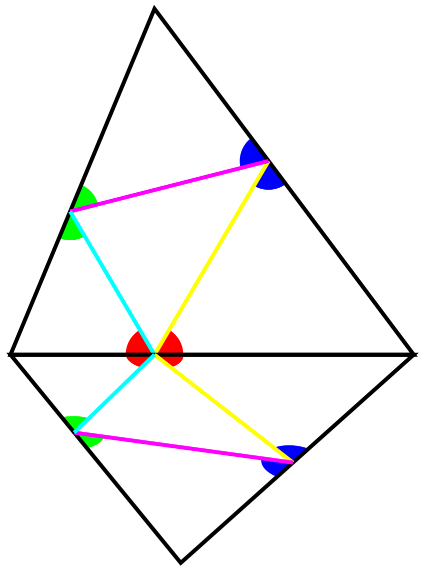

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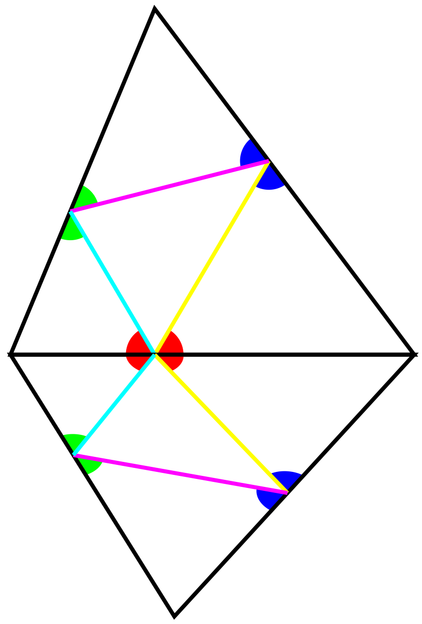

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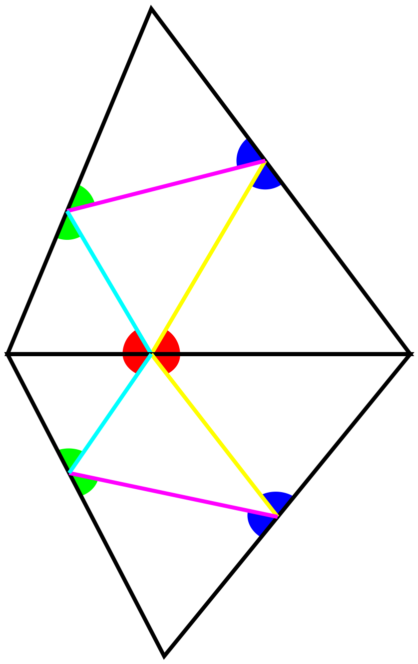

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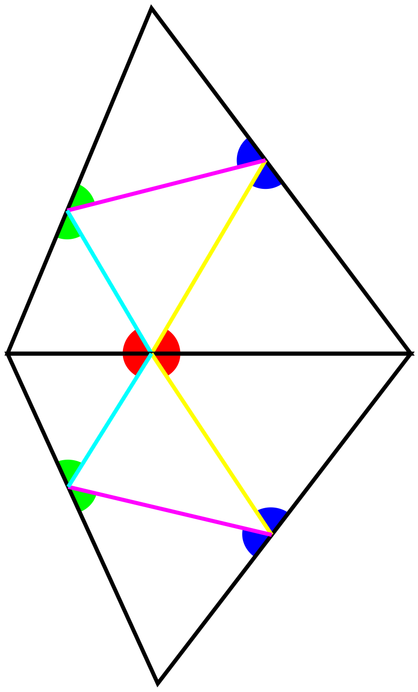

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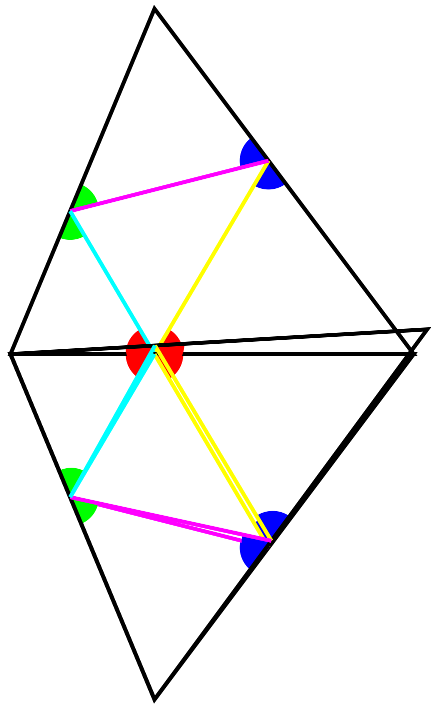

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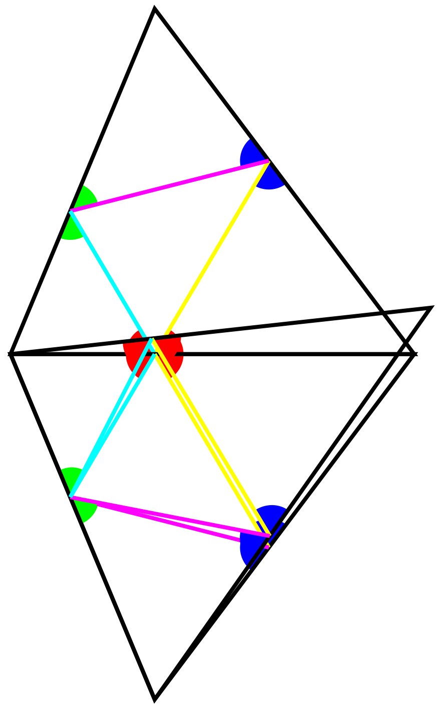

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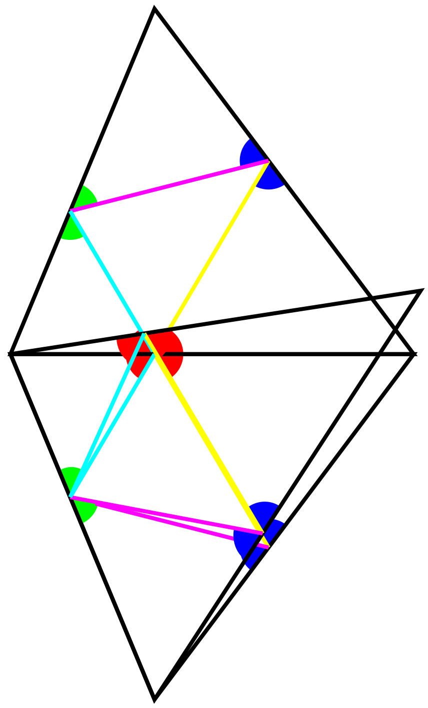

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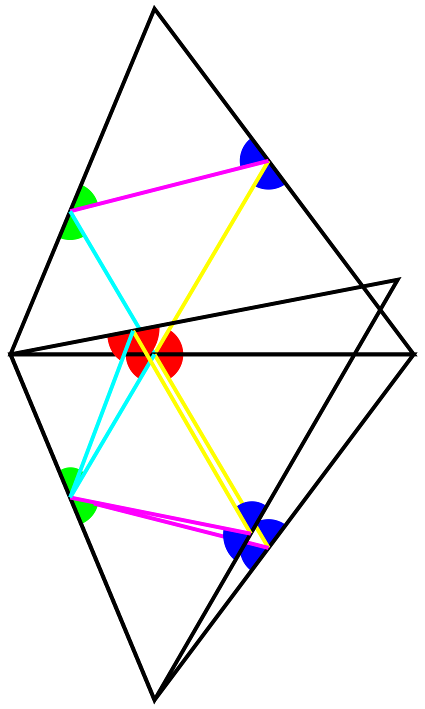

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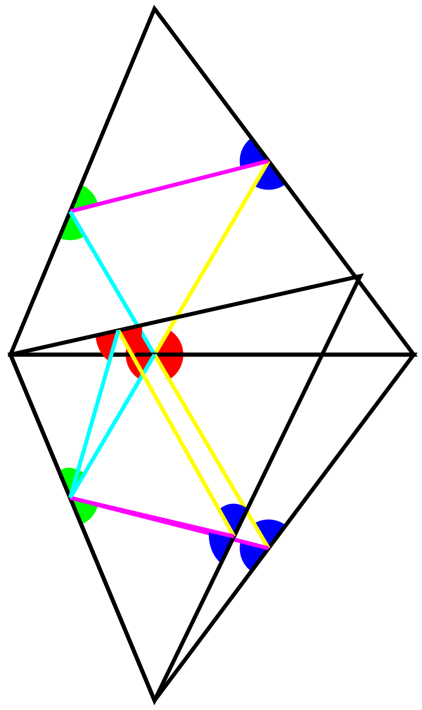

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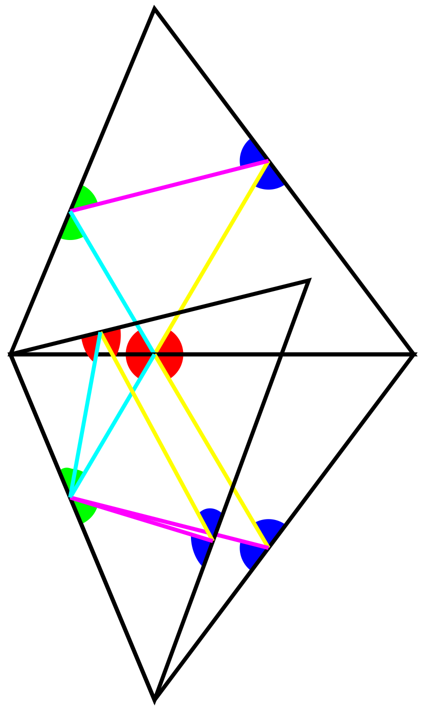

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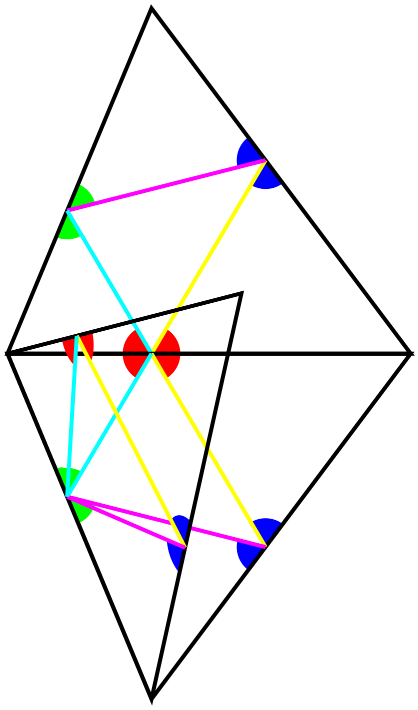

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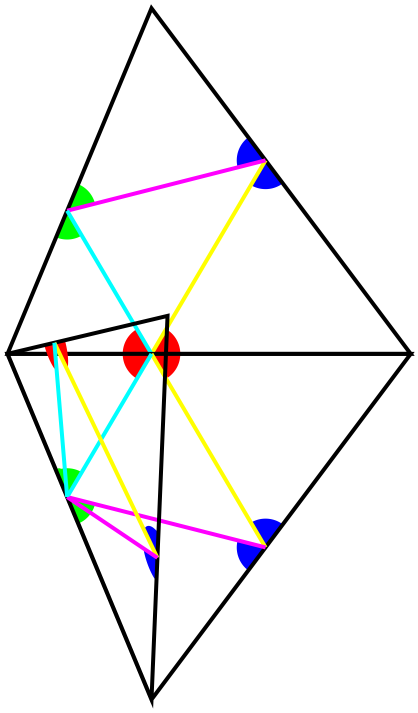

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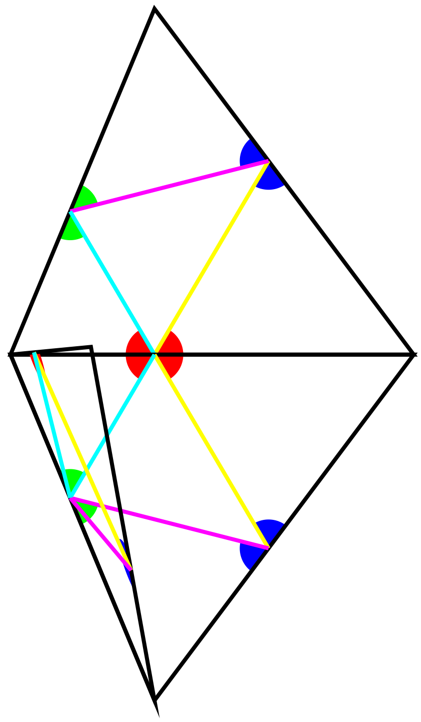

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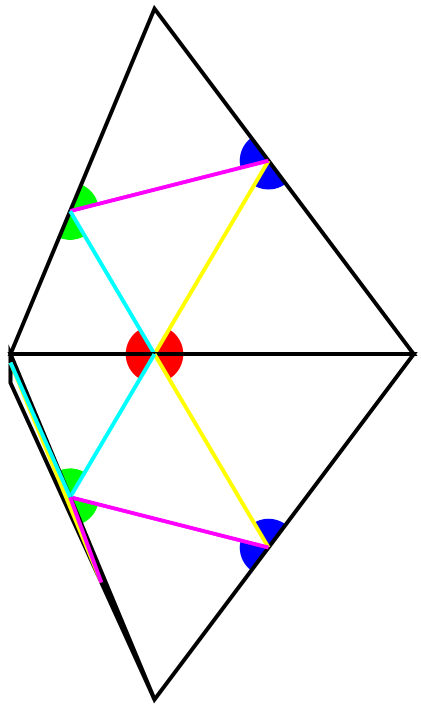

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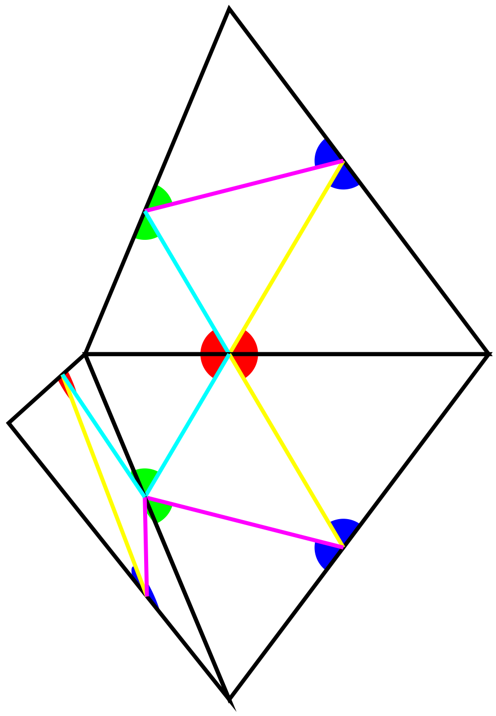

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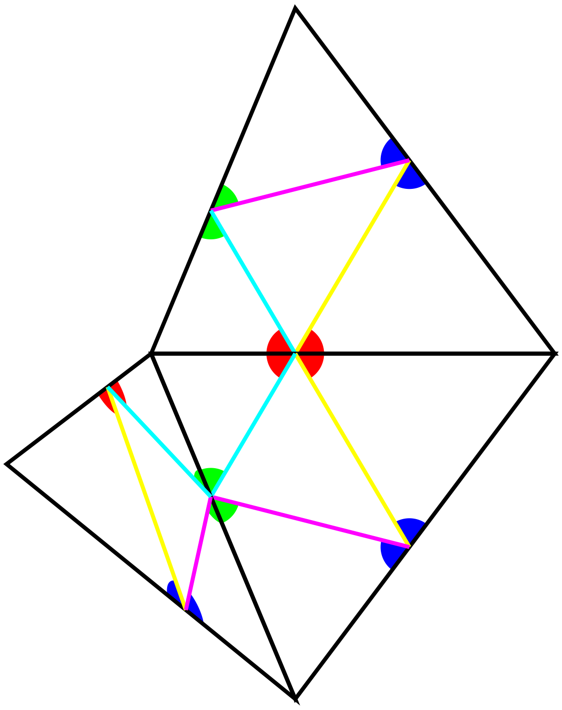

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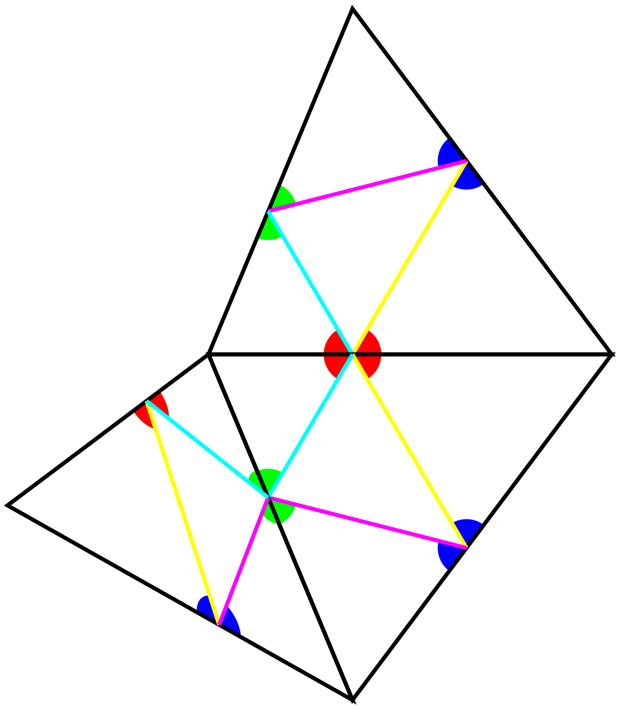

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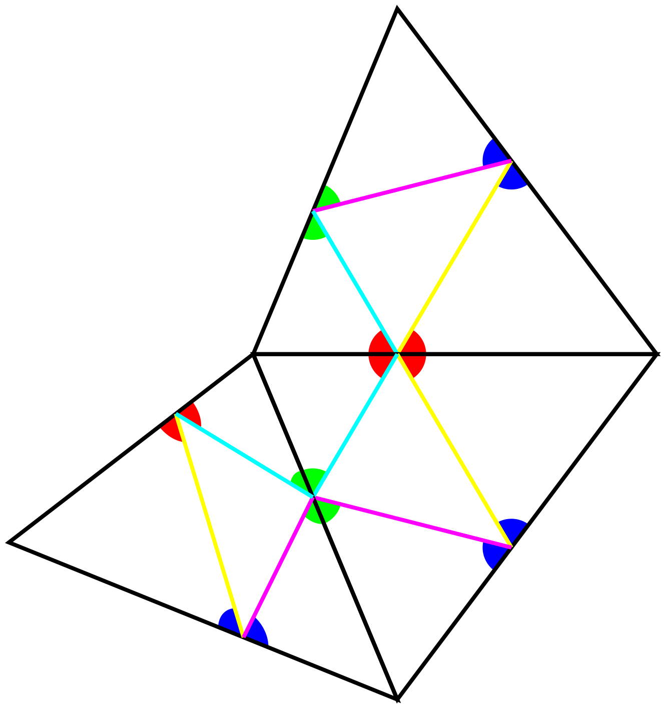

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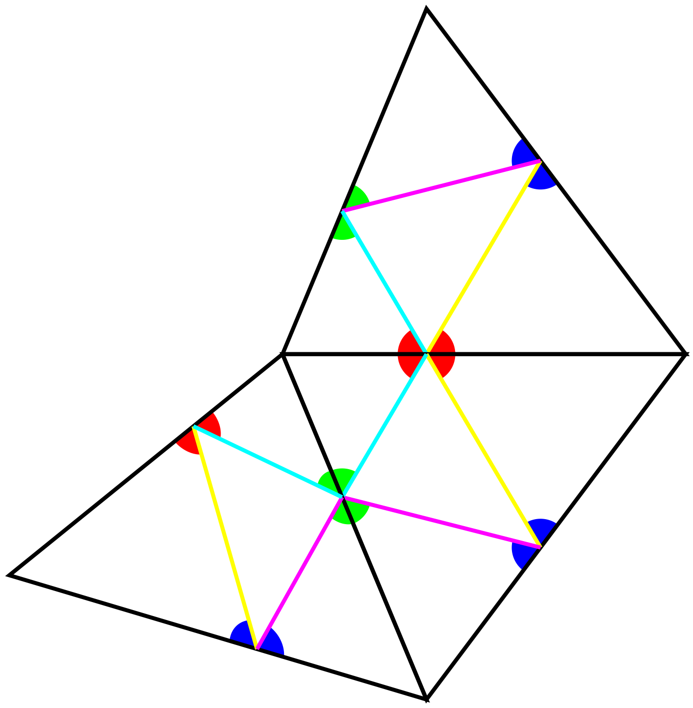

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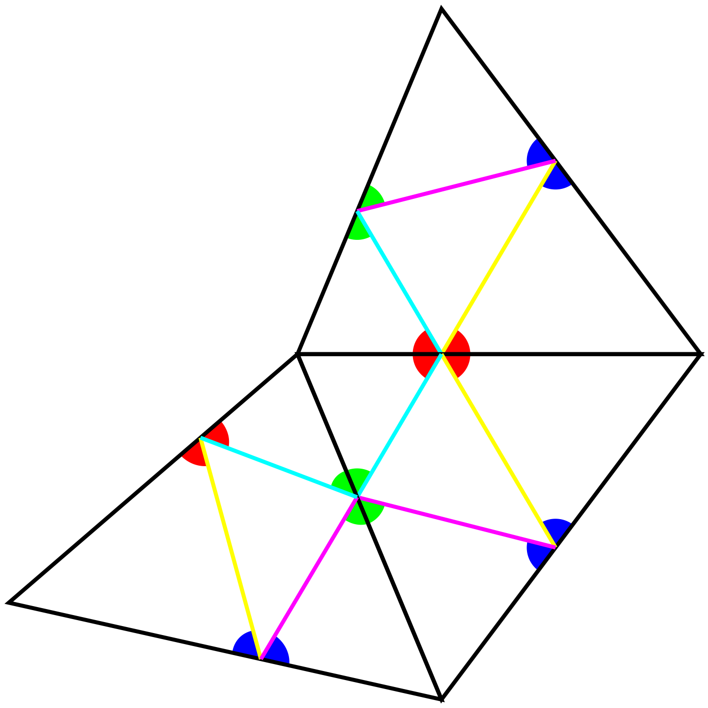

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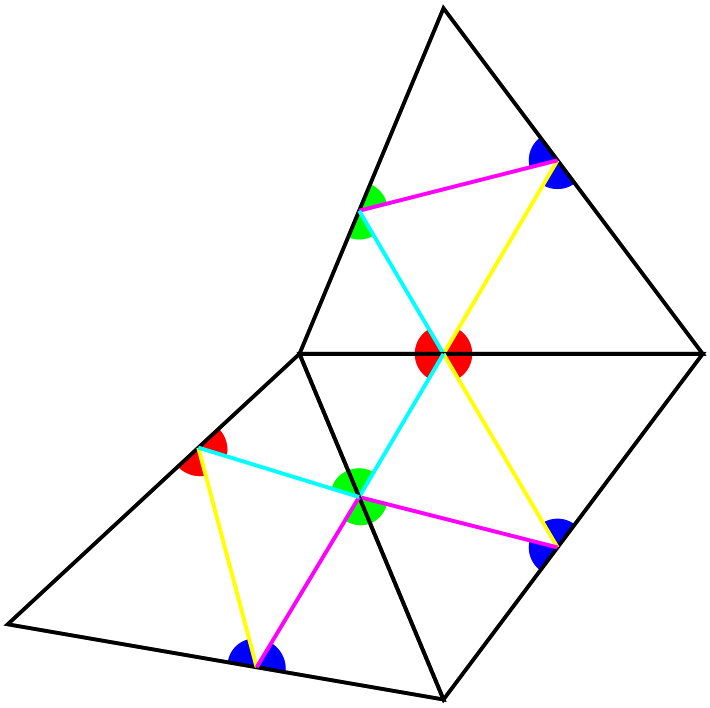

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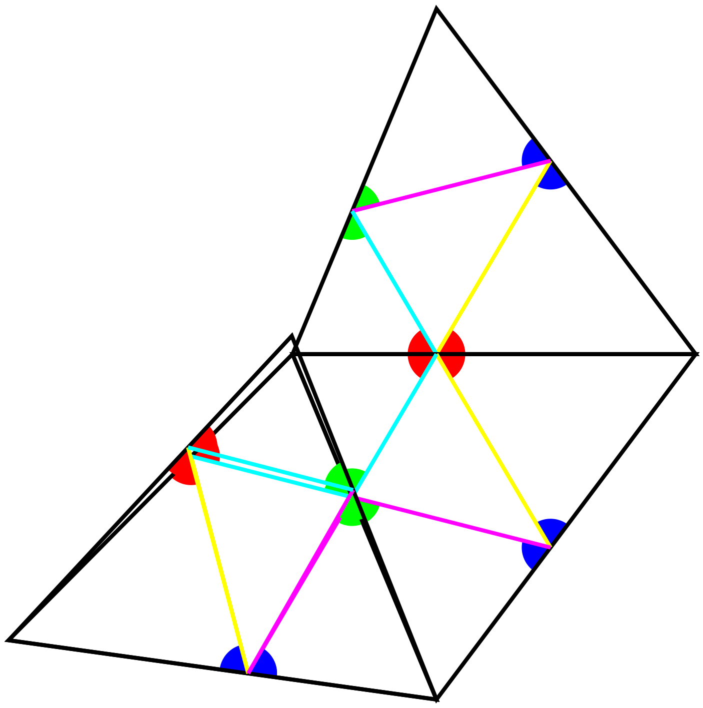

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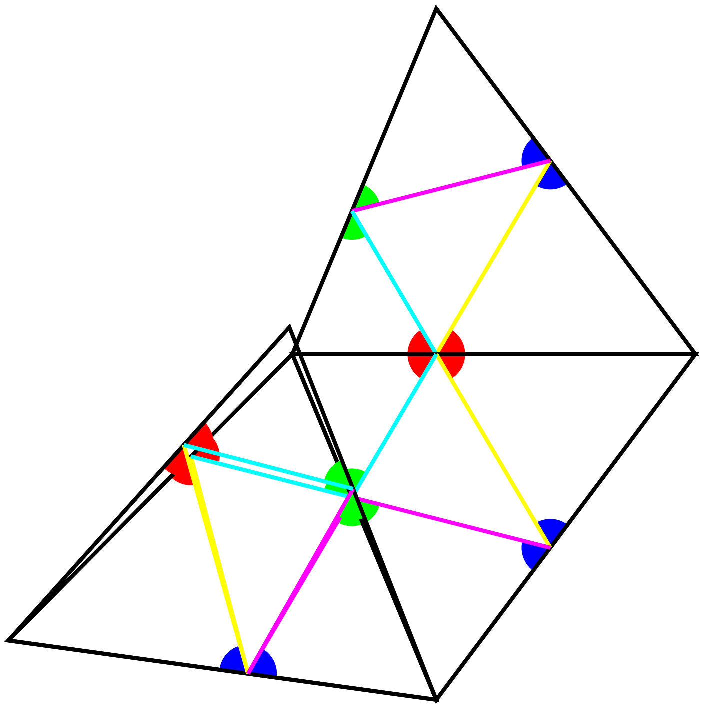

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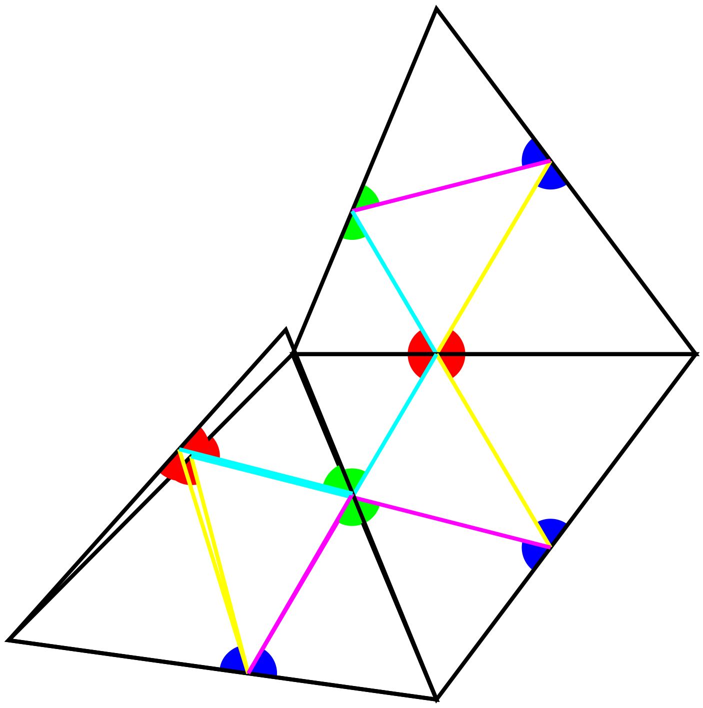

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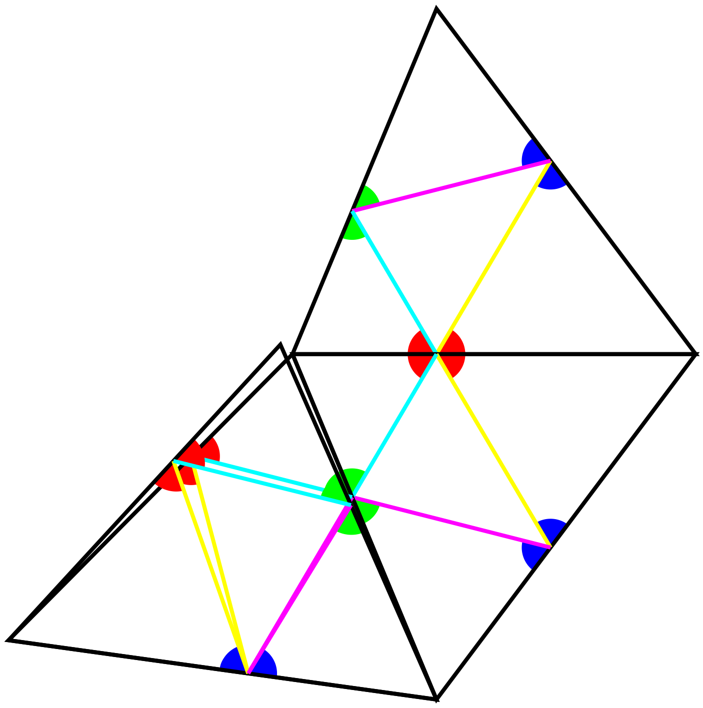

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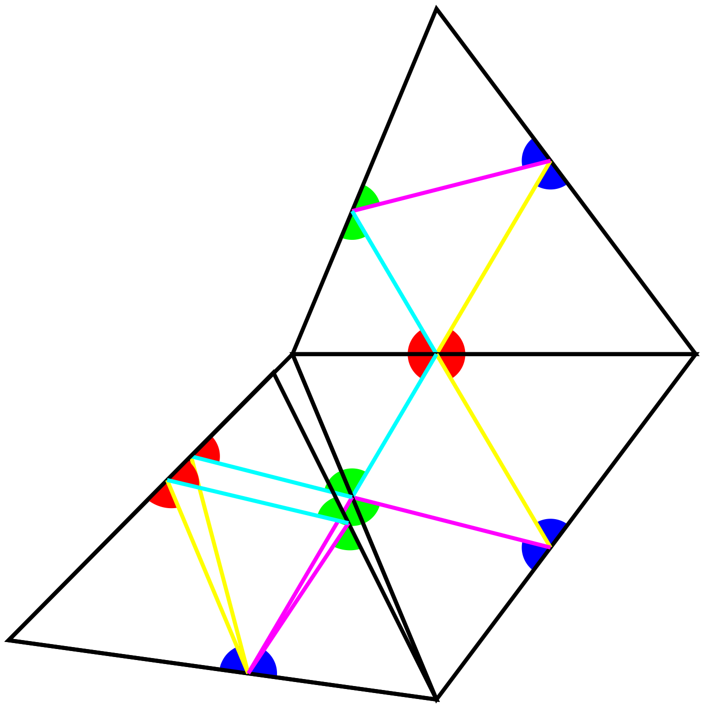

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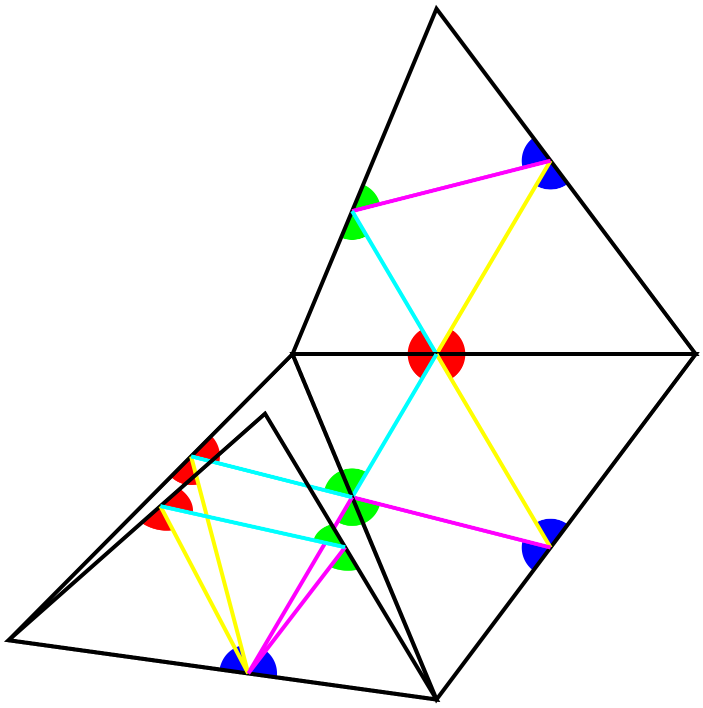

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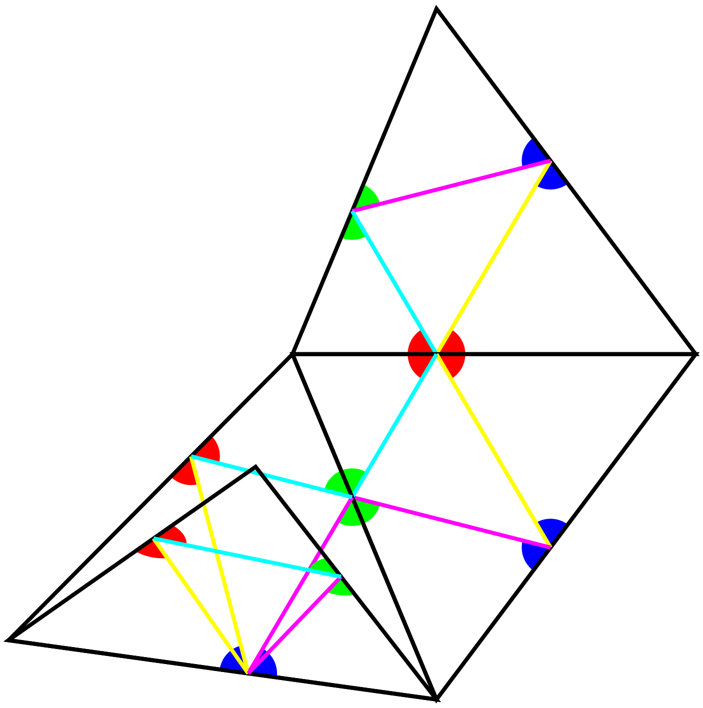

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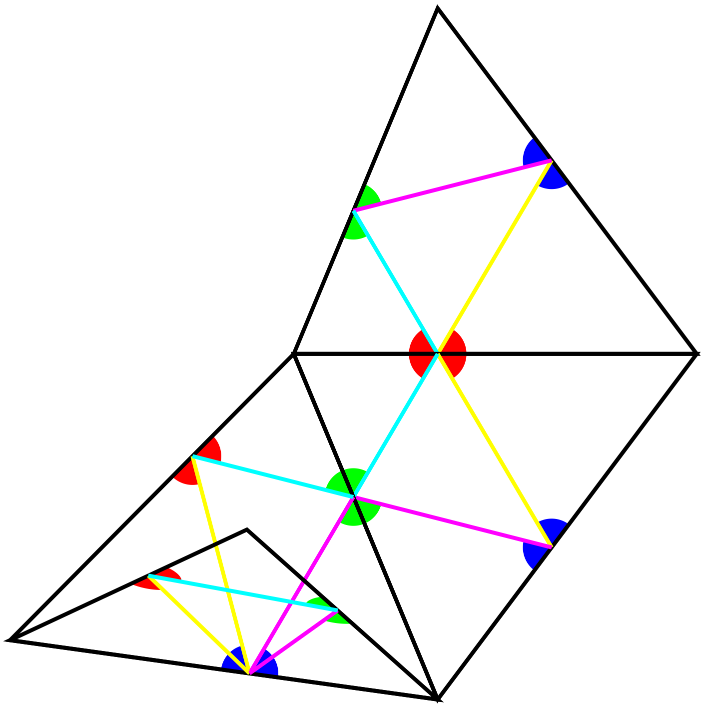

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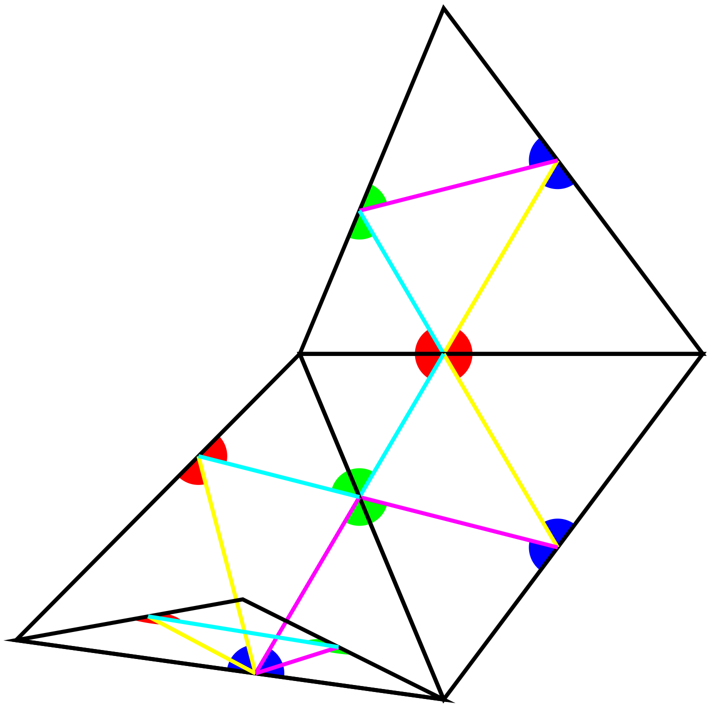

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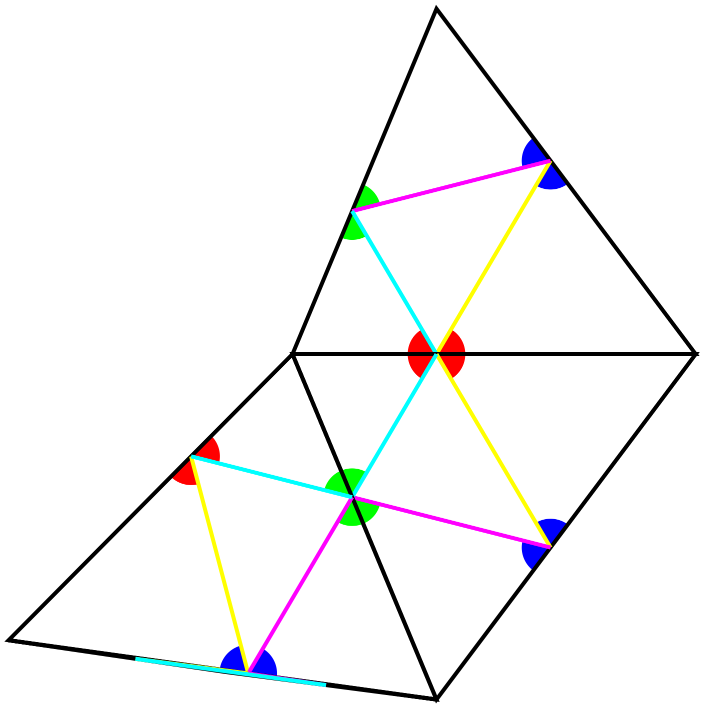

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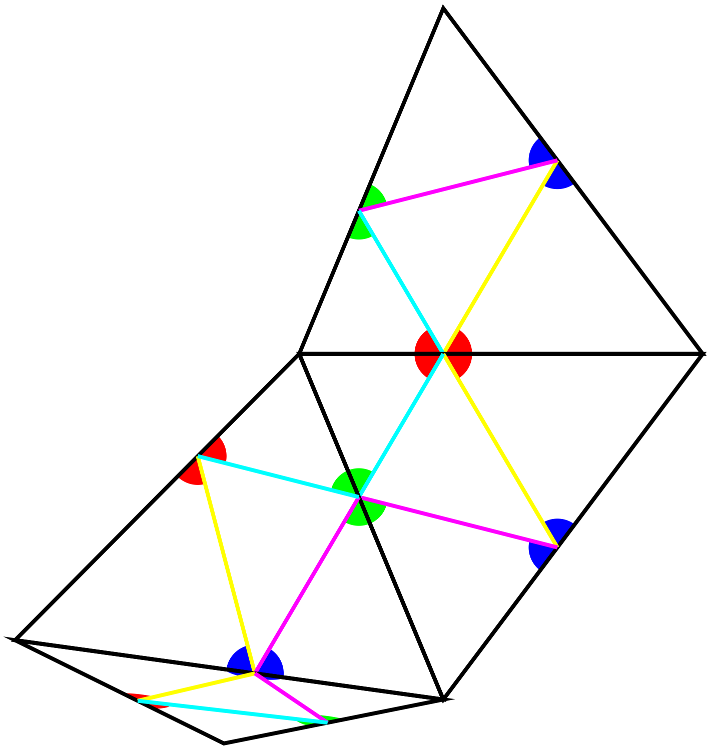

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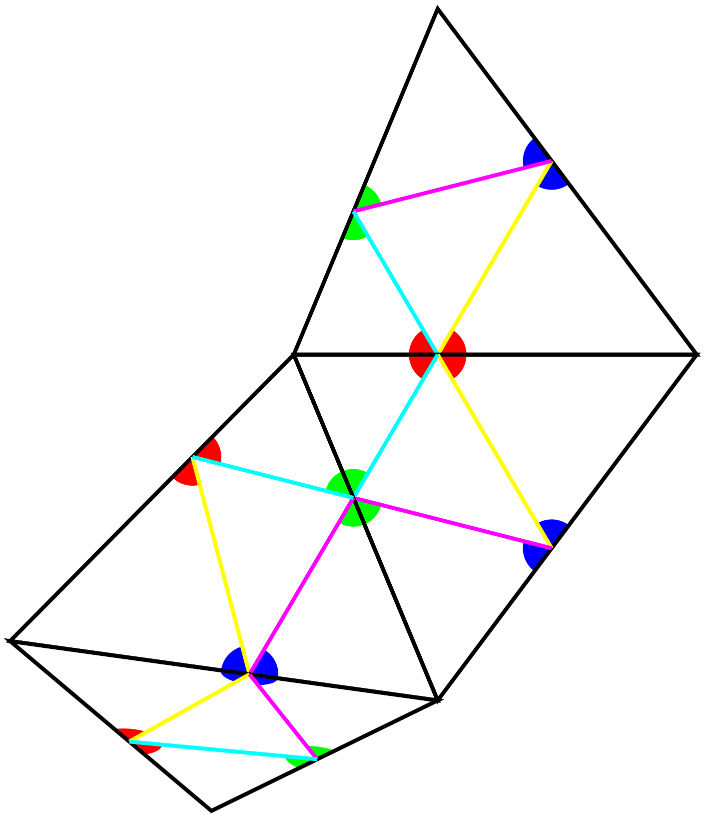

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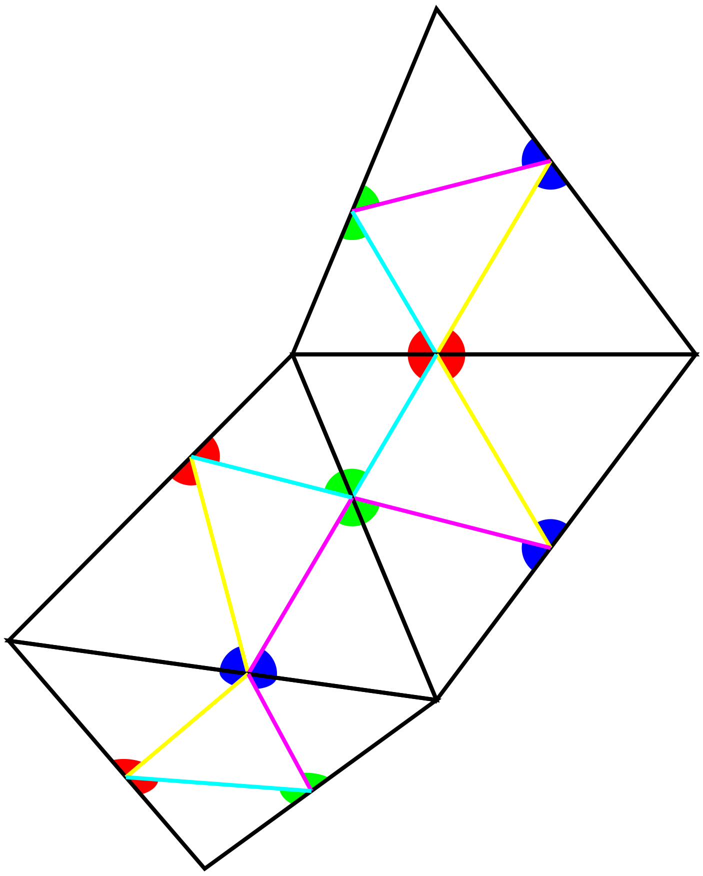

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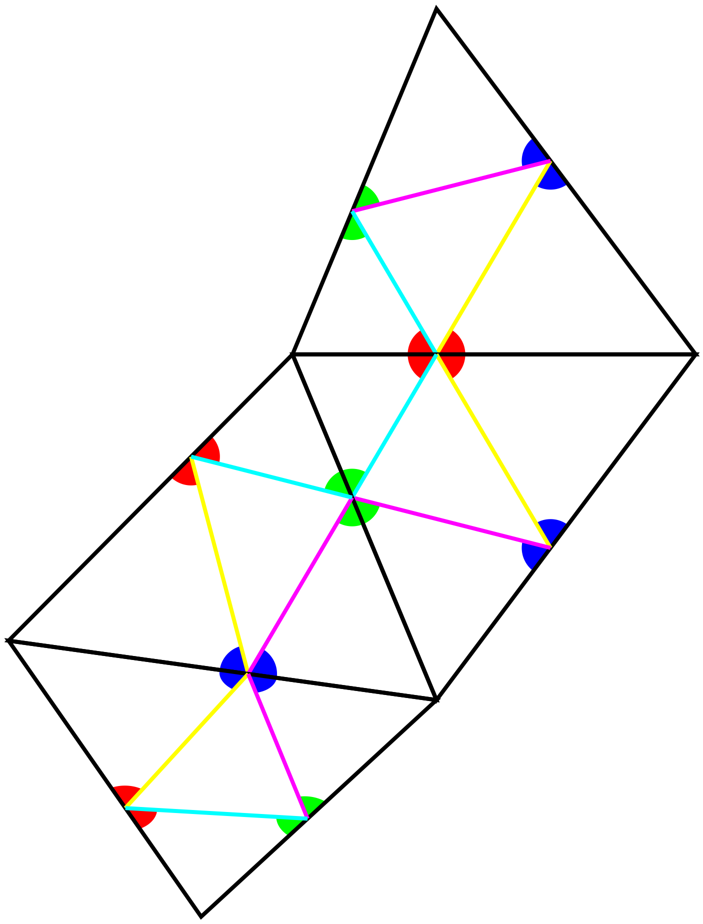

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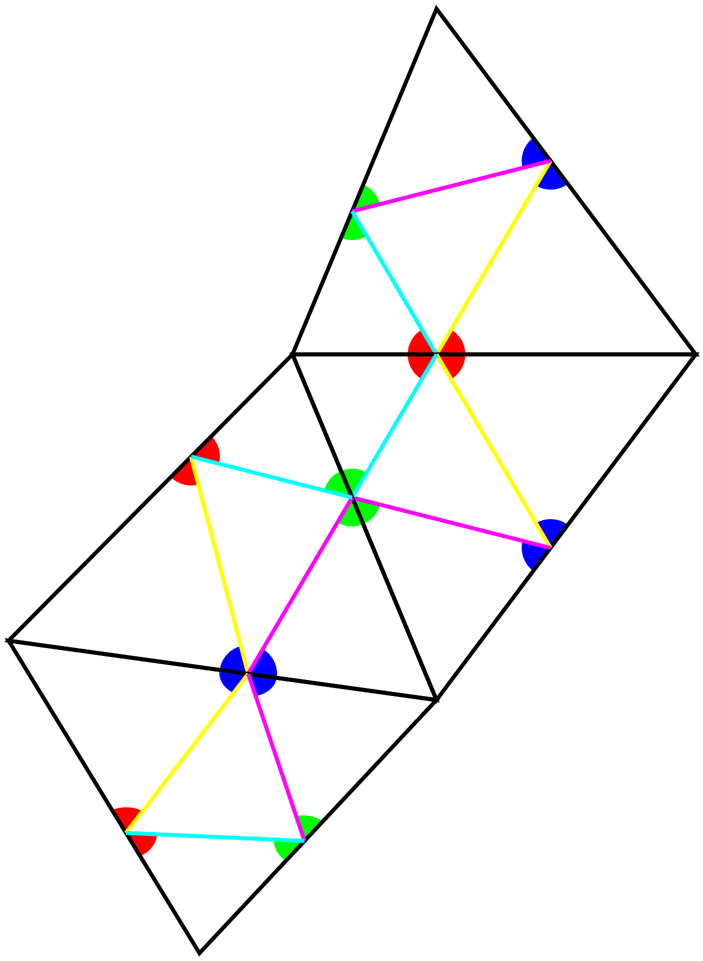

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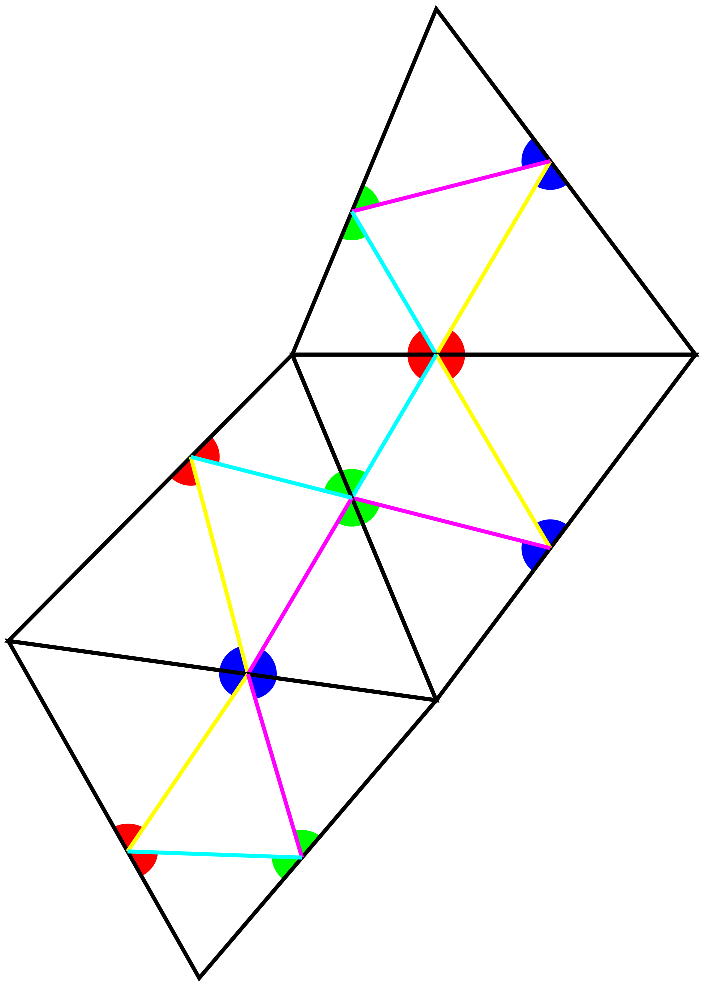

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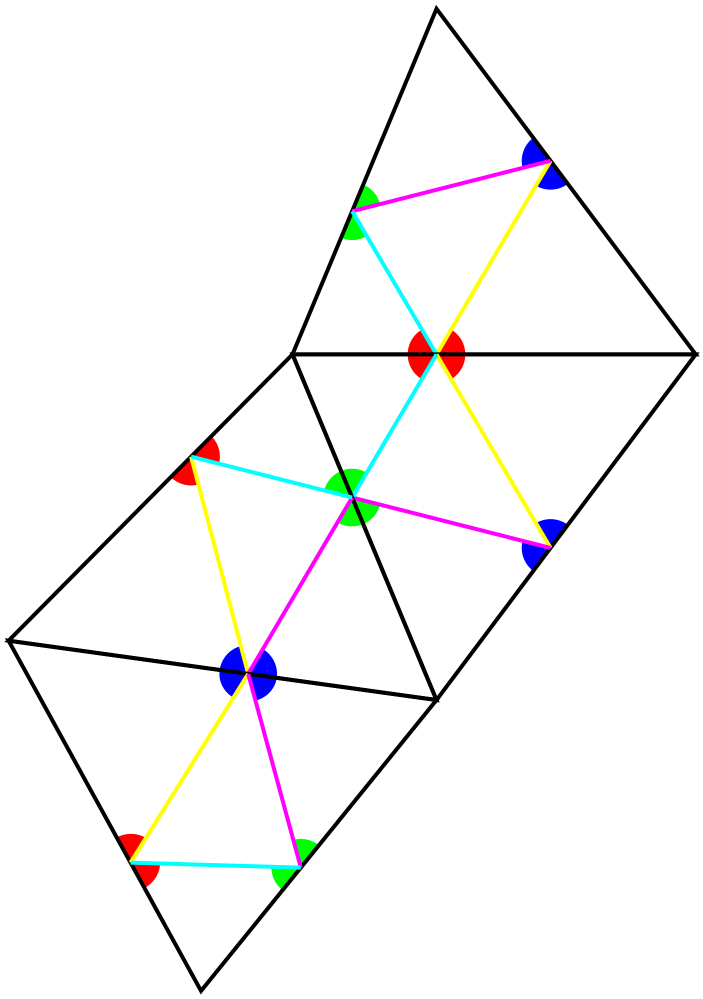

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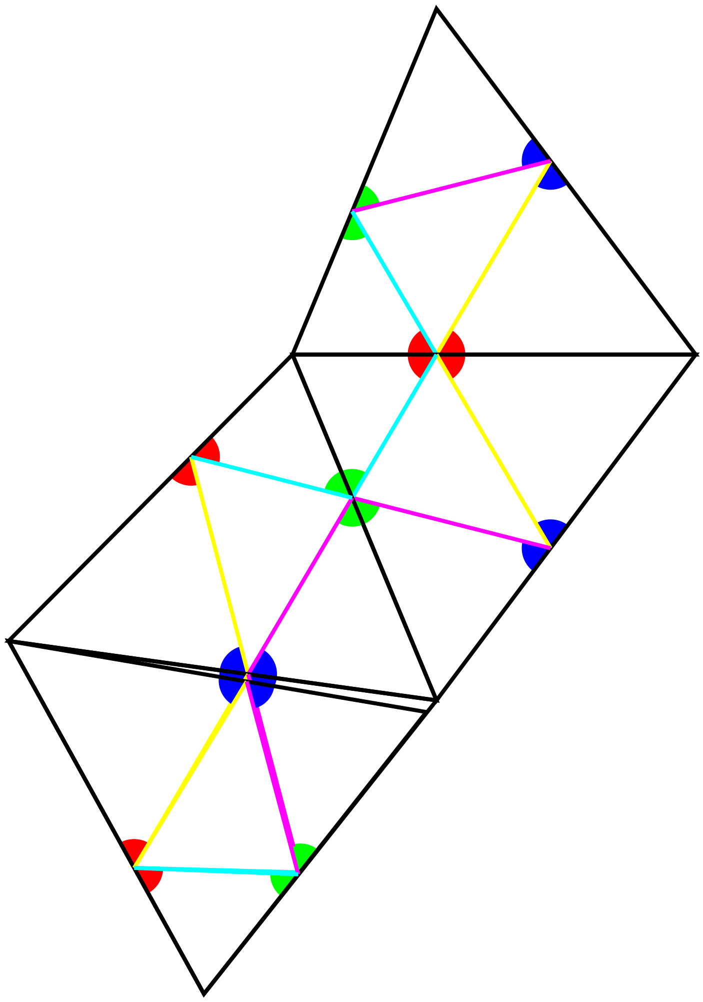

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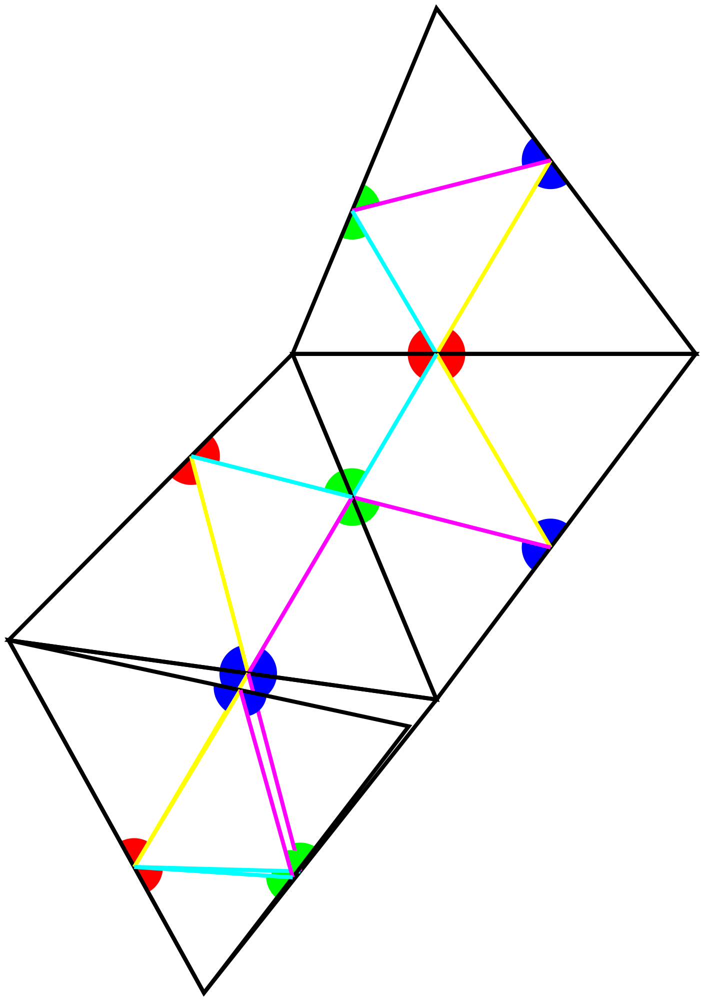

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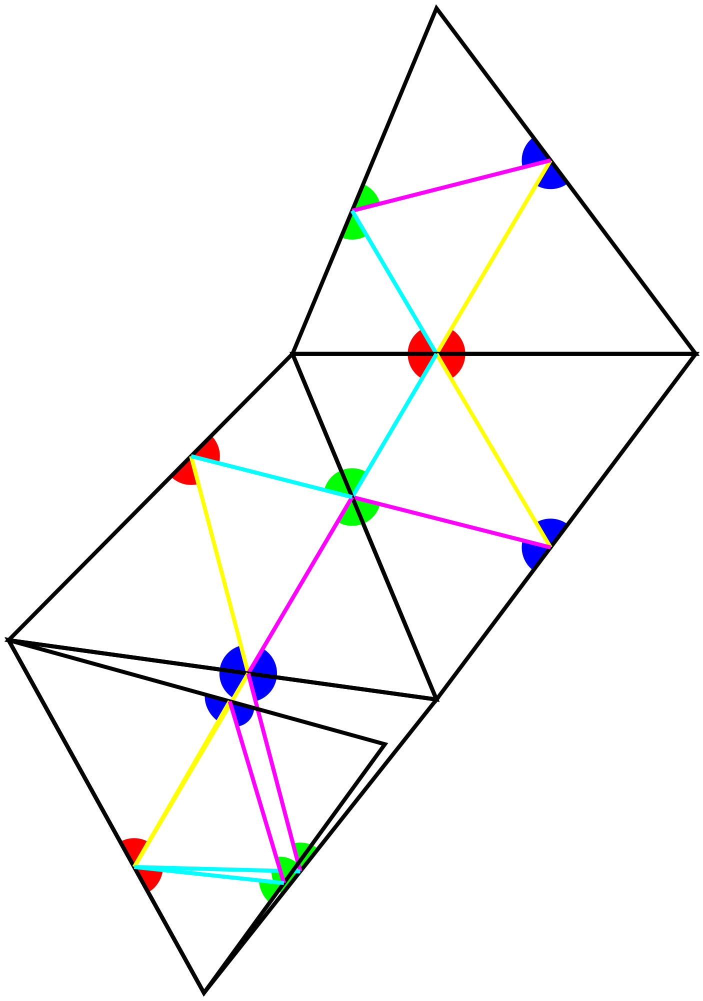

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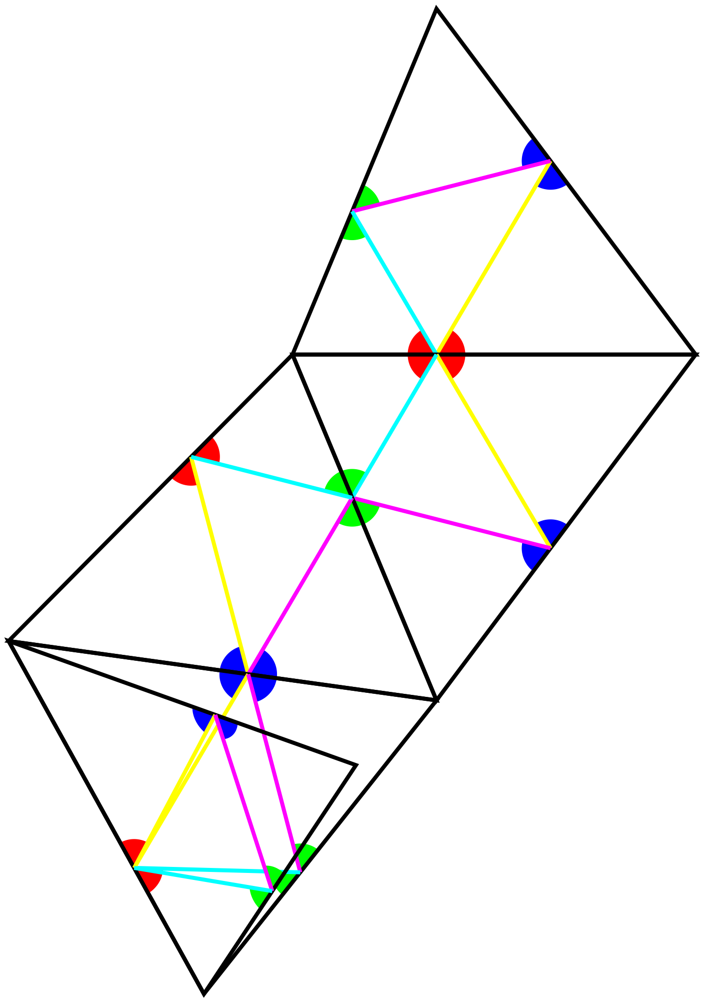

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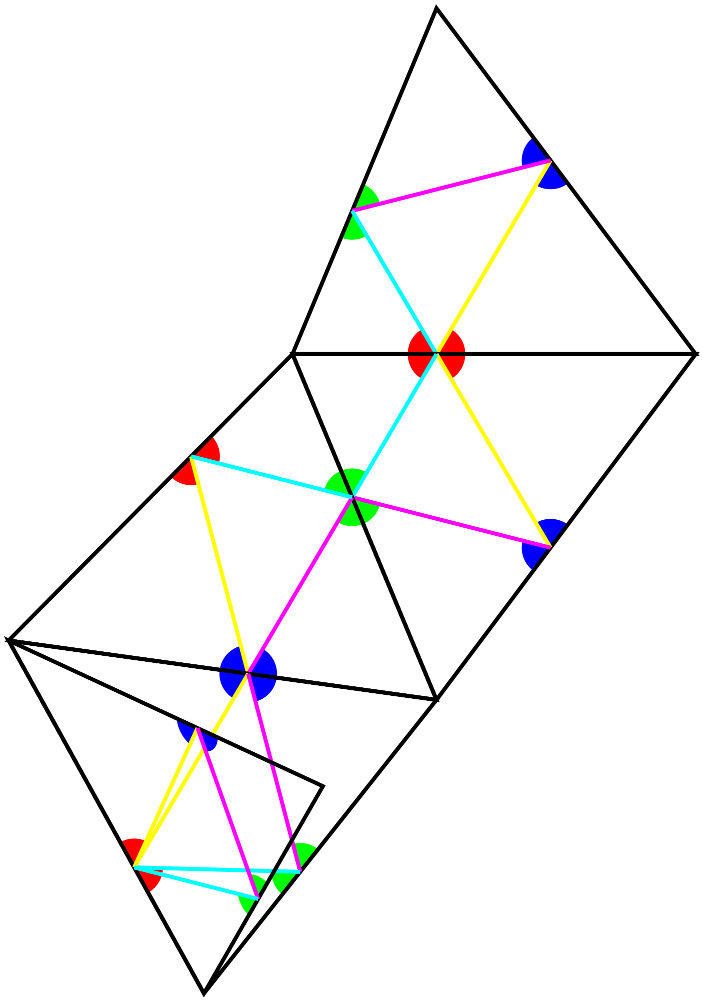

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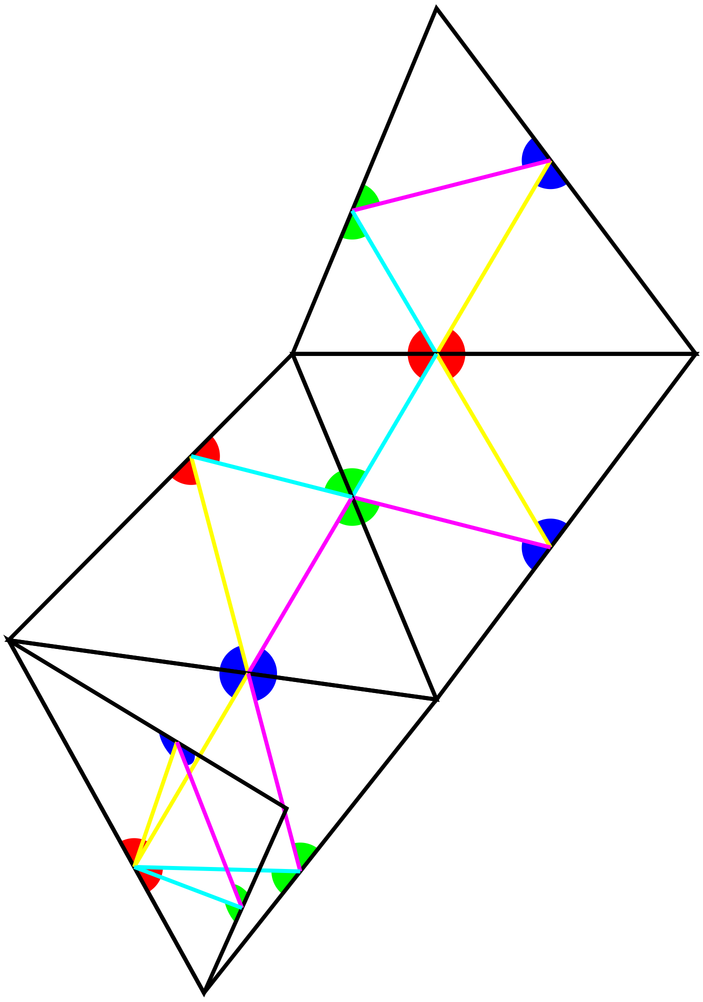

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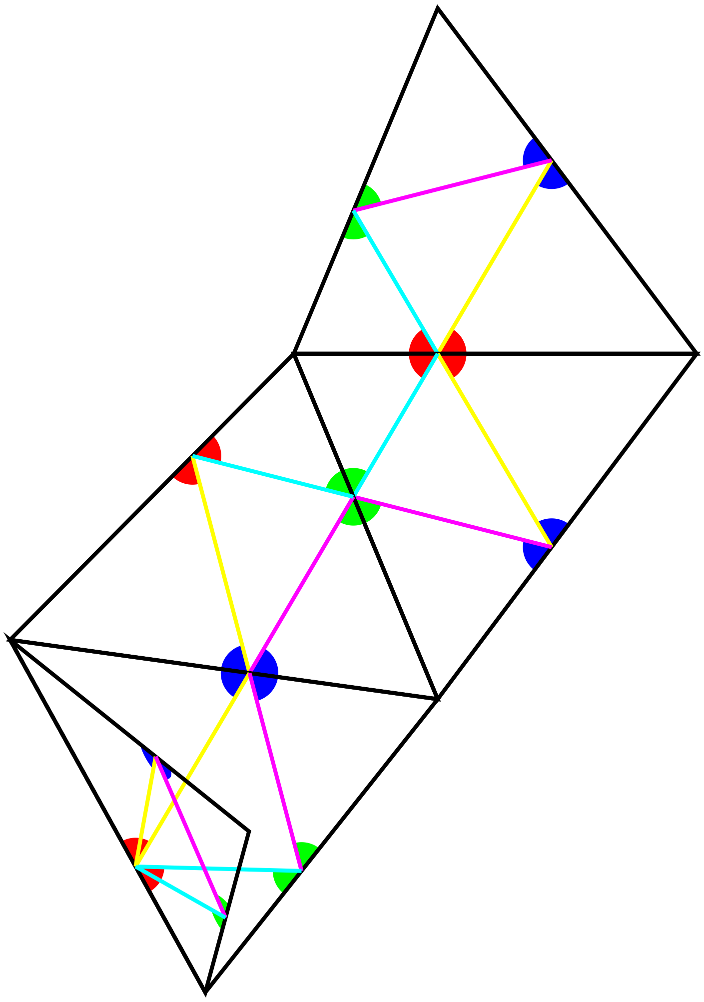

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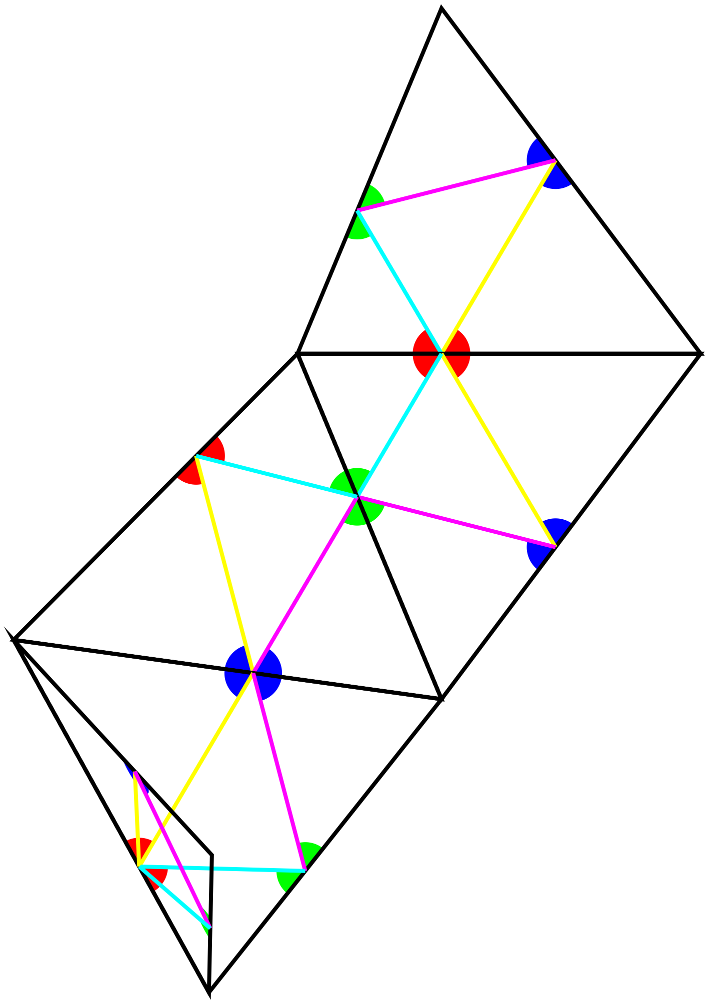

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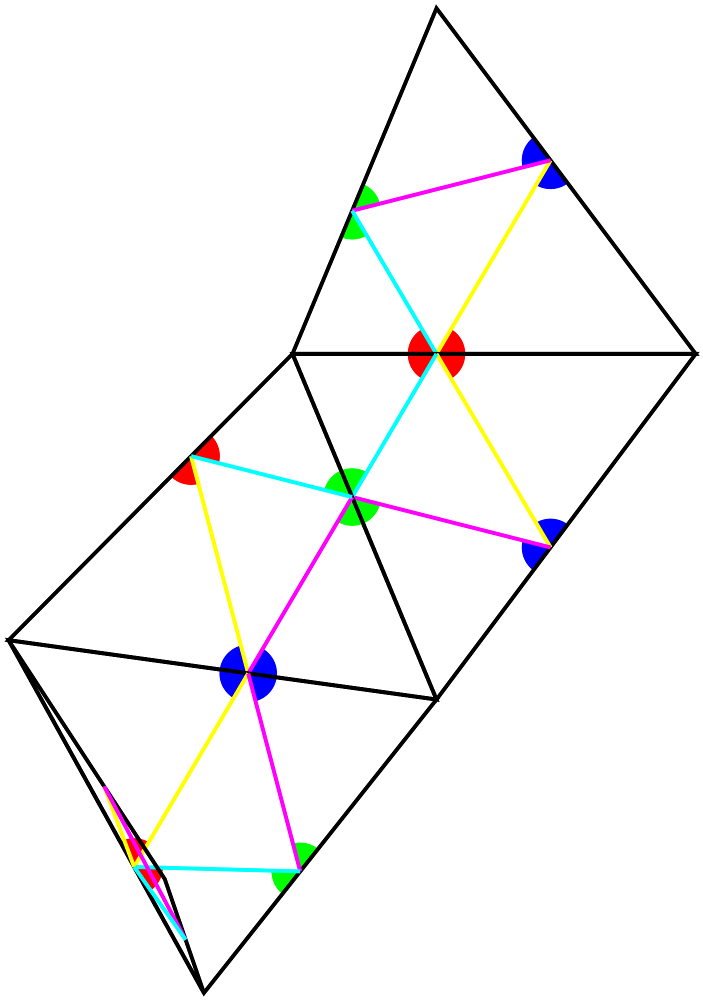

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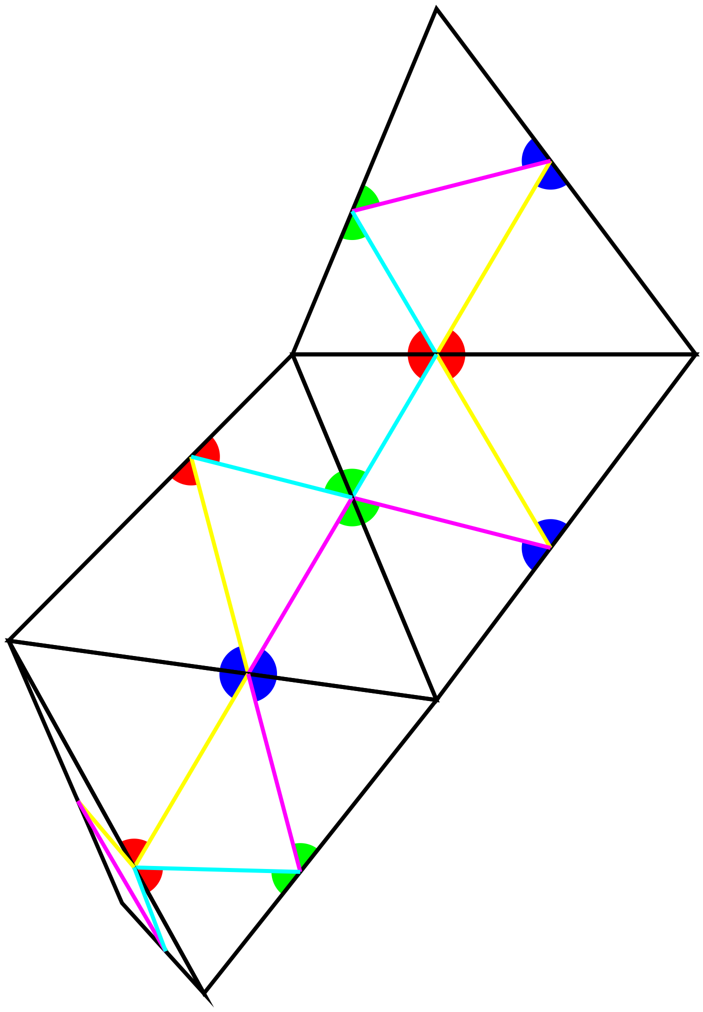

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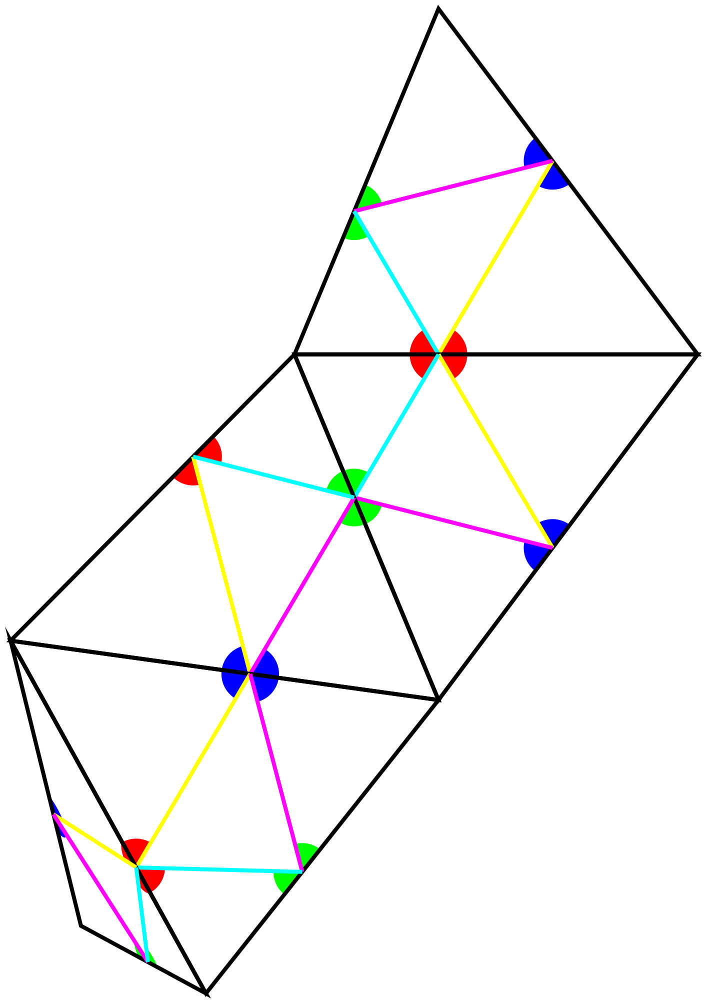

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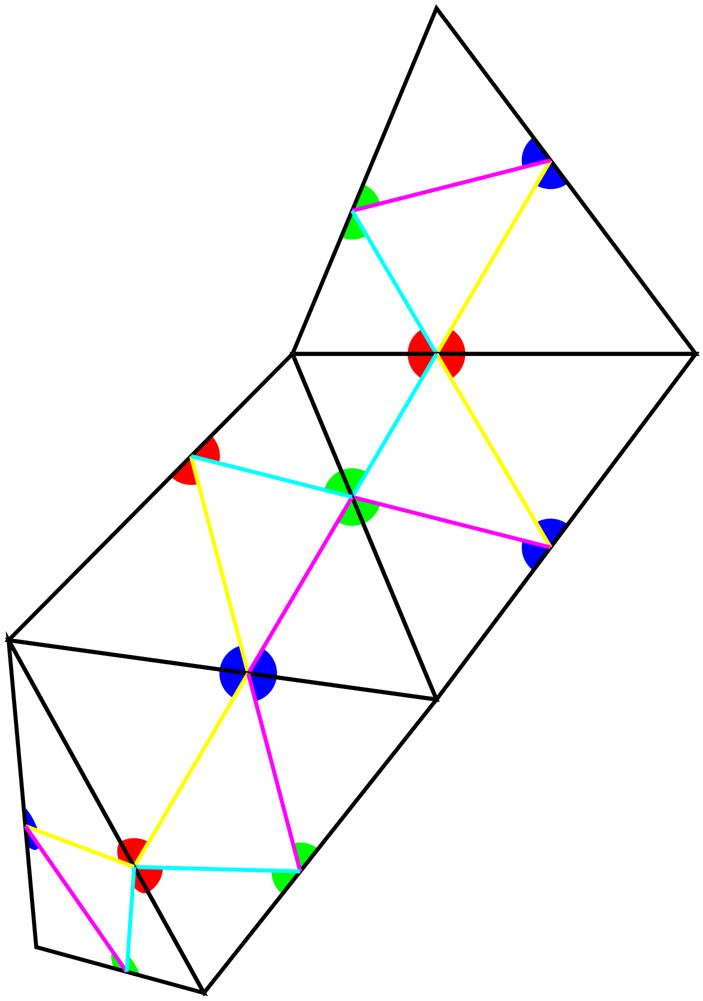

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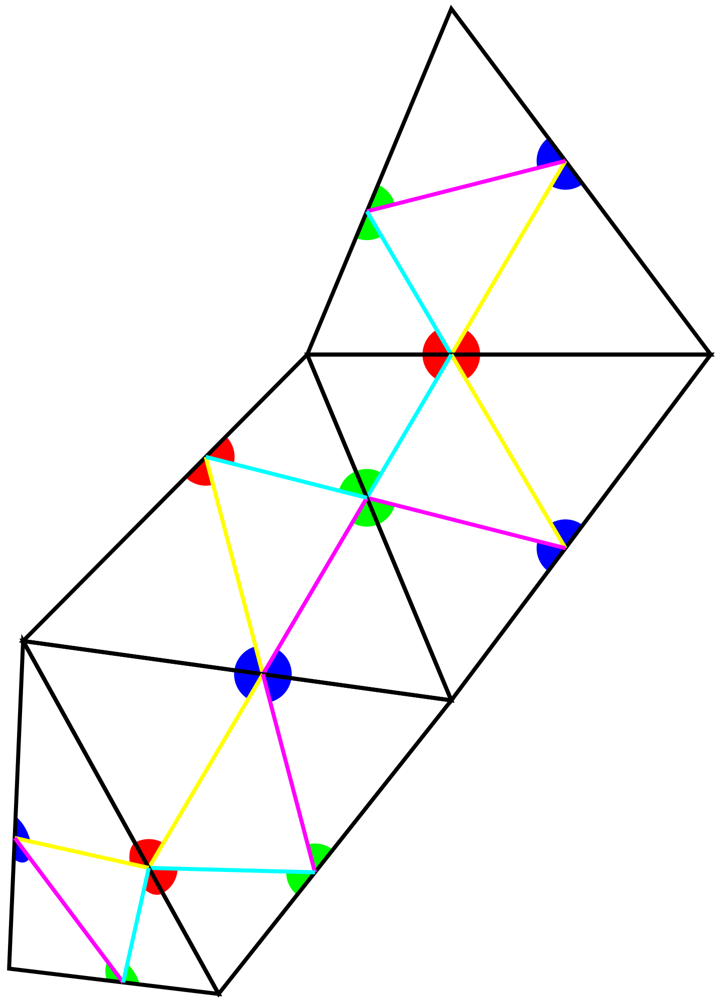

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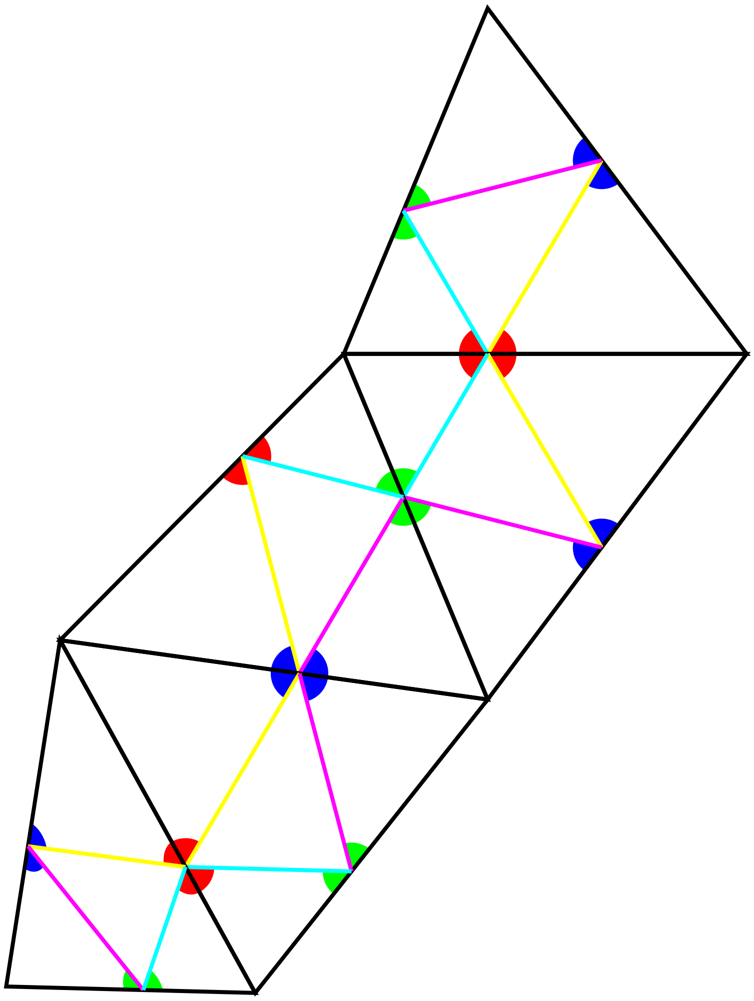

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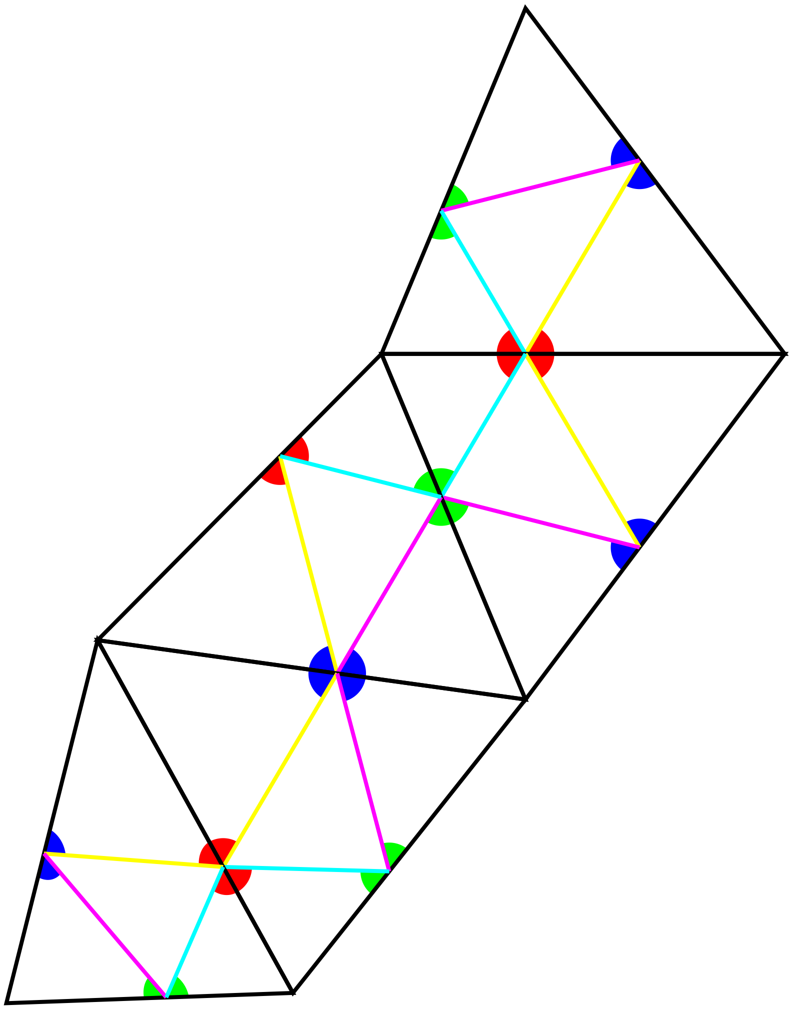

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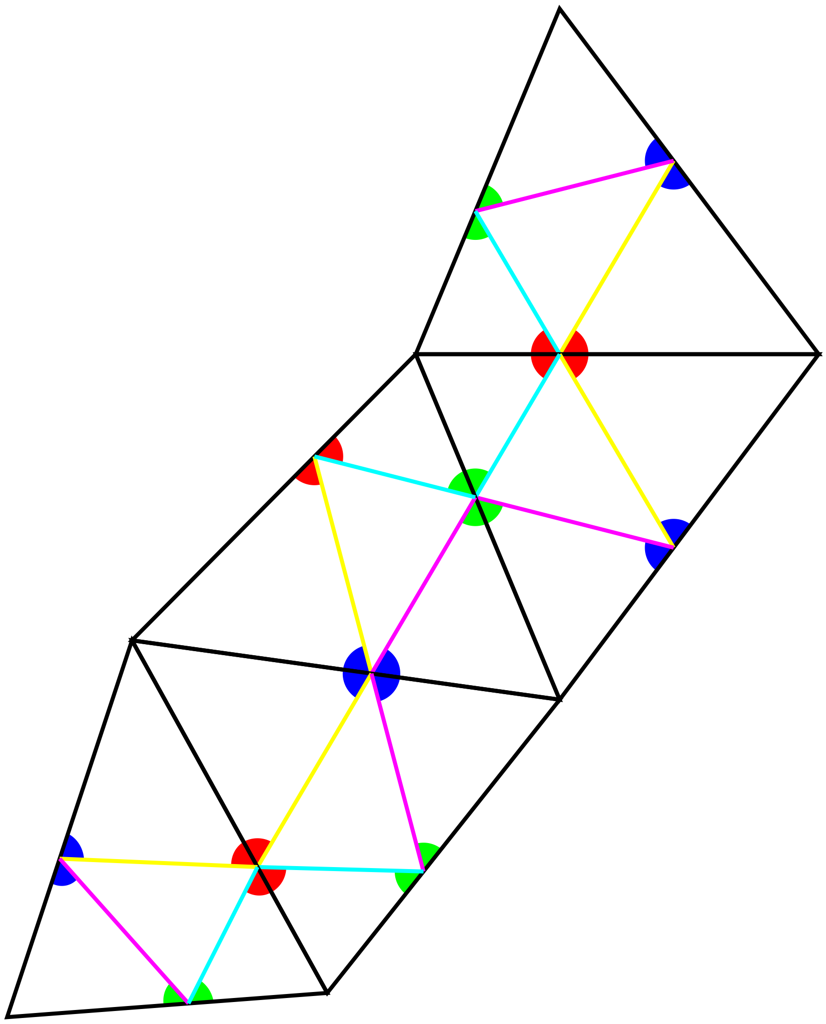

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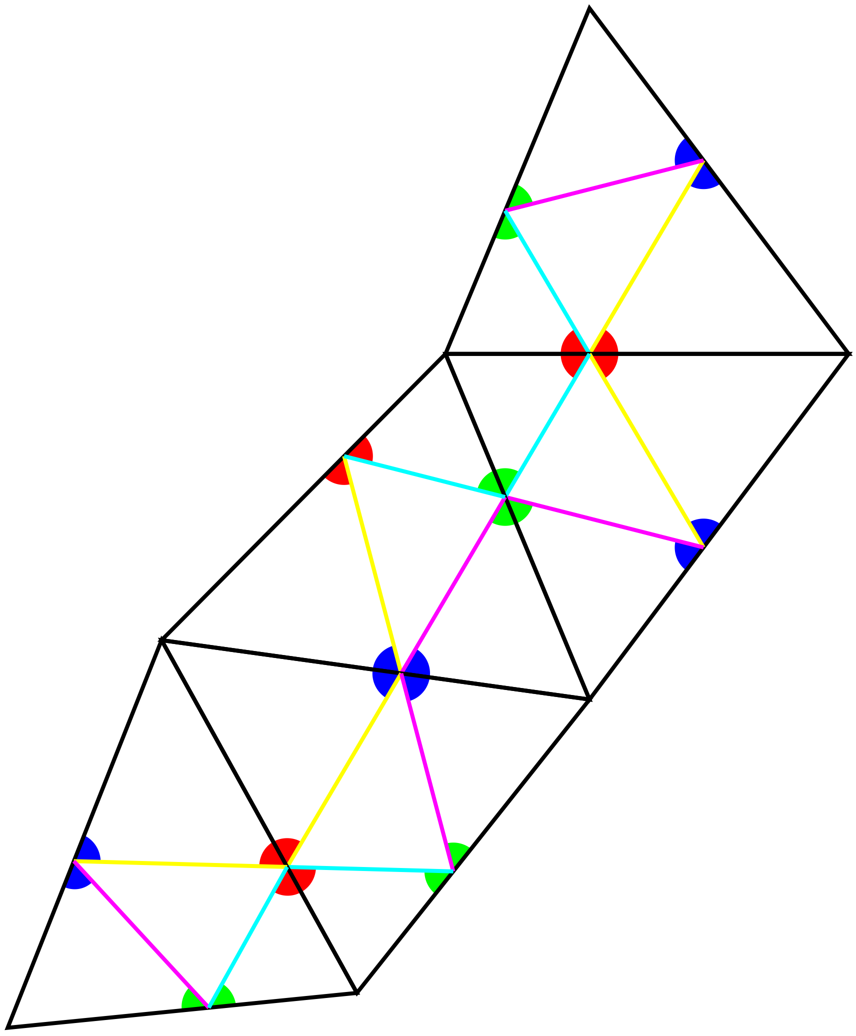

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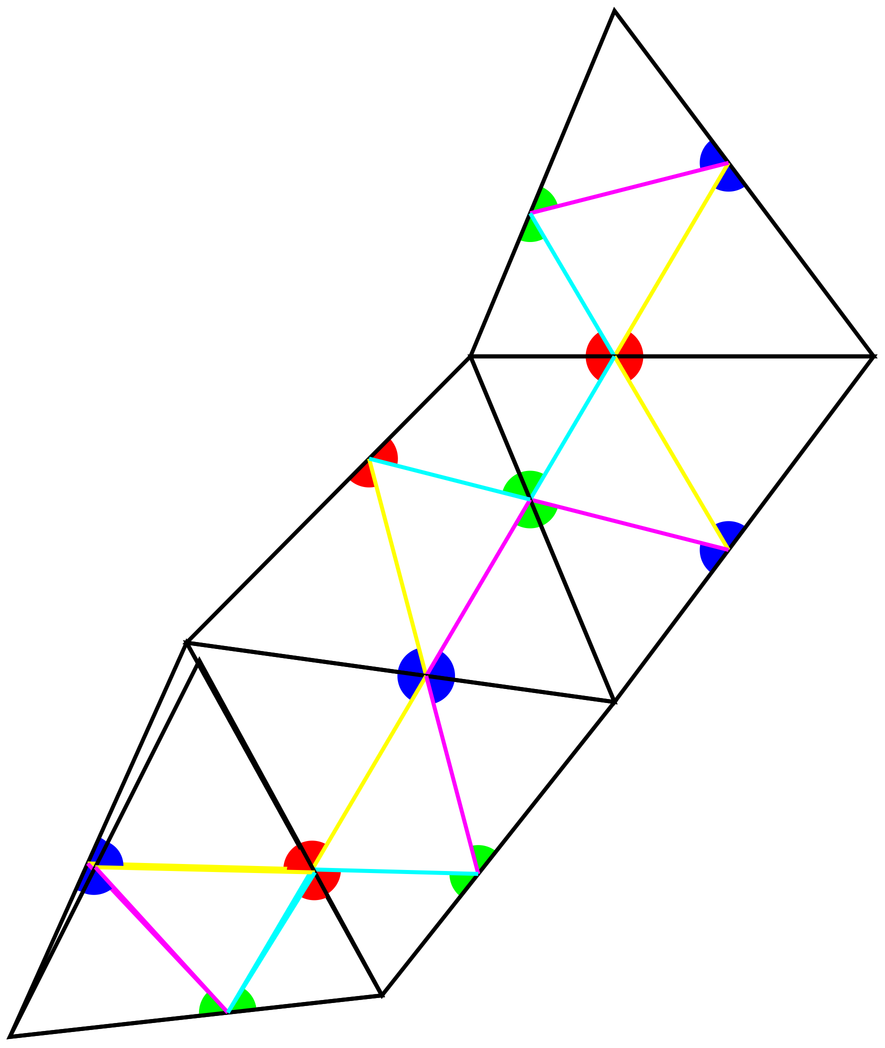

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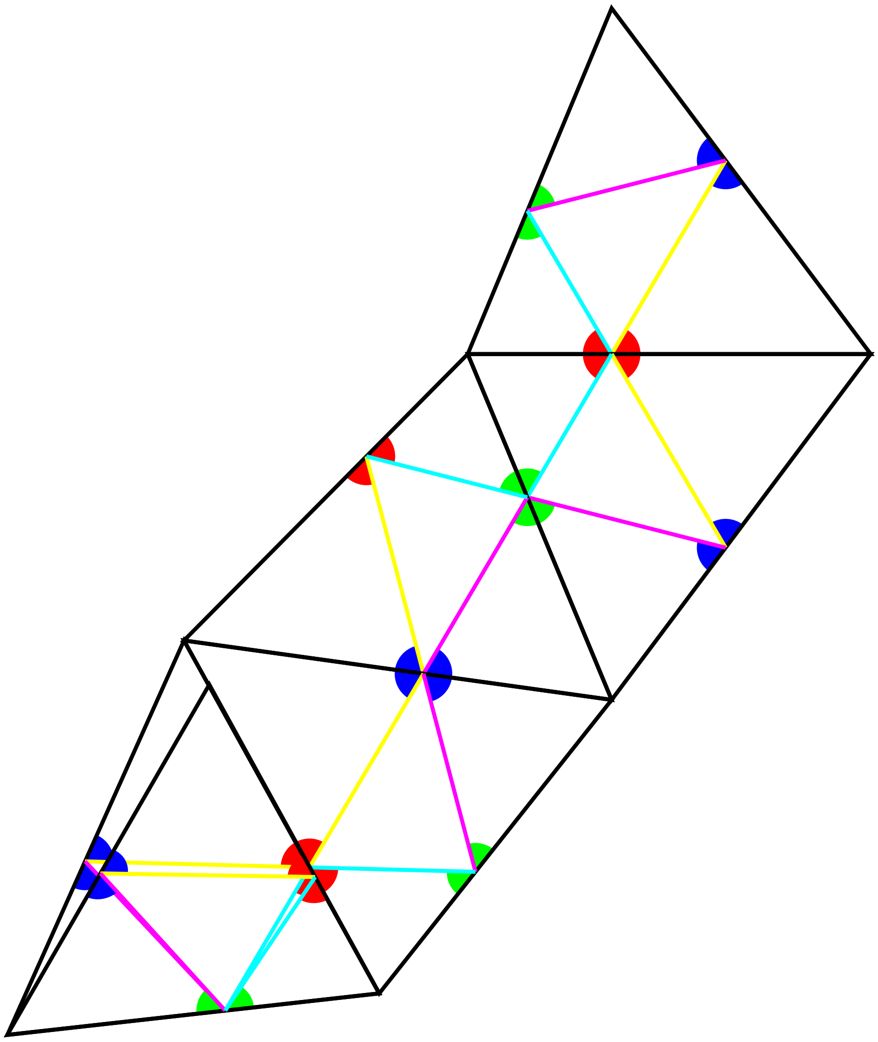

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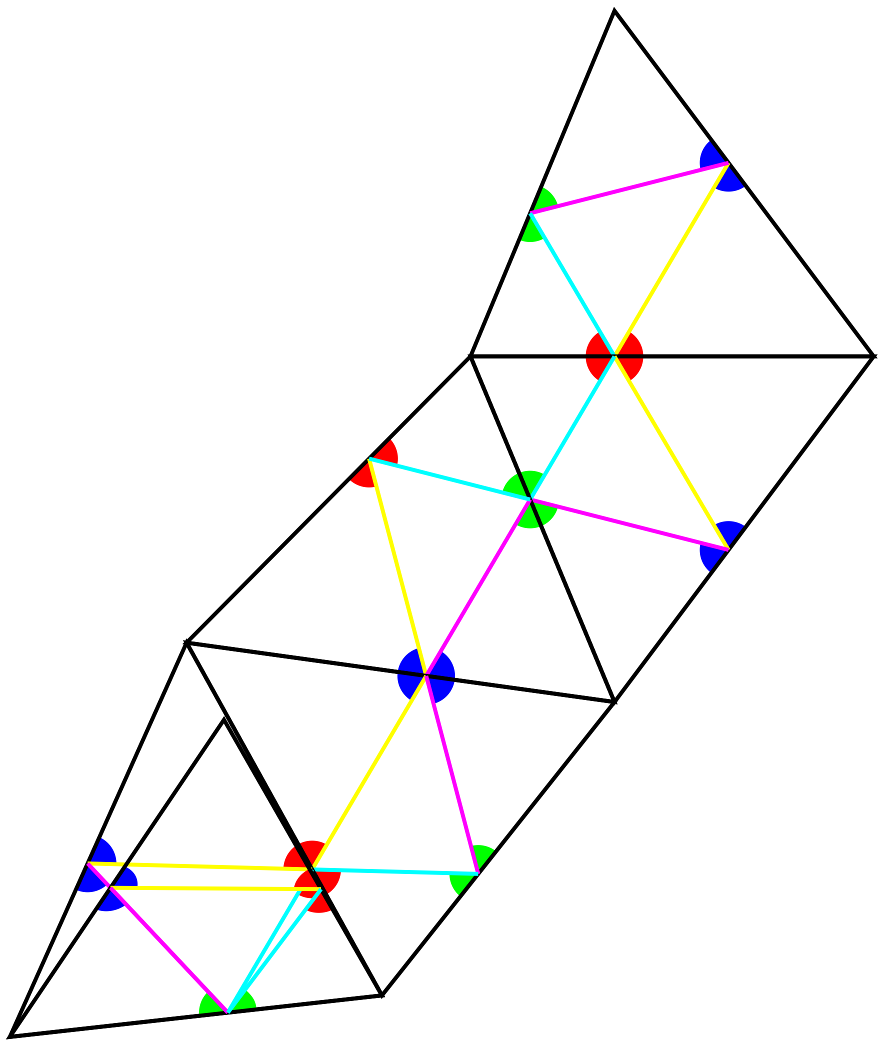

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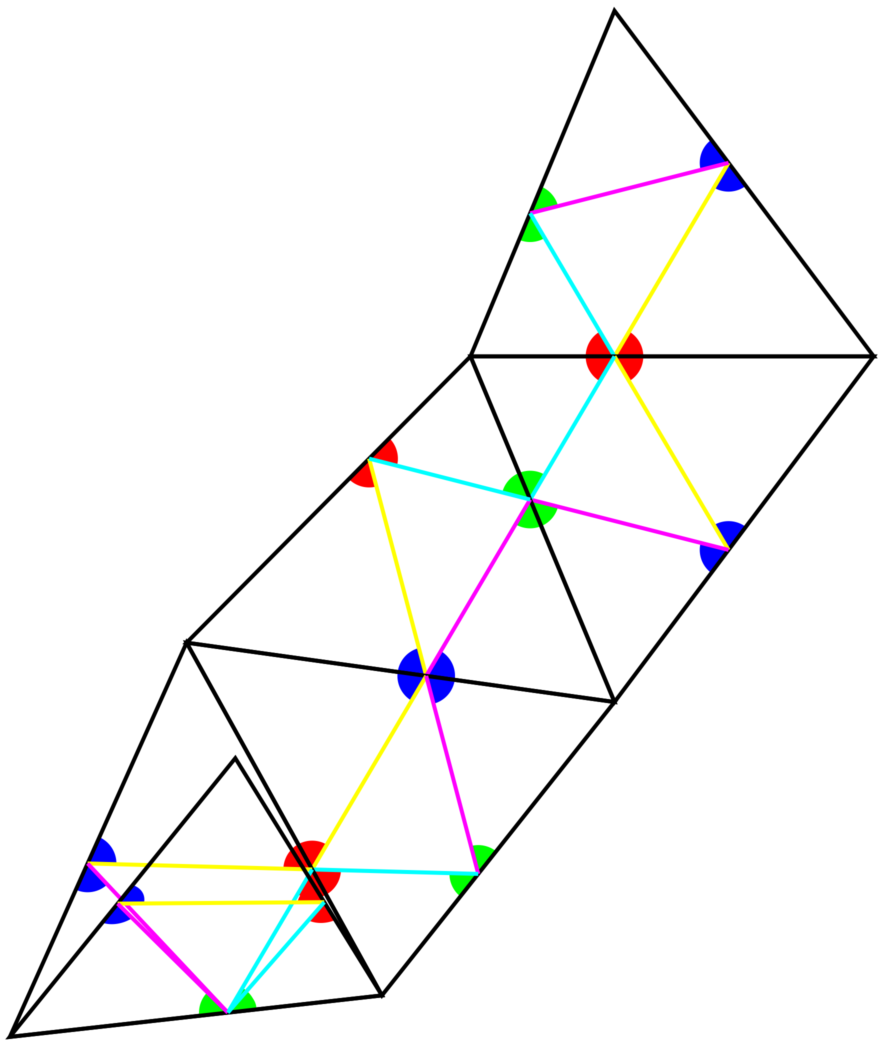

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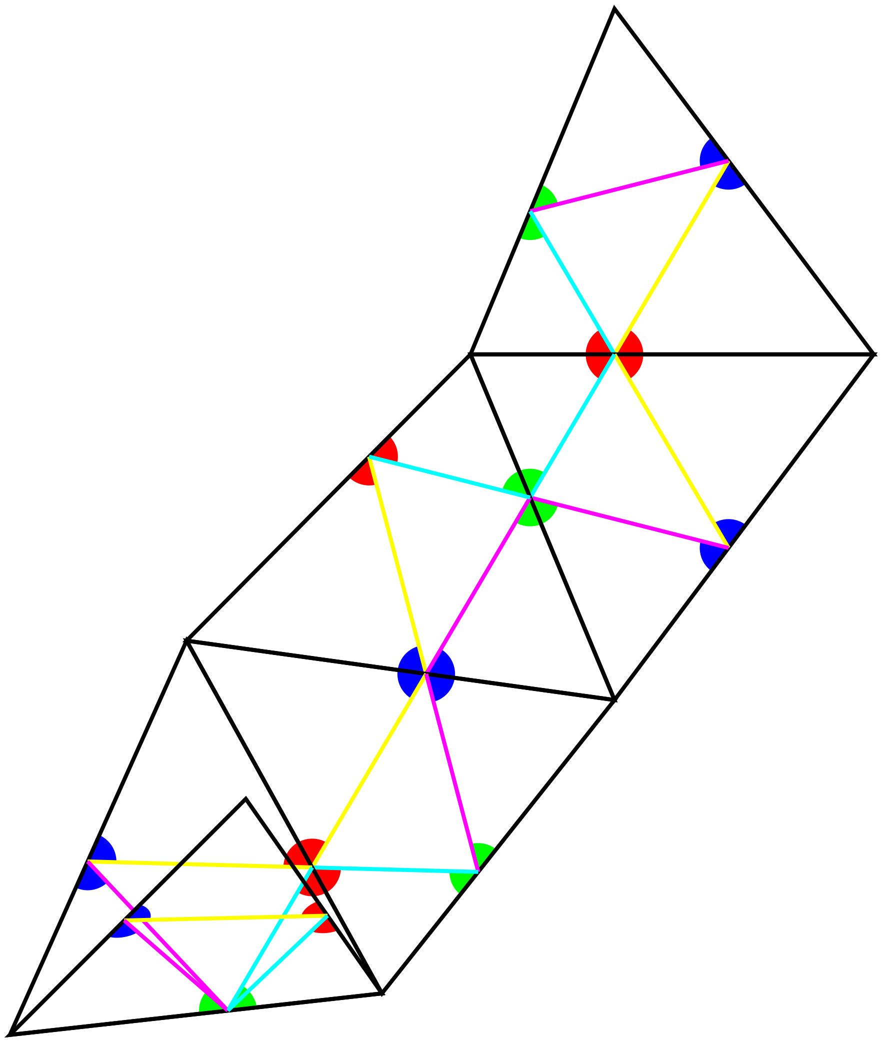

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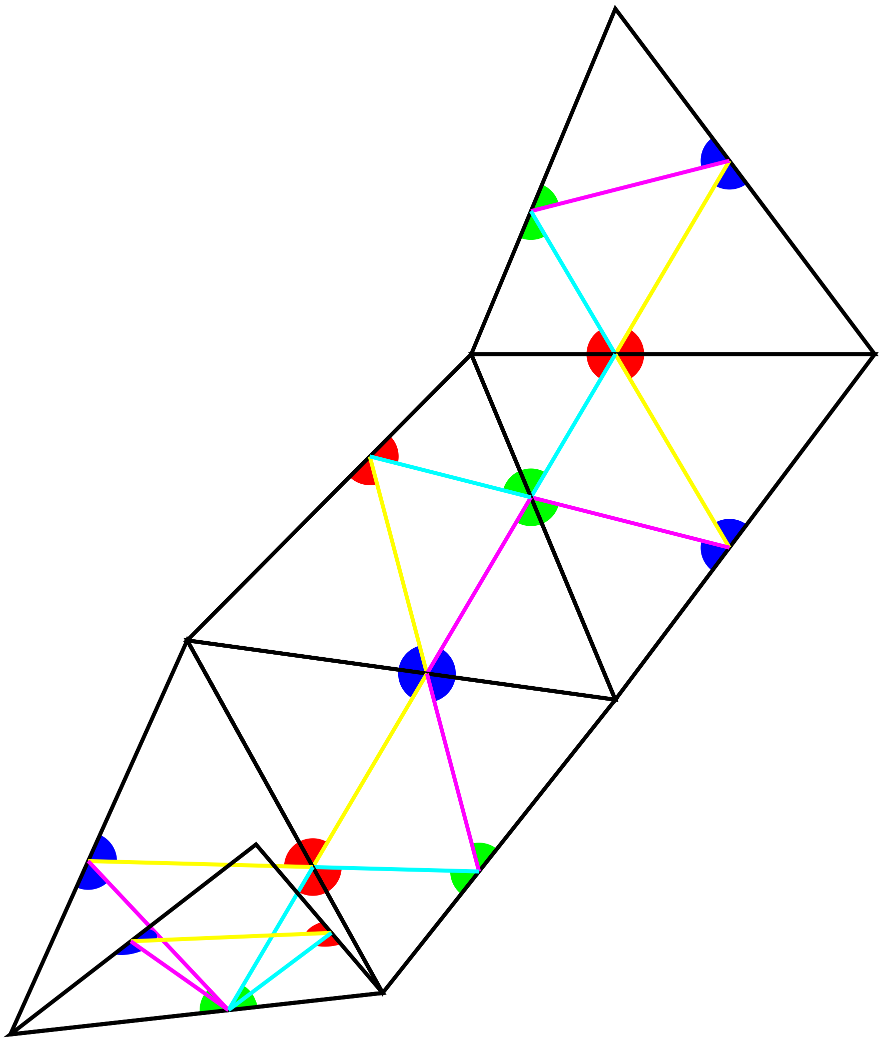

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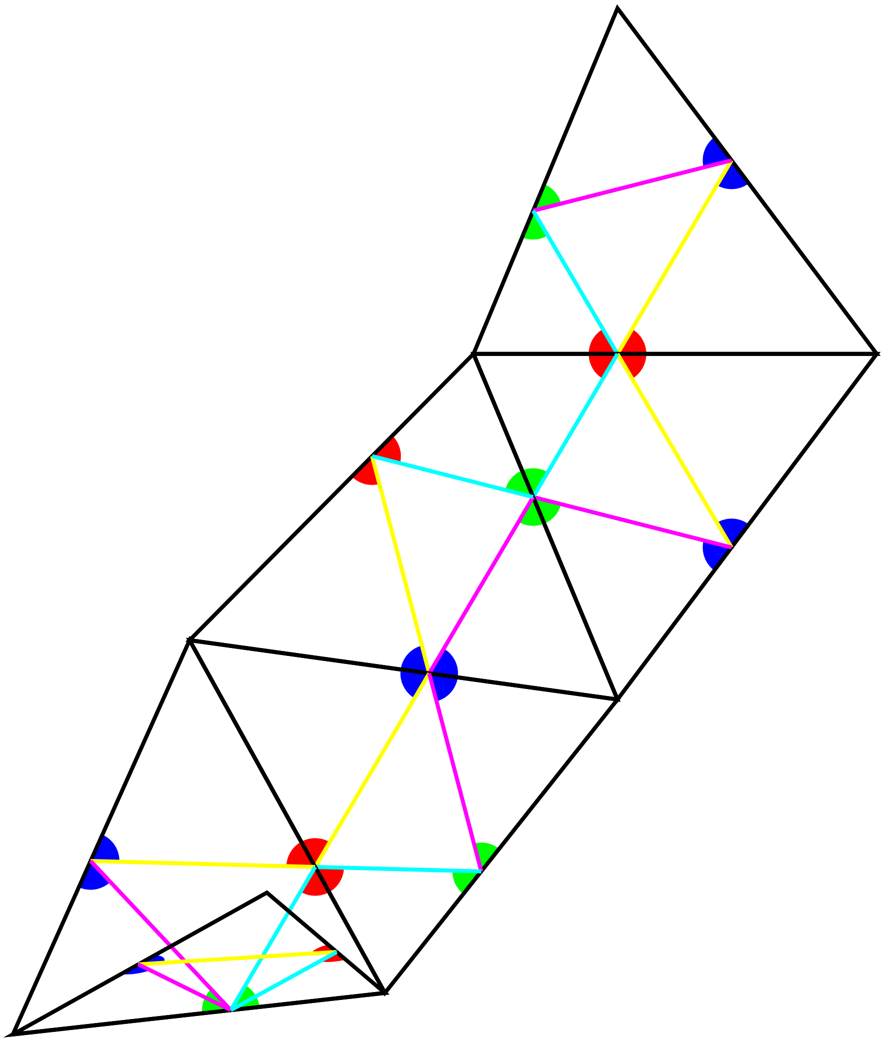

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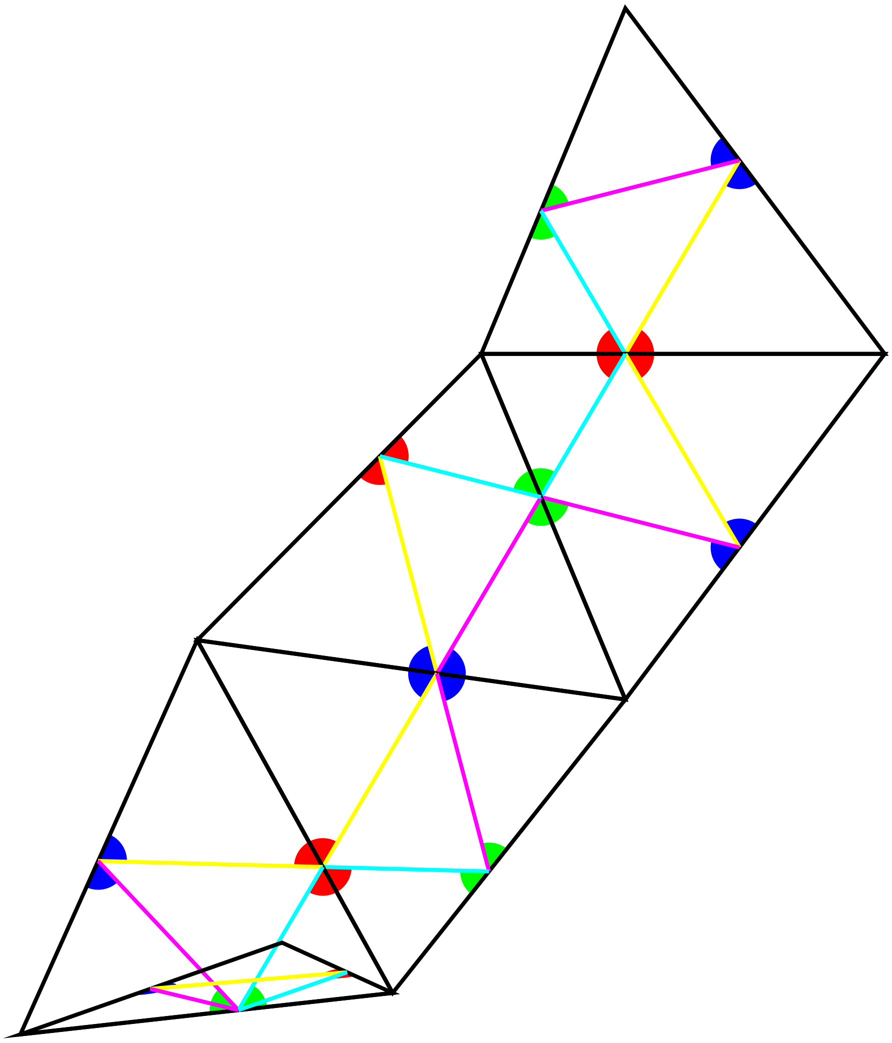

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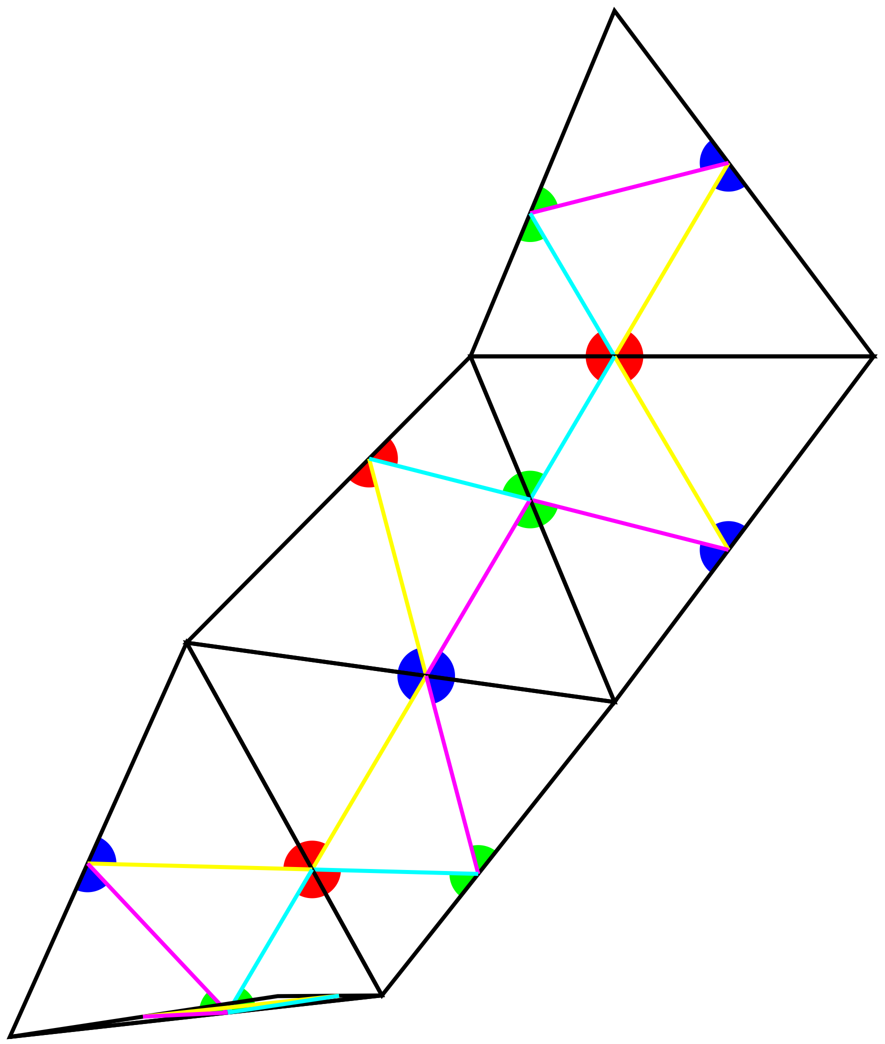

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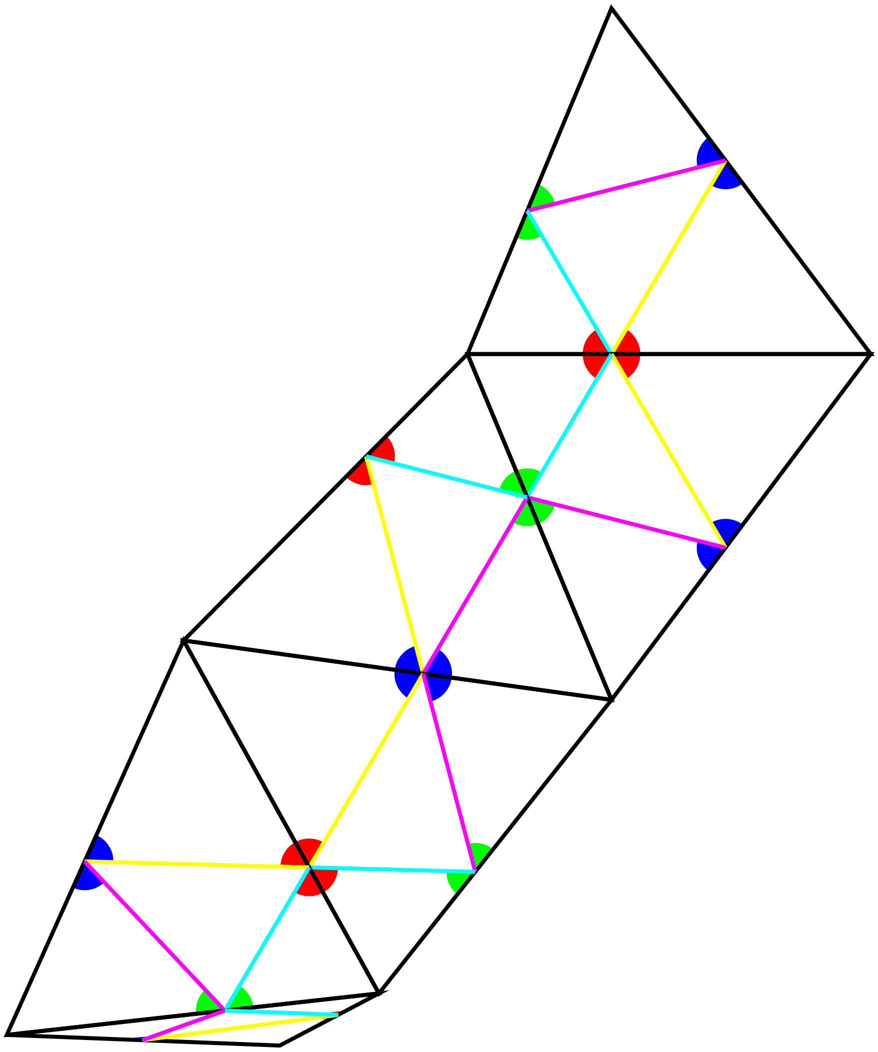

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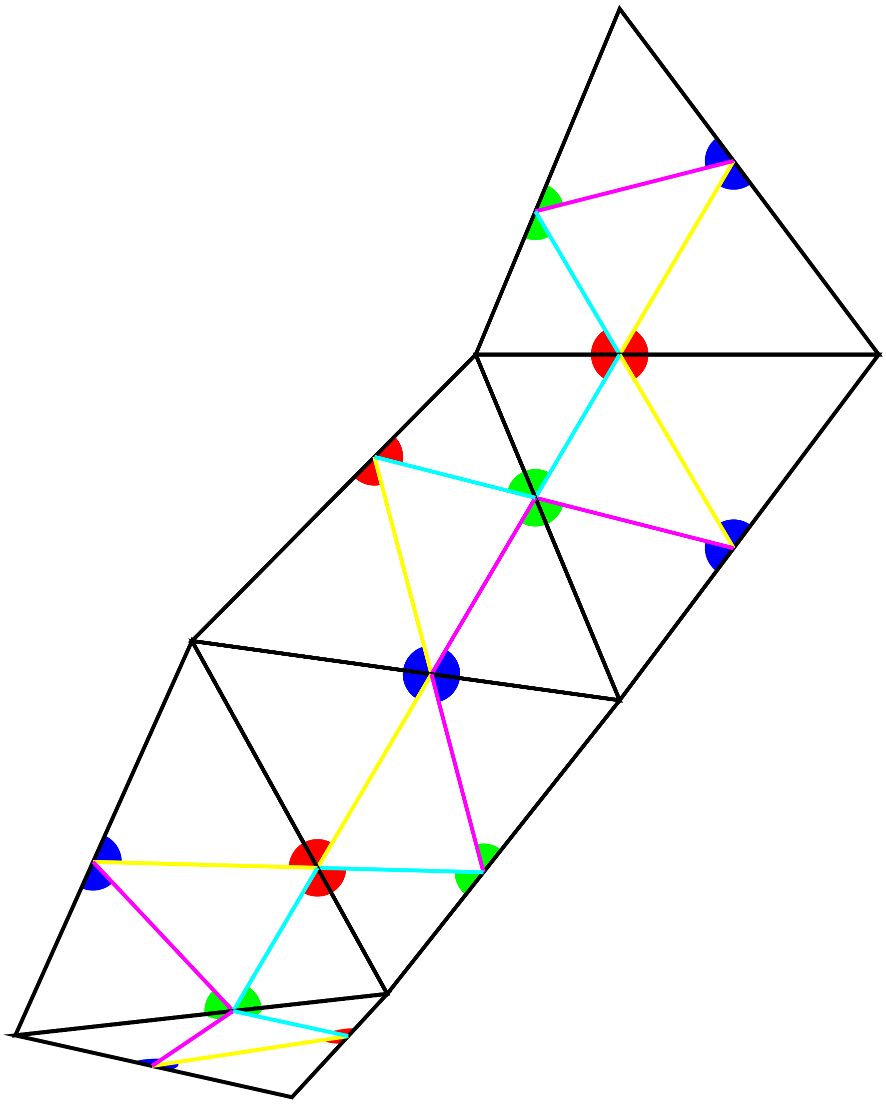

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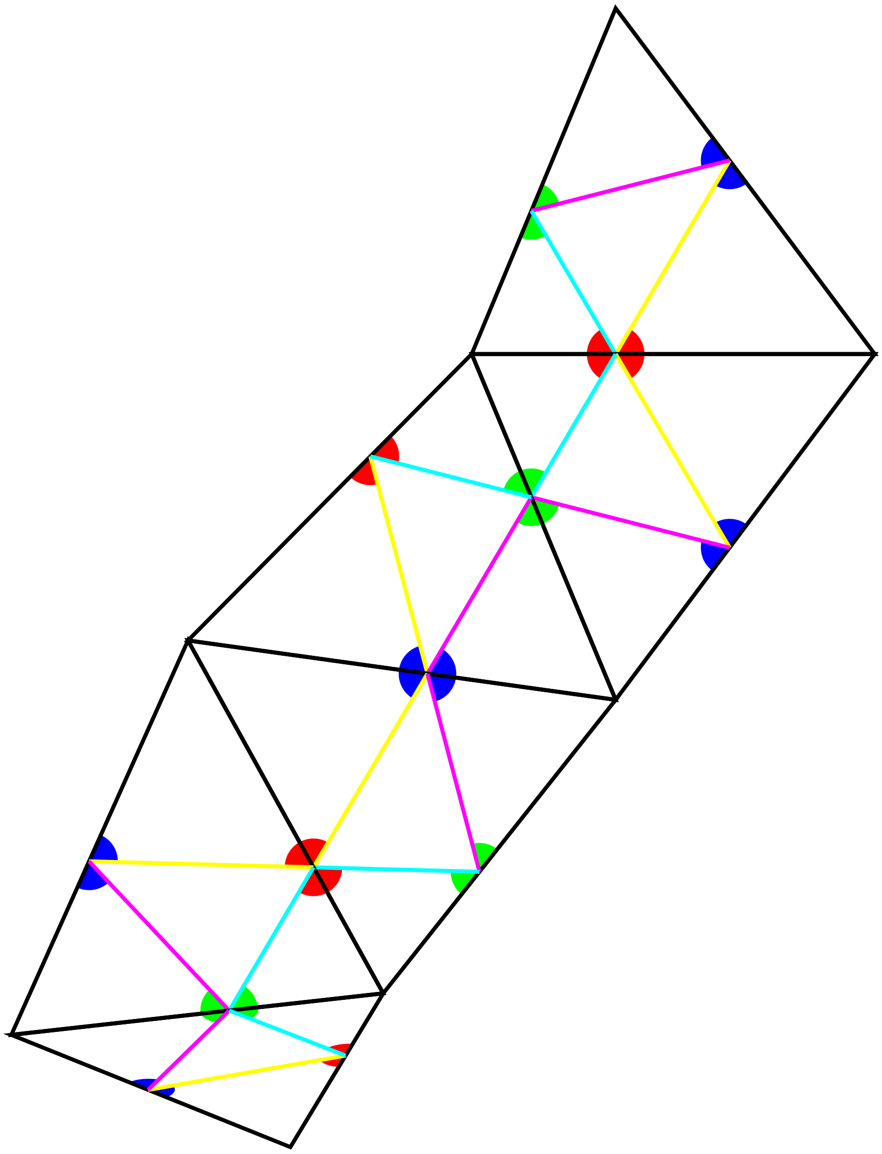

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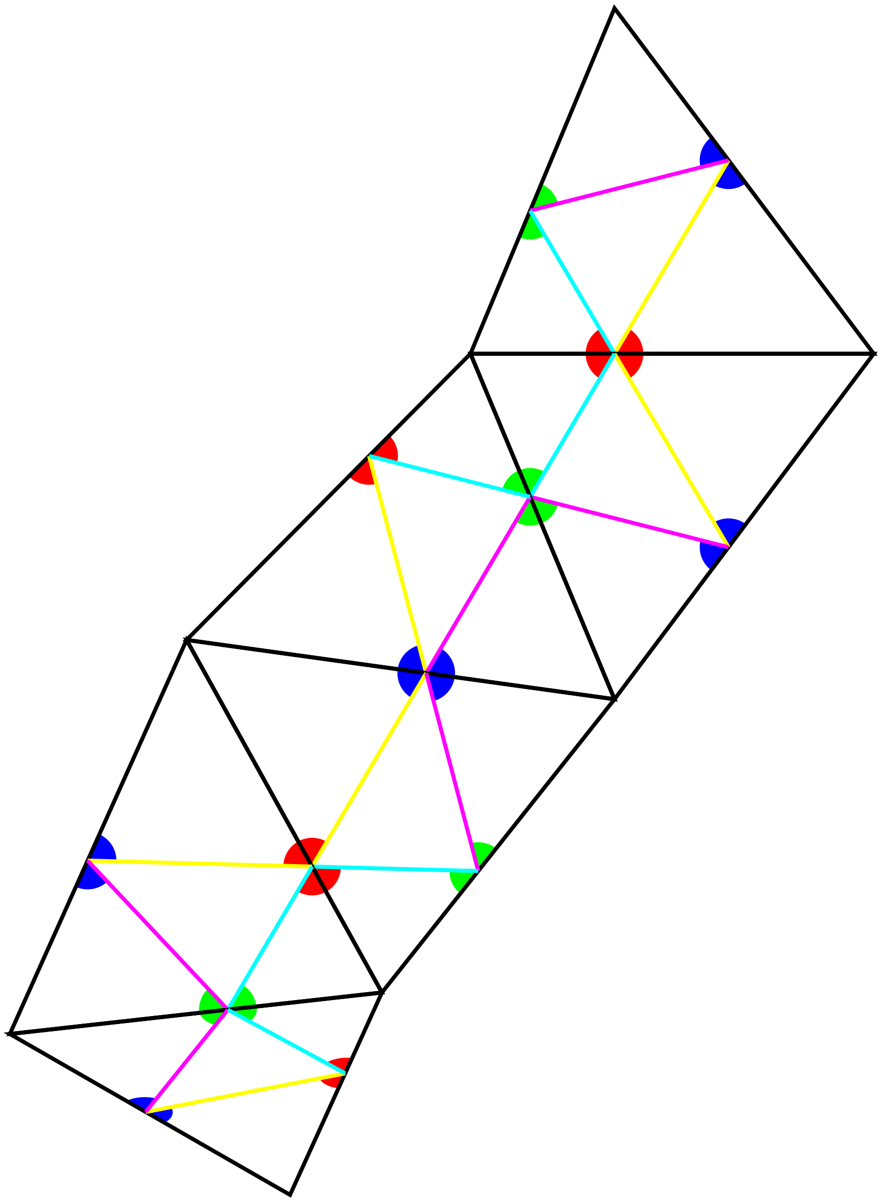

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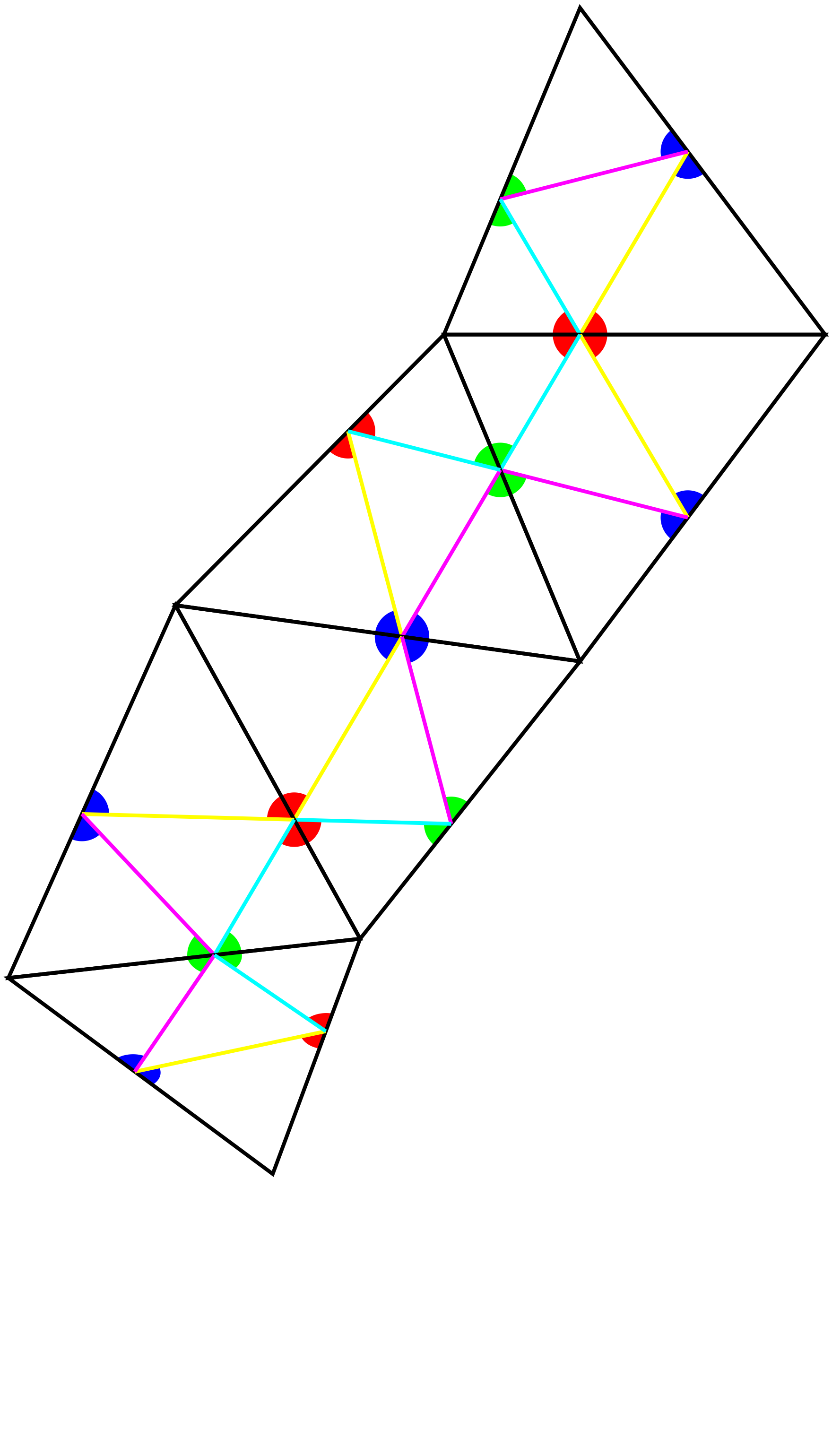

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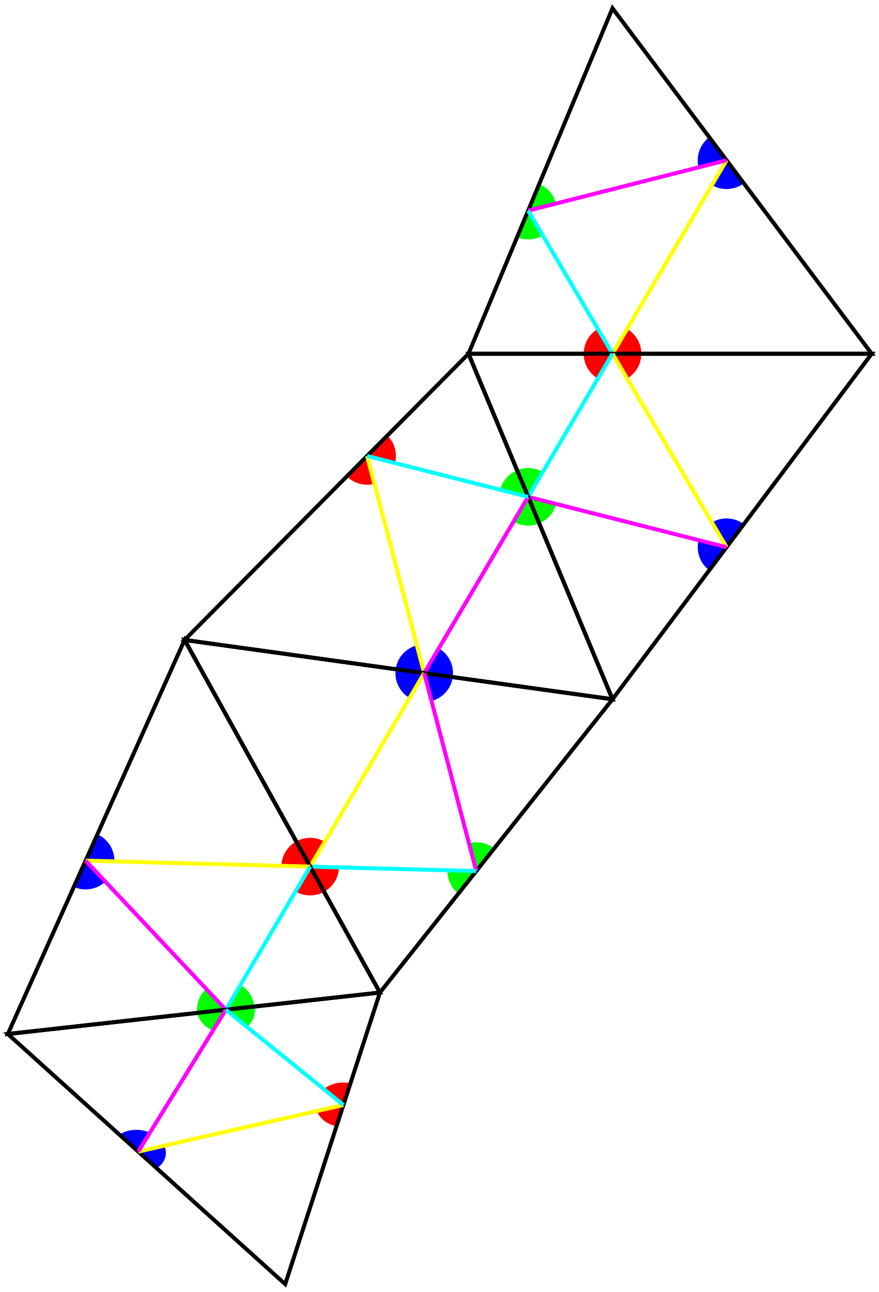

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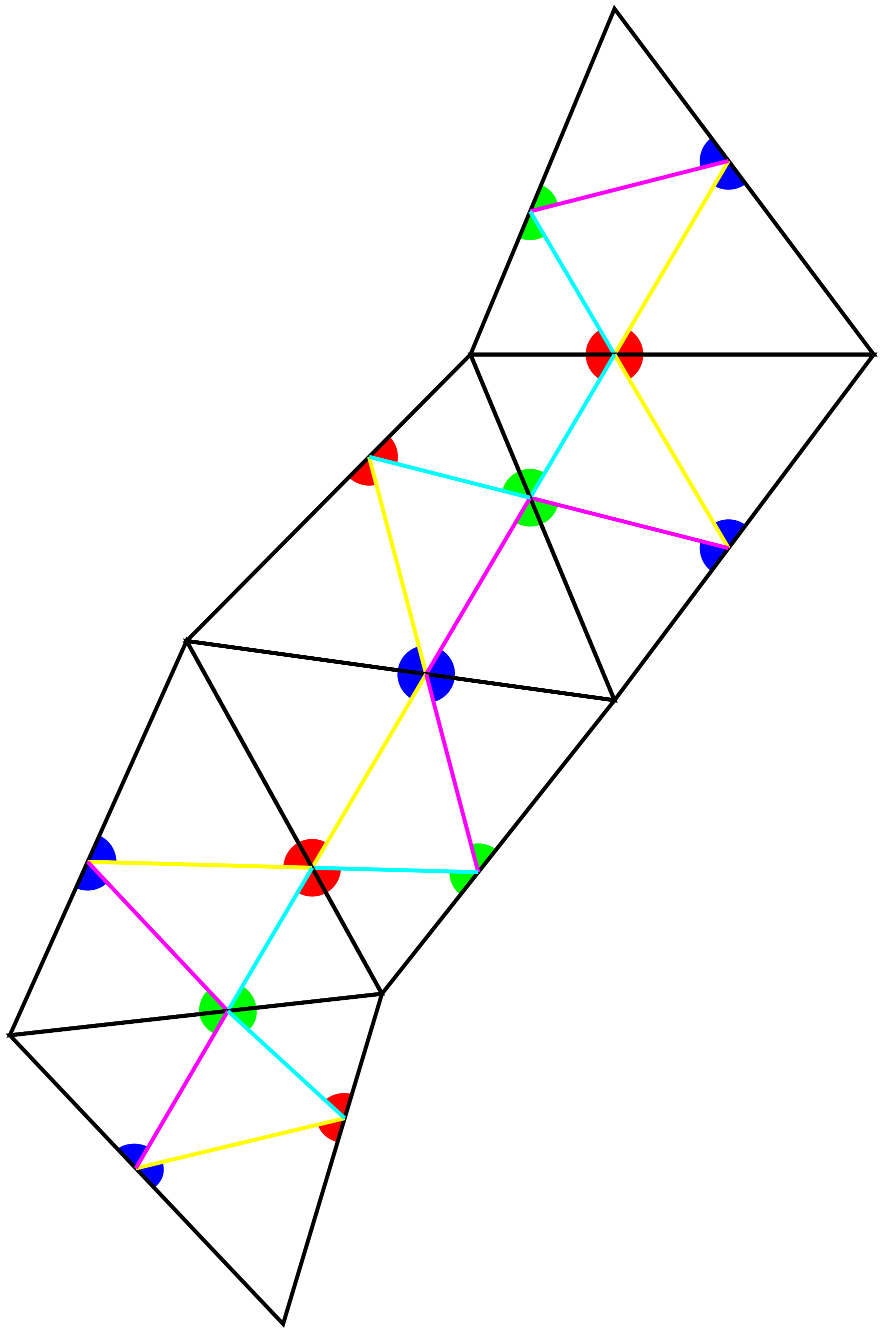

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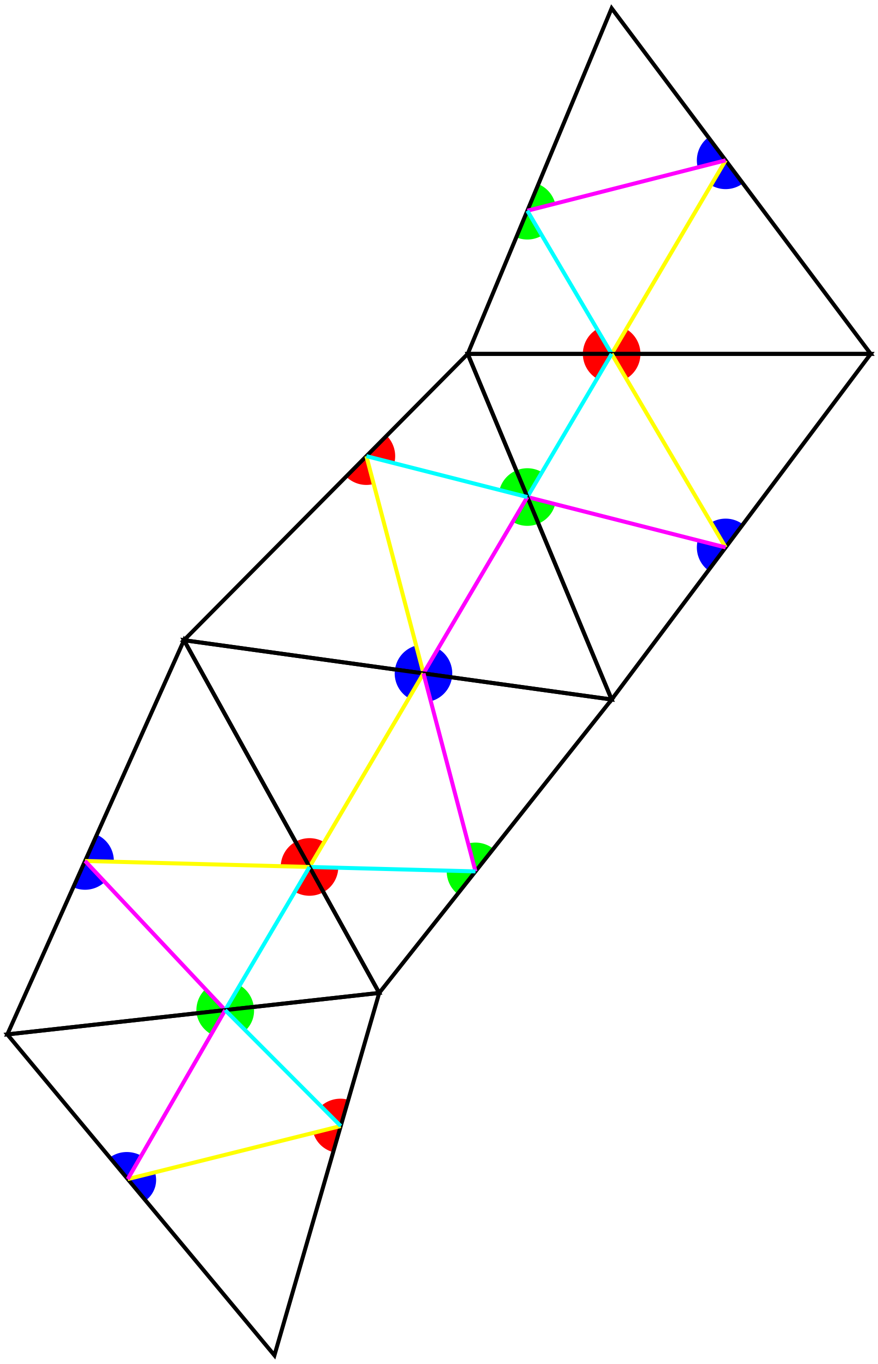

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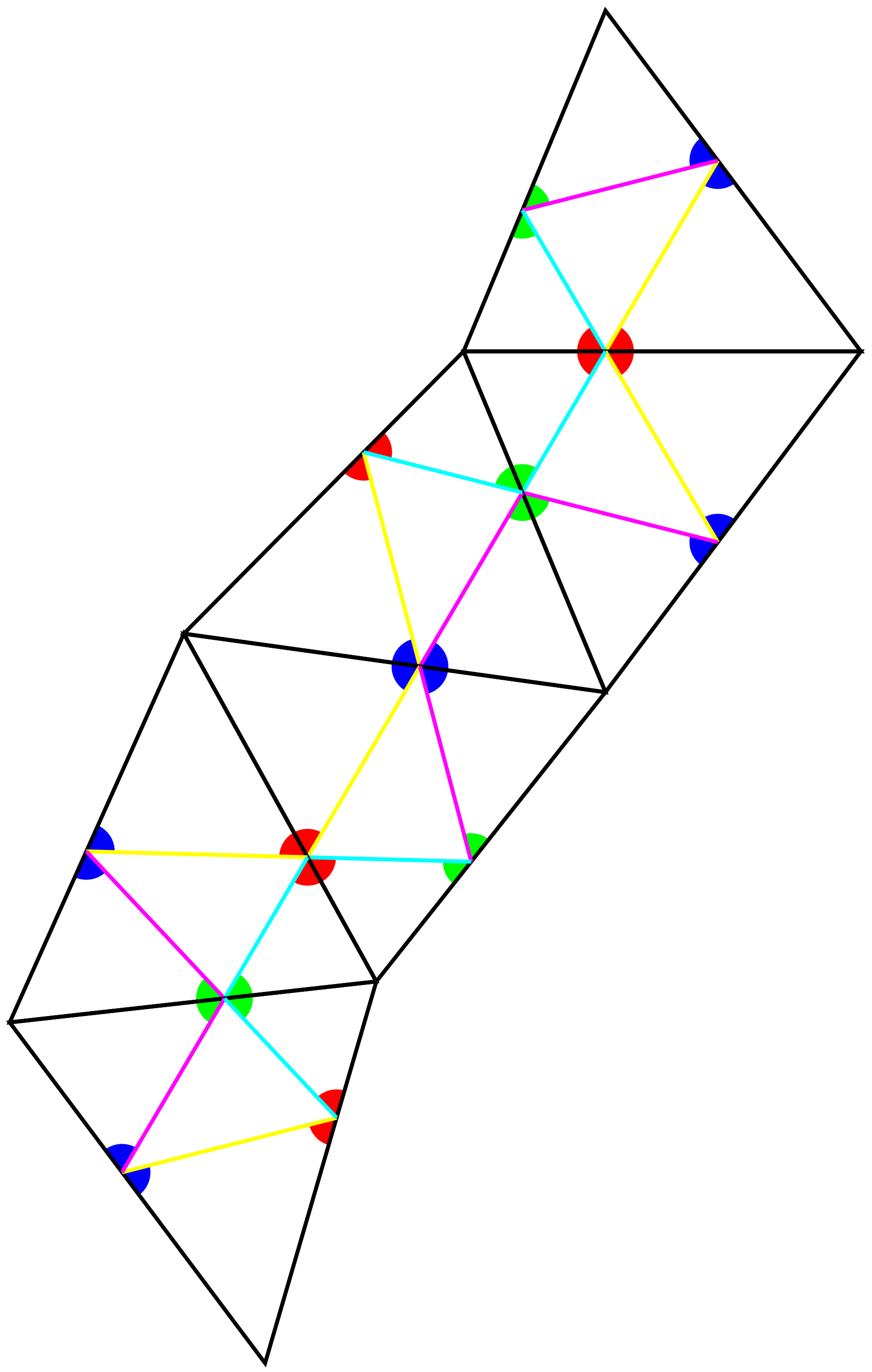
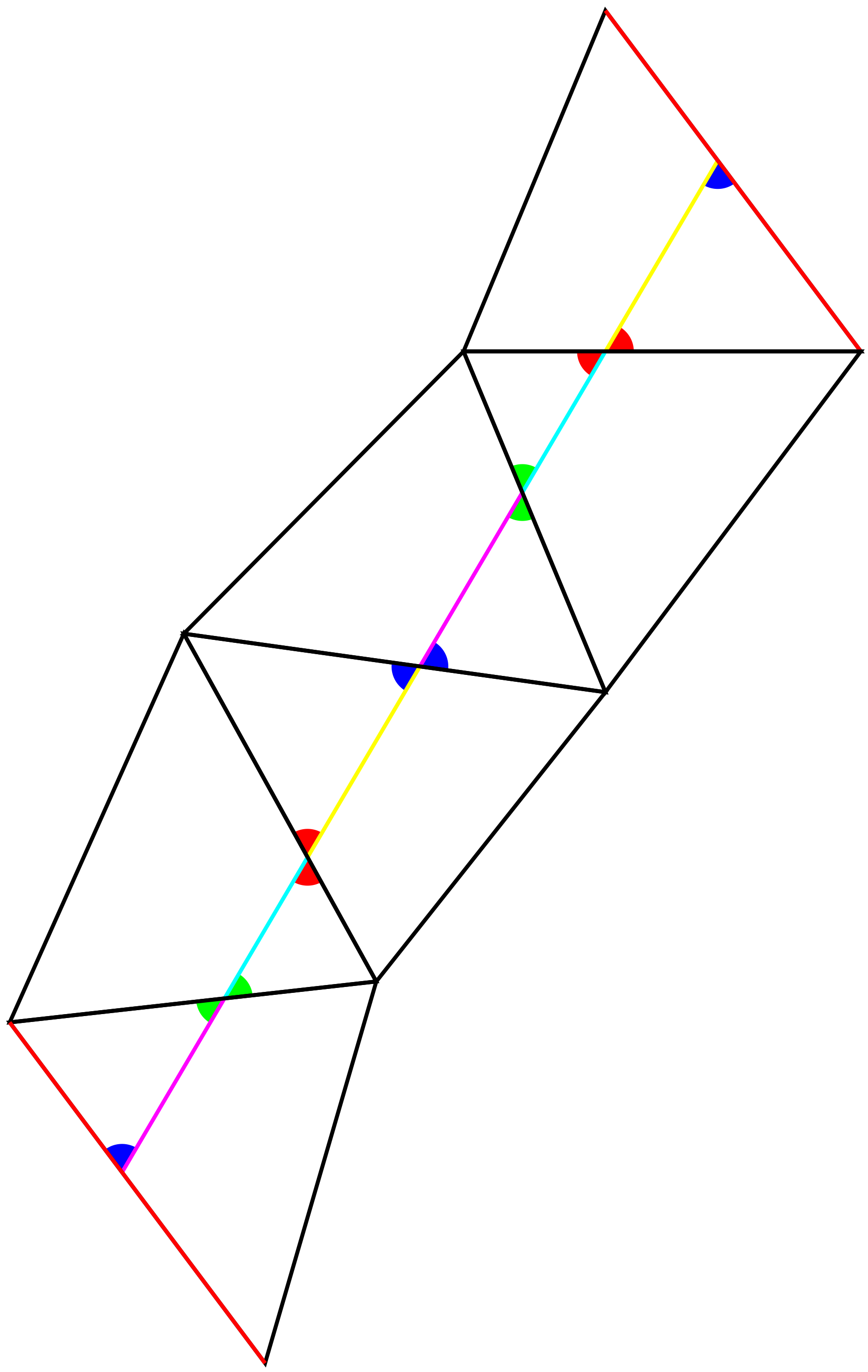
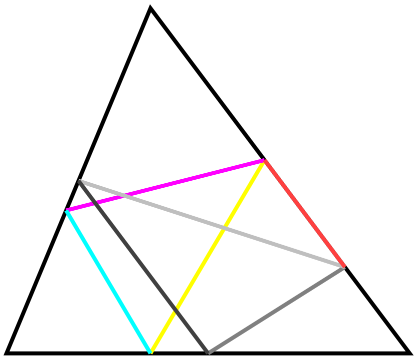


There.

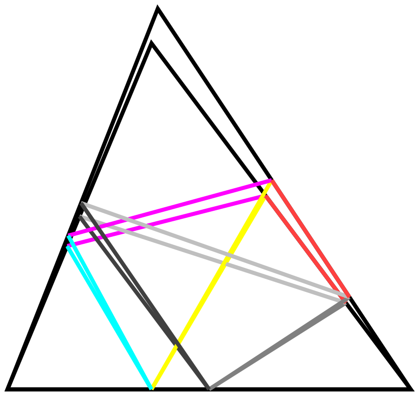


Now take away the distractions. We can clearly see that the multi-colored line is a straight line (opposite angles =) AND the two red lines are parallel. (alternate interior angles =)



Now draw an arbitrary inscribed triangle for comparison purposes. Again flip the triangle five times.

(Pay attention to the dark red line on the upper right side.)



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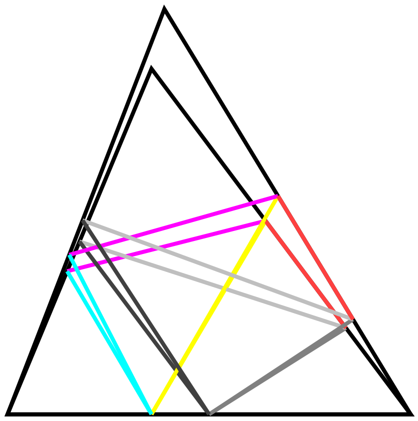

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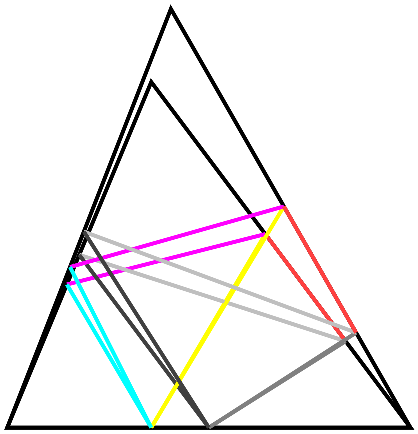

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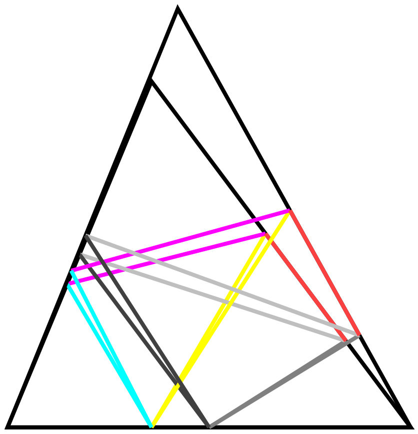

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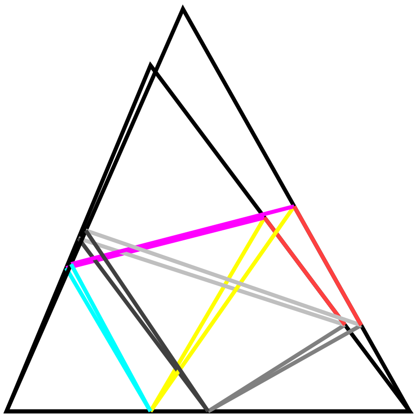

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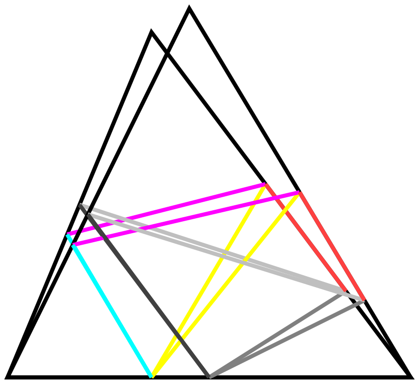

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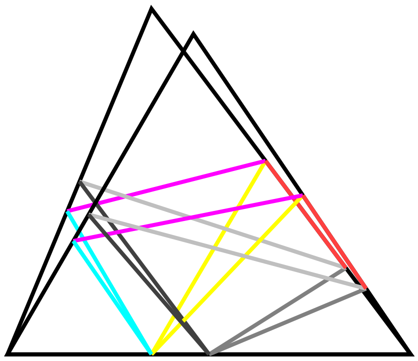

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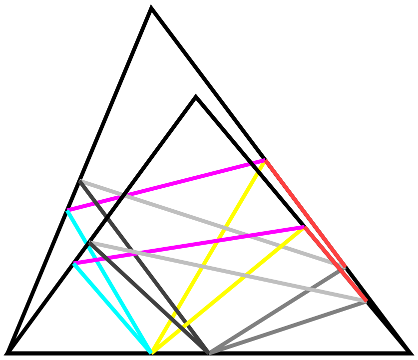

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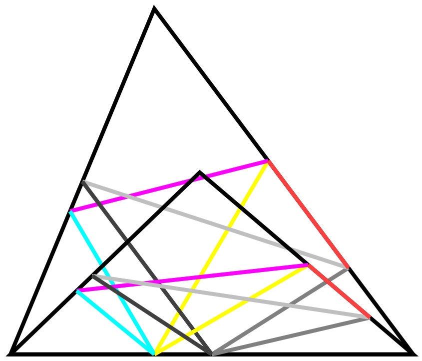

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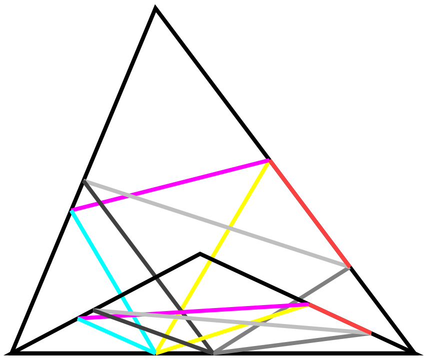

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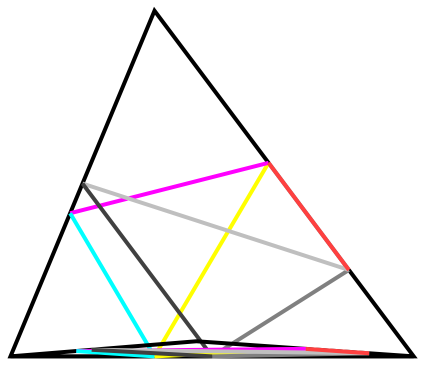

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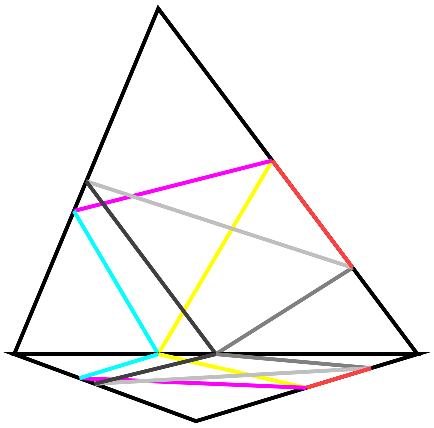

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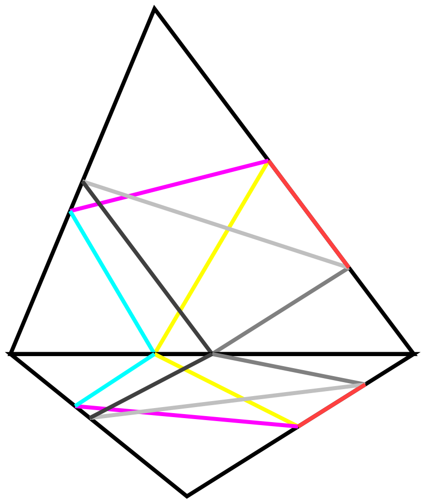

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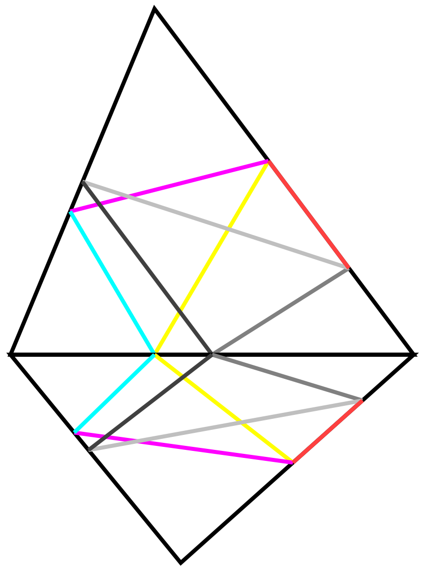

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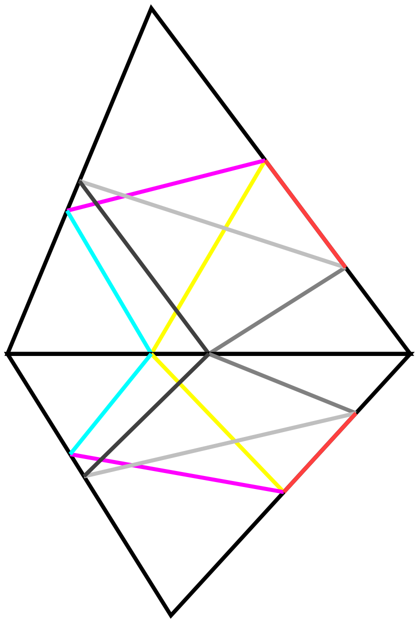

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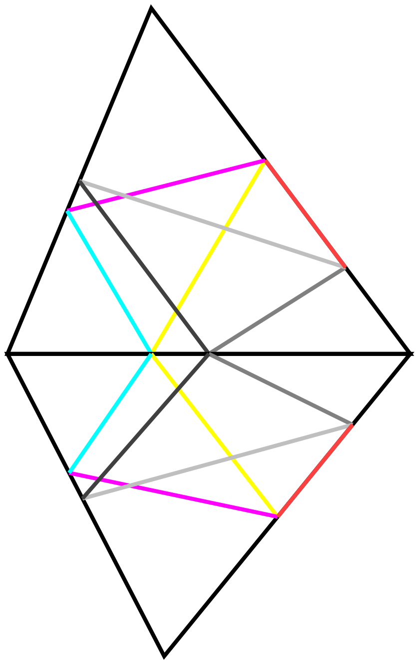

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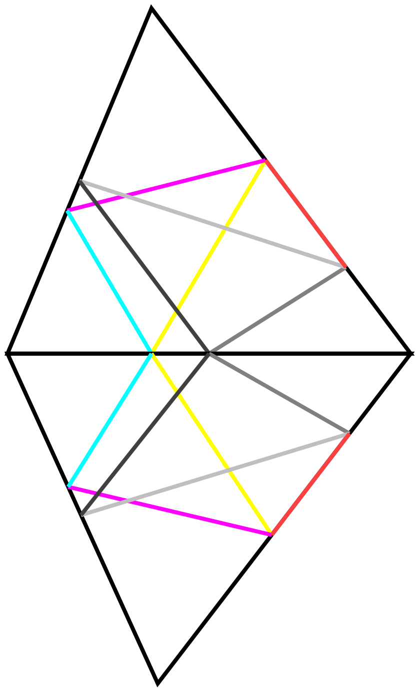

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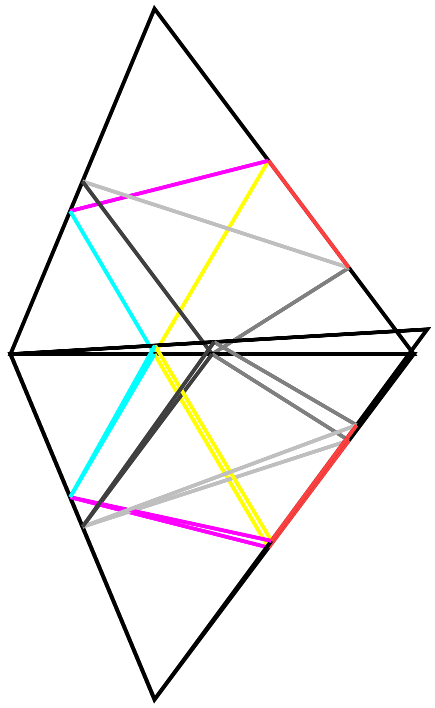

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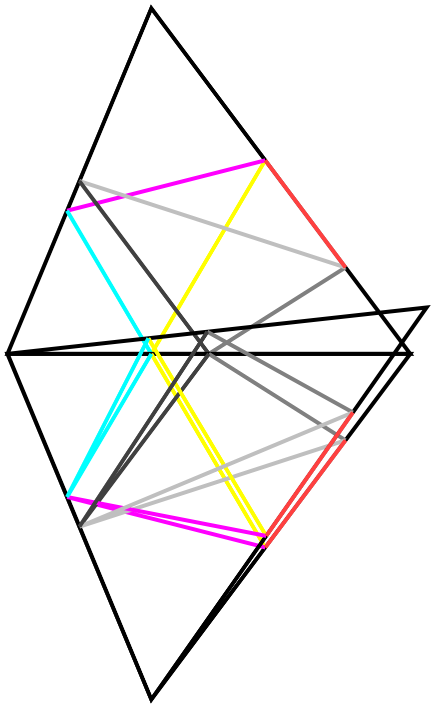

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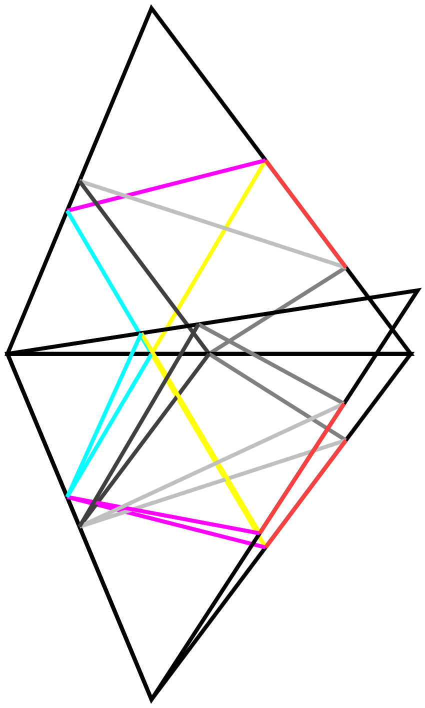

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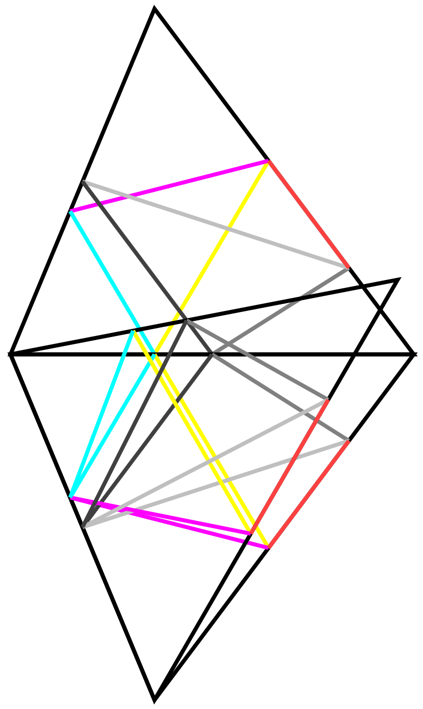

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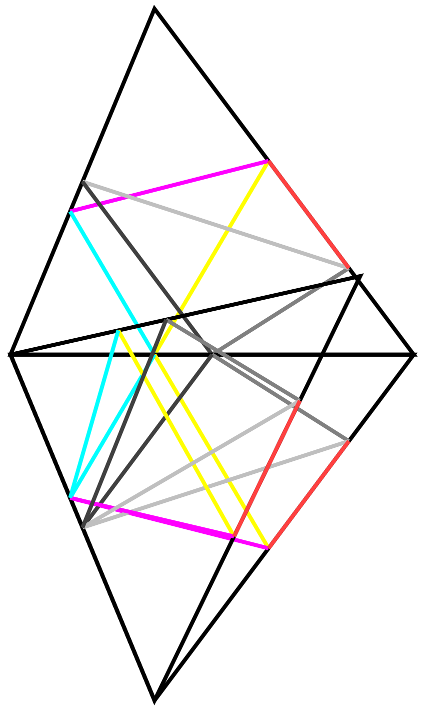

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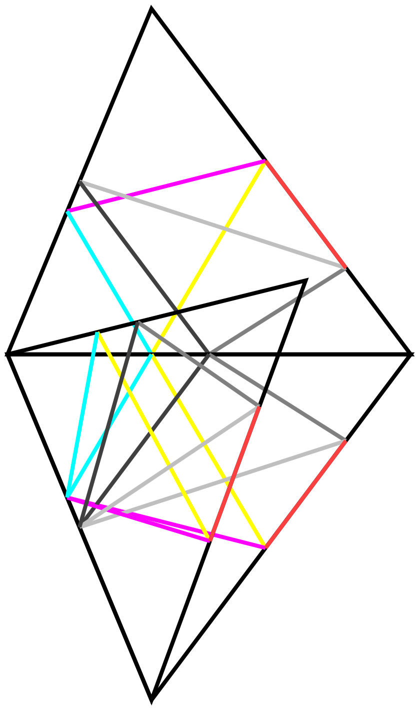

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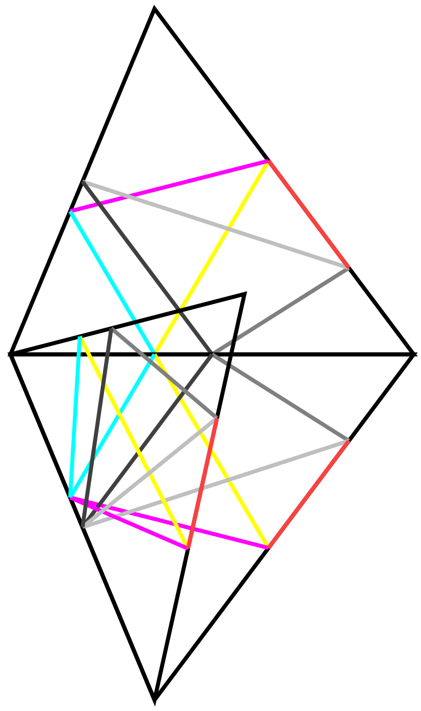

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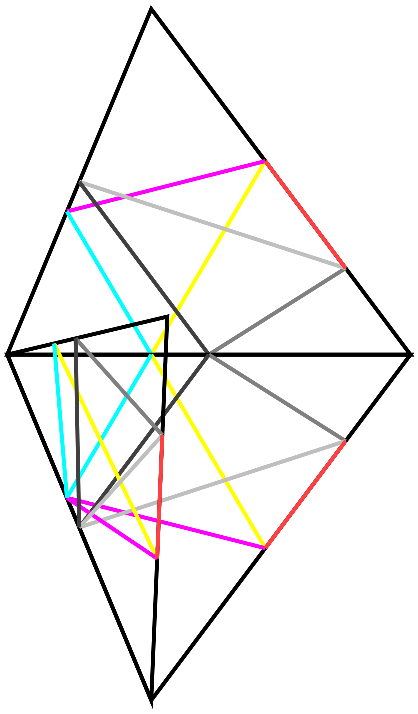

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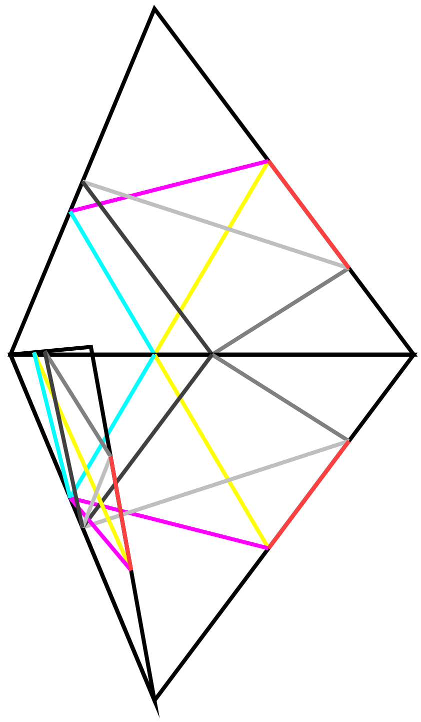

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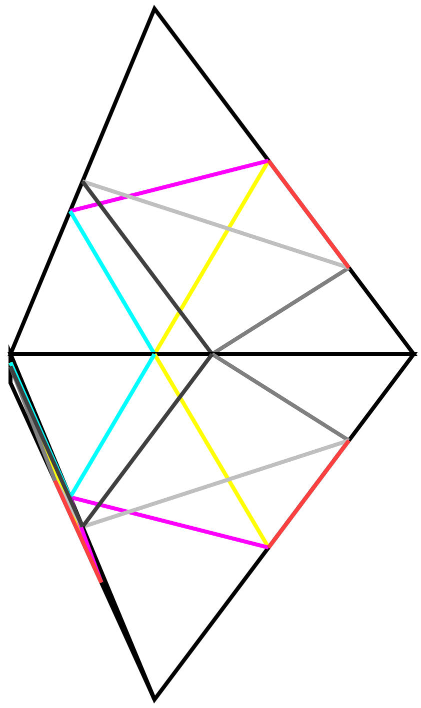

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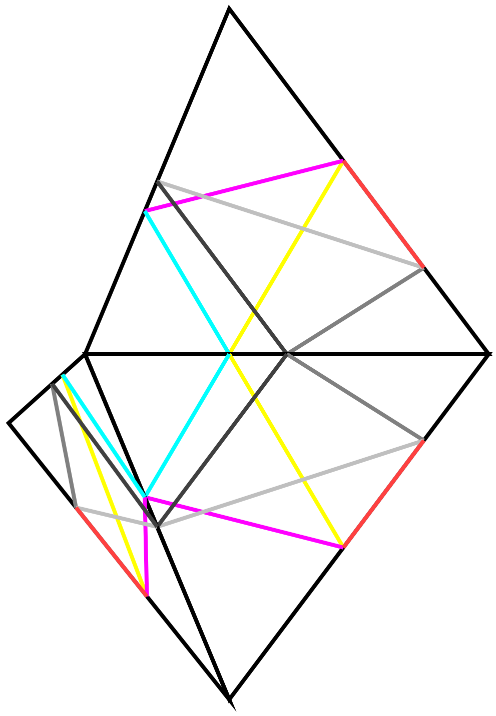

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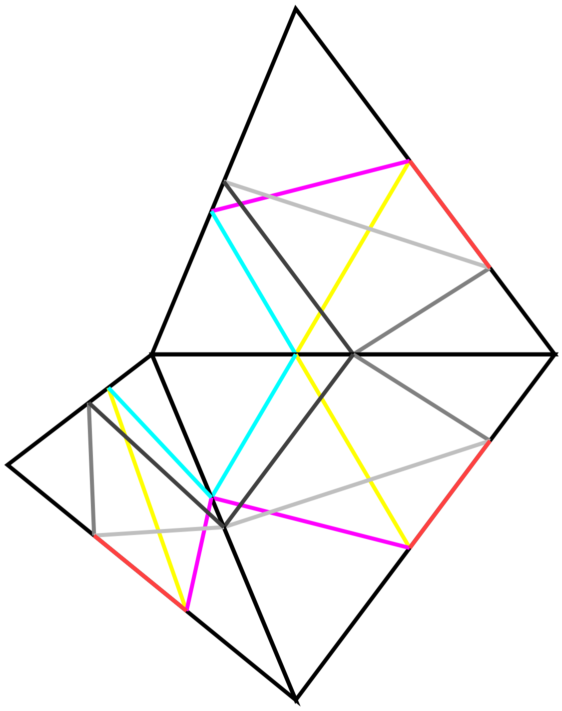

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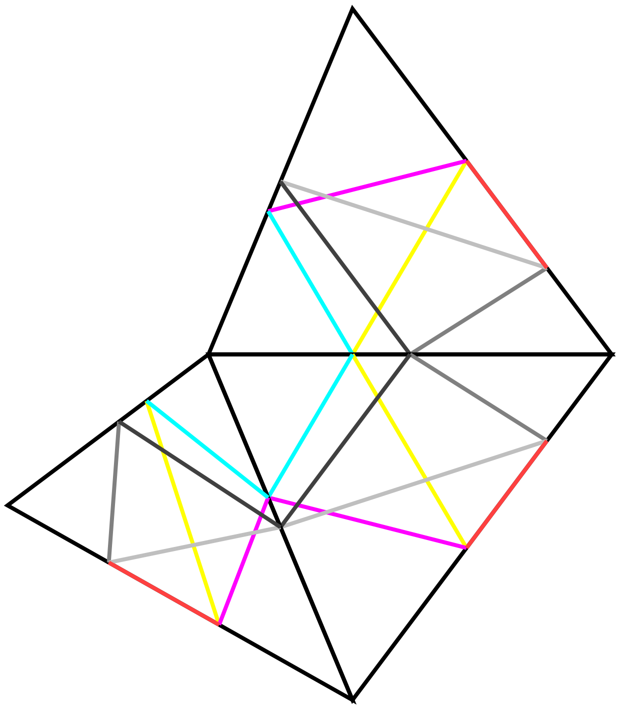

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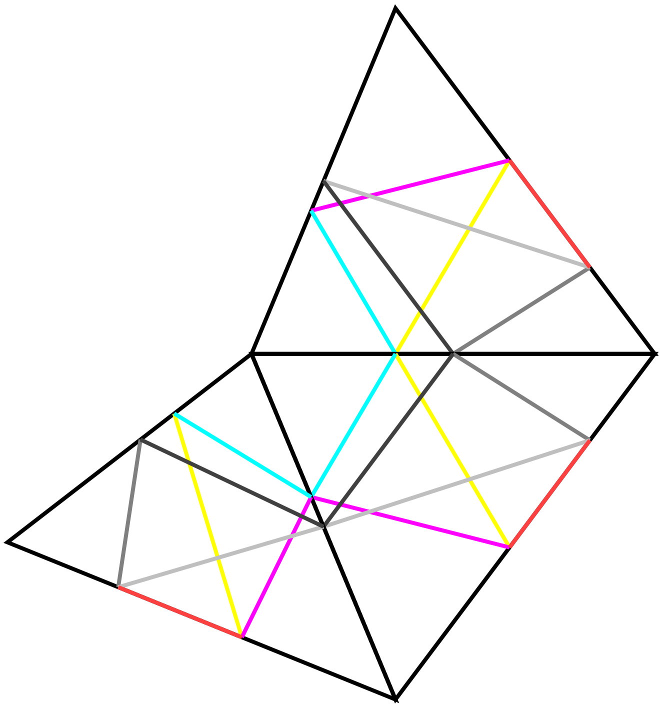

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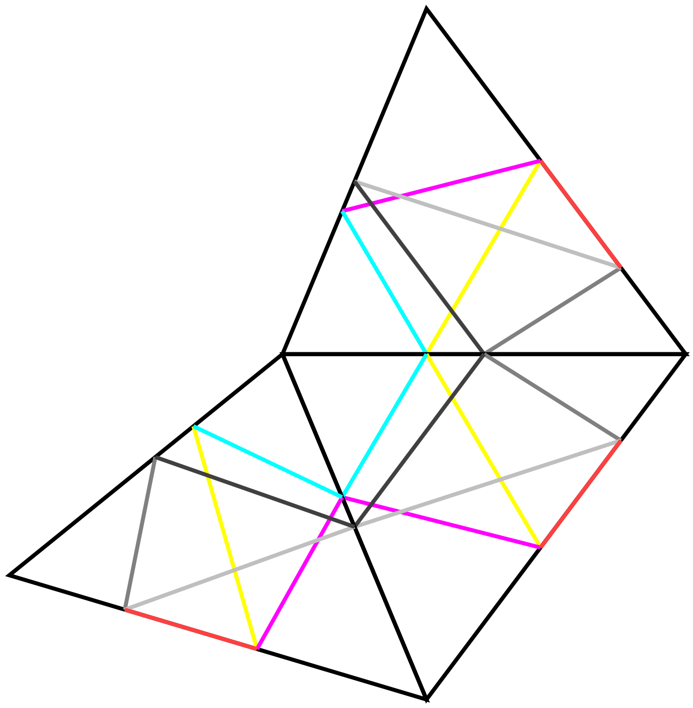

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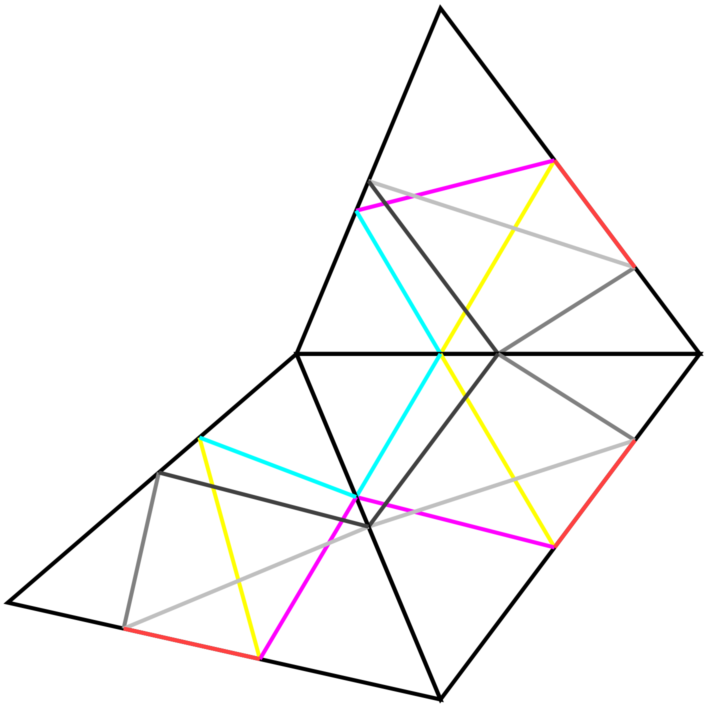

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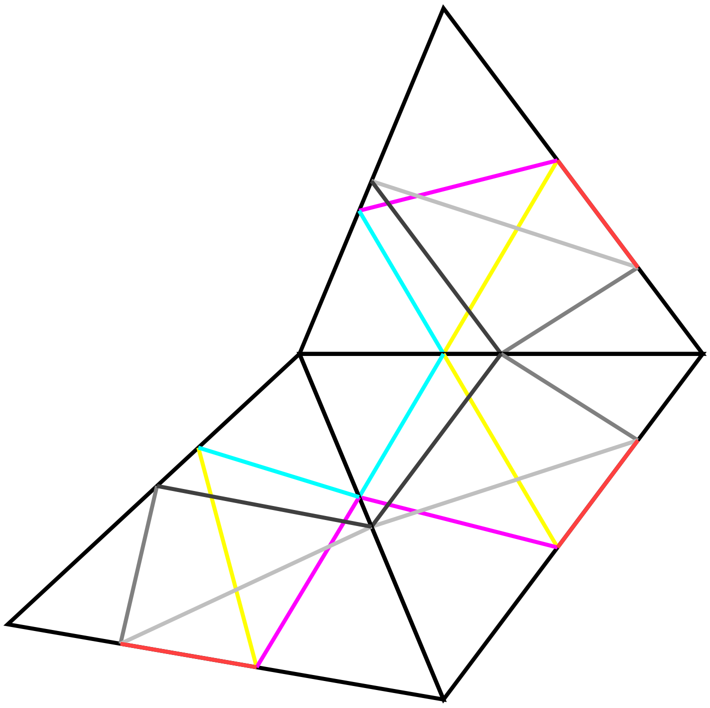

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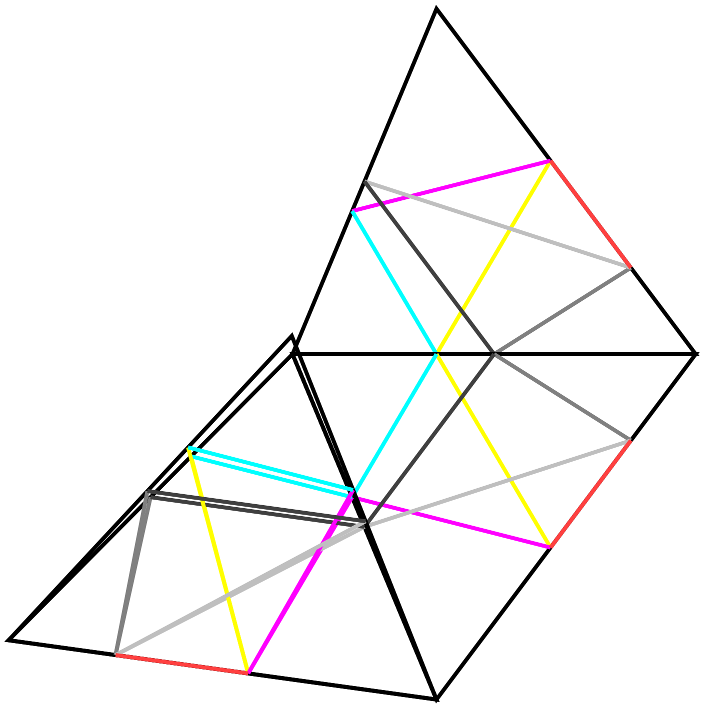

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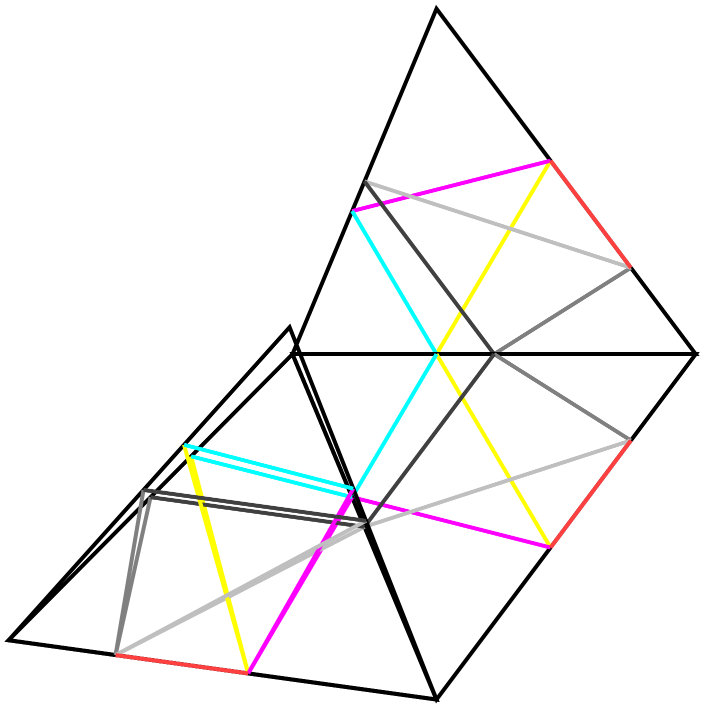

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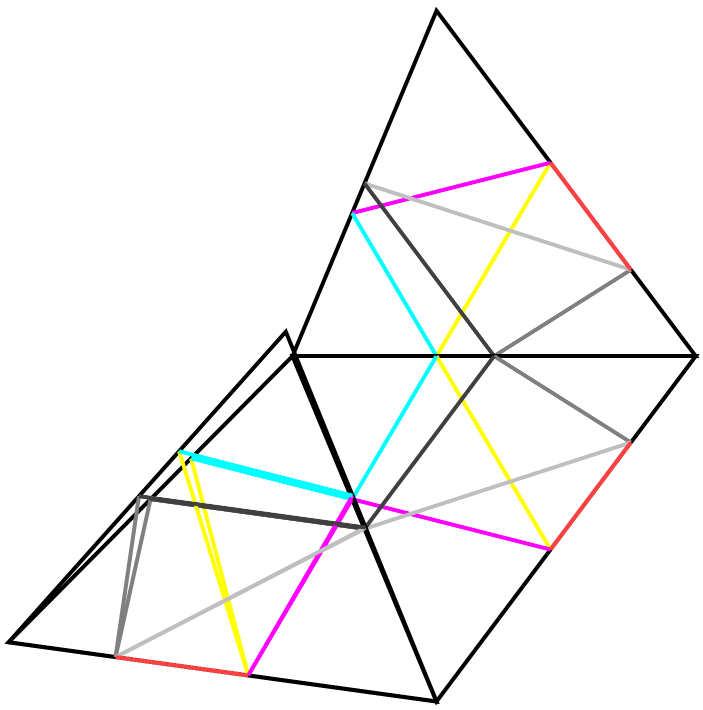

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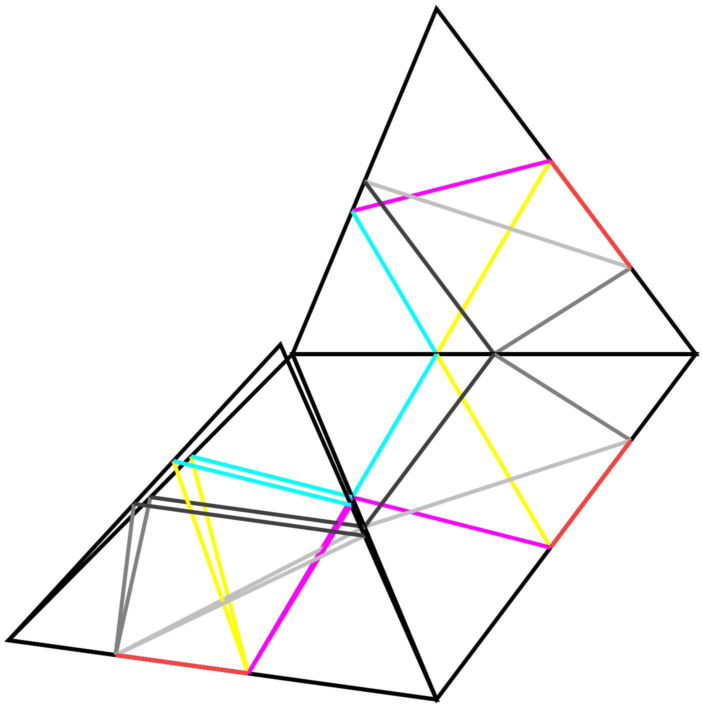

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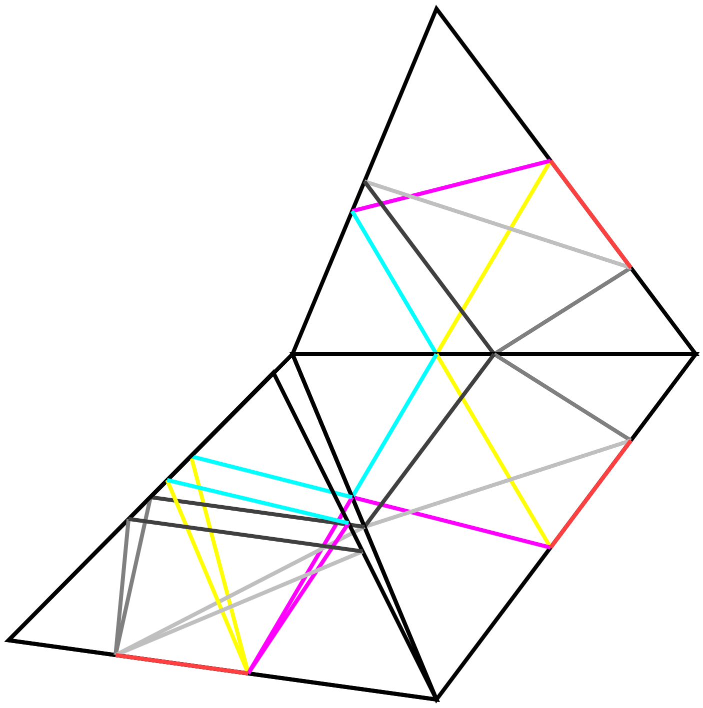

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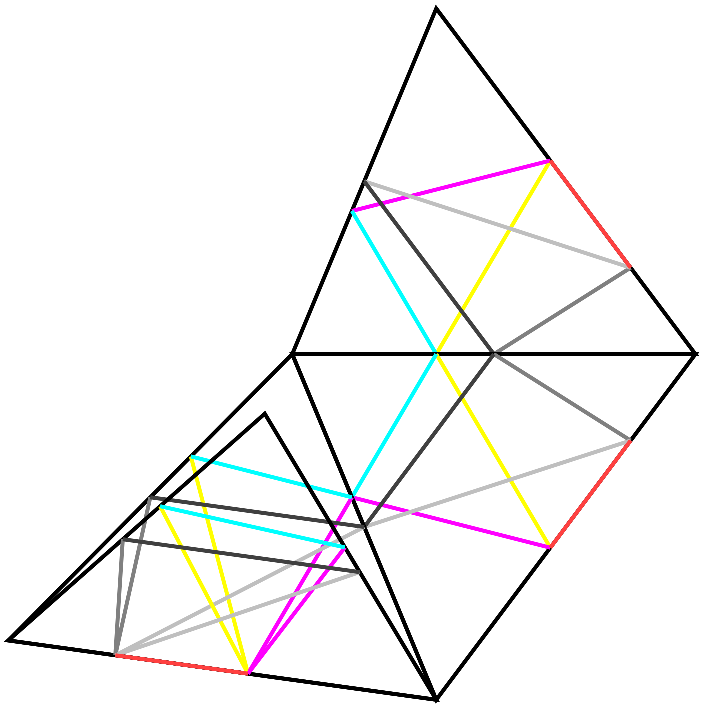

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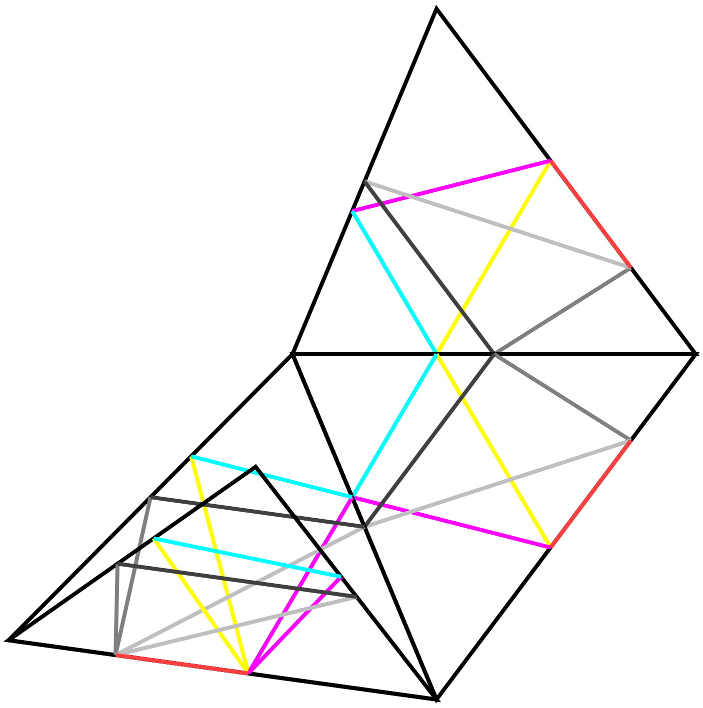

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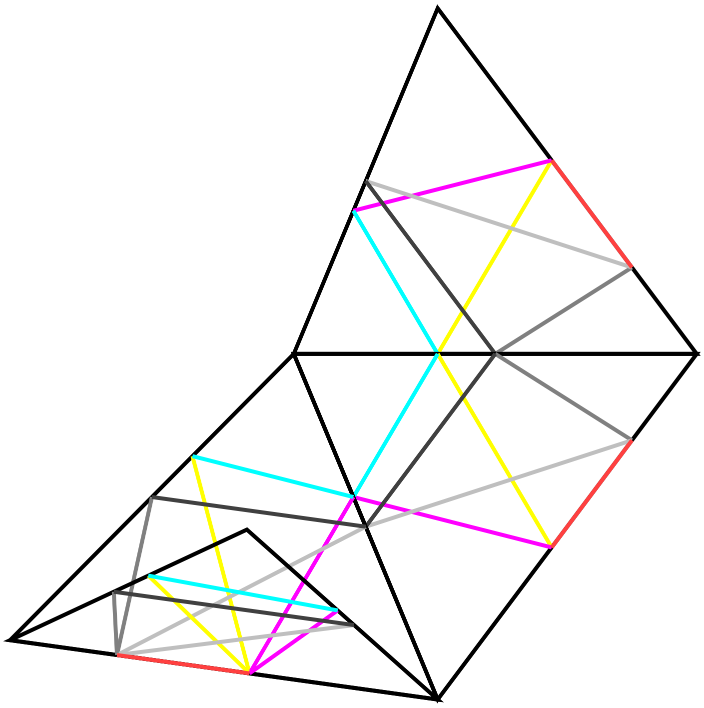

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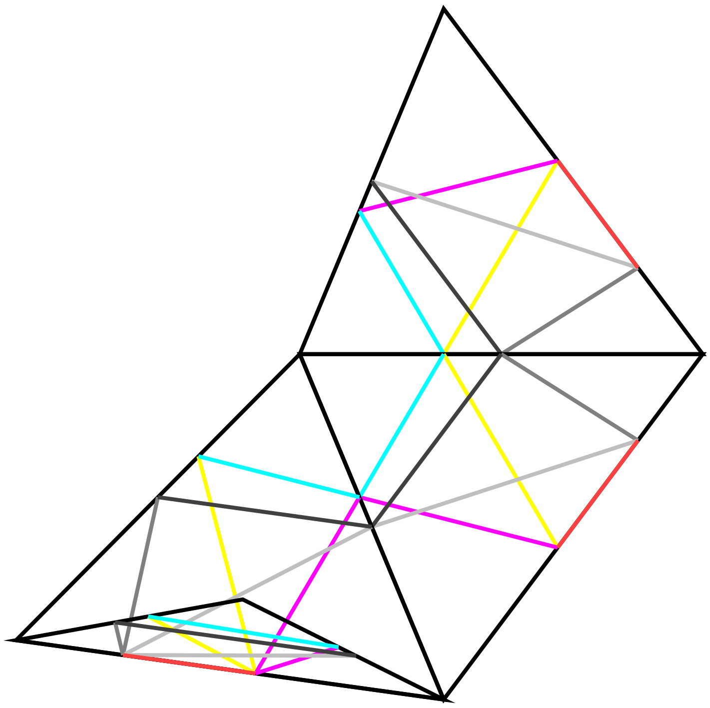

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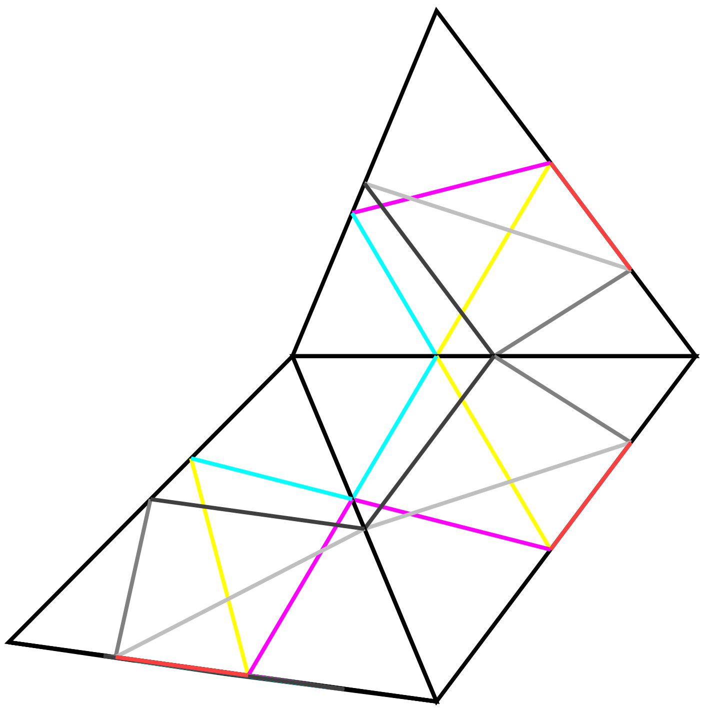

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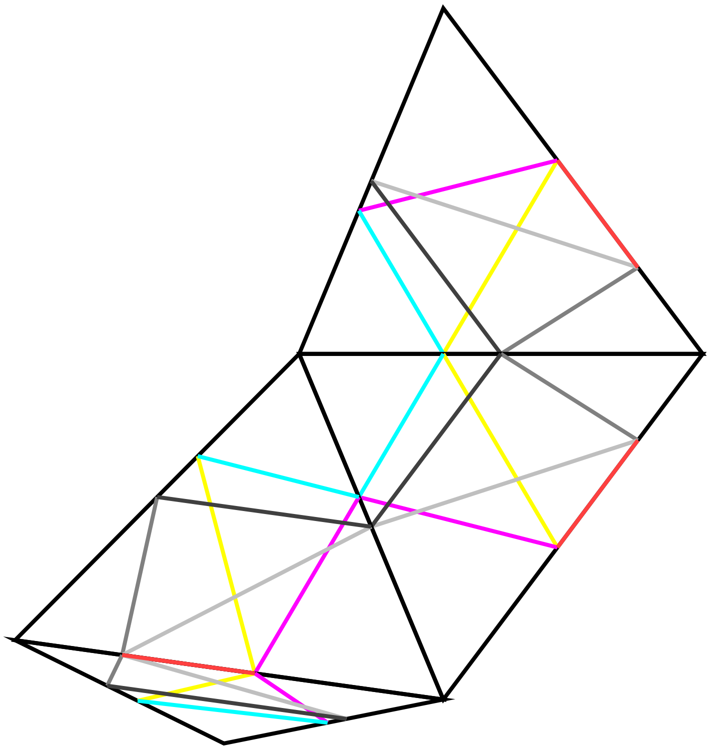

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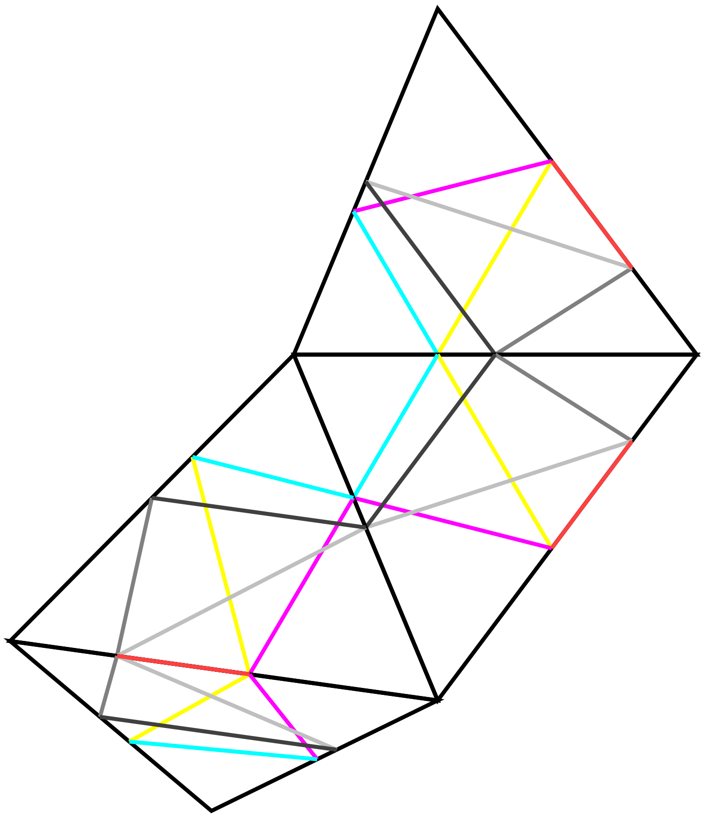

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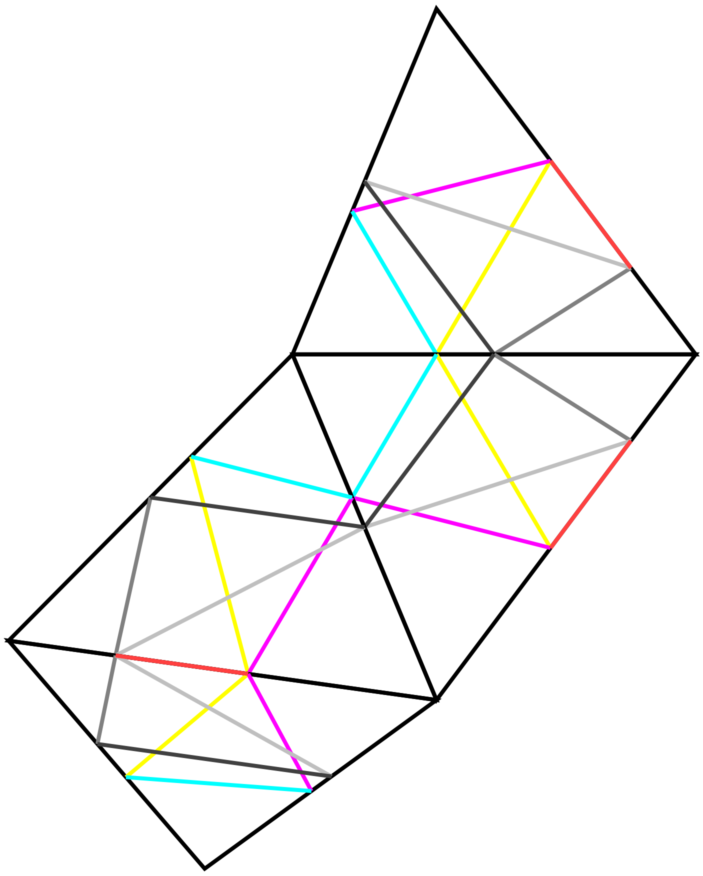

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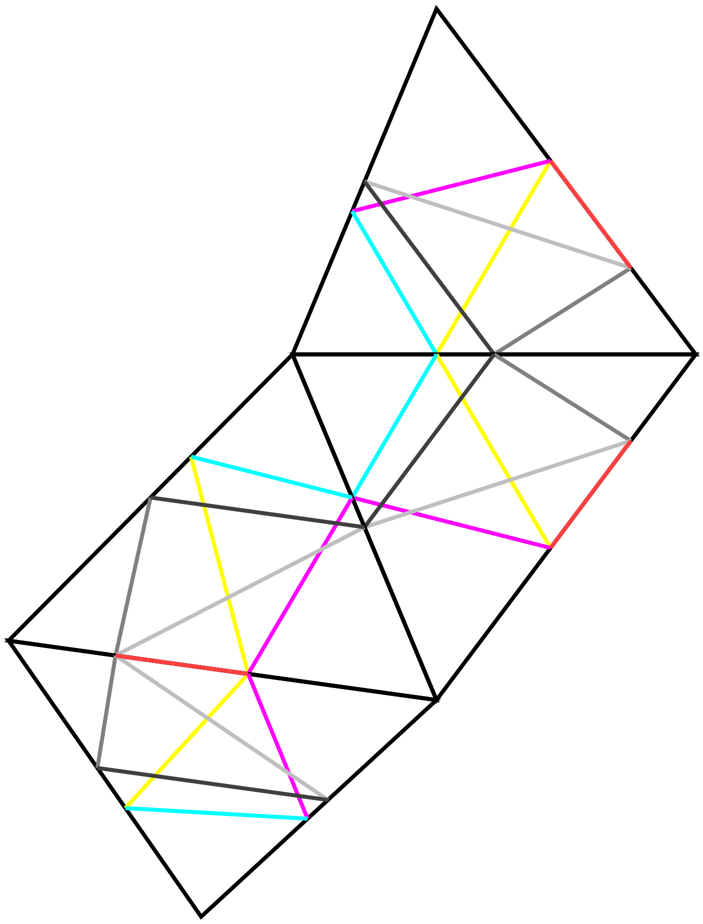

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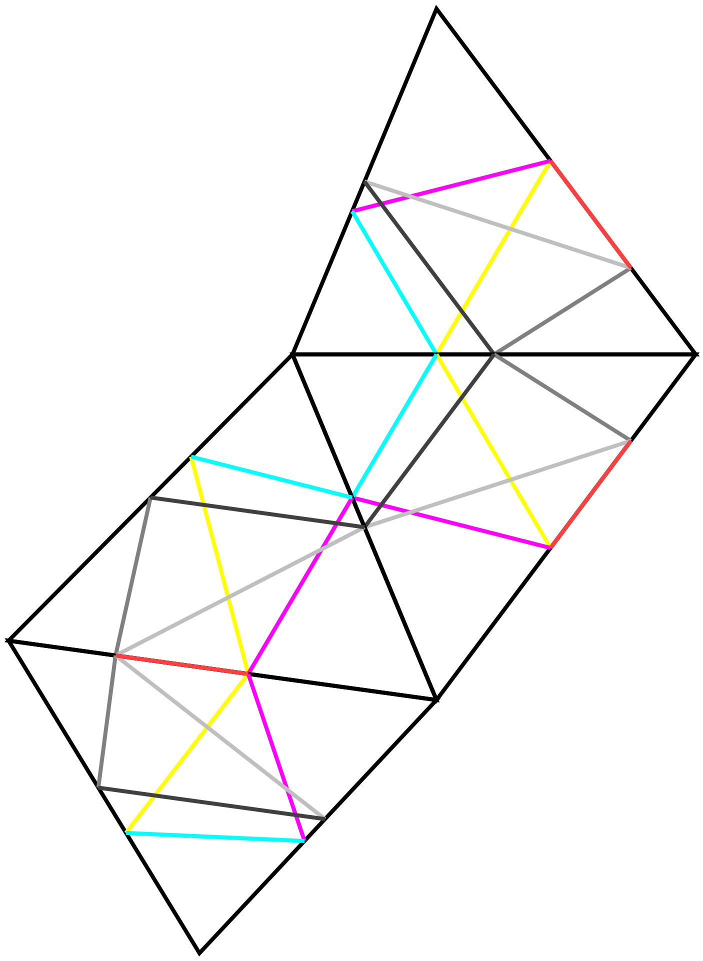

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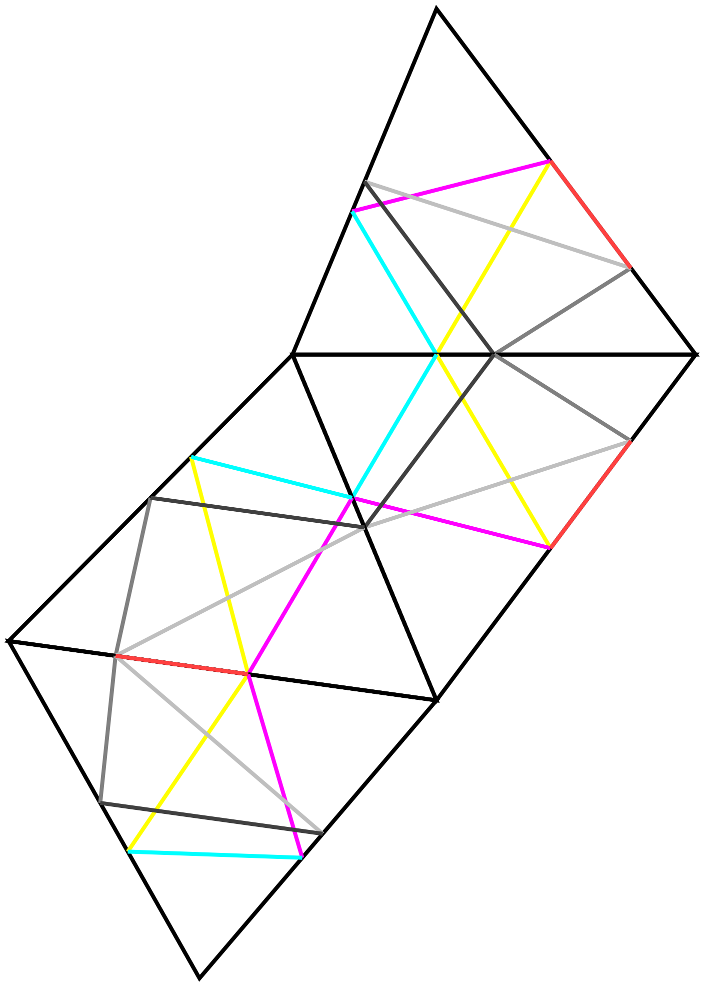

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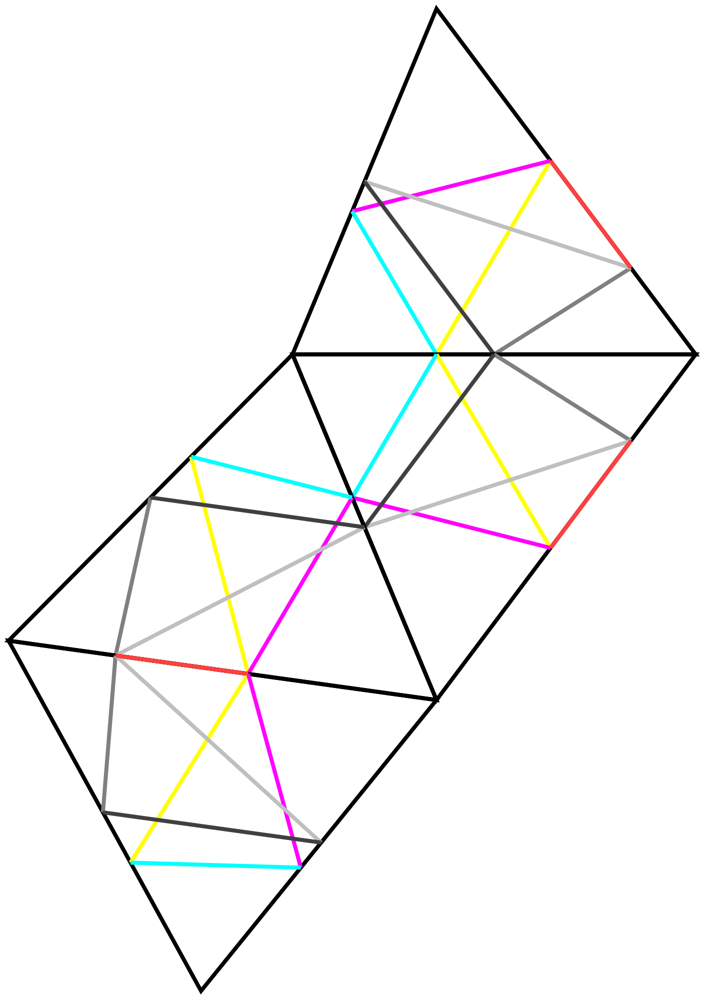

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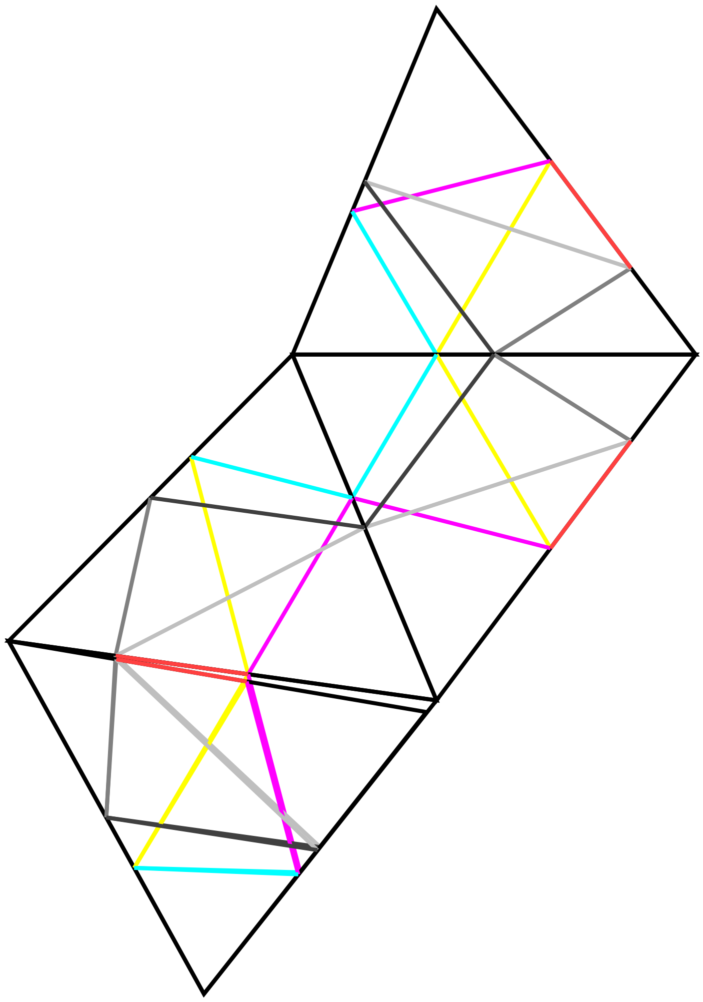

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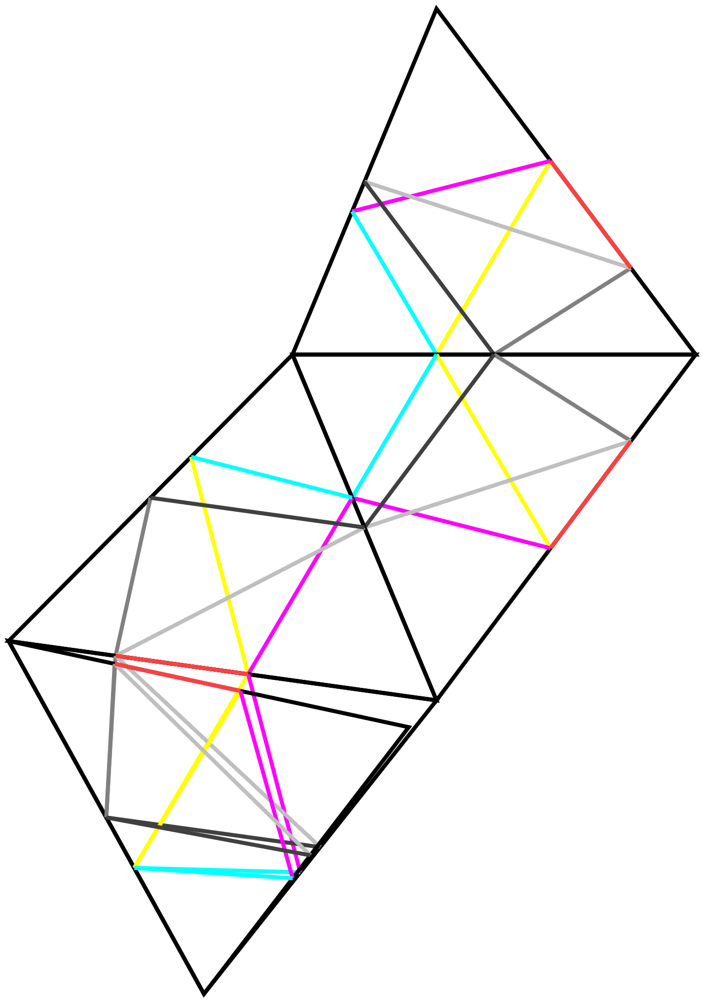

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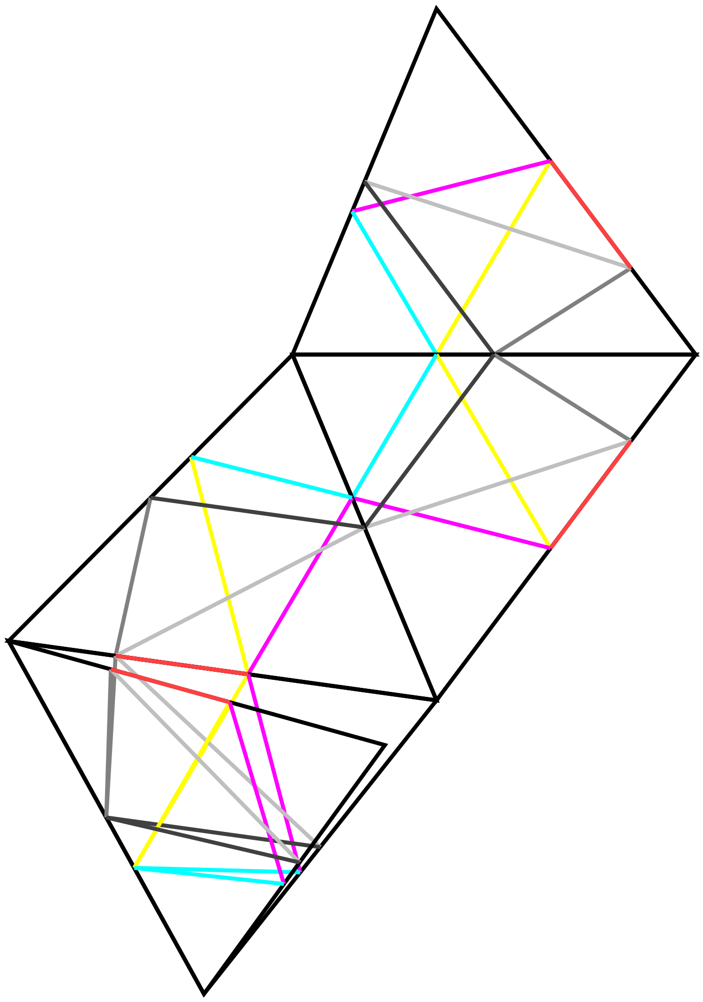

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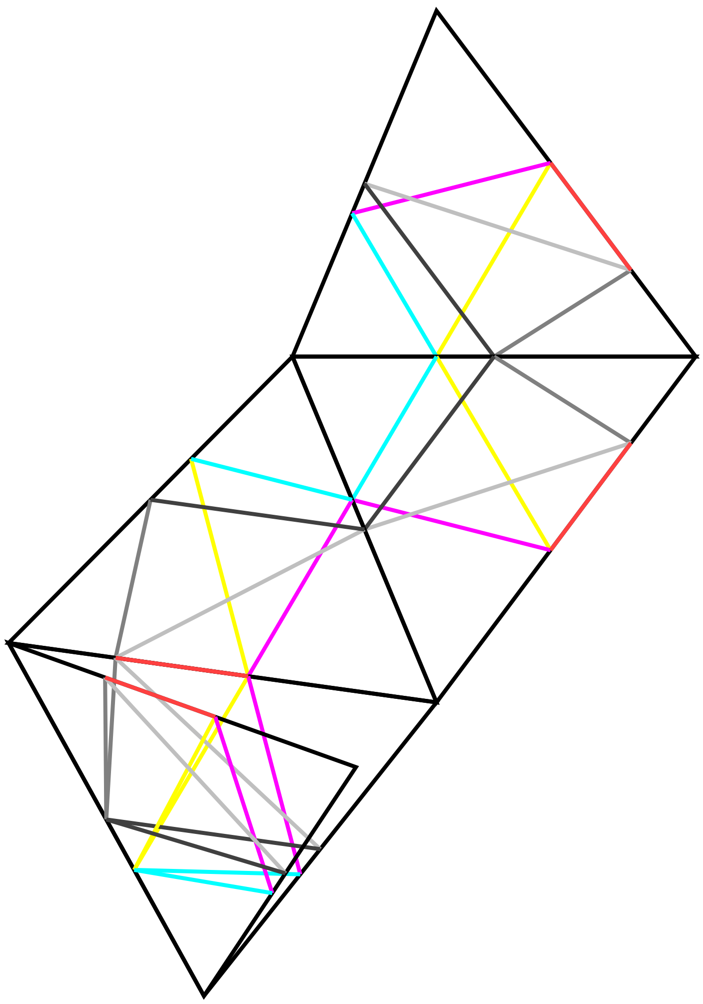

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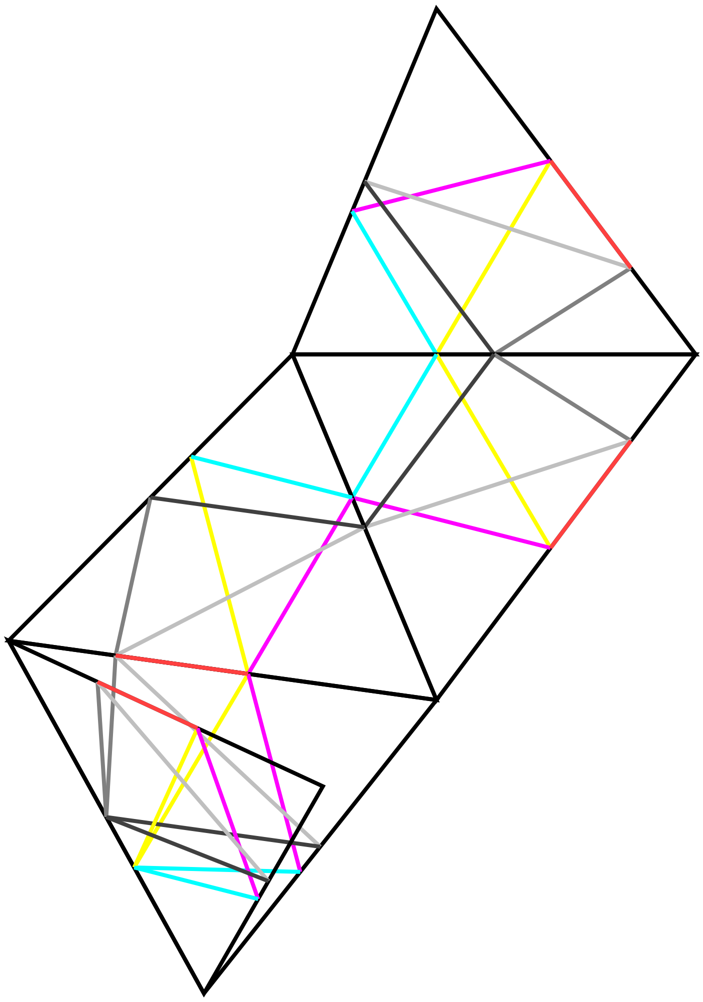

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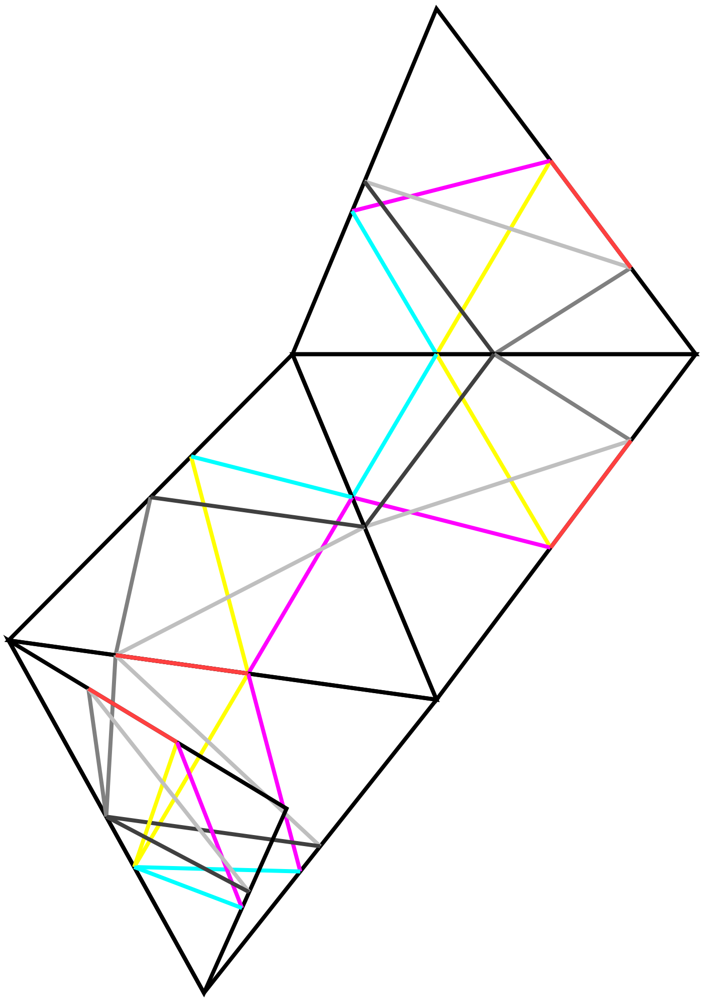

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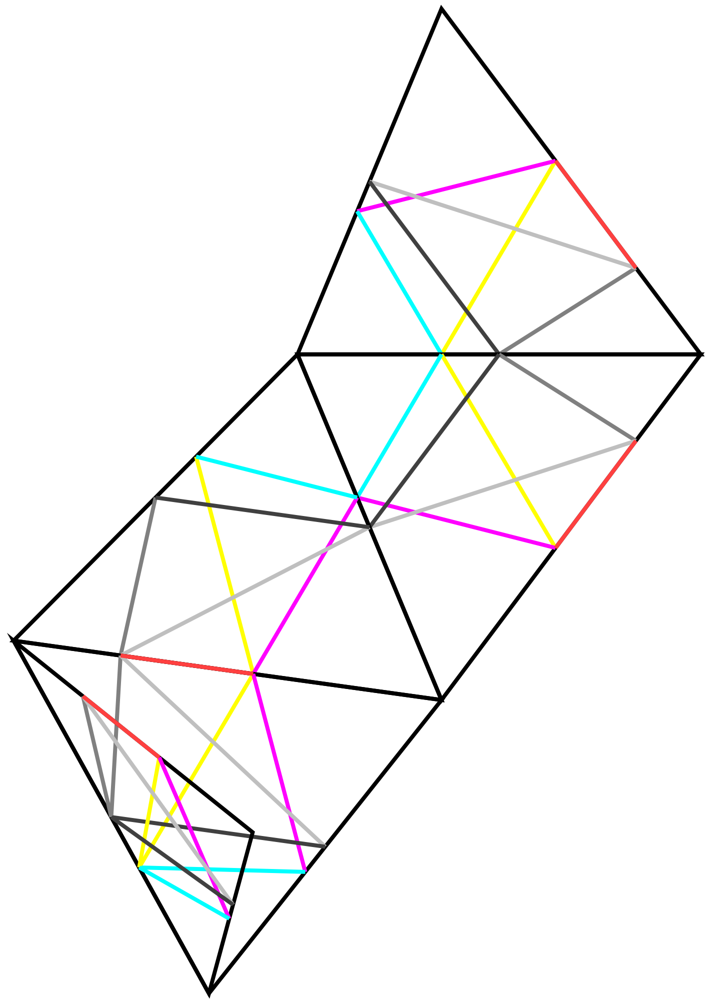

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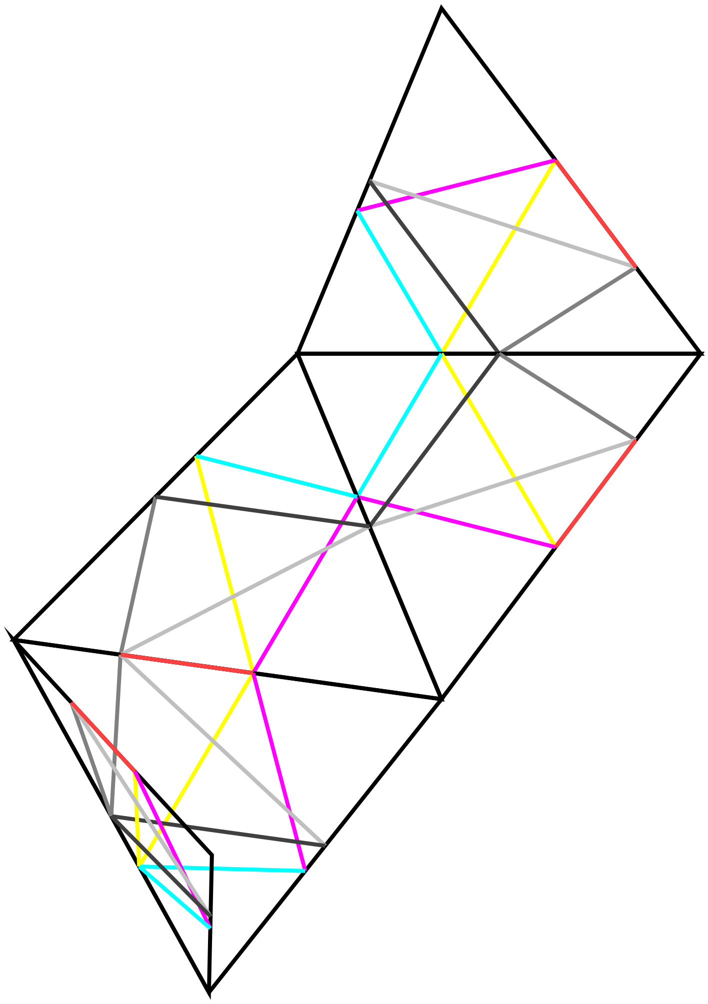

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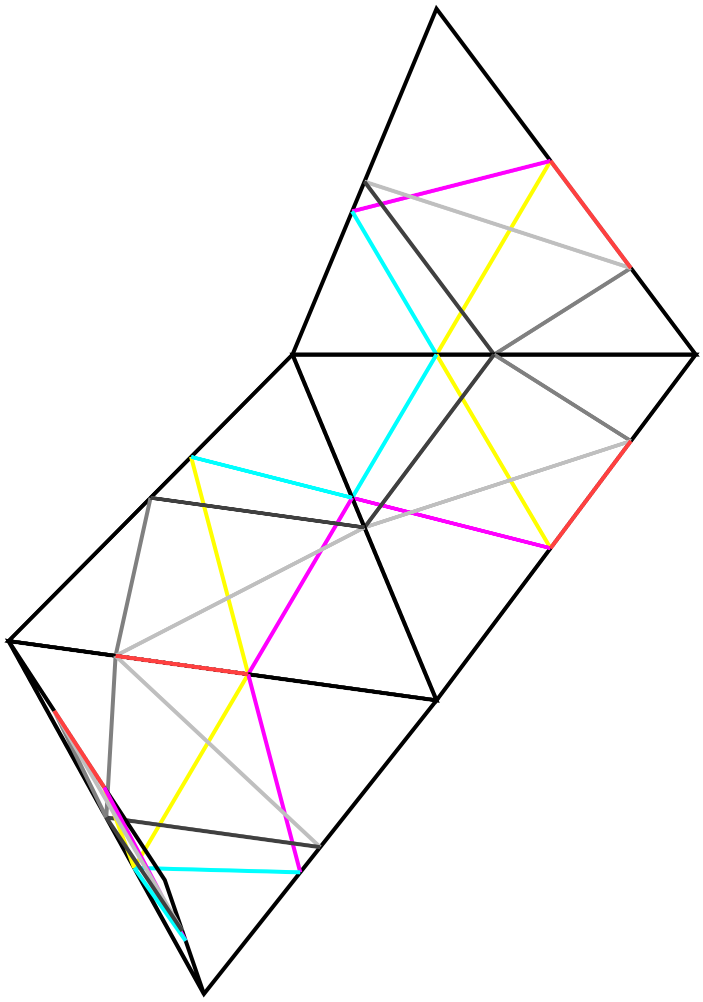

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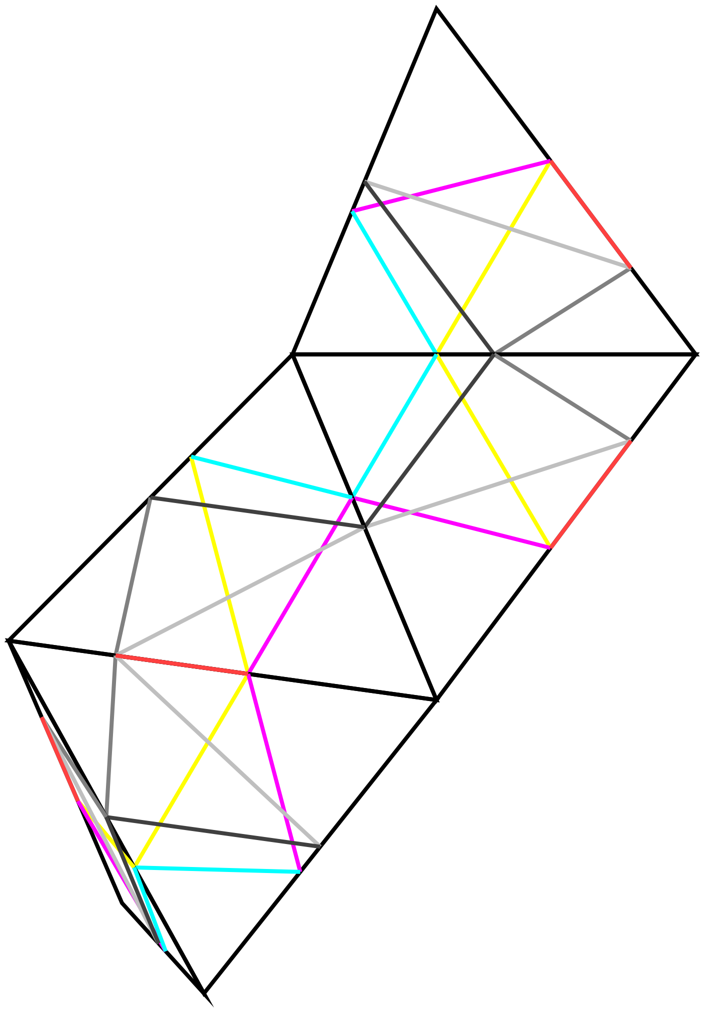

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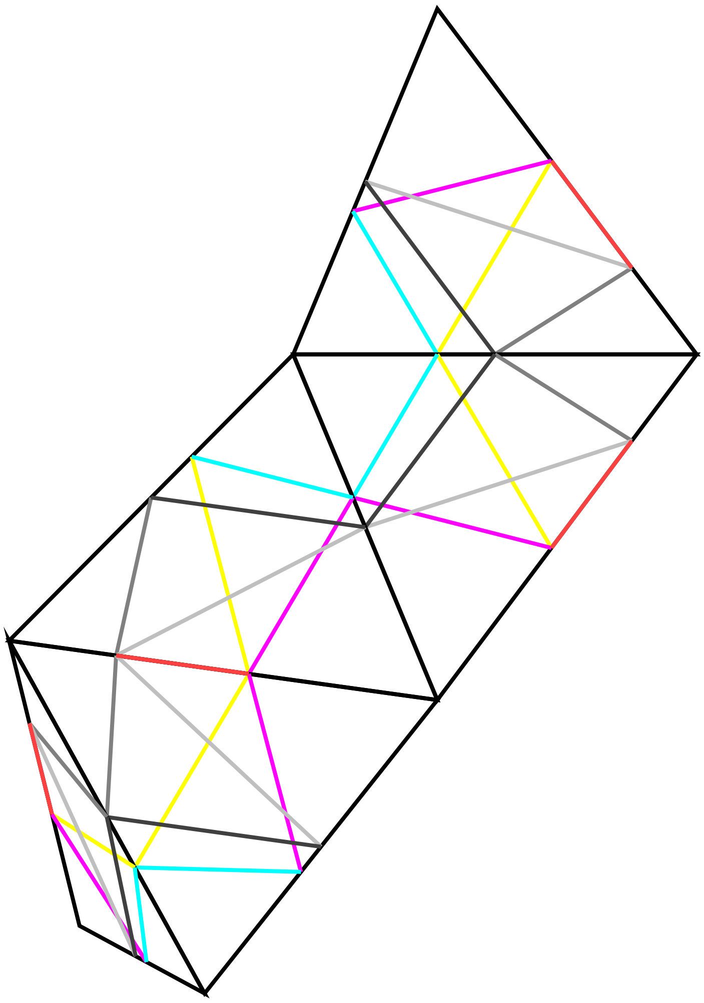

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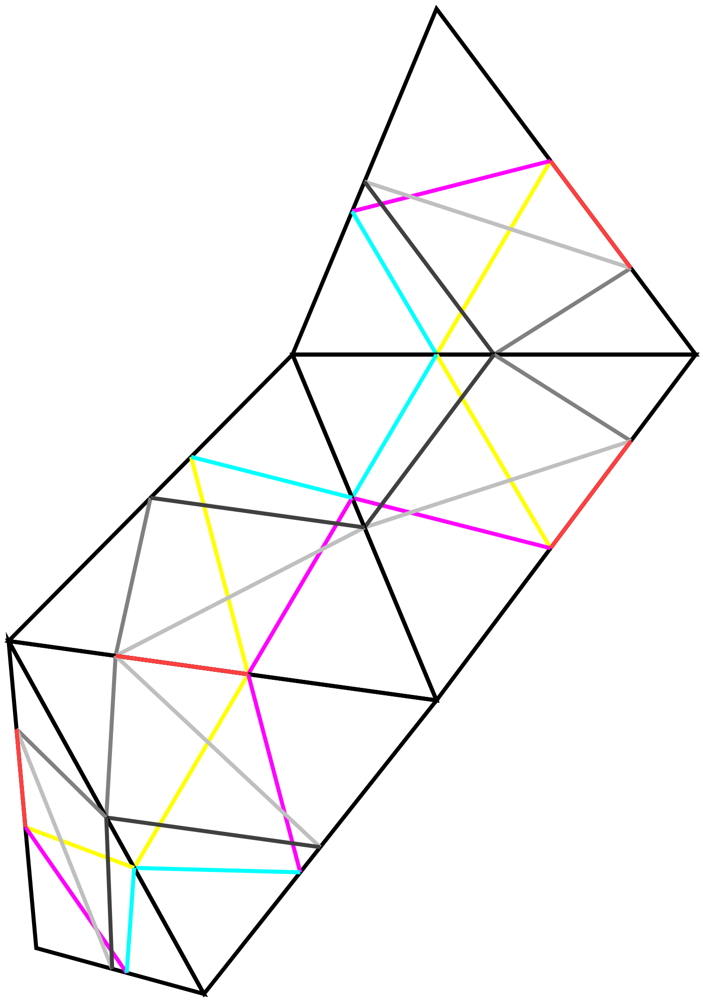

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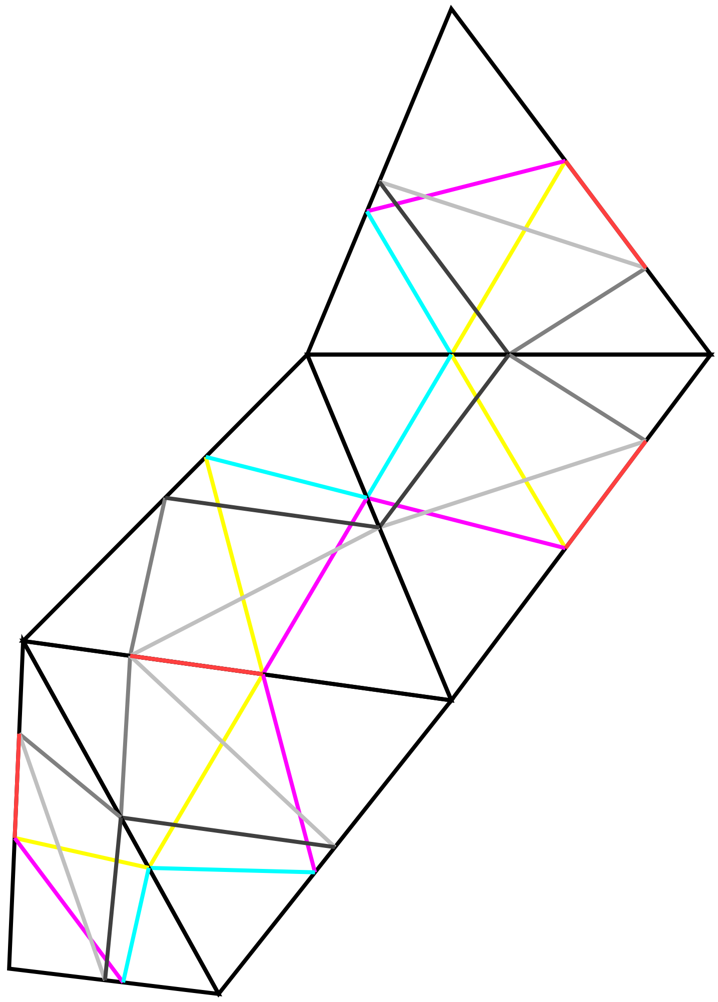

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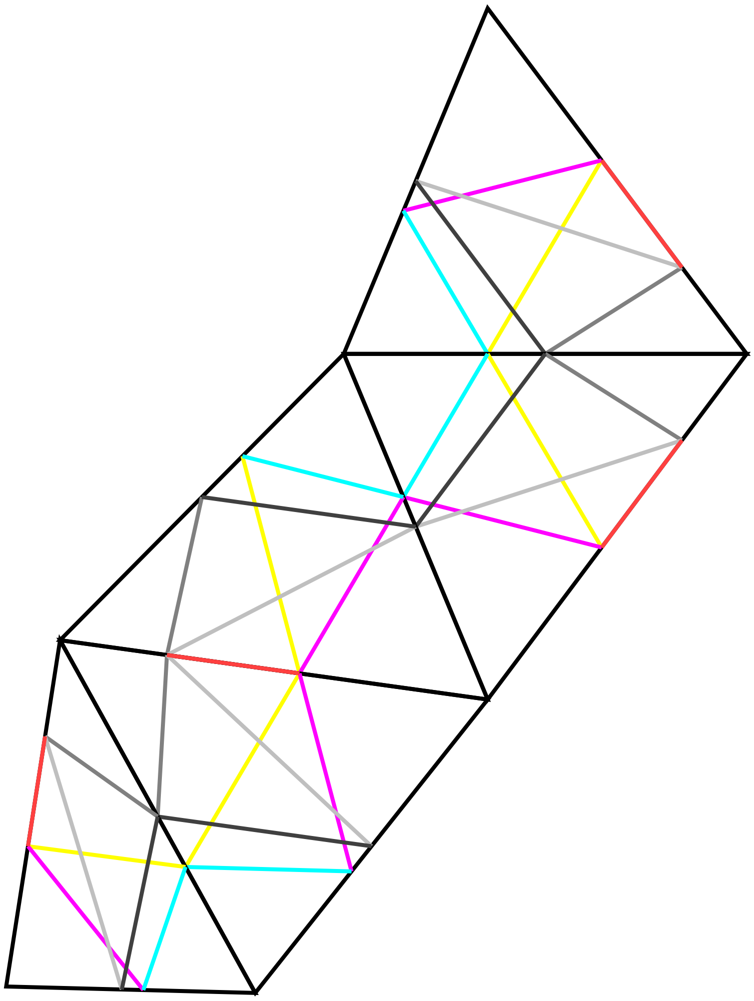

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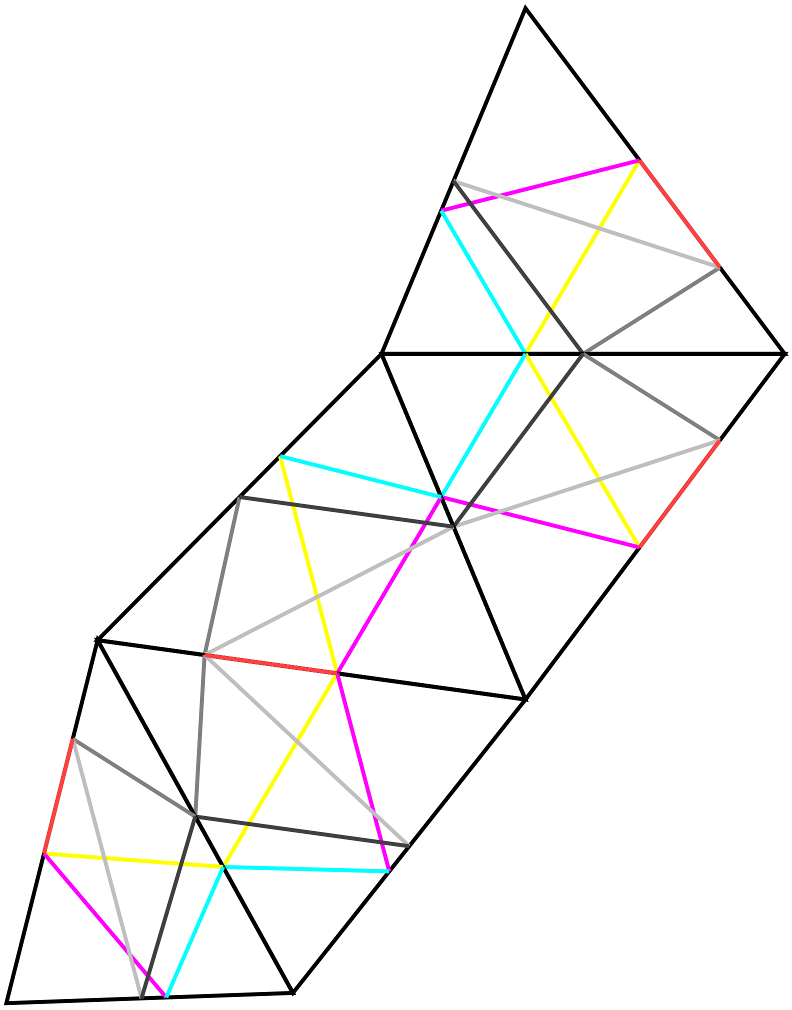

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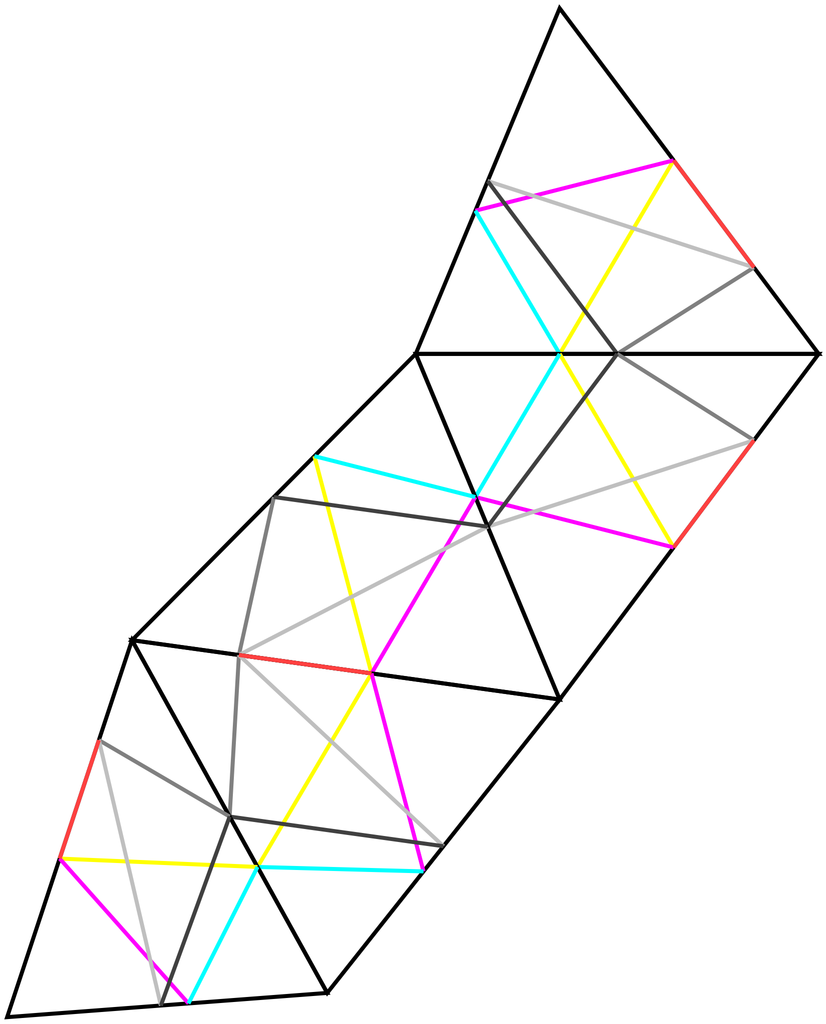

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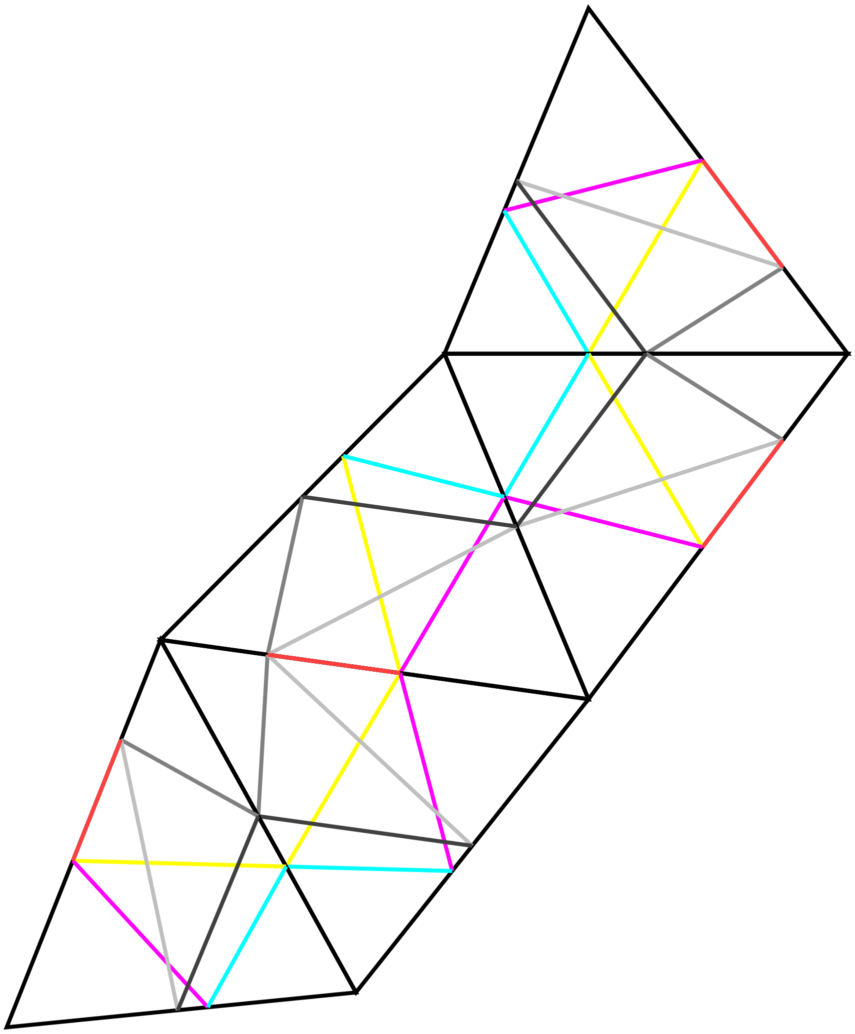

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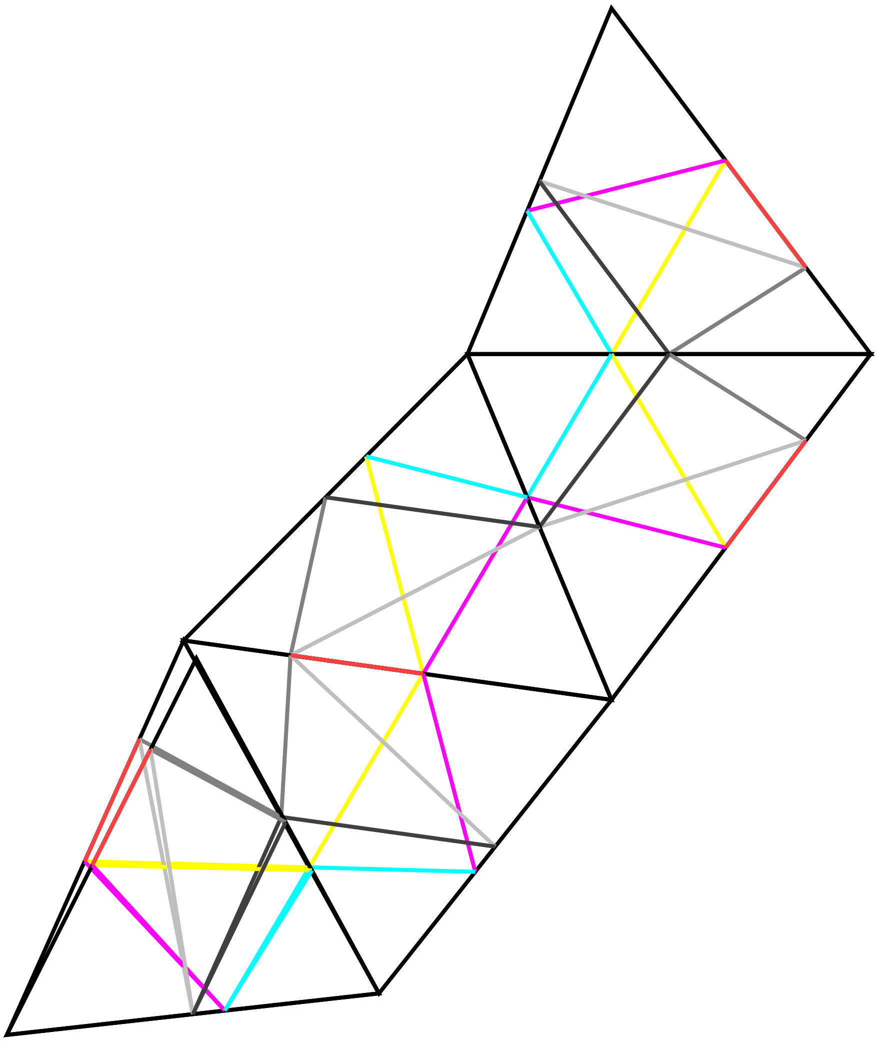

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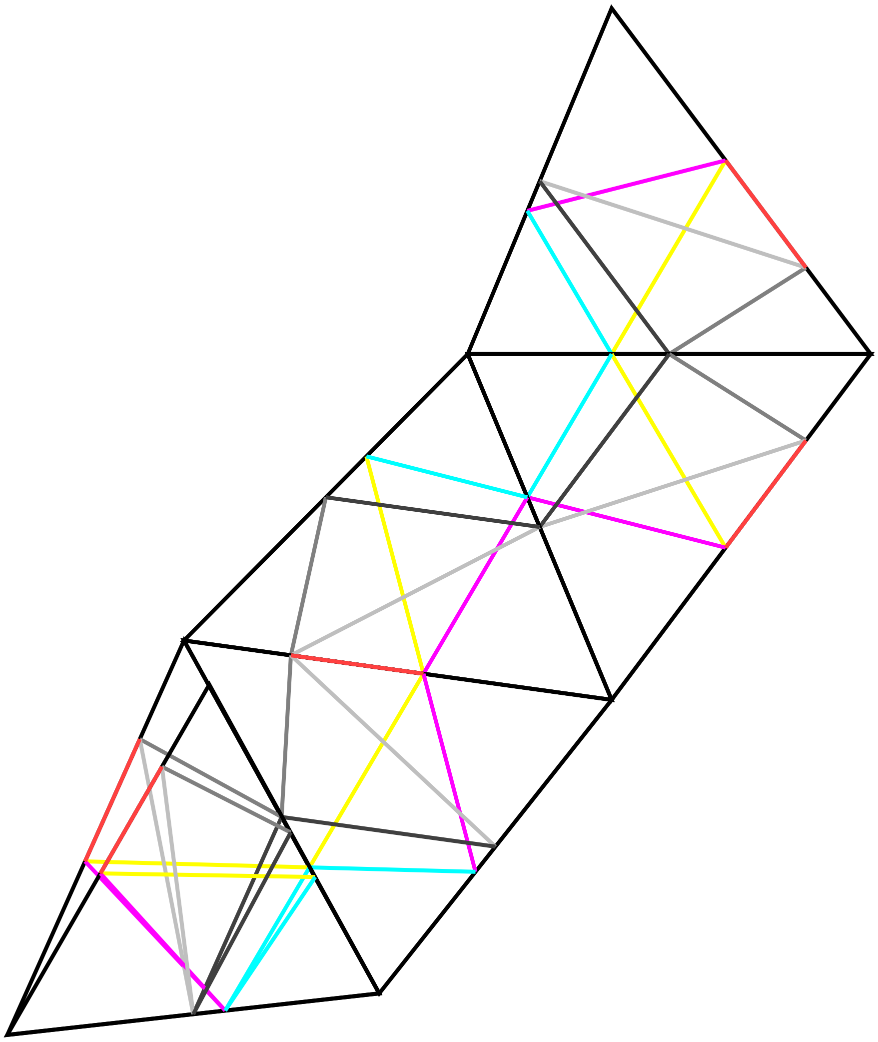

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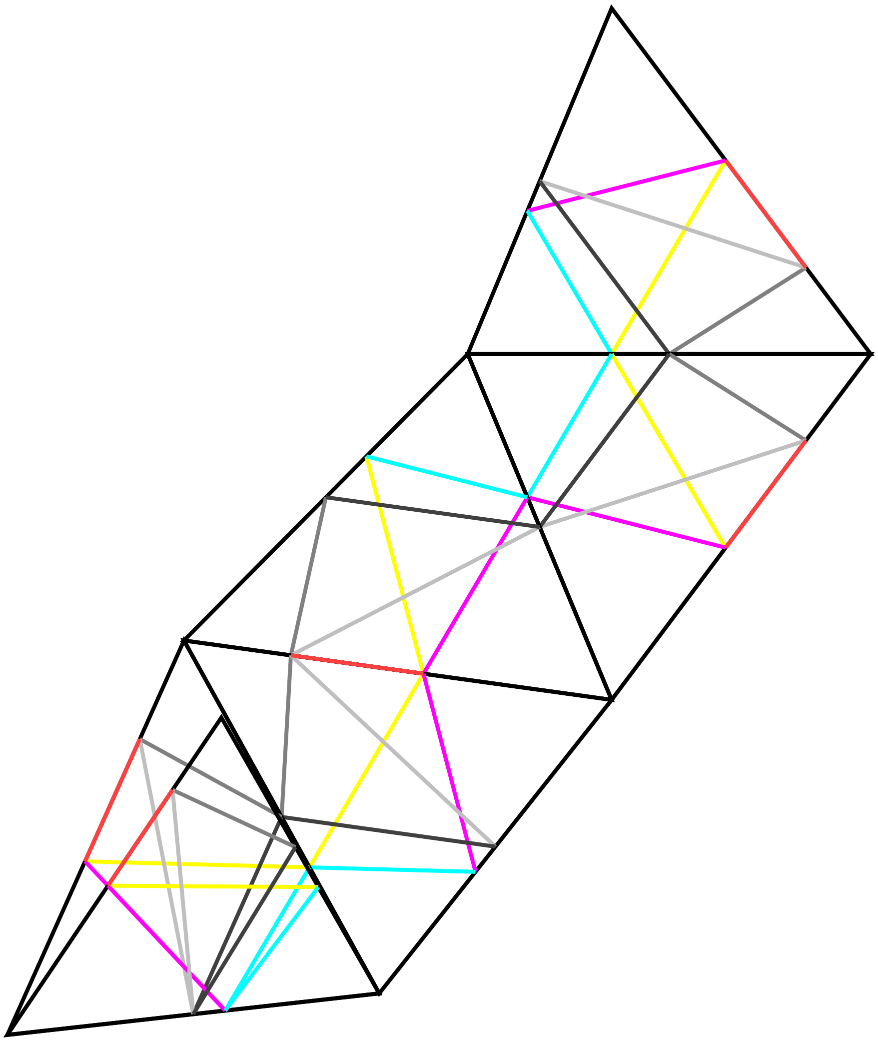

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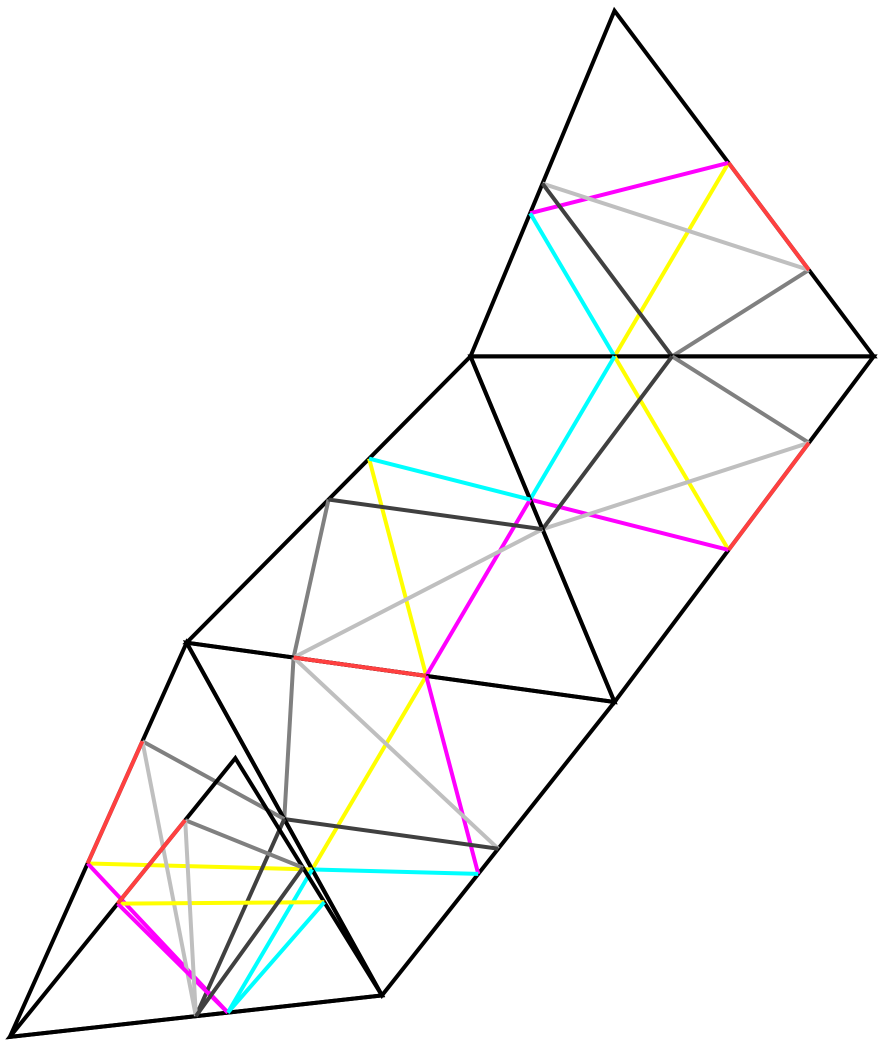

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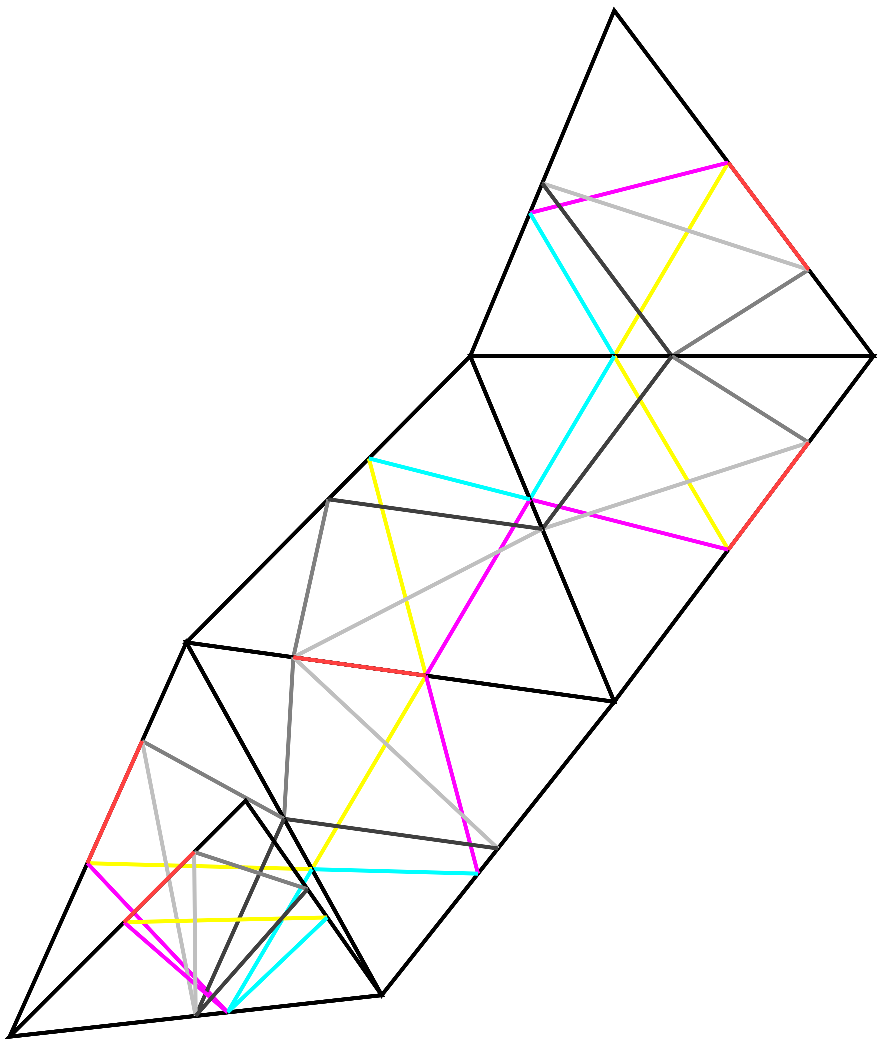

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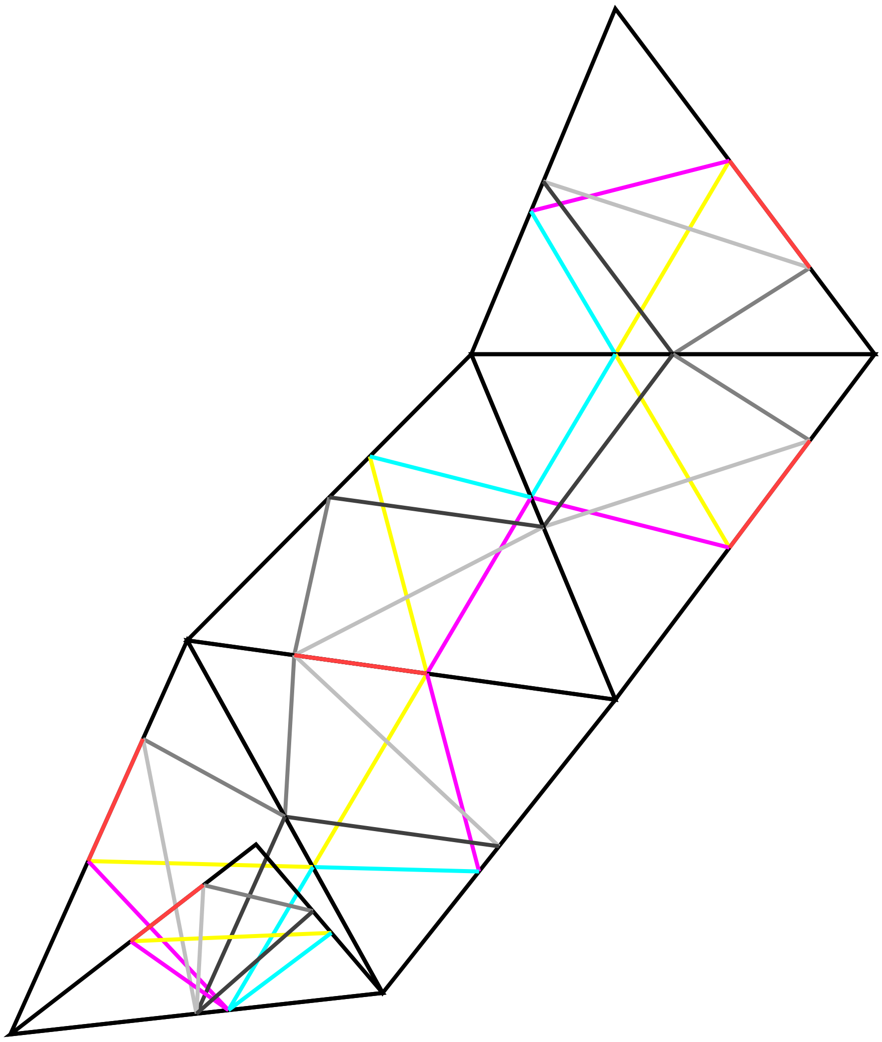

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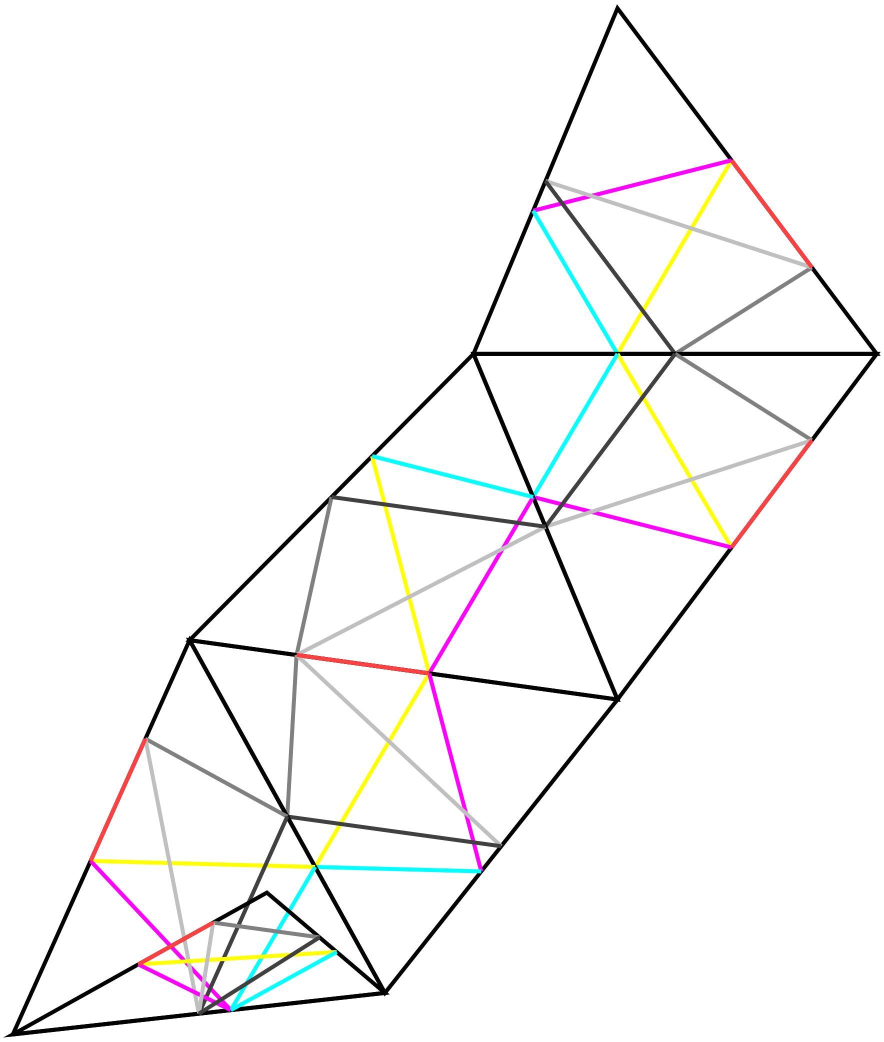

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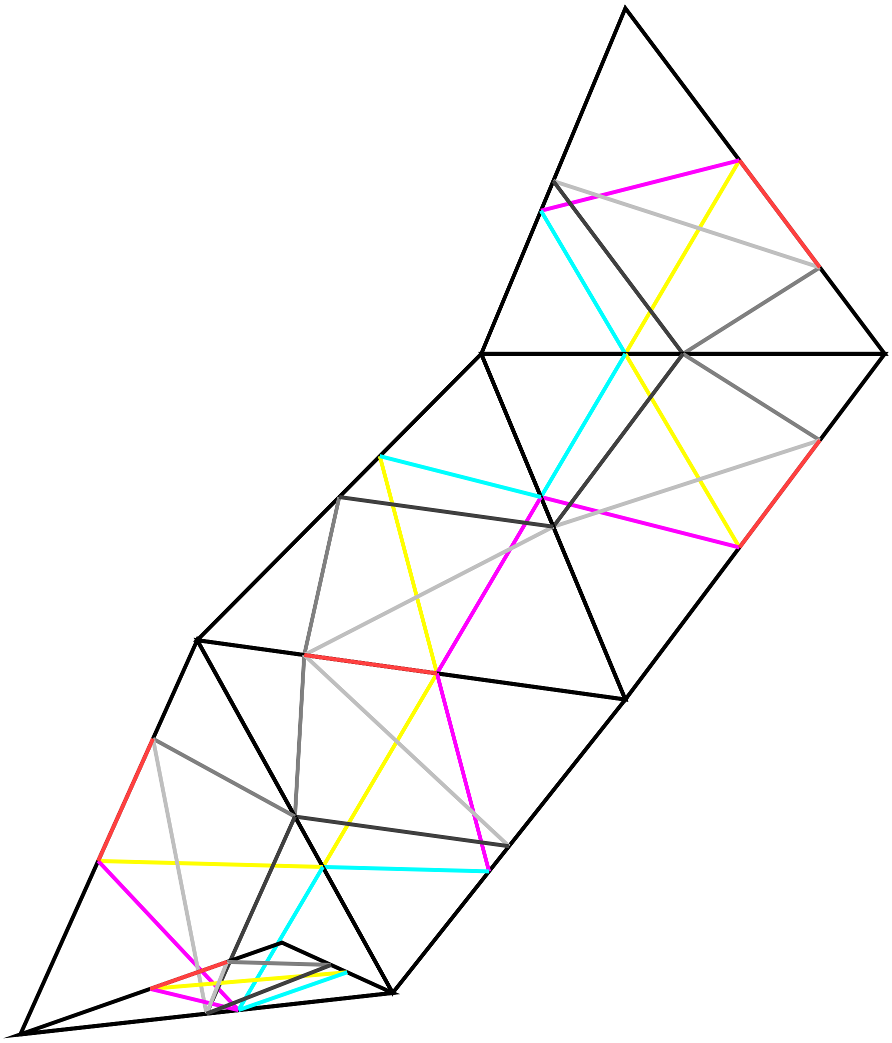

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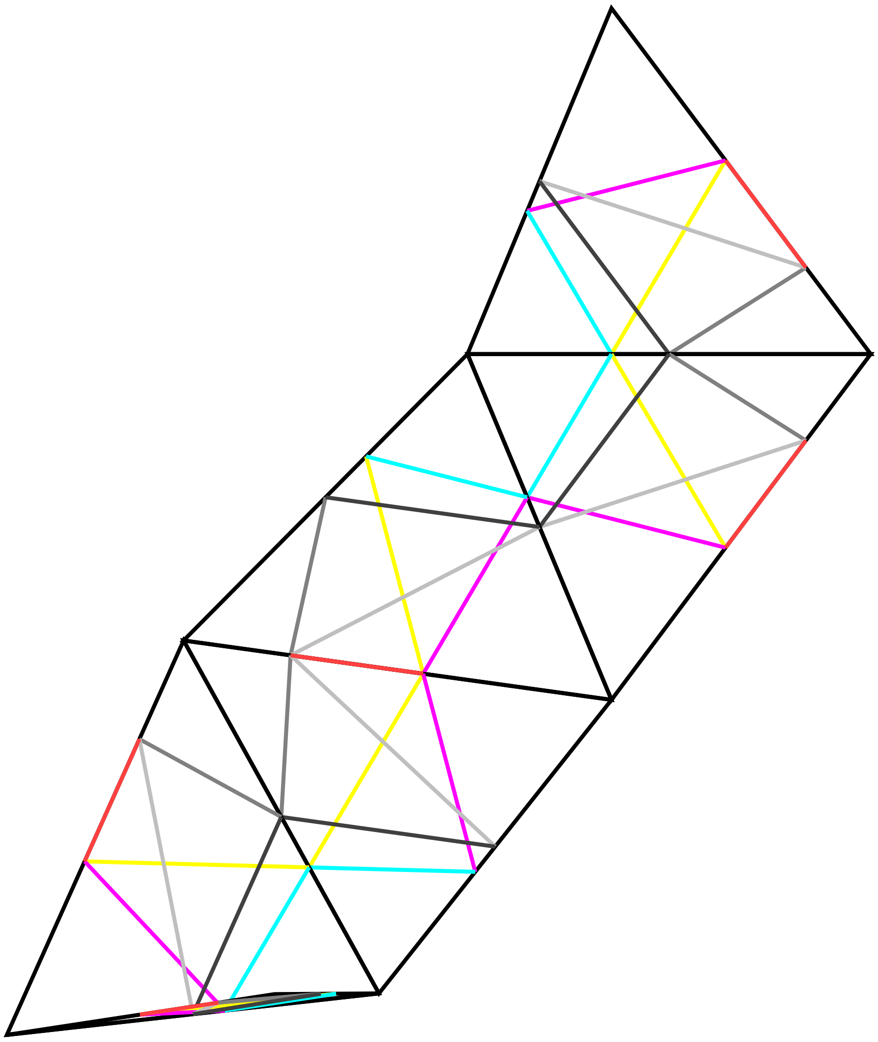

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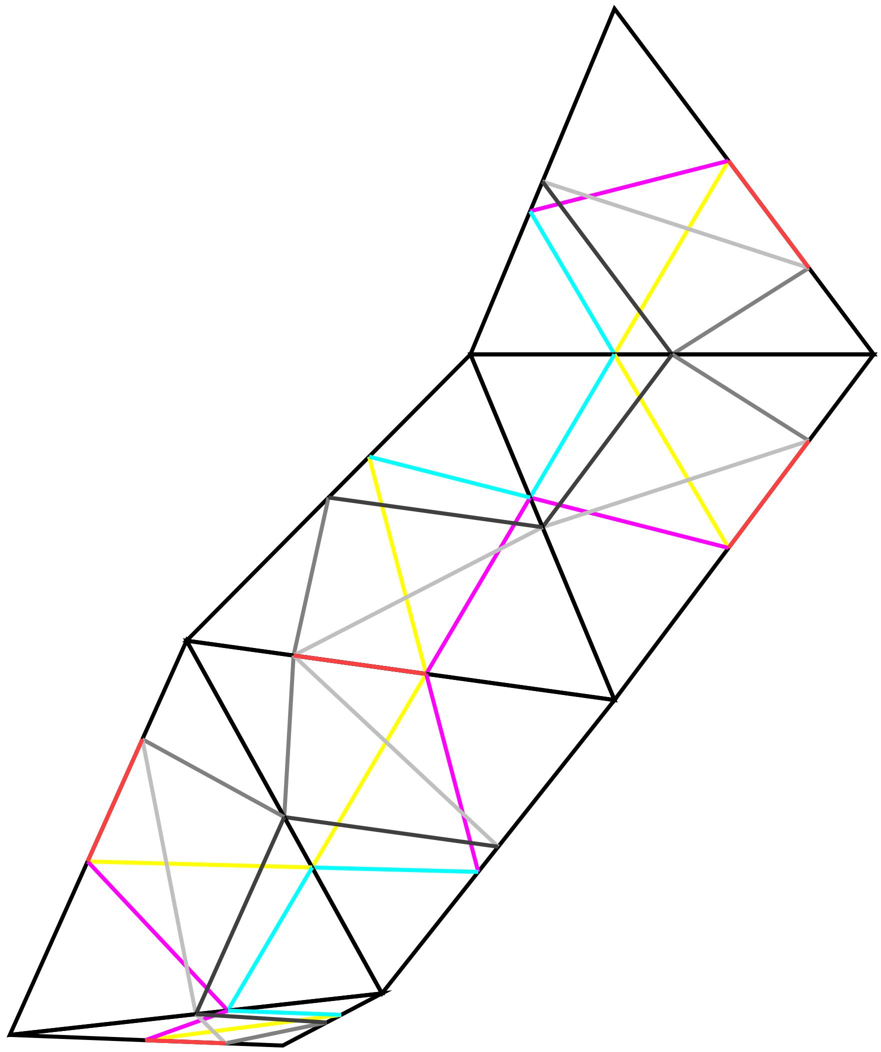

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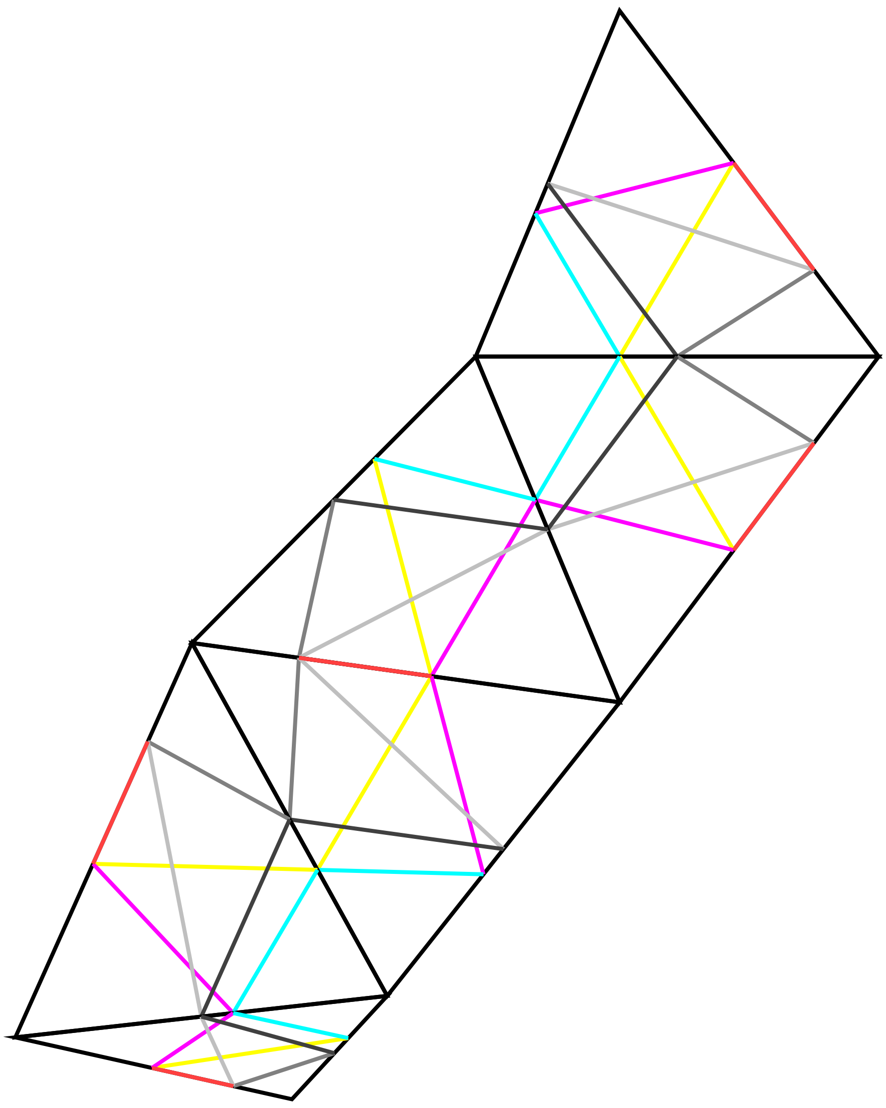

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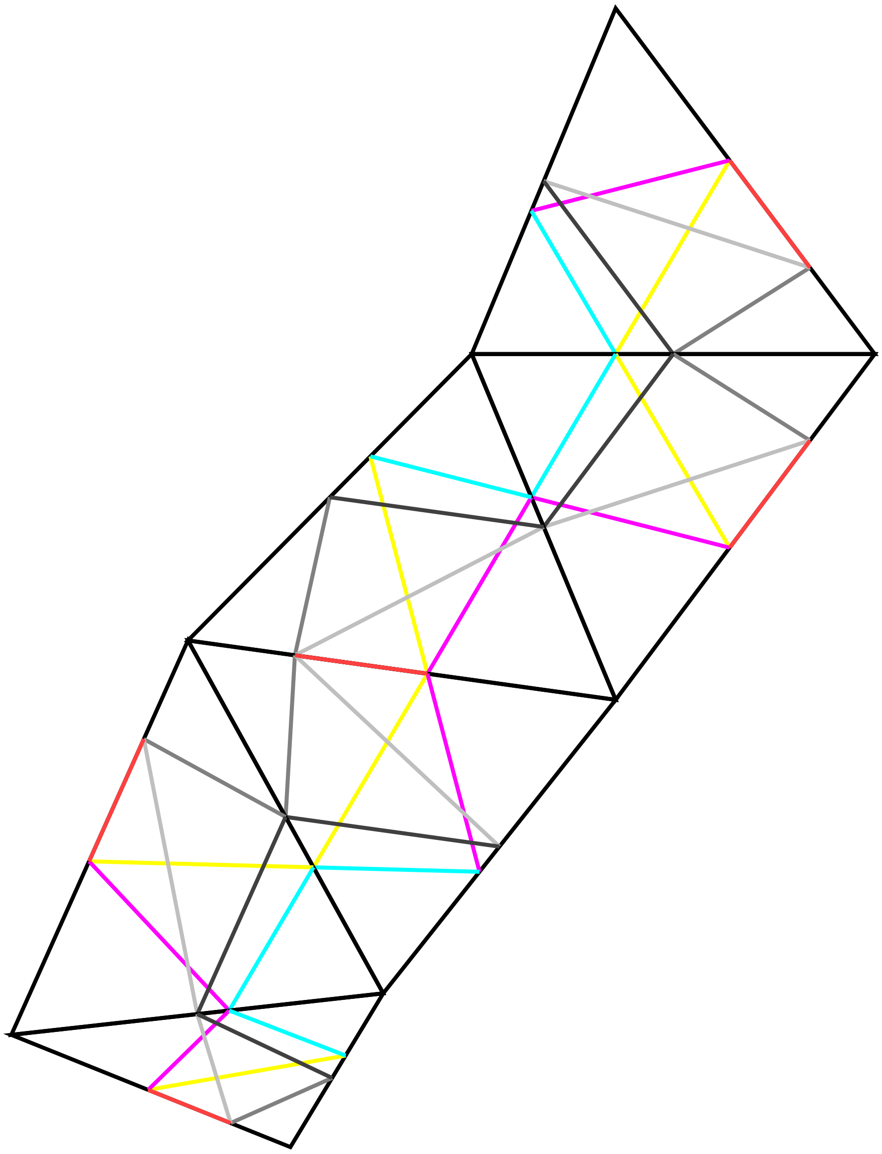

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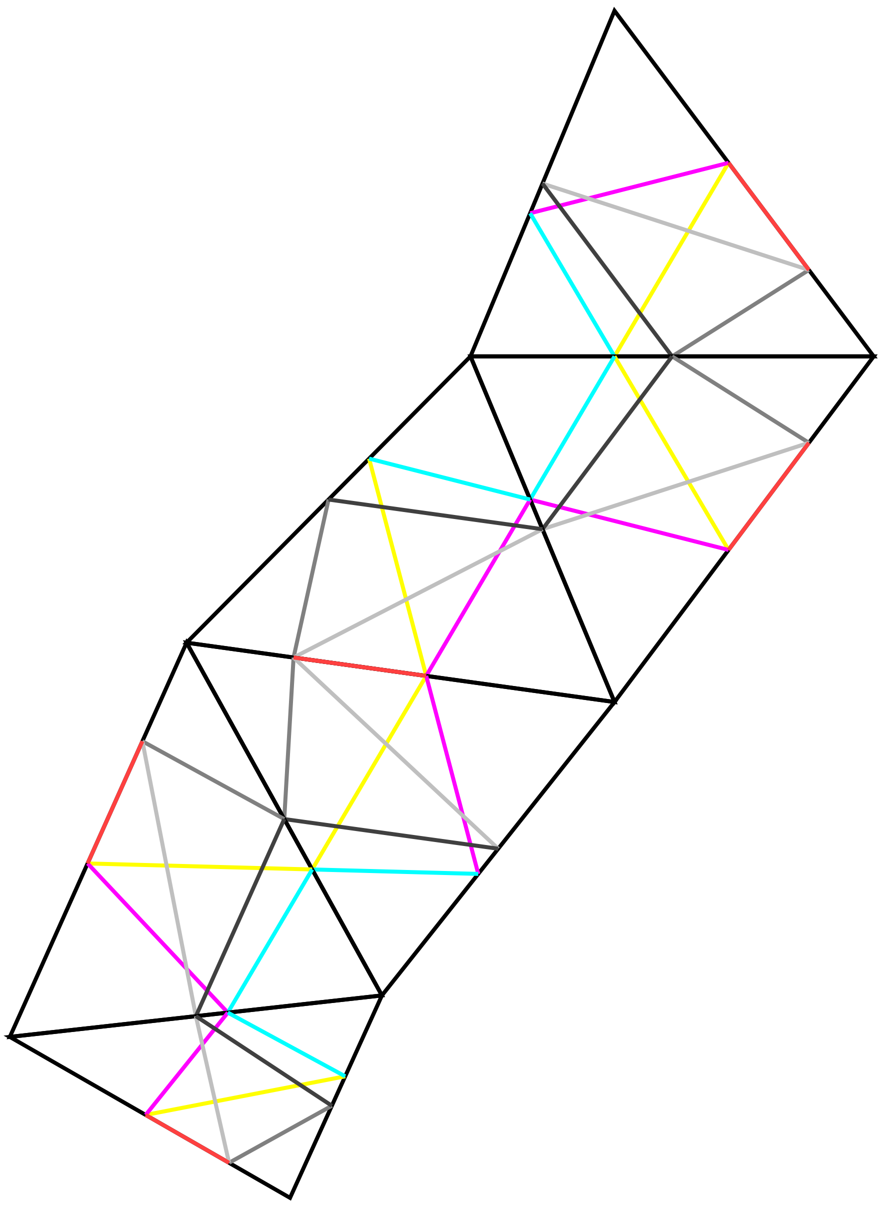

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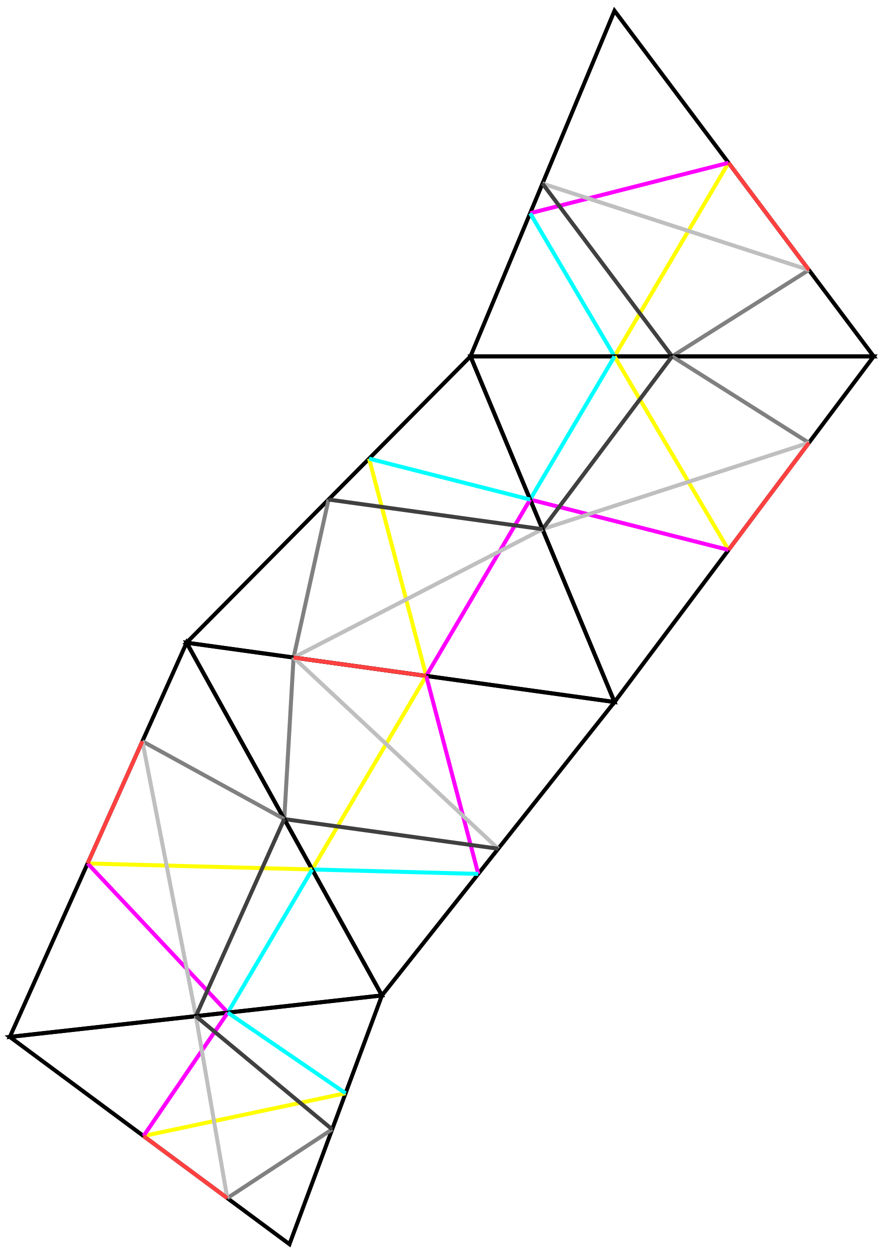

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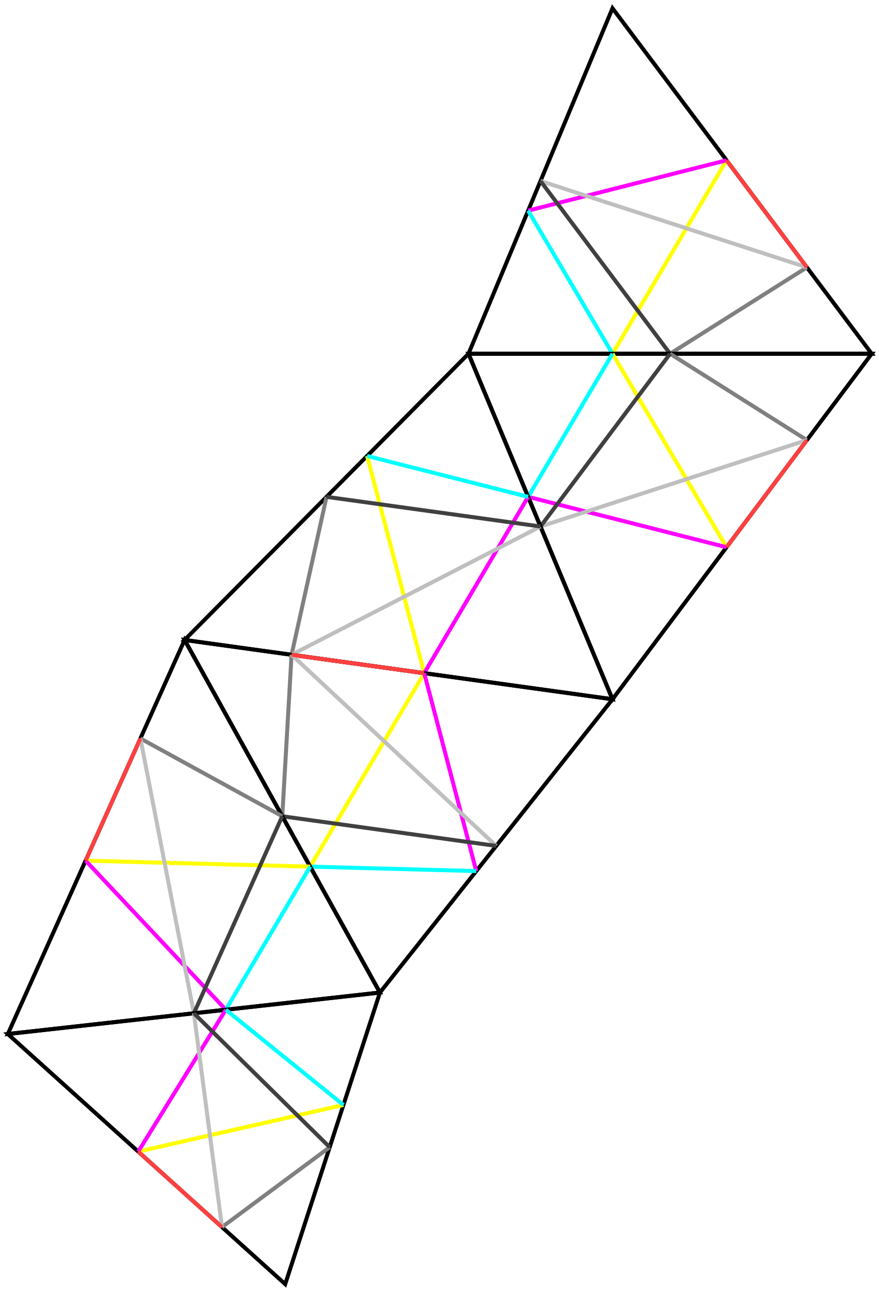

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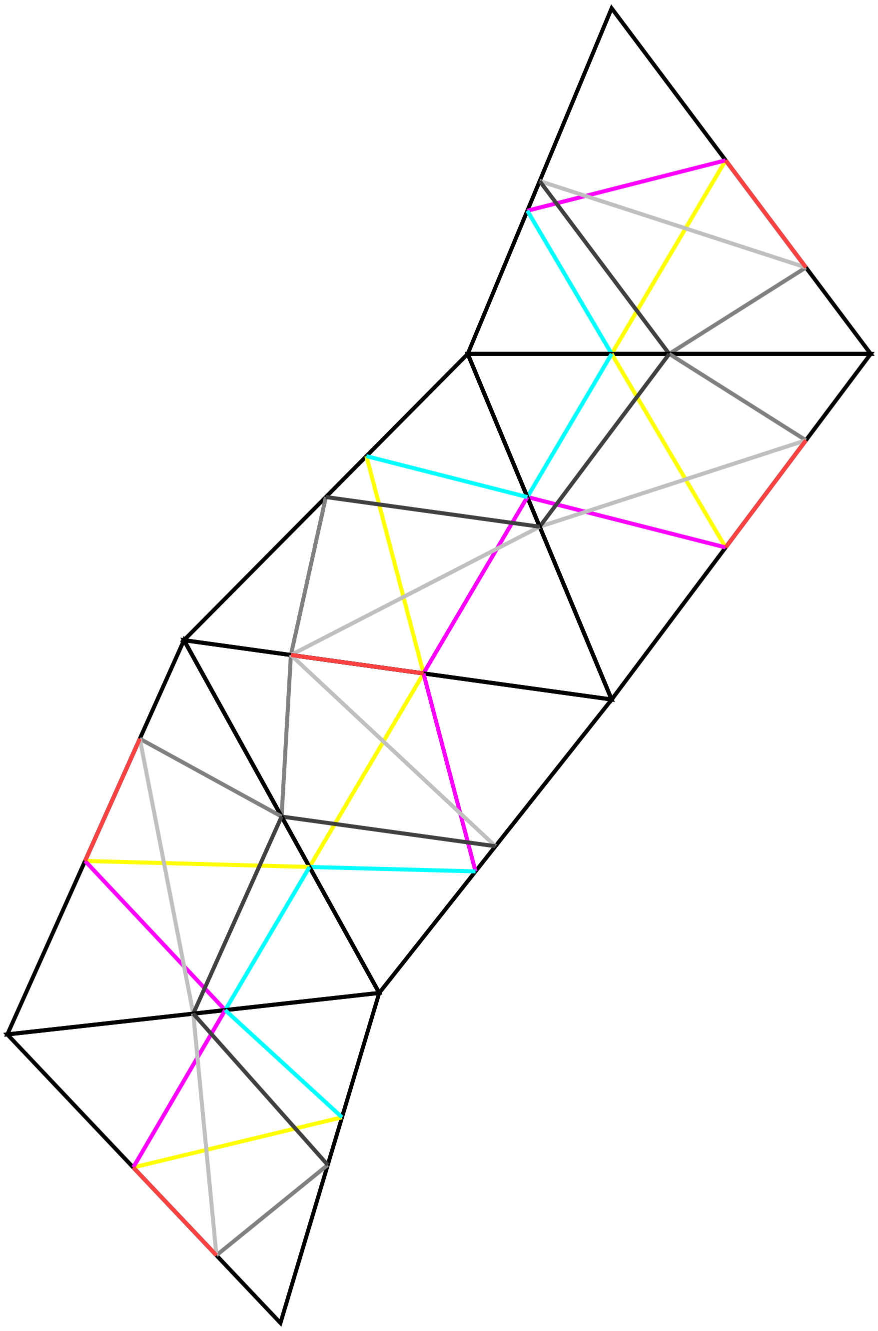

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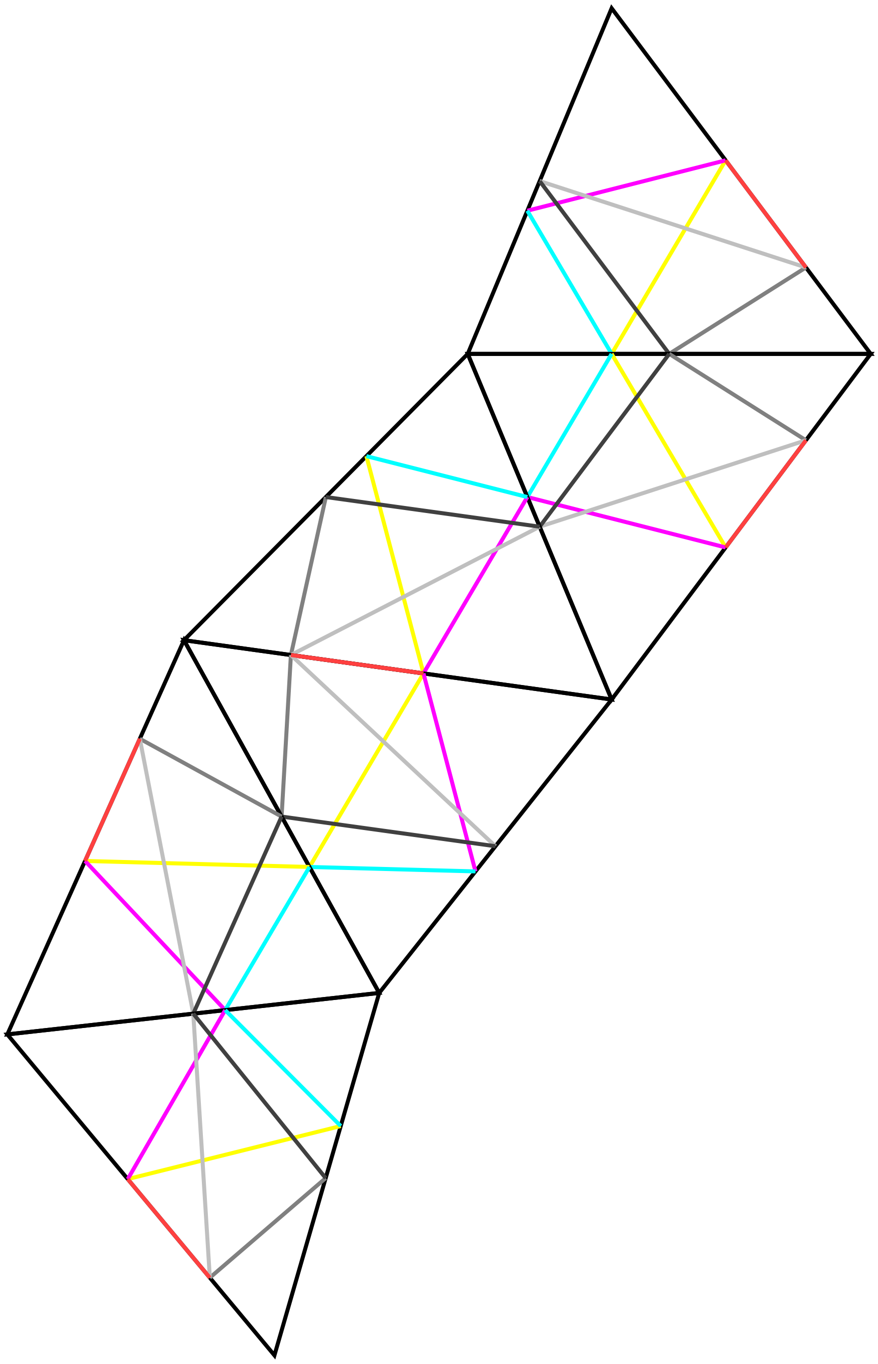

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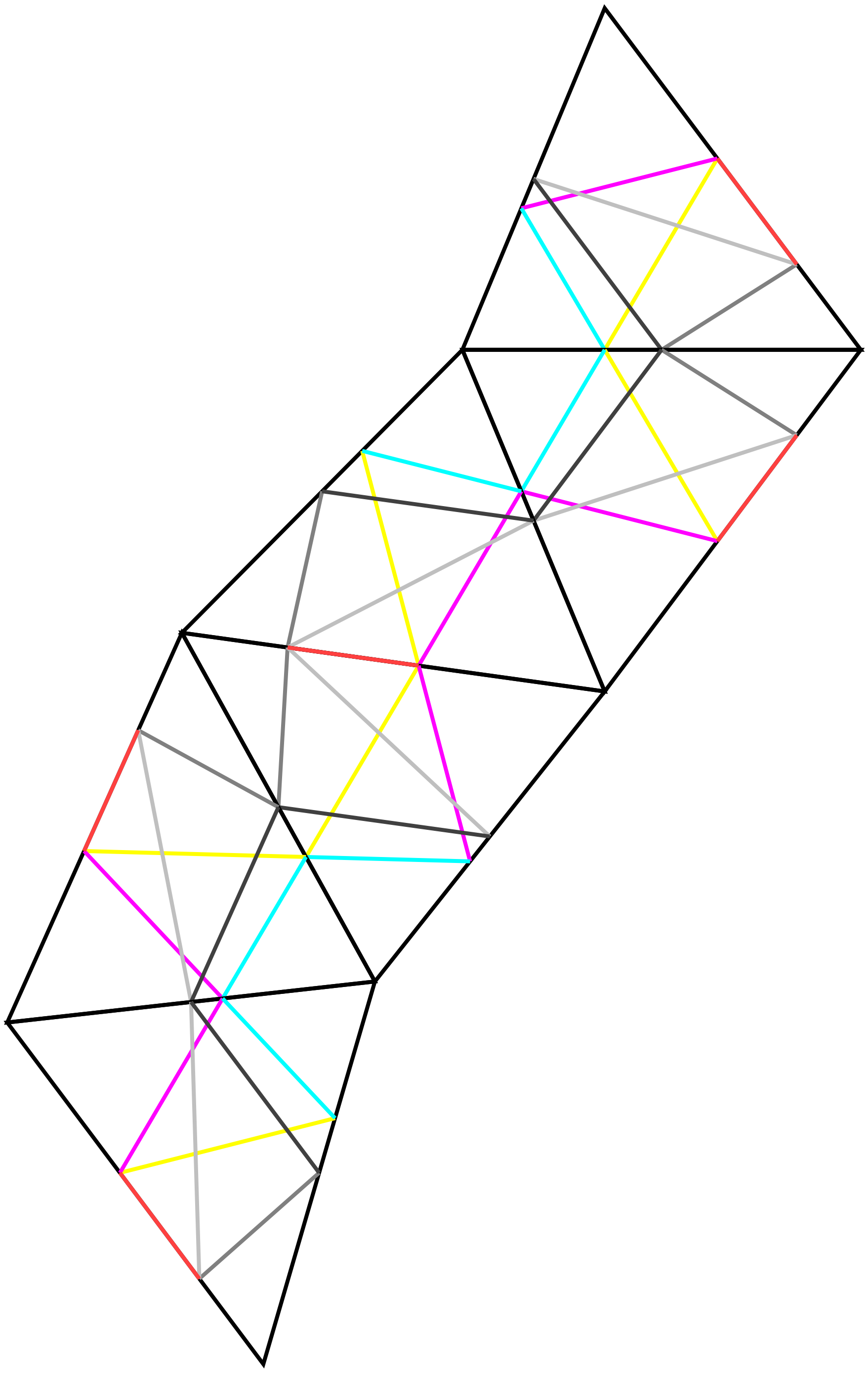
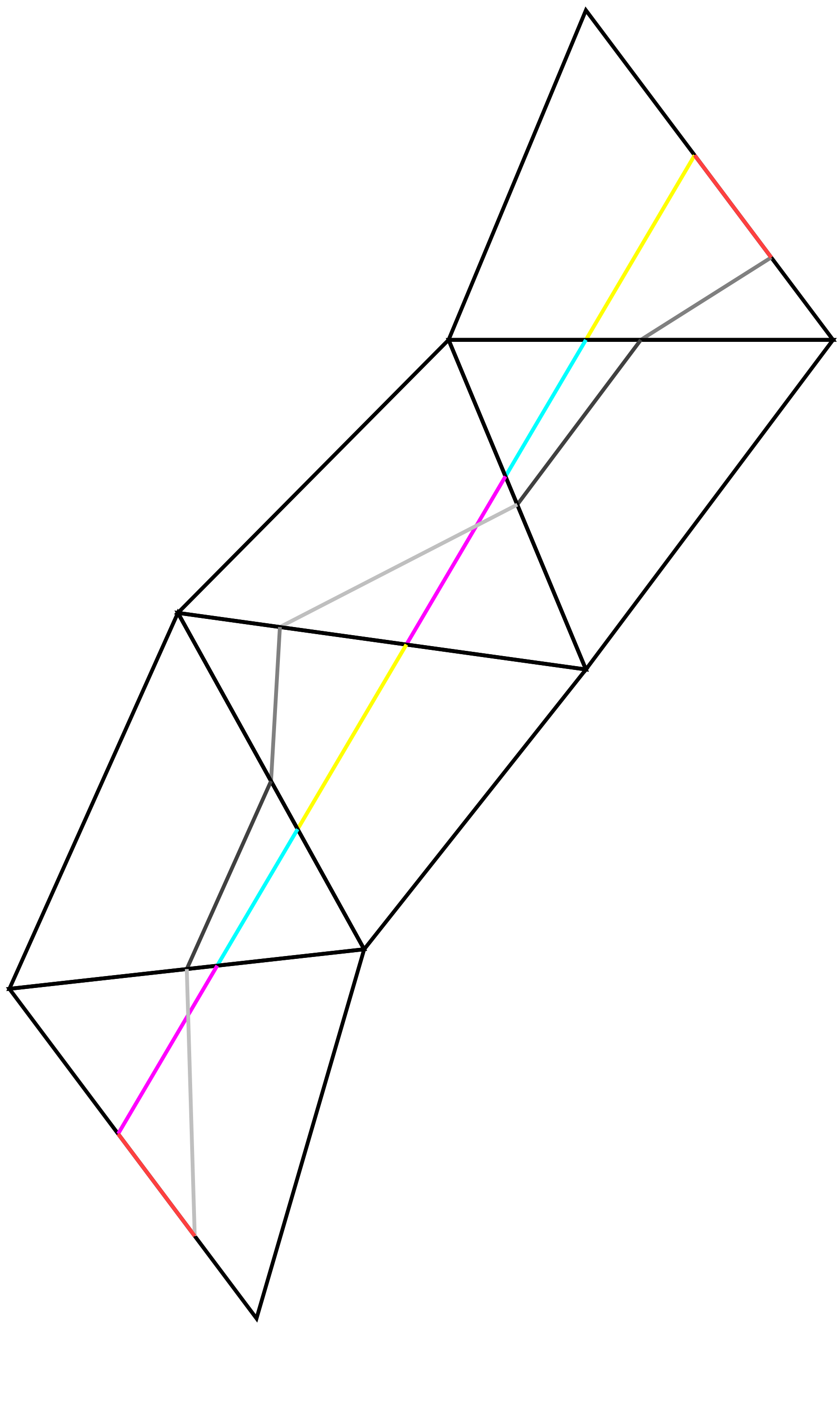
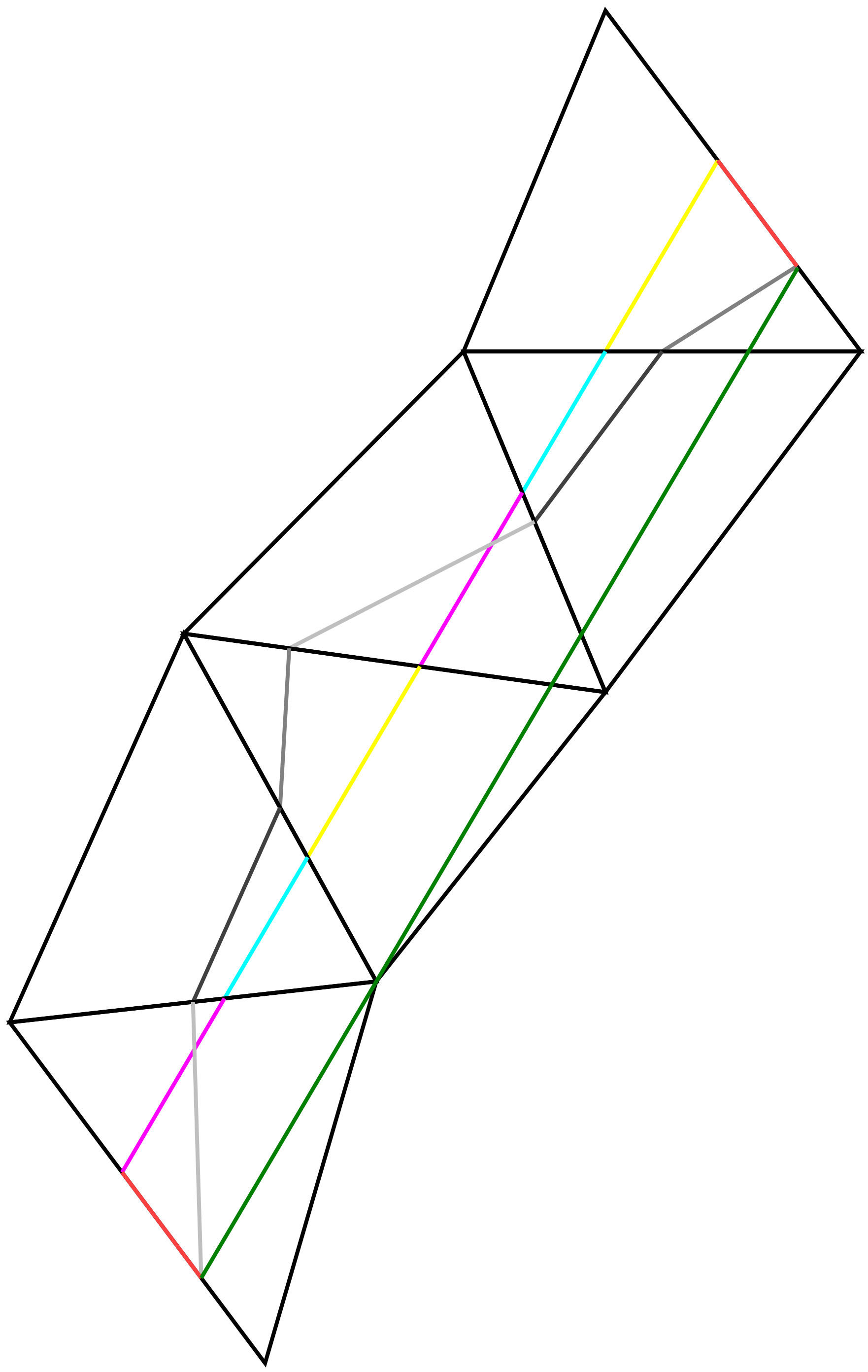


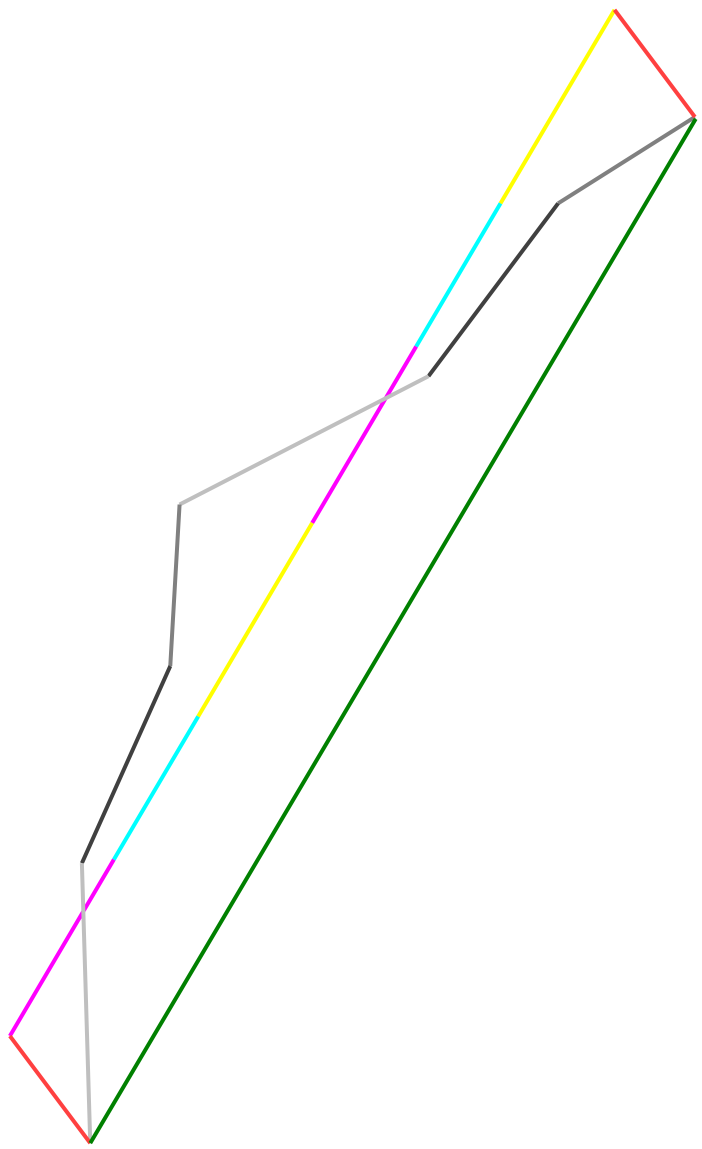
There.



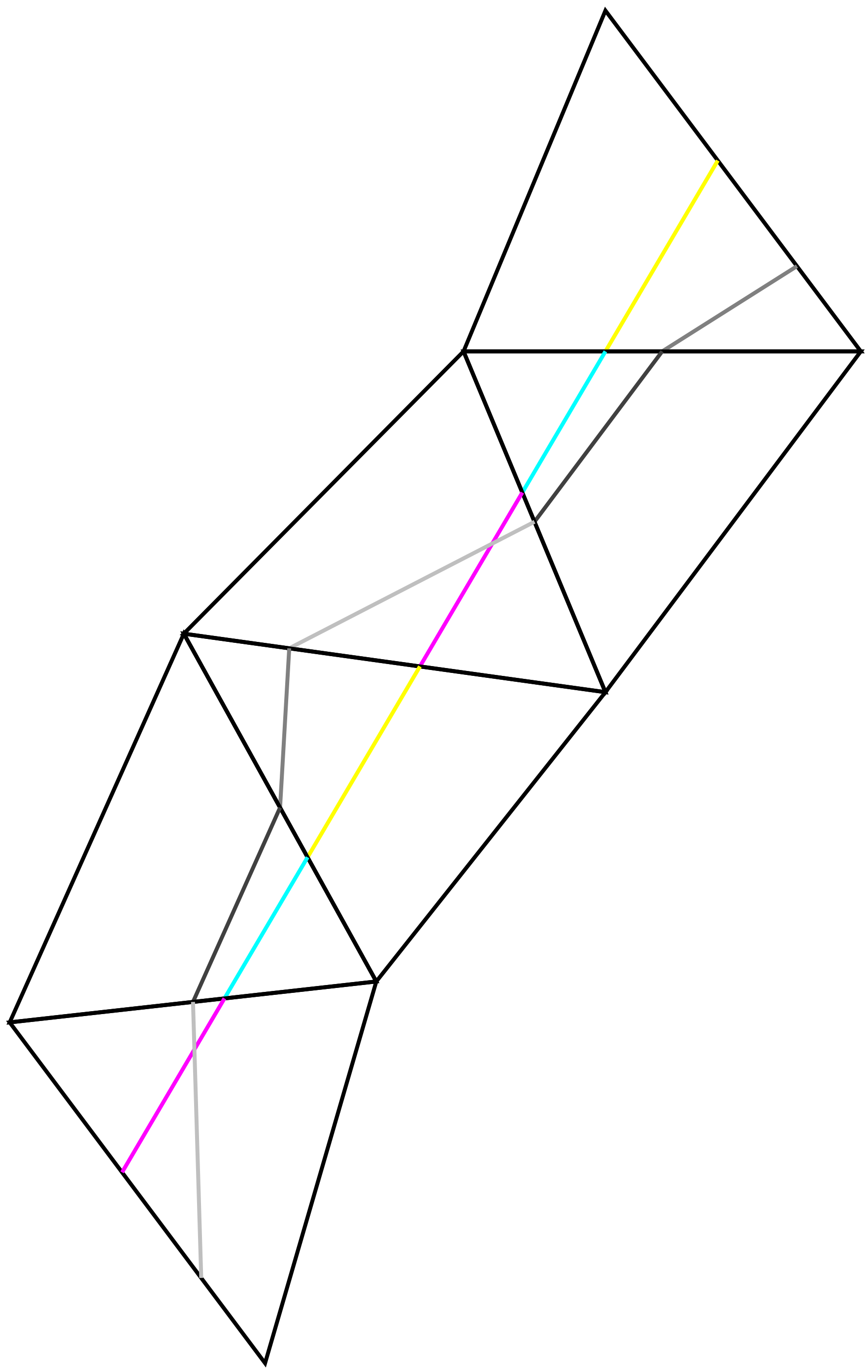
Now take away the distractions.



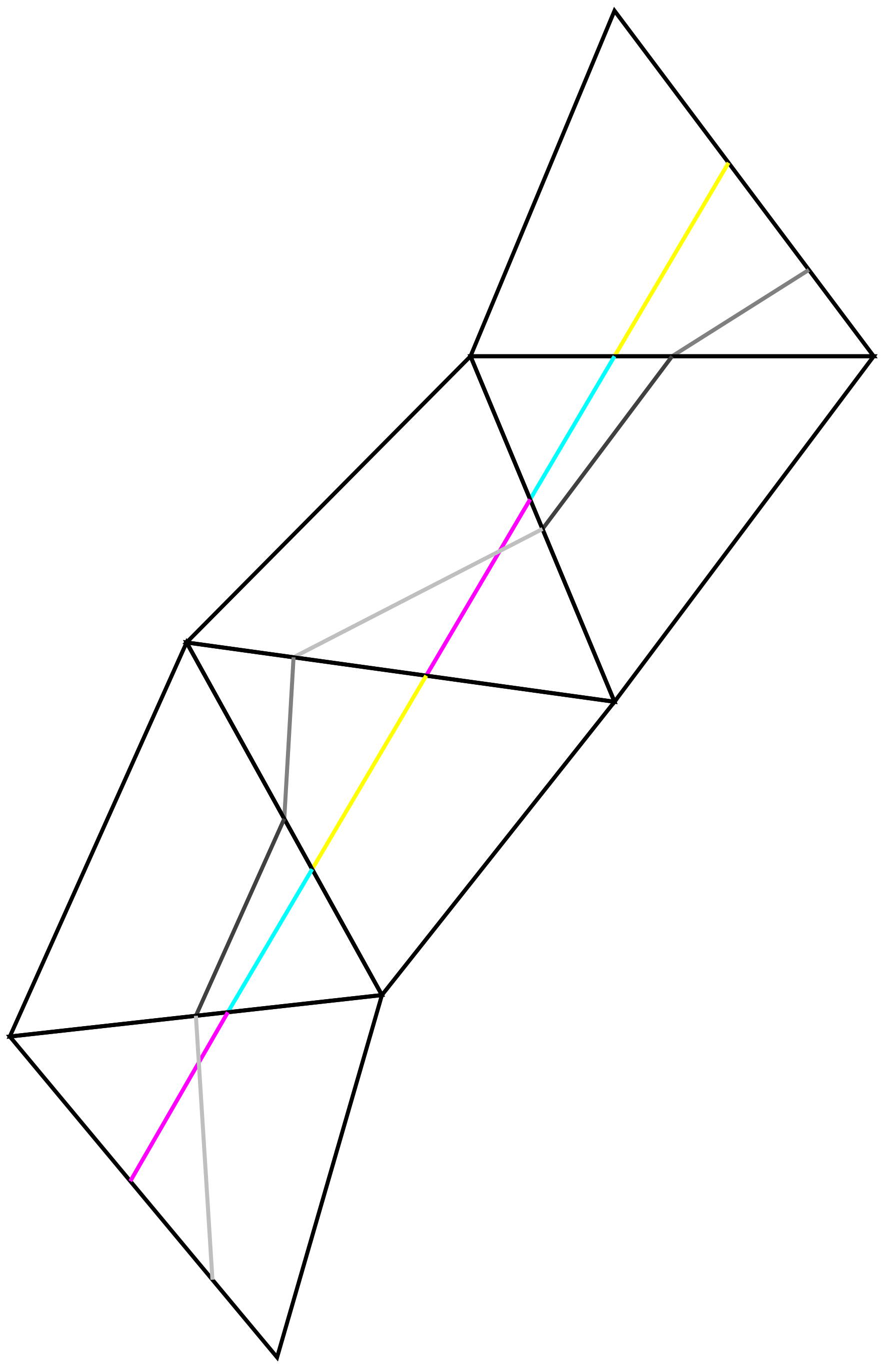
Draw a line here.

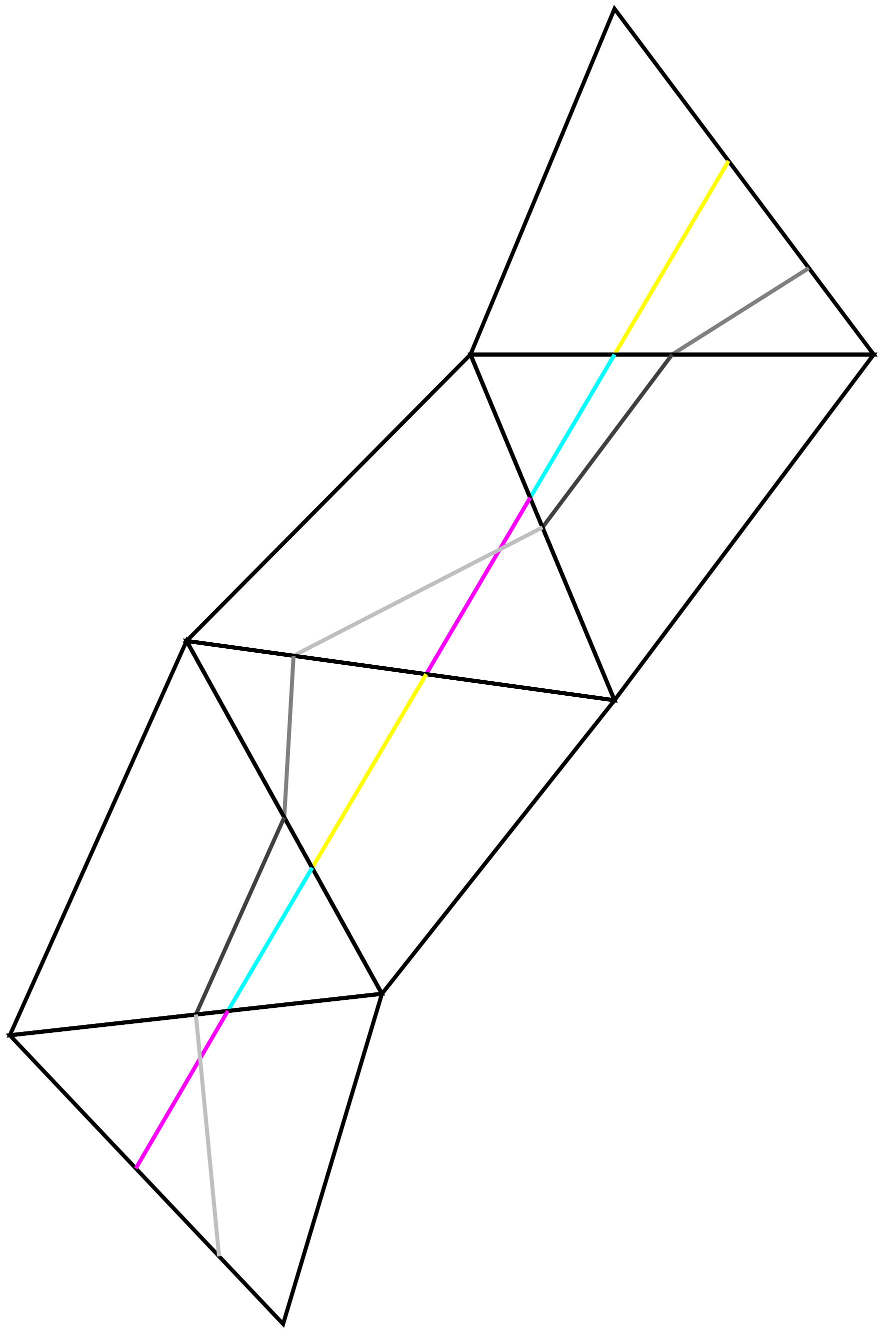


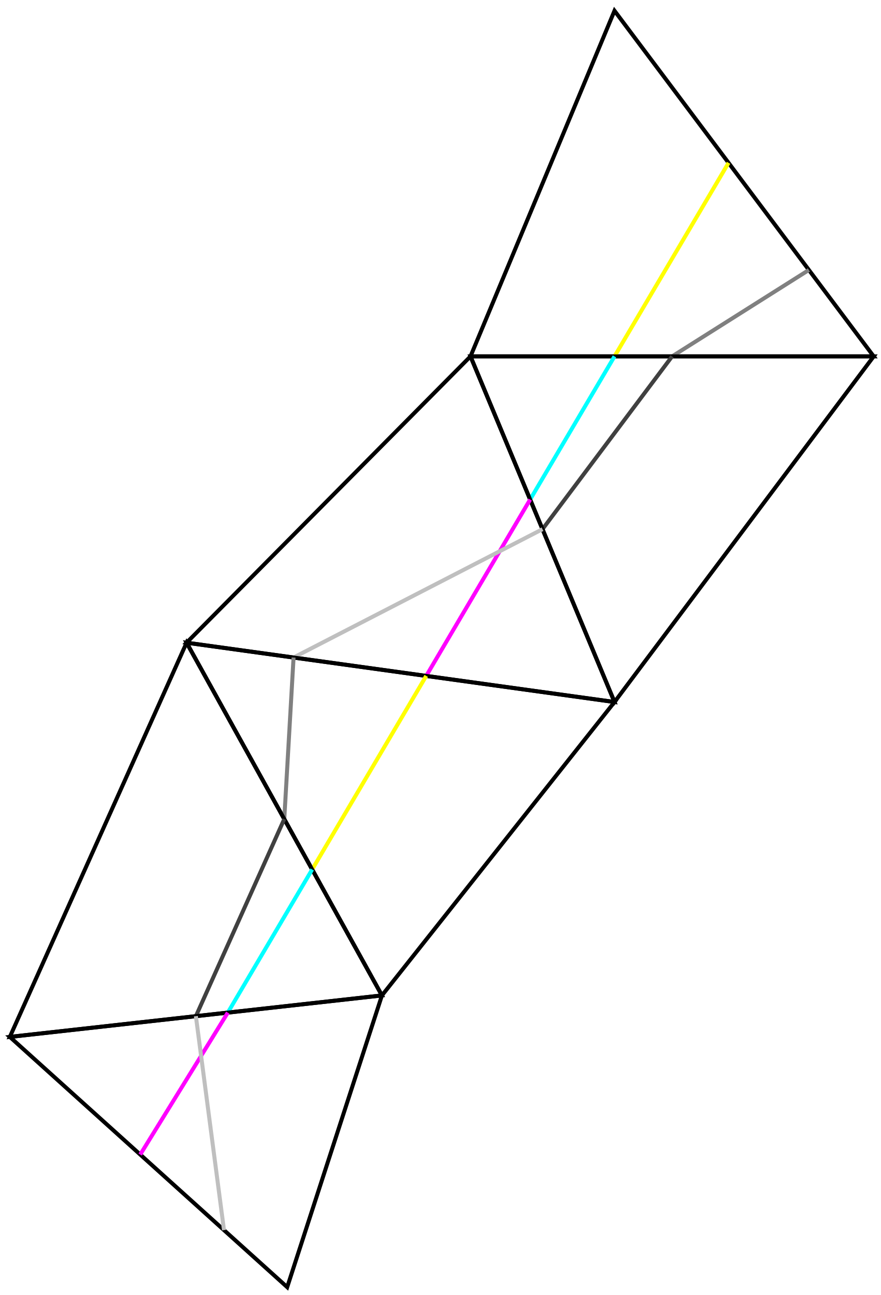
Again, take away the distraction. Since we know the two dark red lines are equal in length and parallel to each other, this is a parallelogram. This means the multi-colored line and the dark green one are equal in length. Plus, we know the length of the monochrome line segments is greater than the green one. (the shortest distance between two points is a straight line.) We can therefore conclude the multi-colored line is shorter in length than the monochrome one.

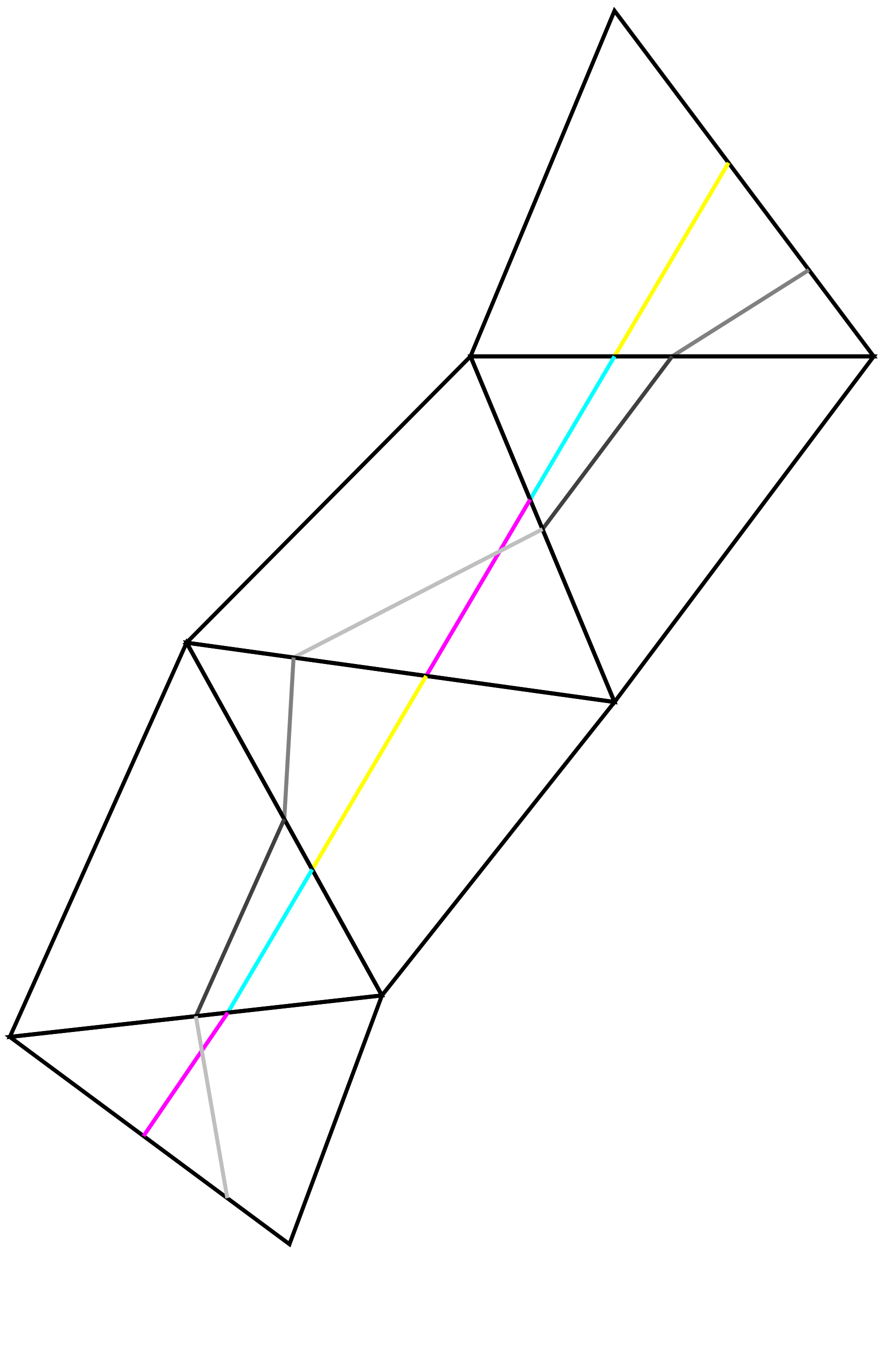


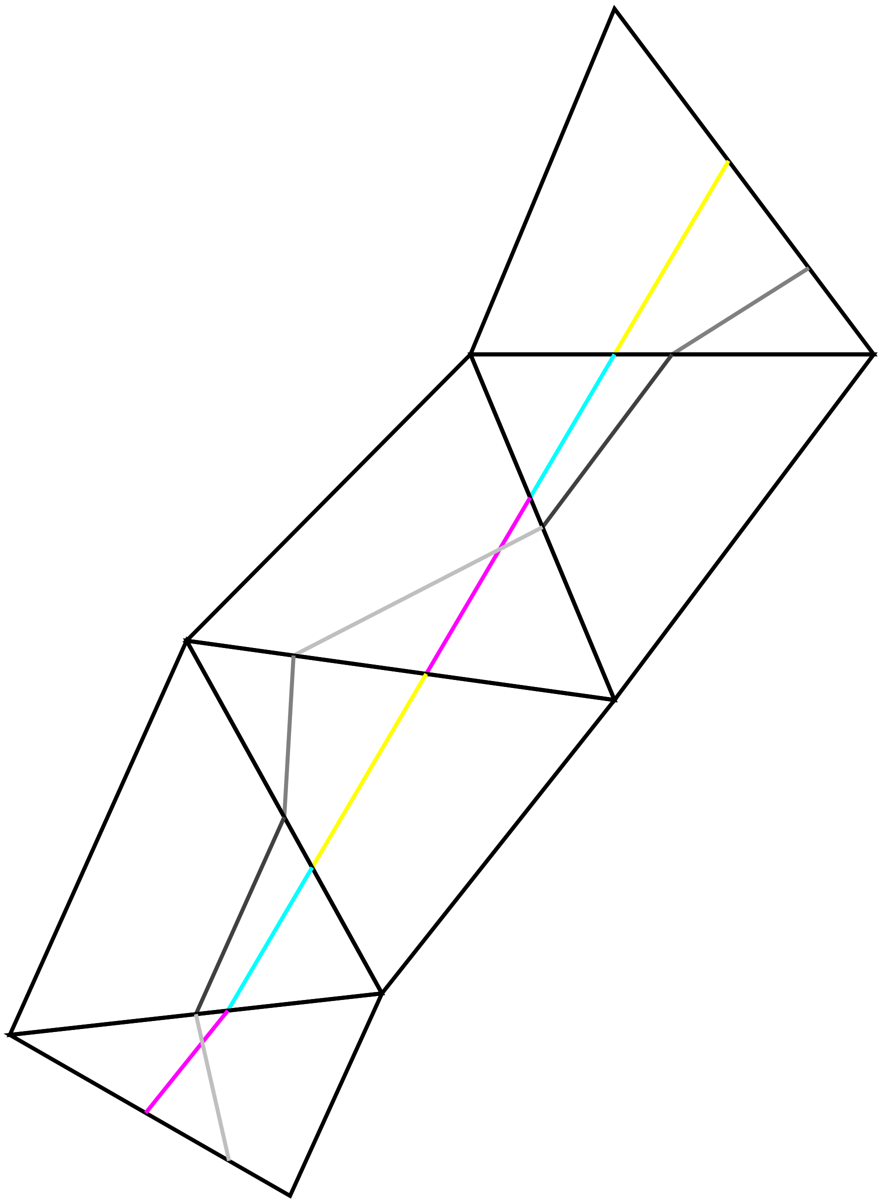
The significance of this is that the multi-colored line and the monochrome line are actually twice the lengths of the perimeters of their respective original triangles. Let's flip them back and see.

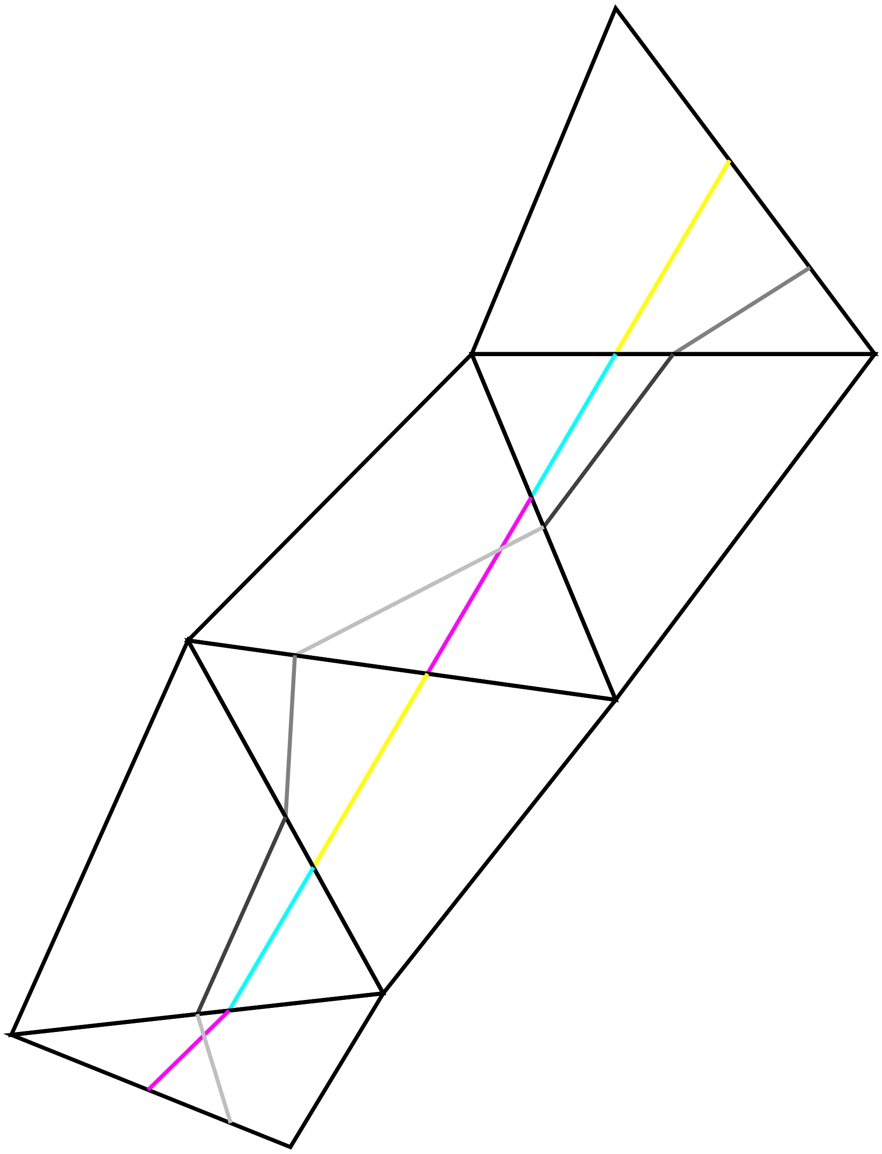


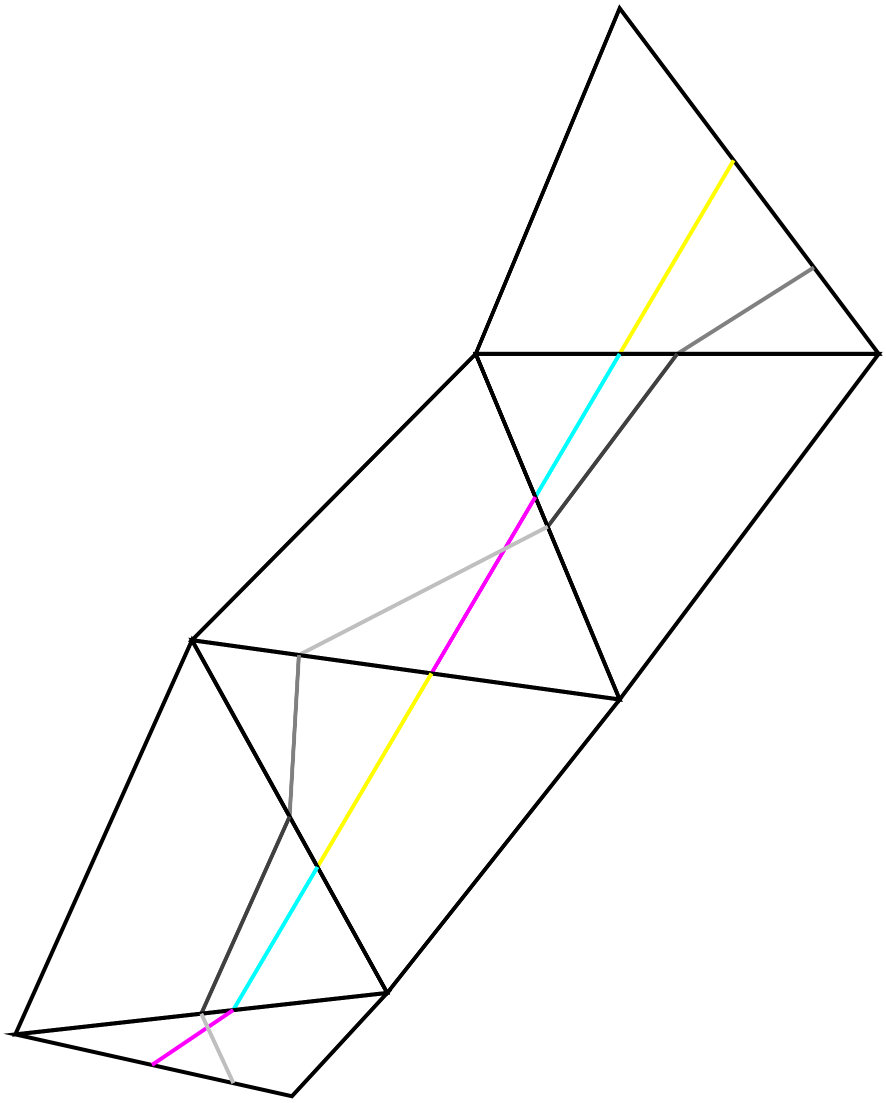


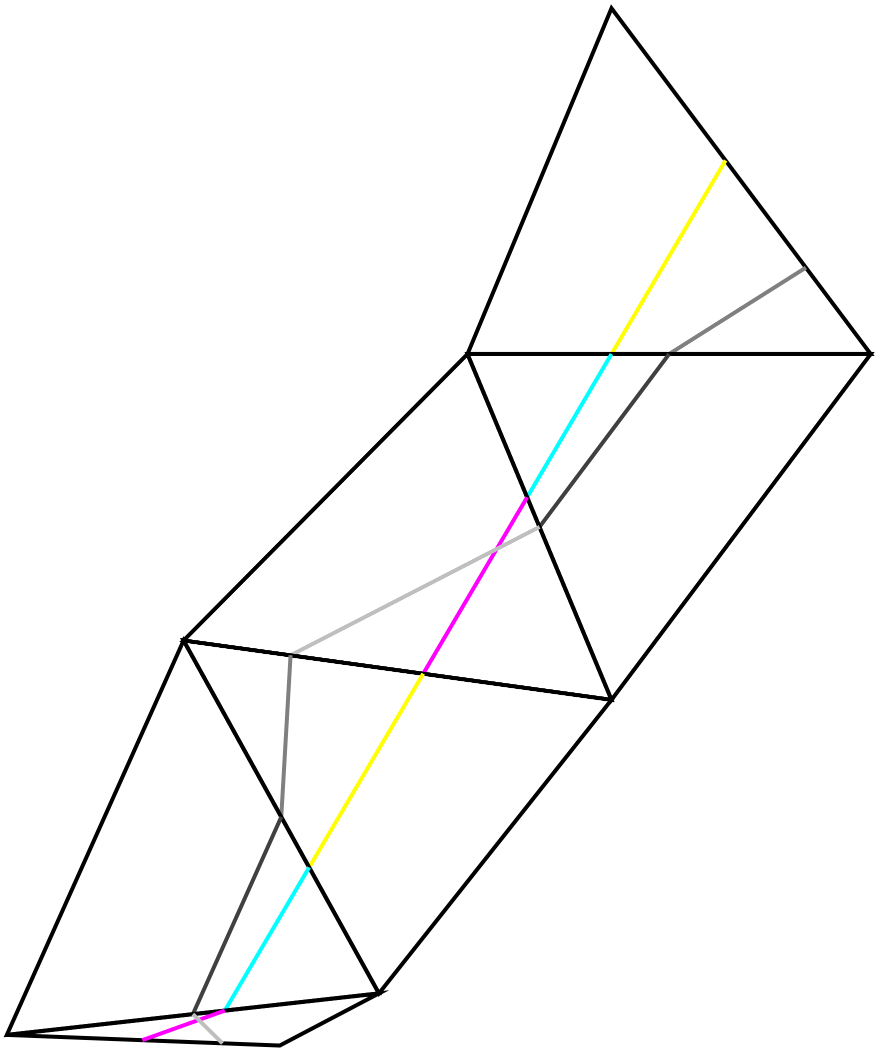


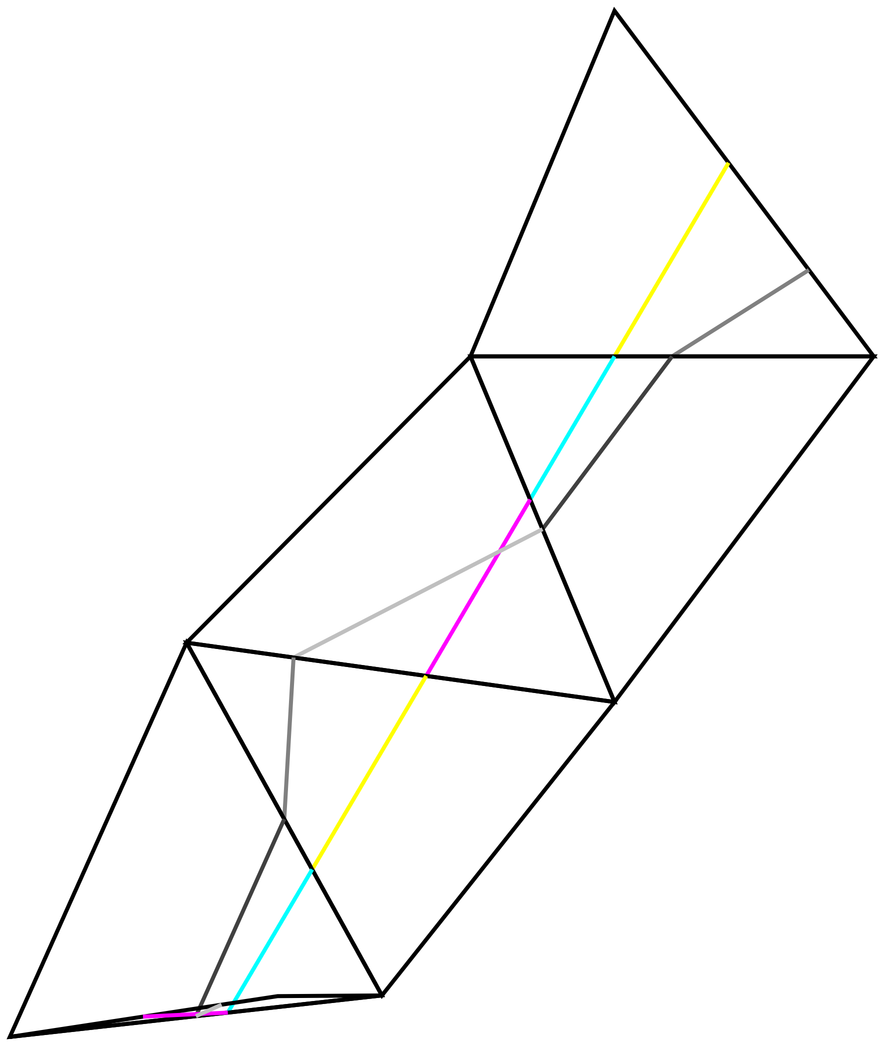


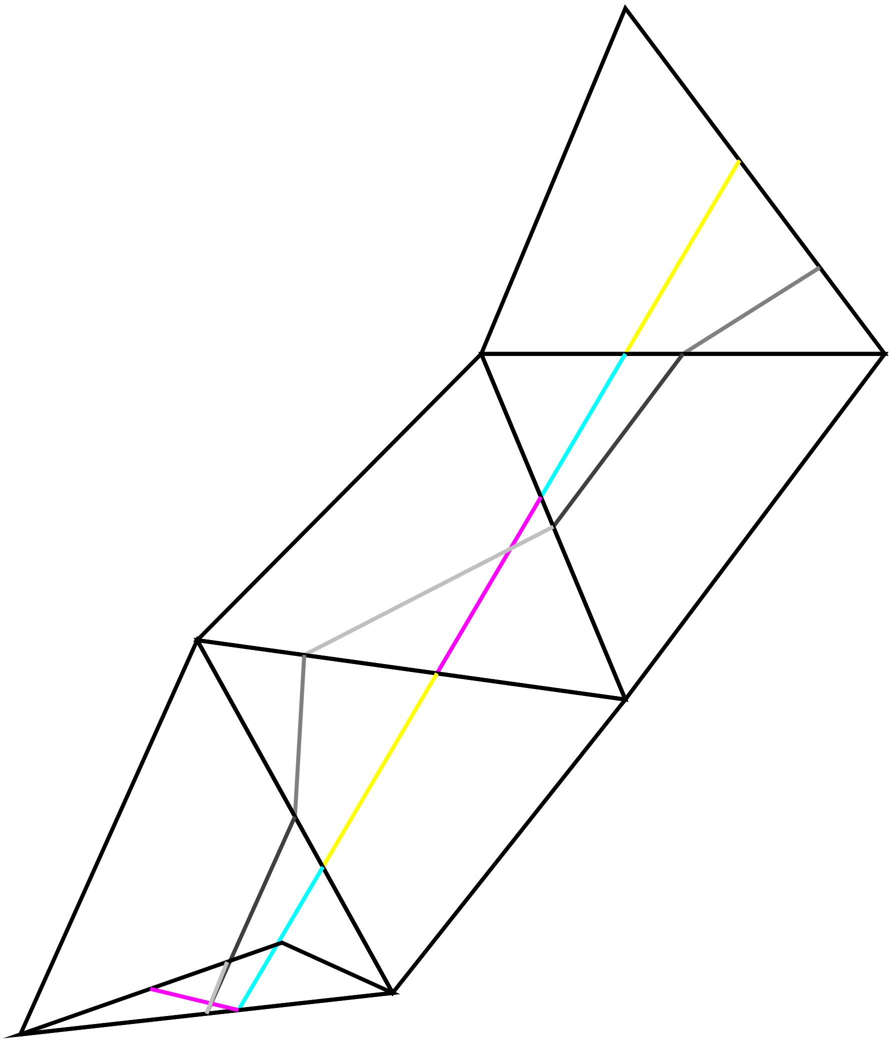


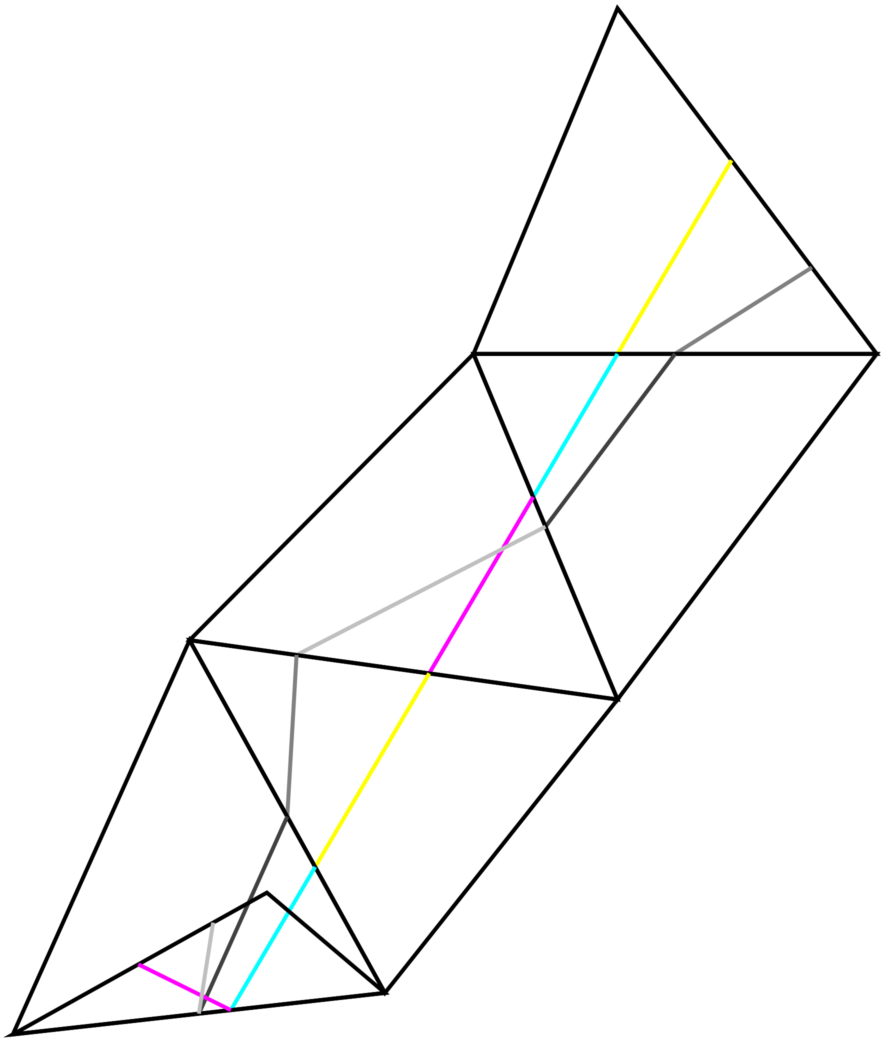


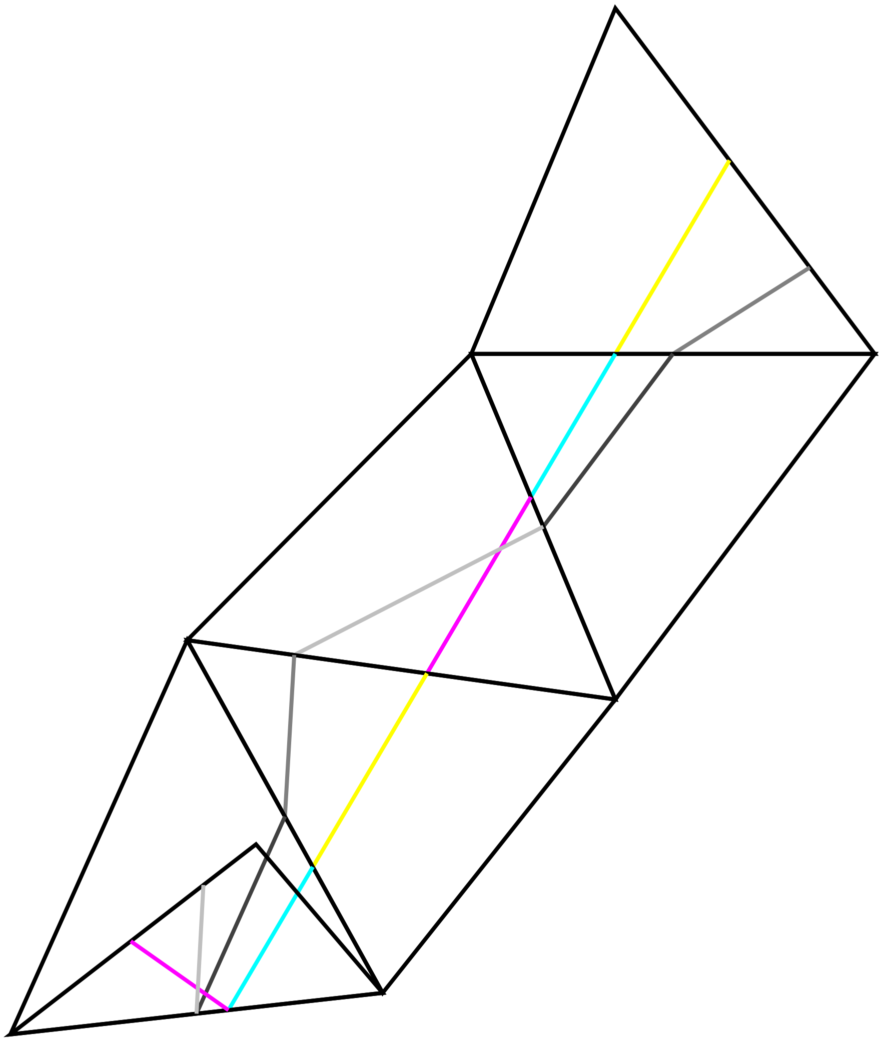


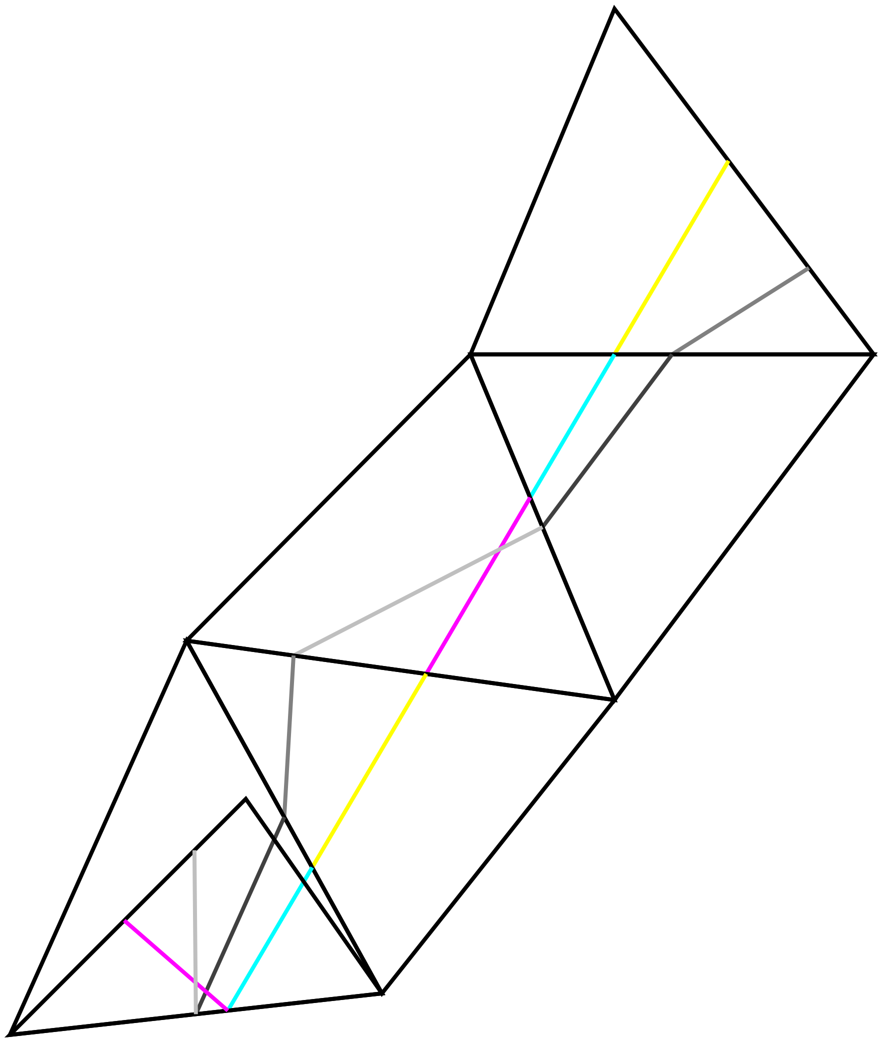


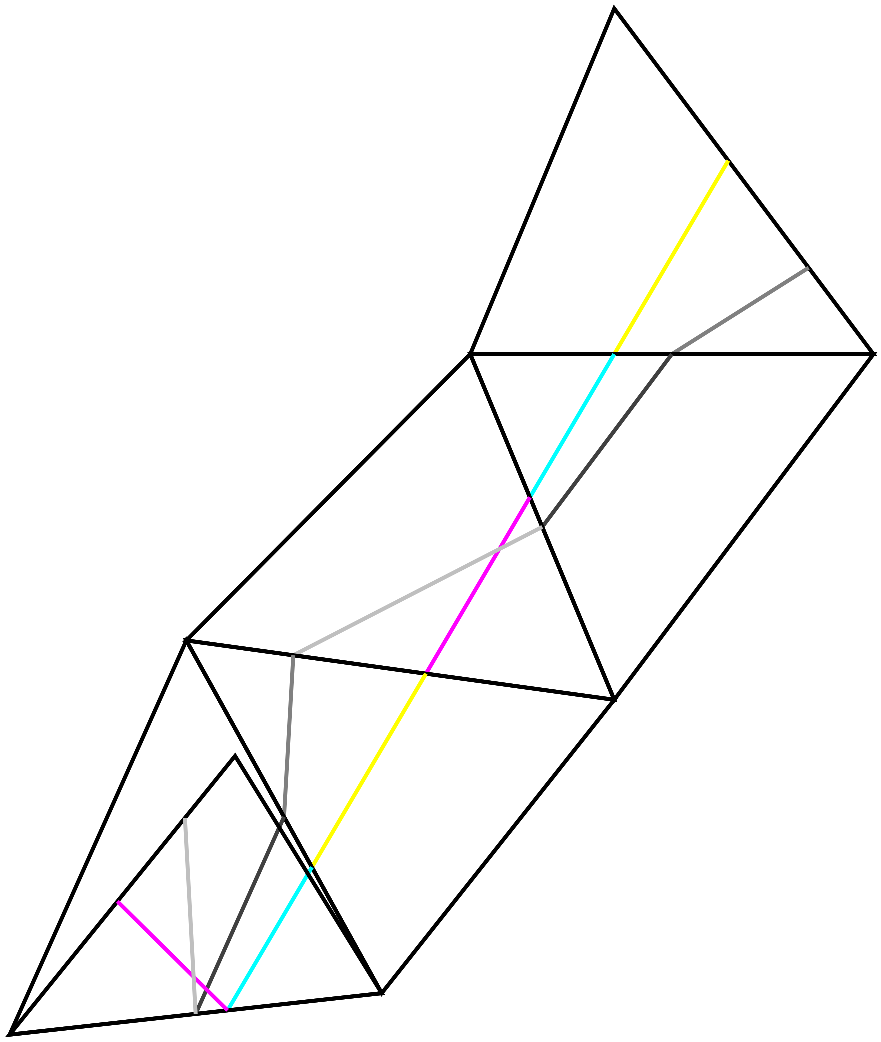


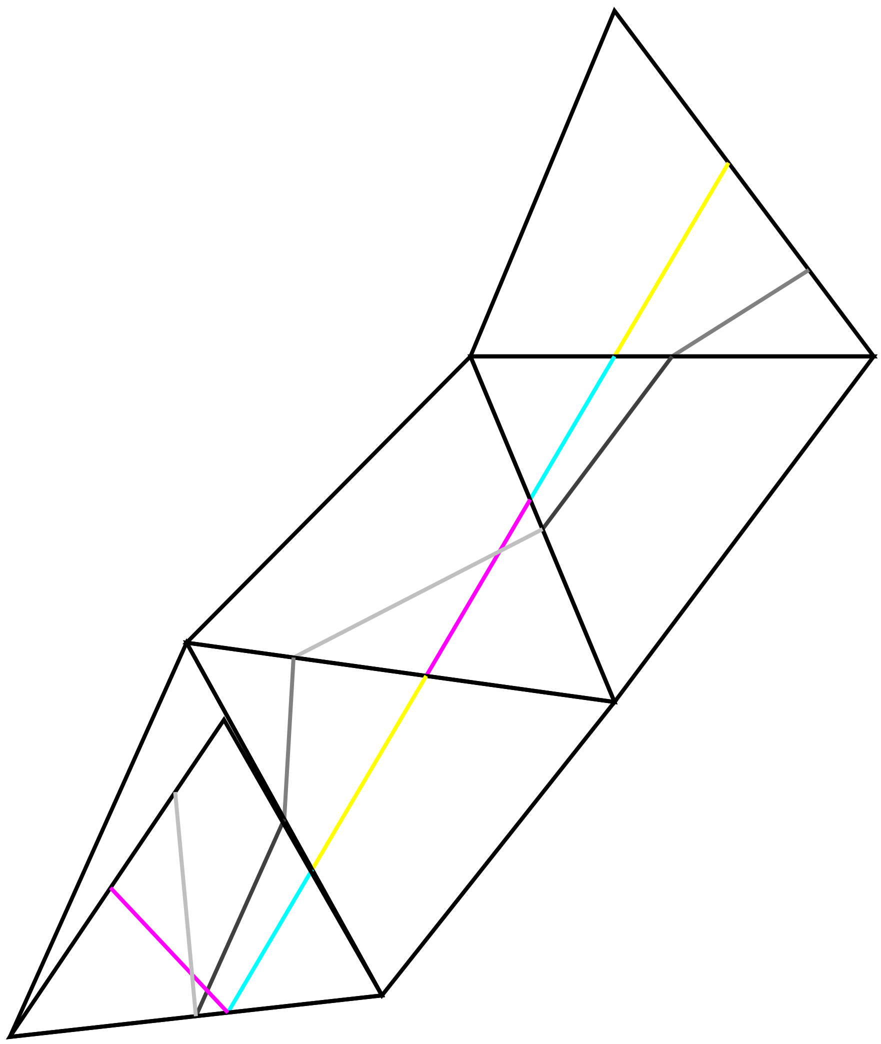


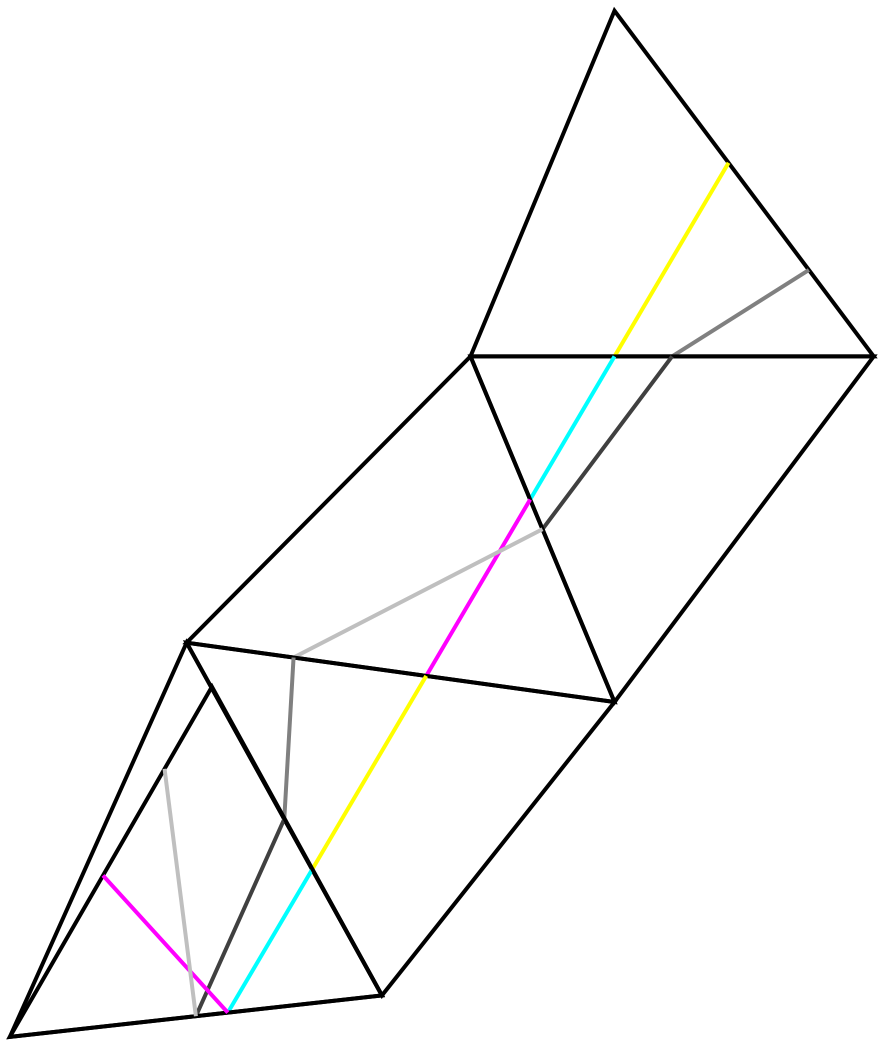


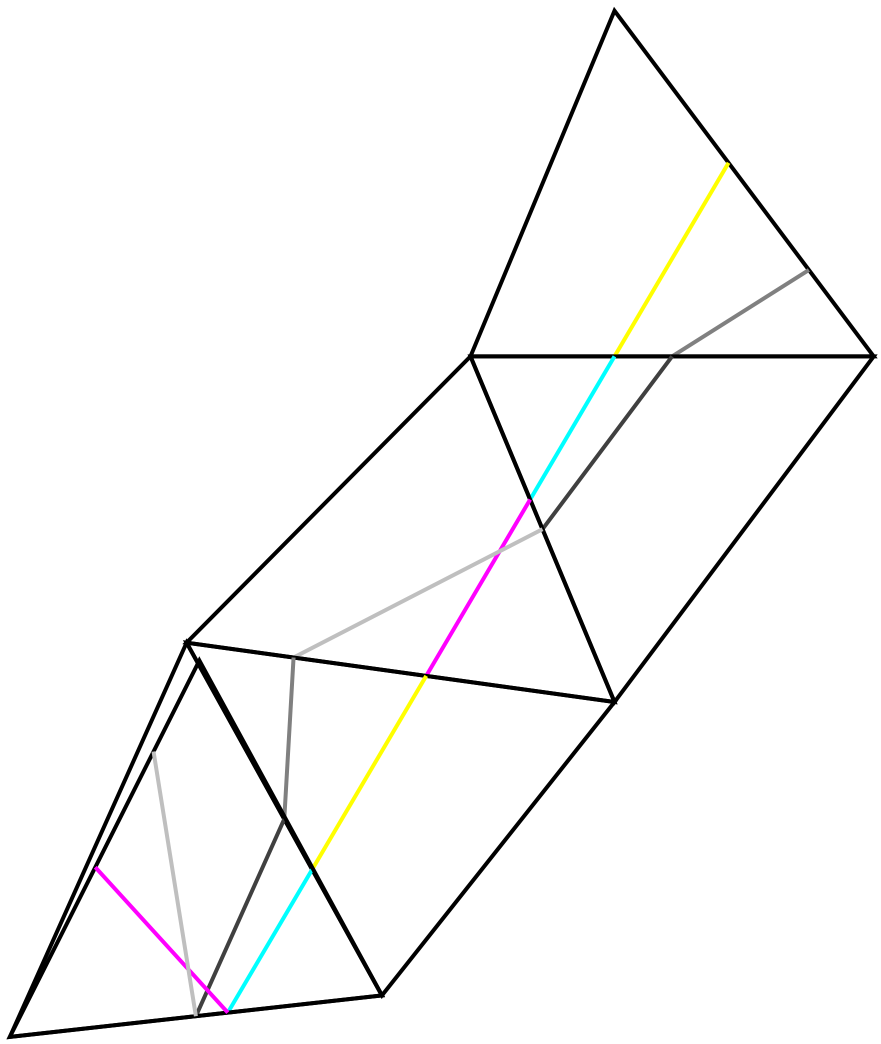


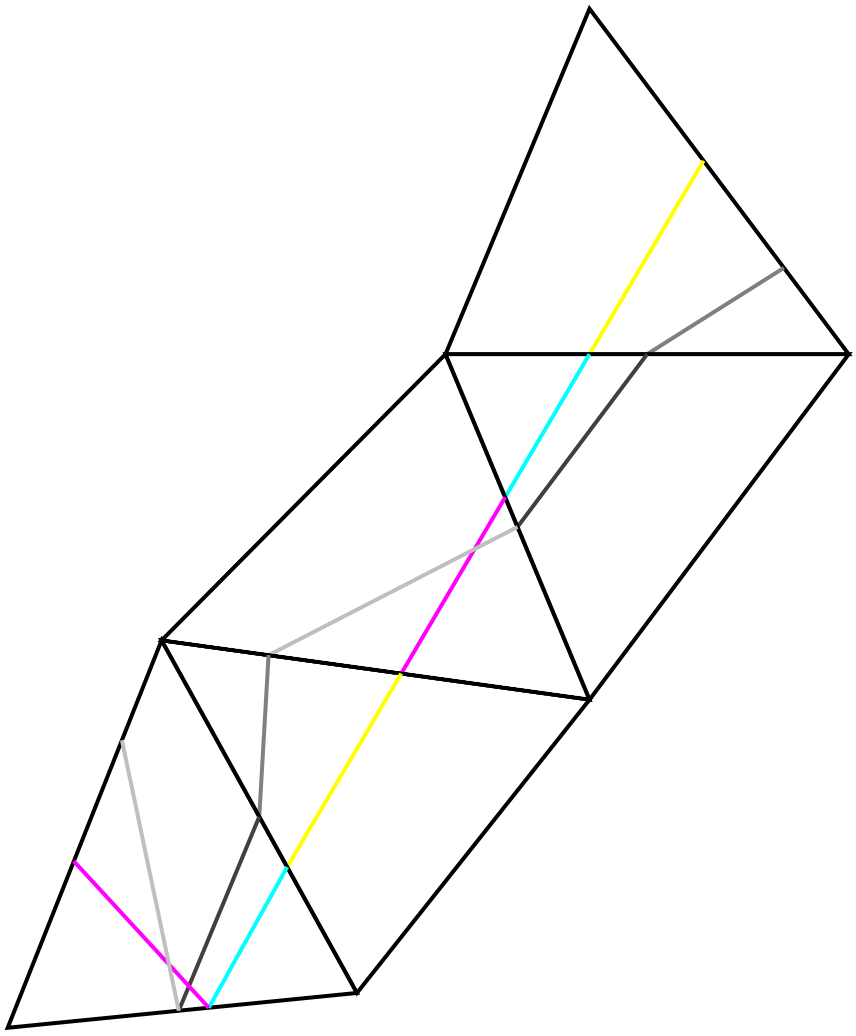


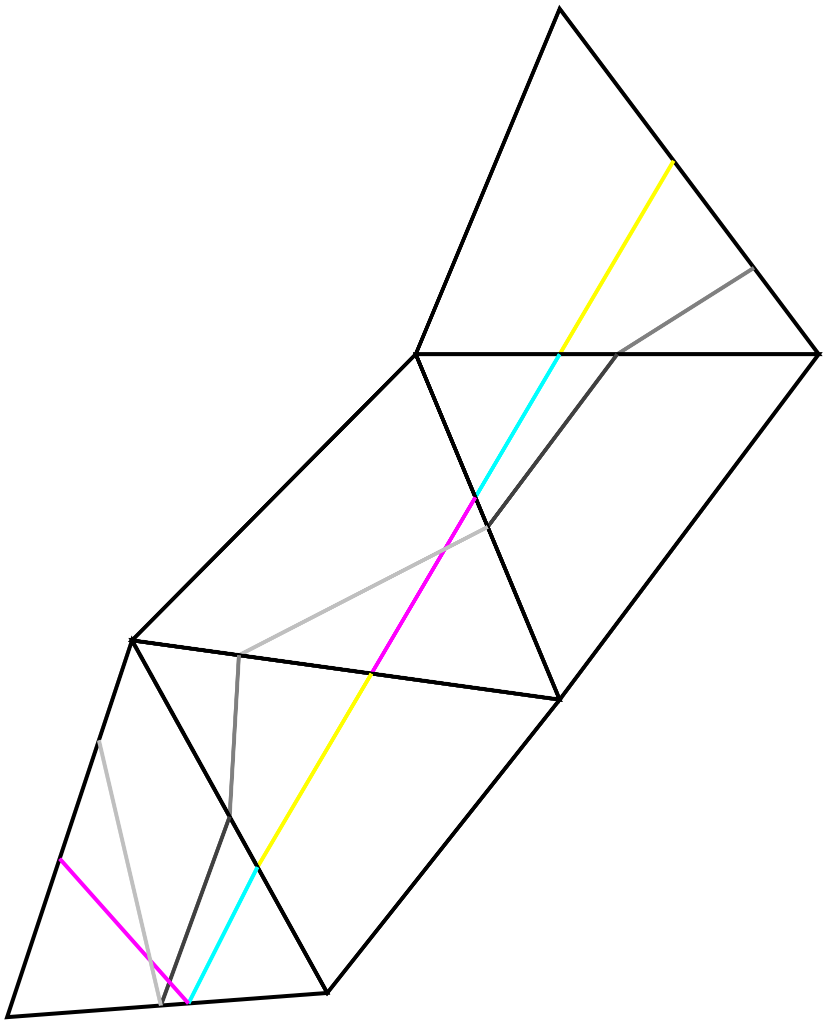


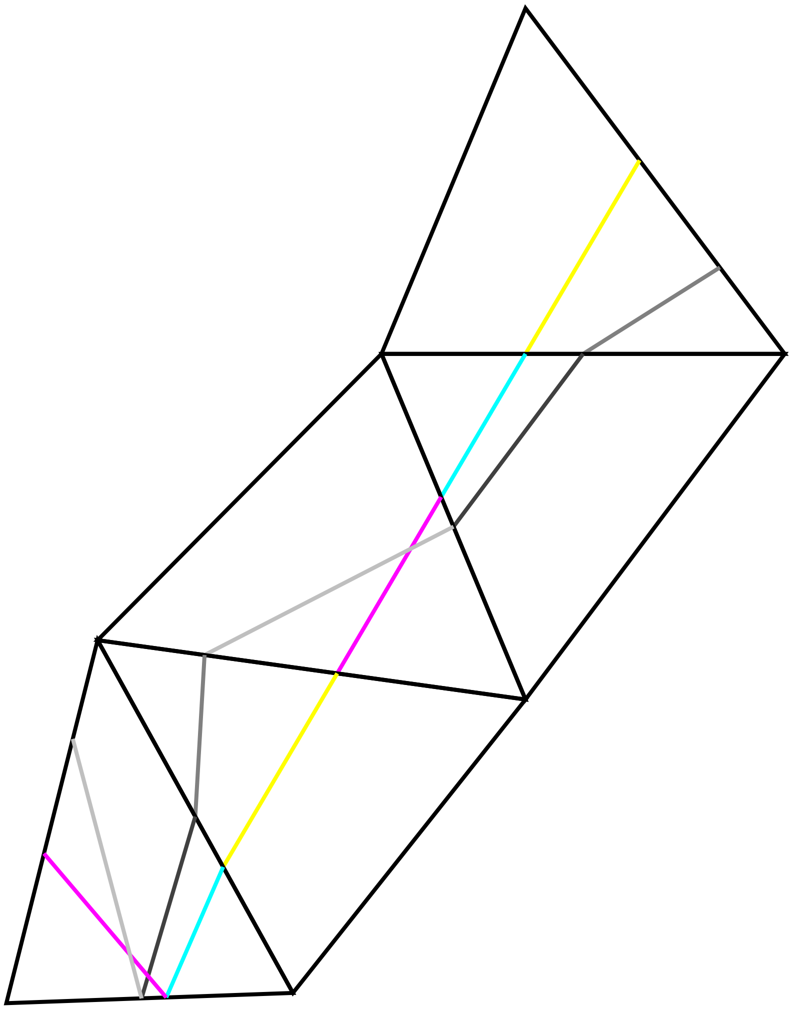


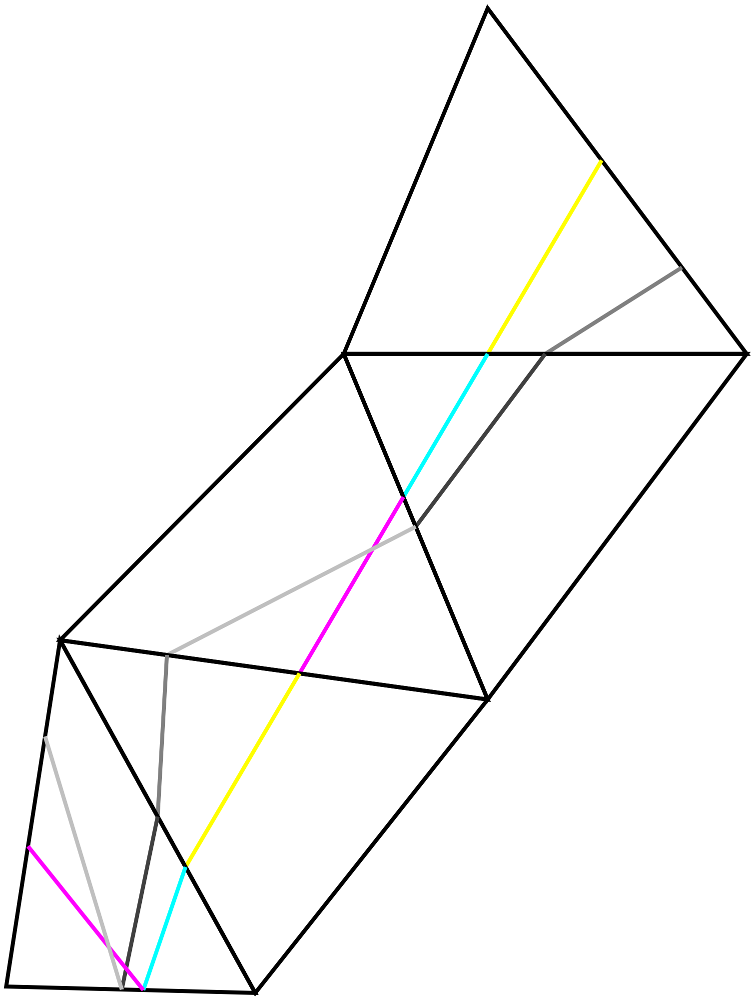


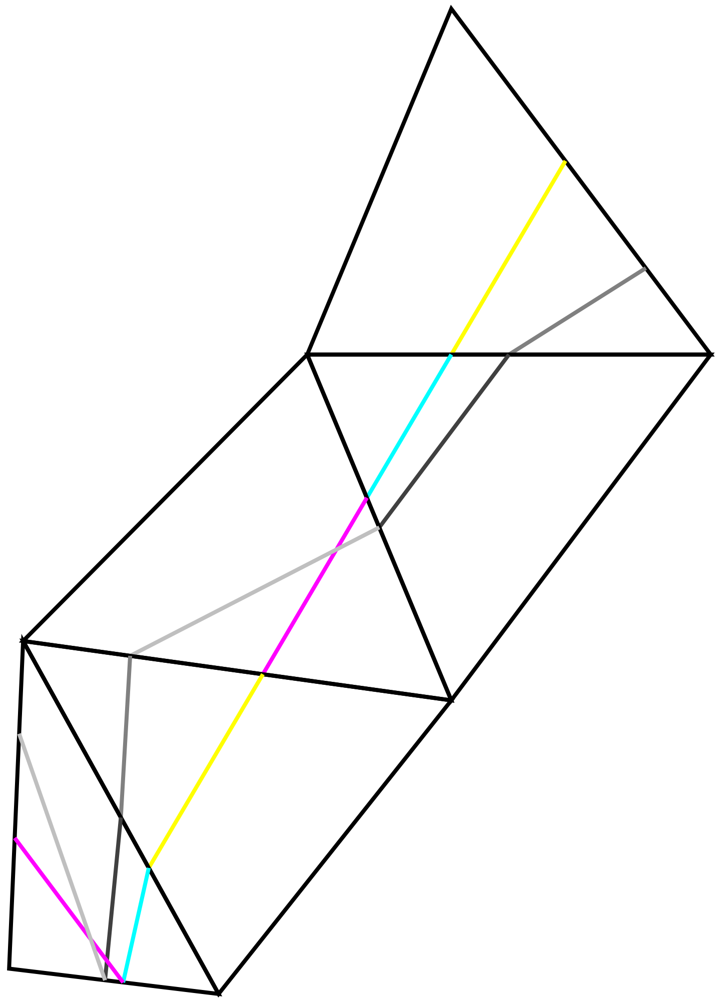


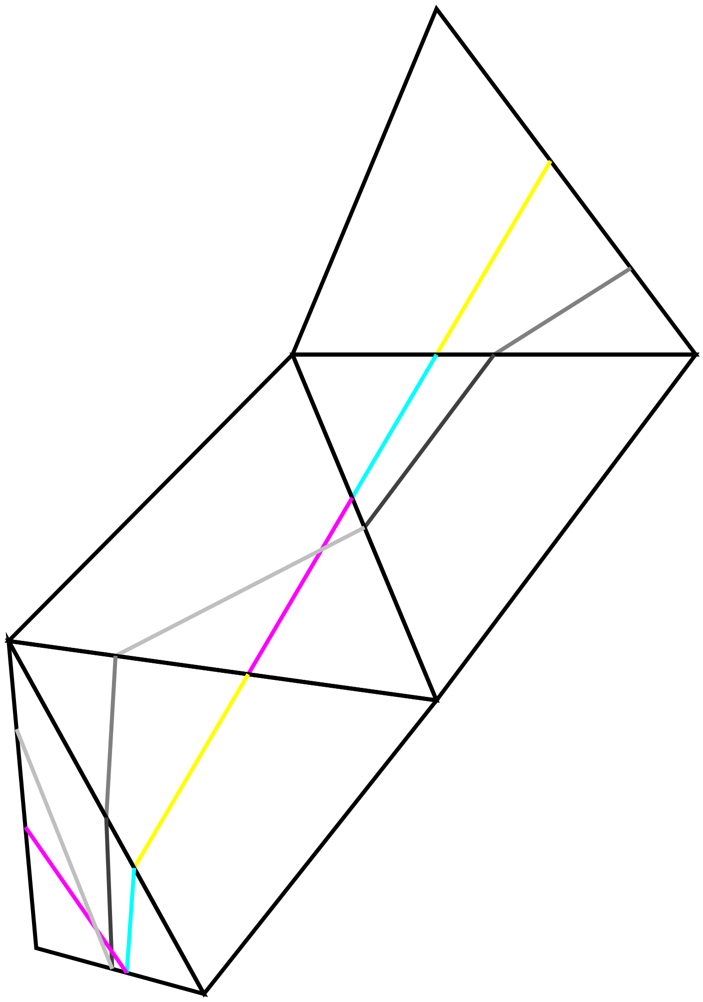


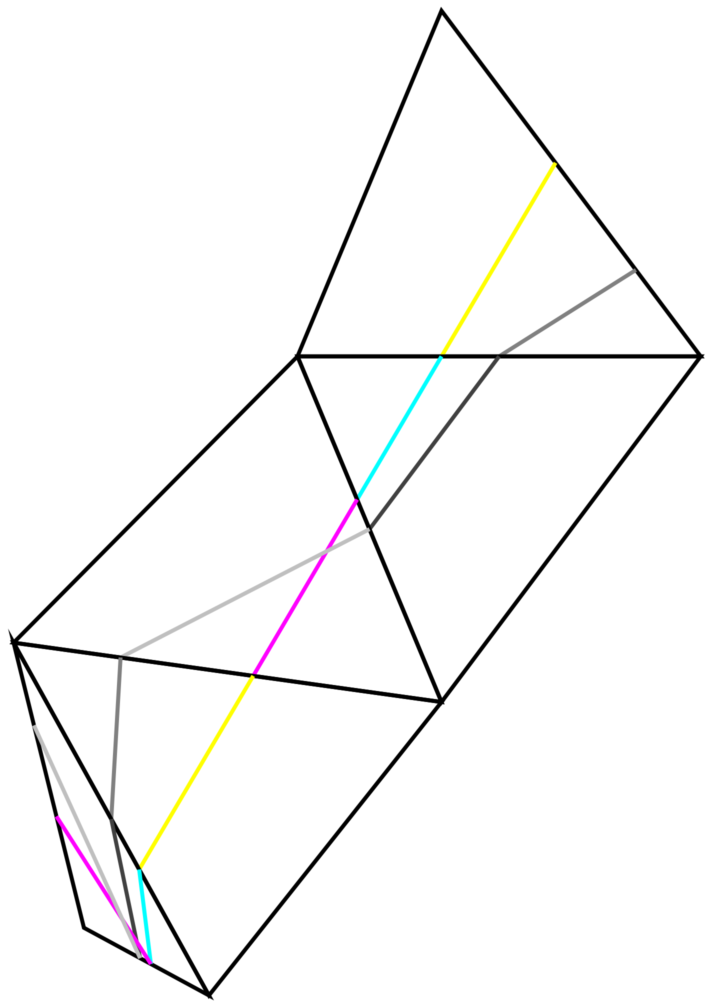


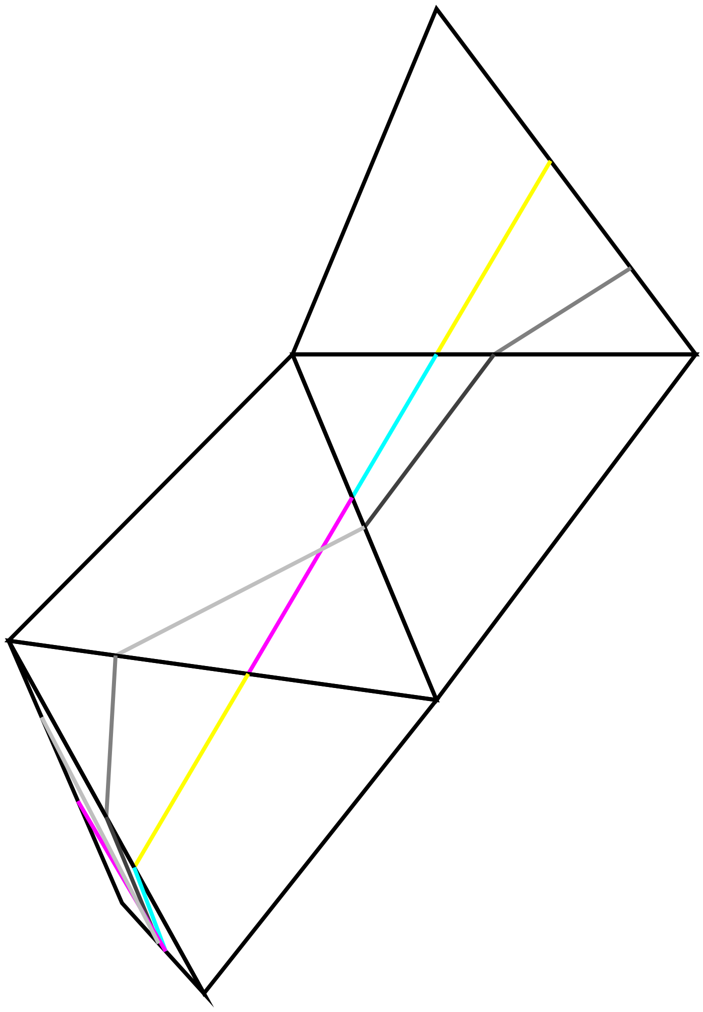


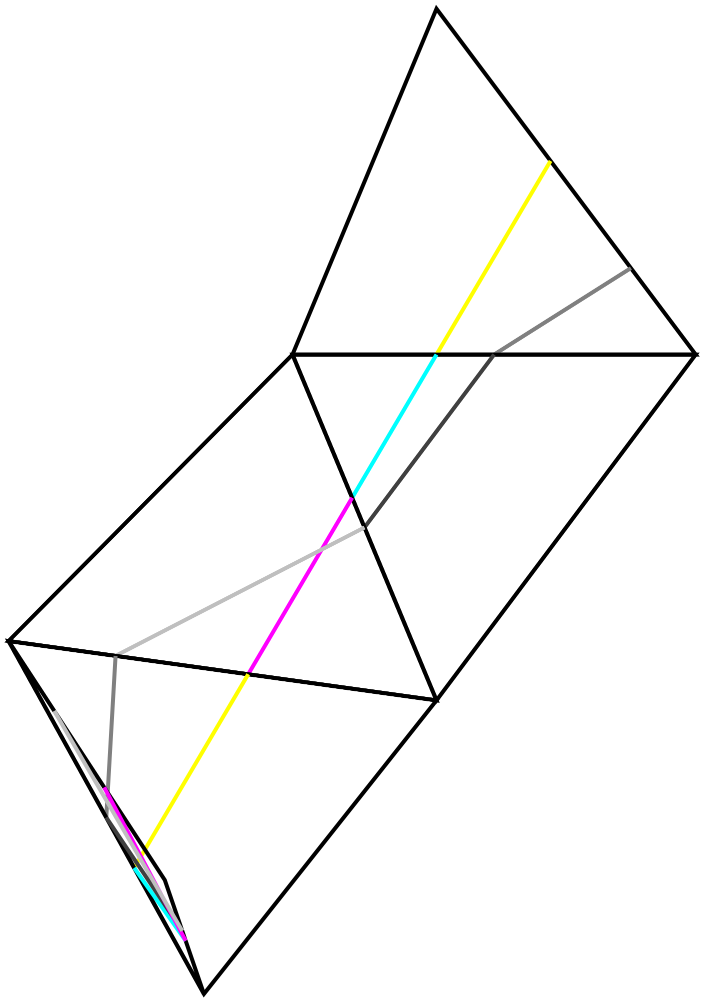


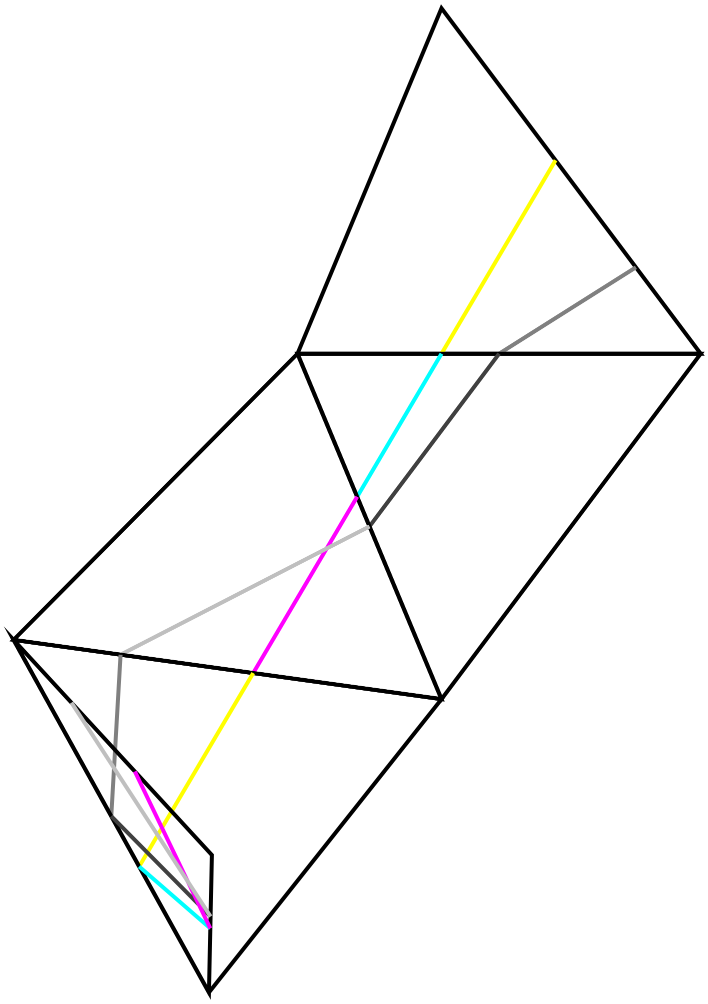


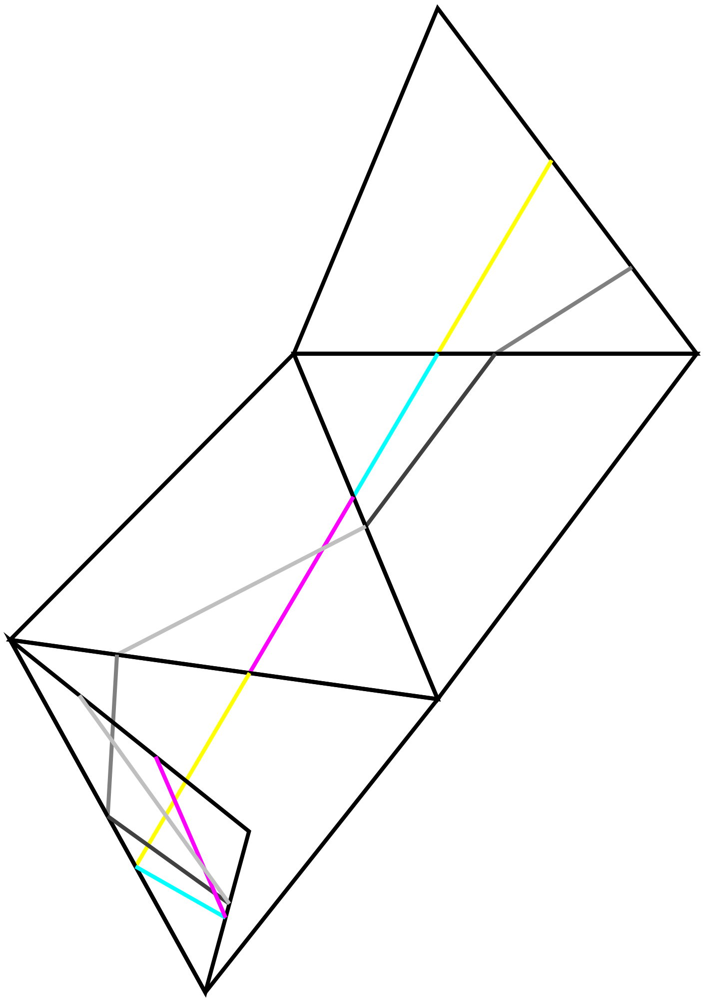


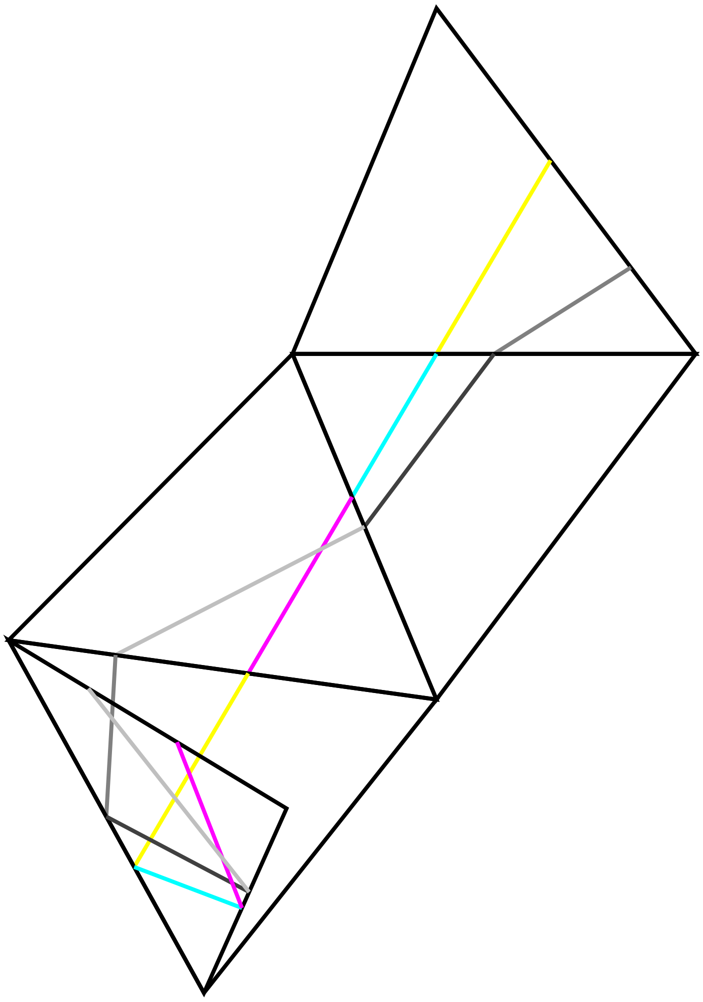


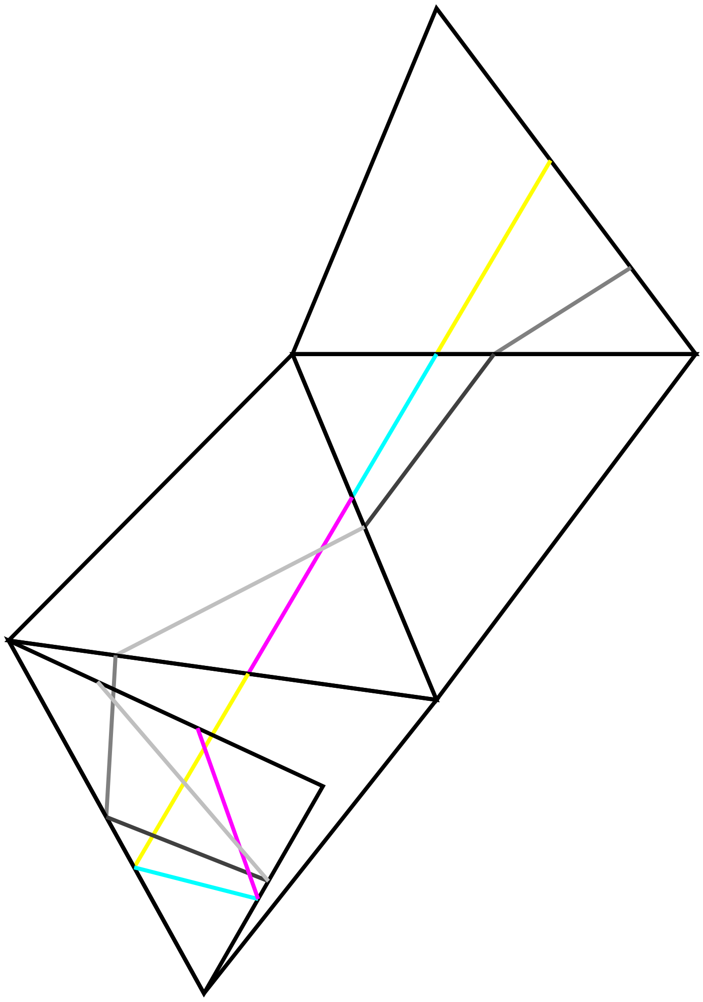


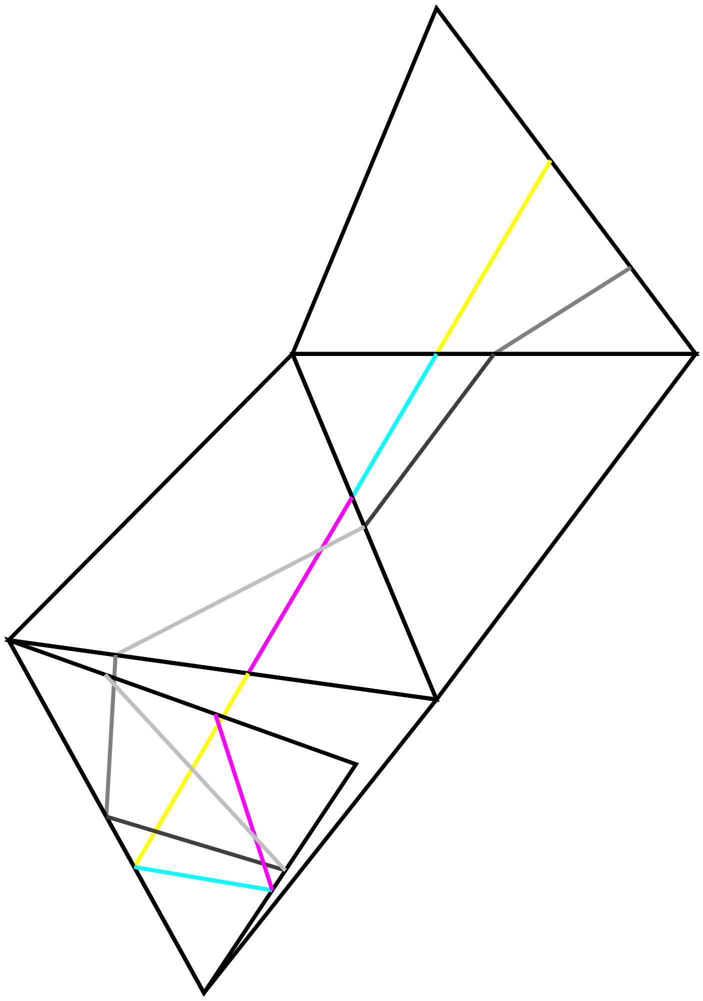


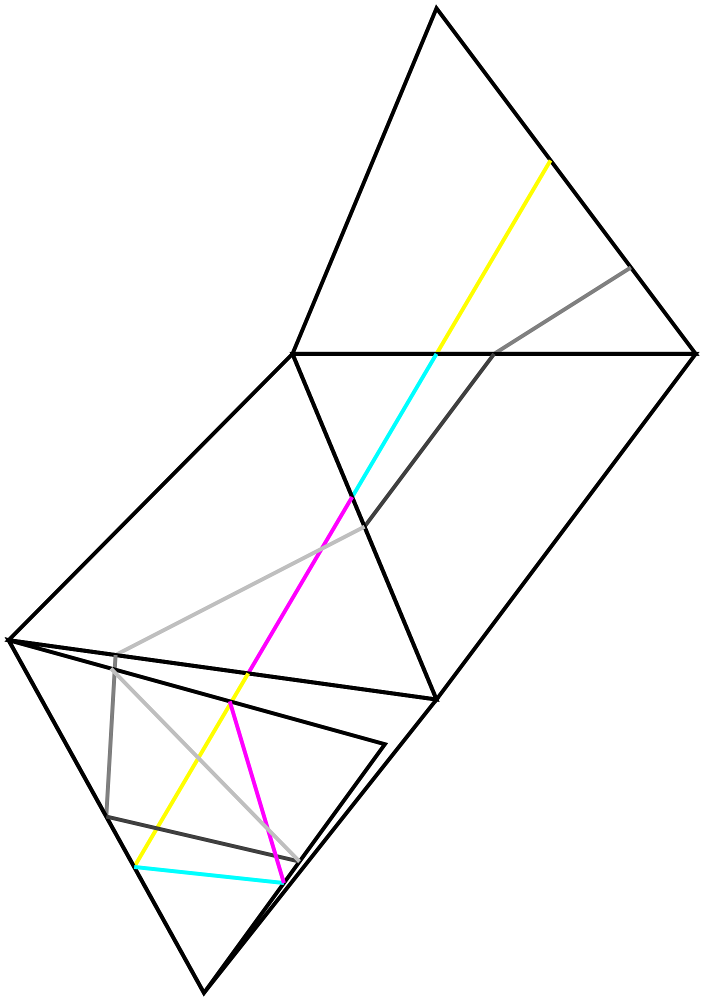


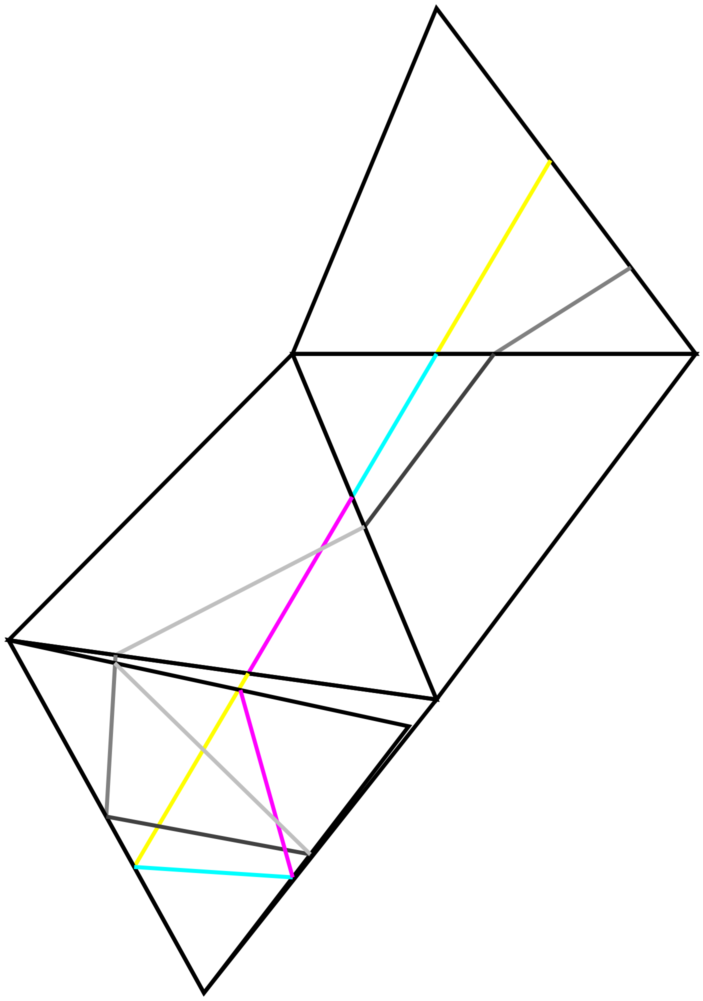


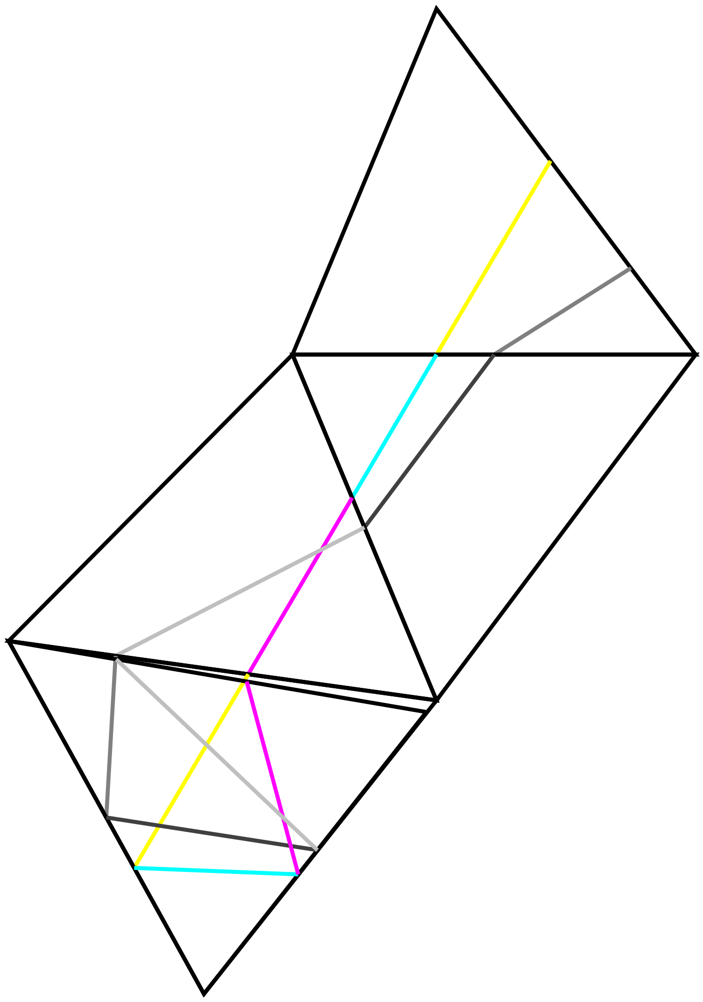


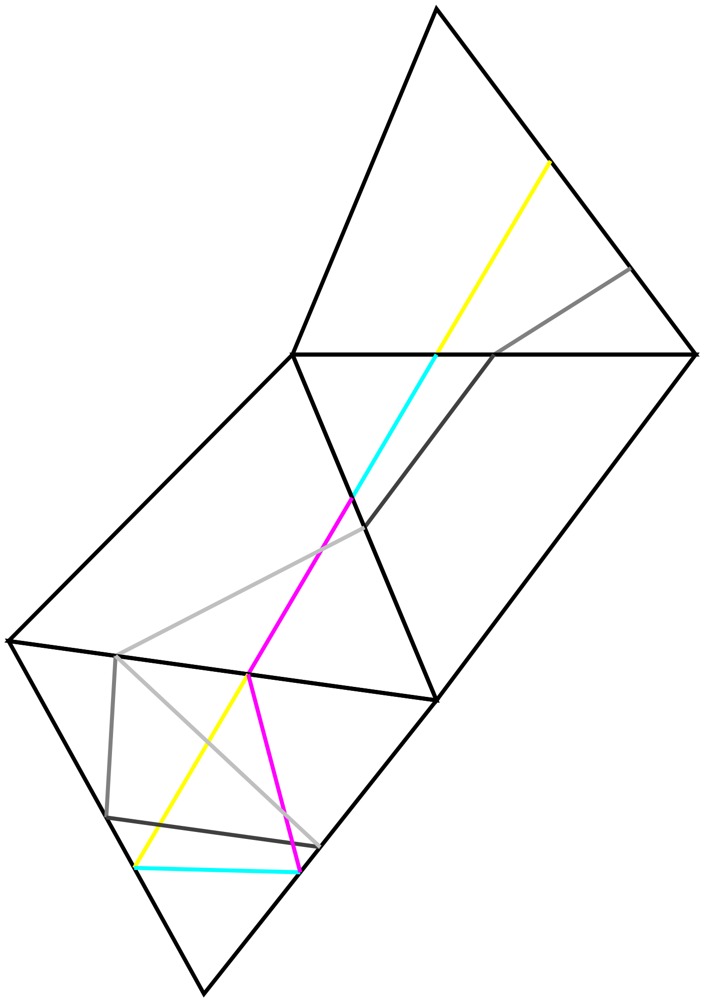


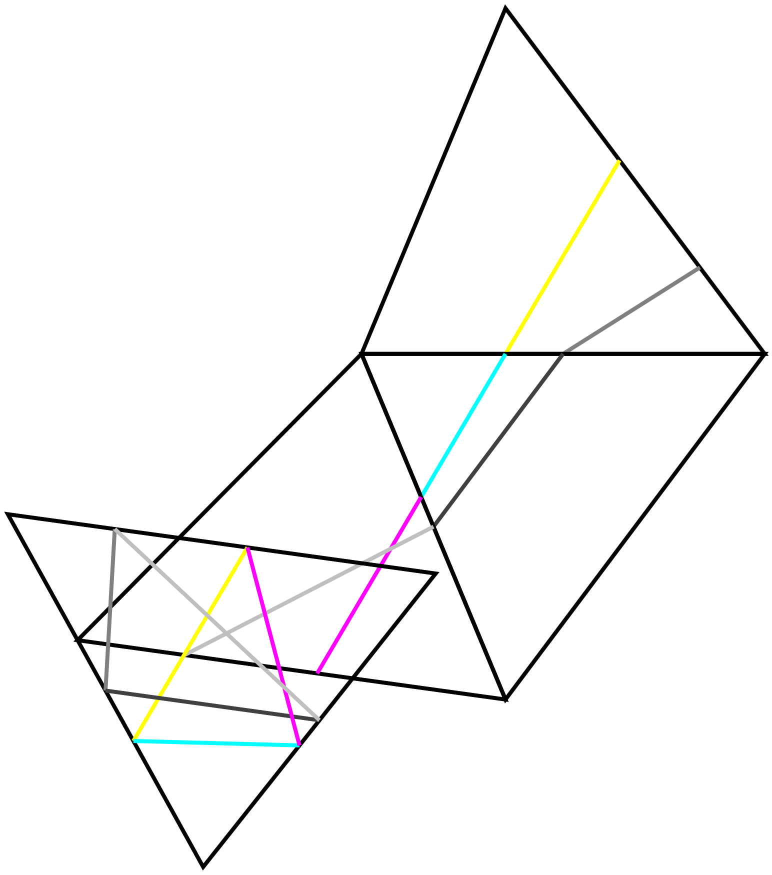


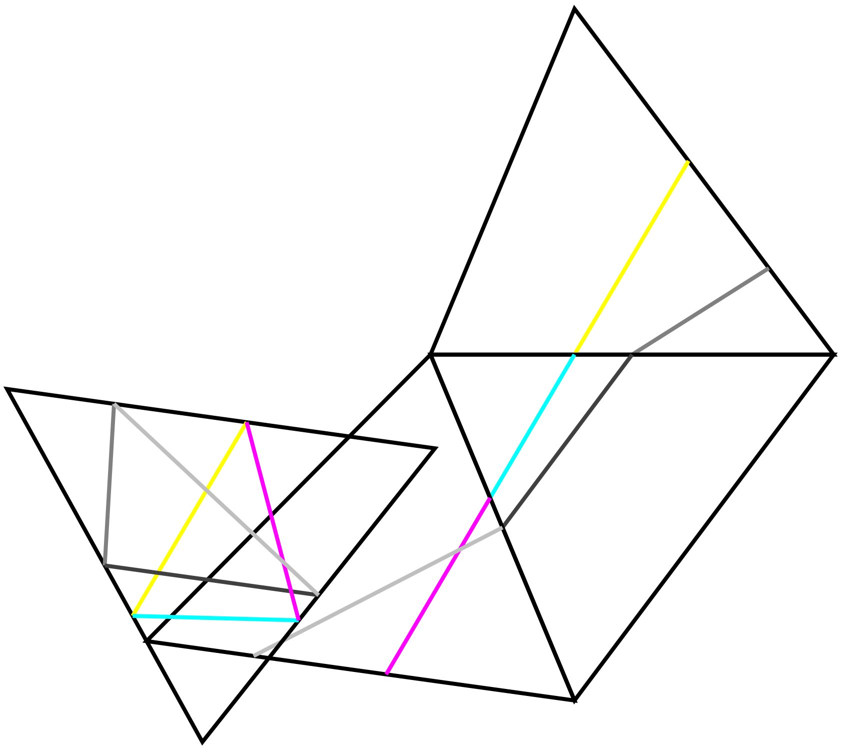


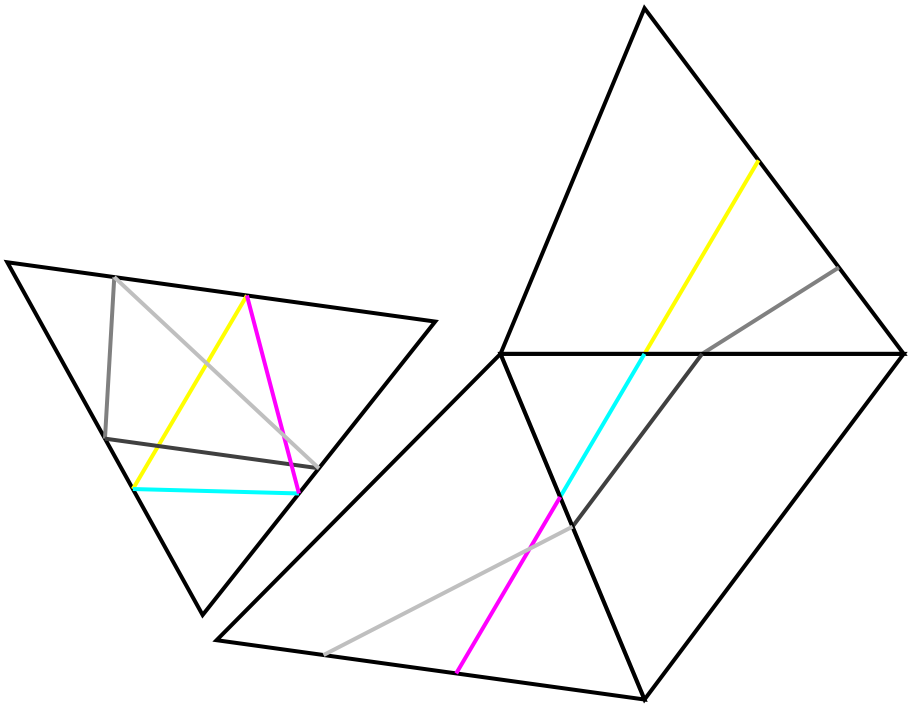


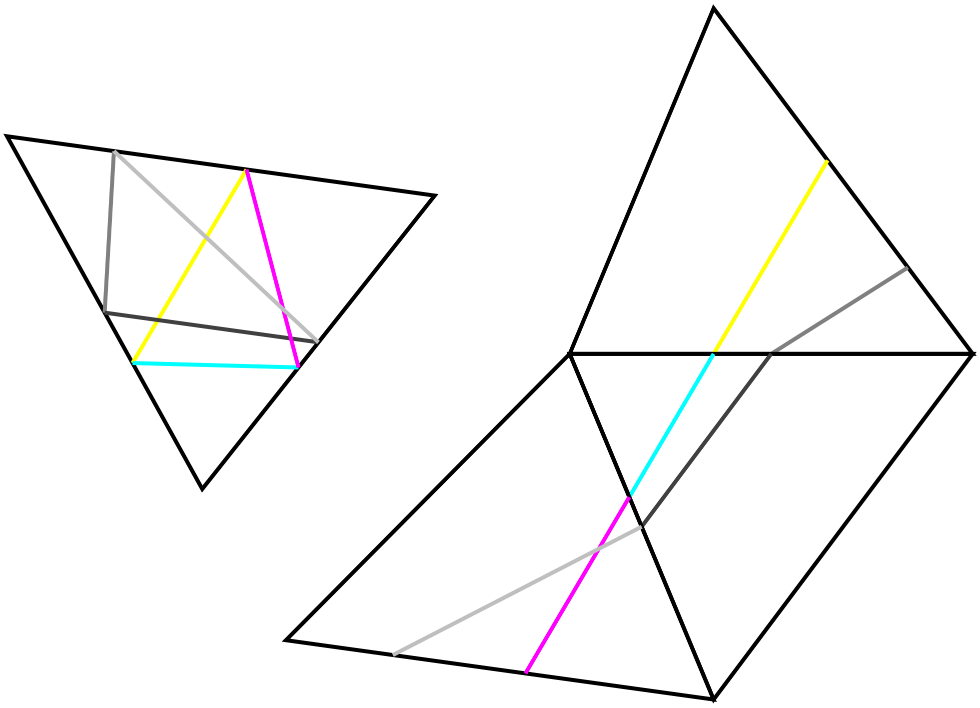


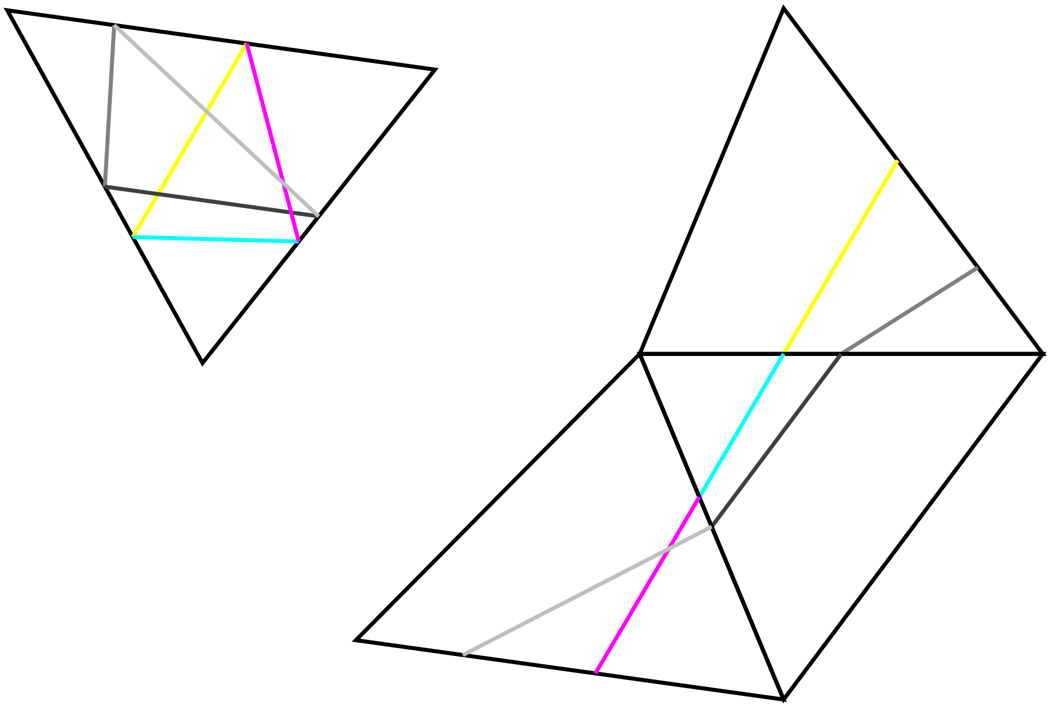


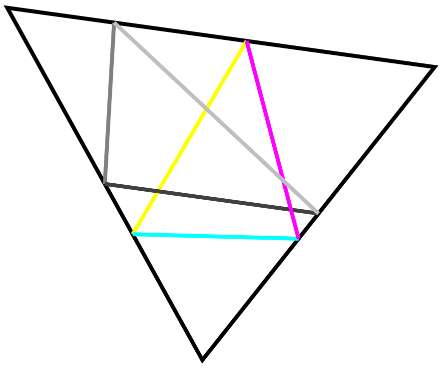


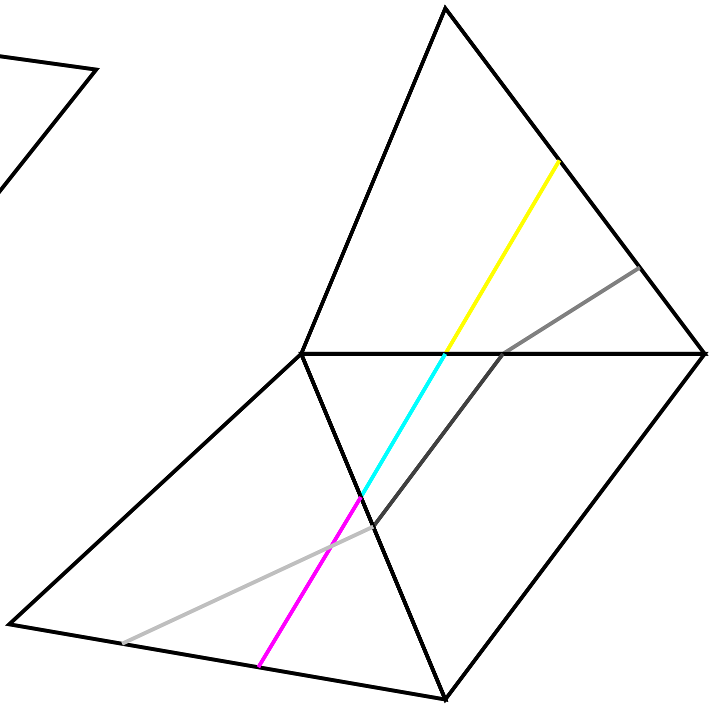


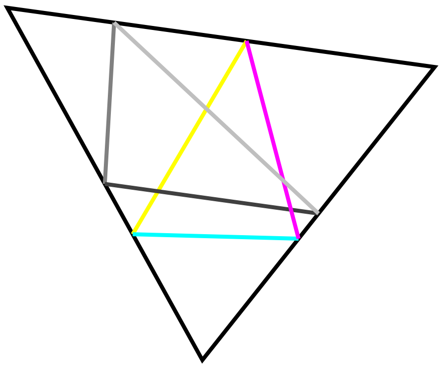


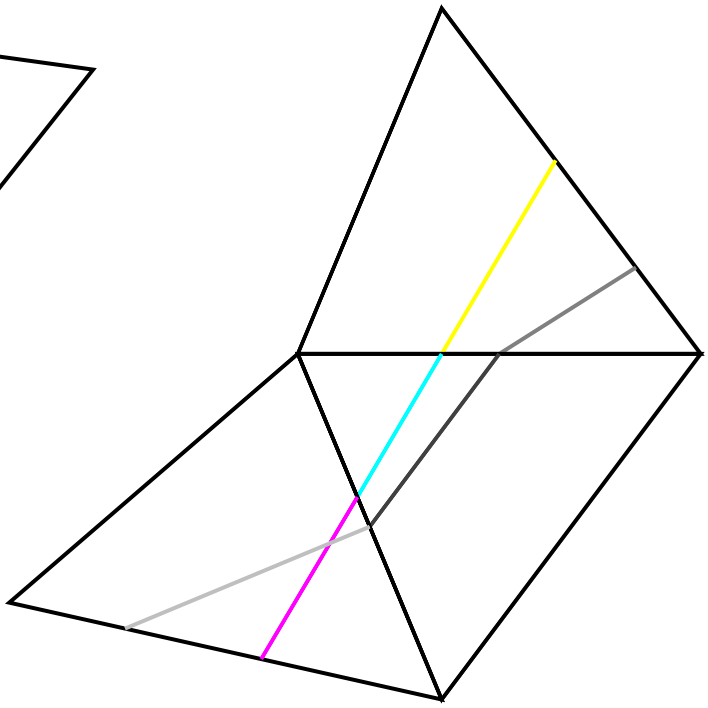


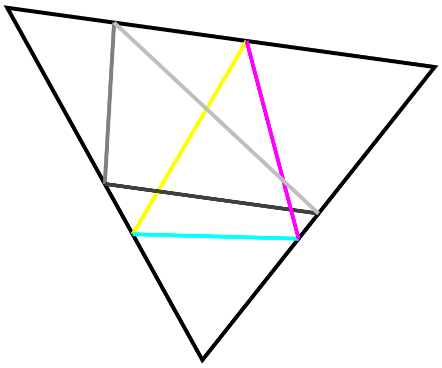


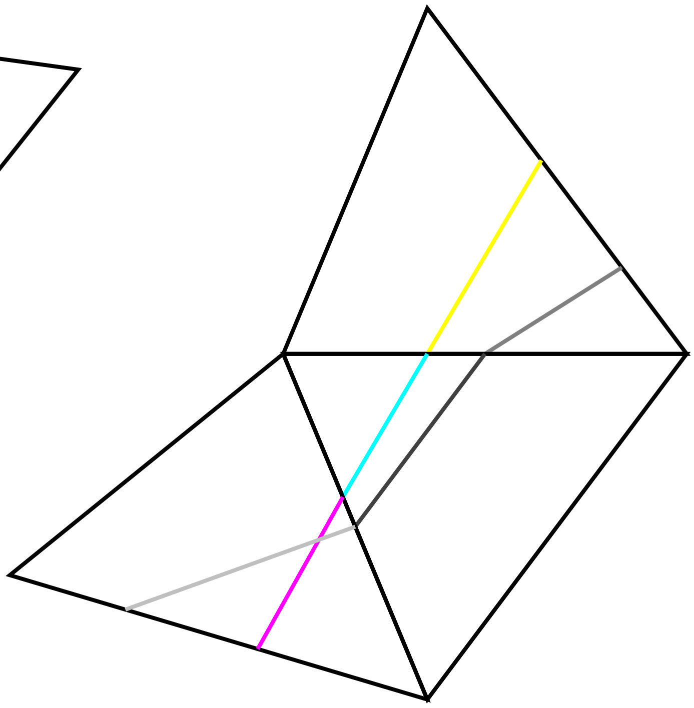


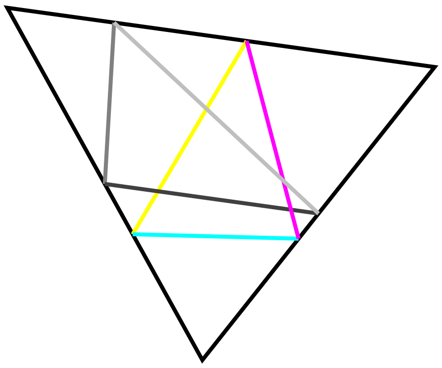


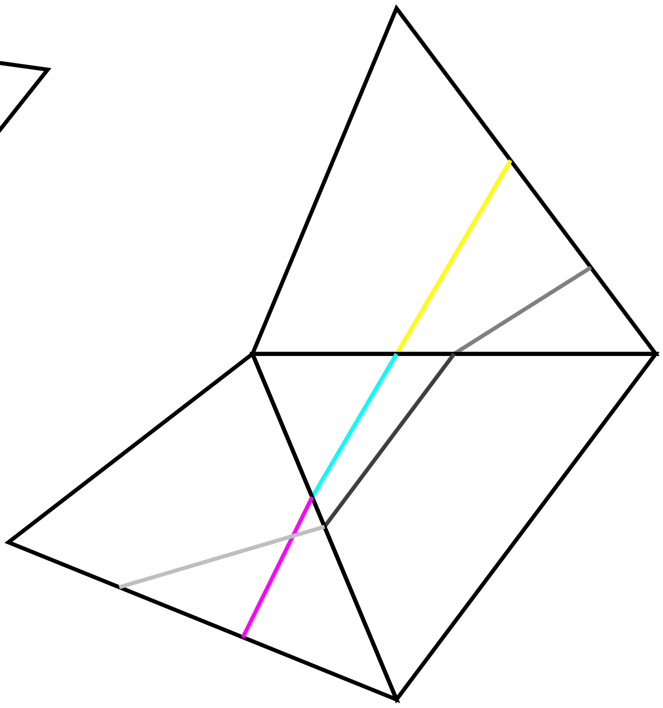


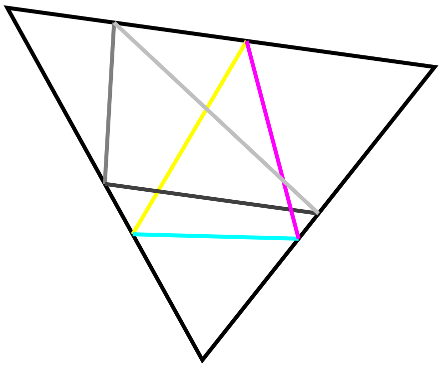


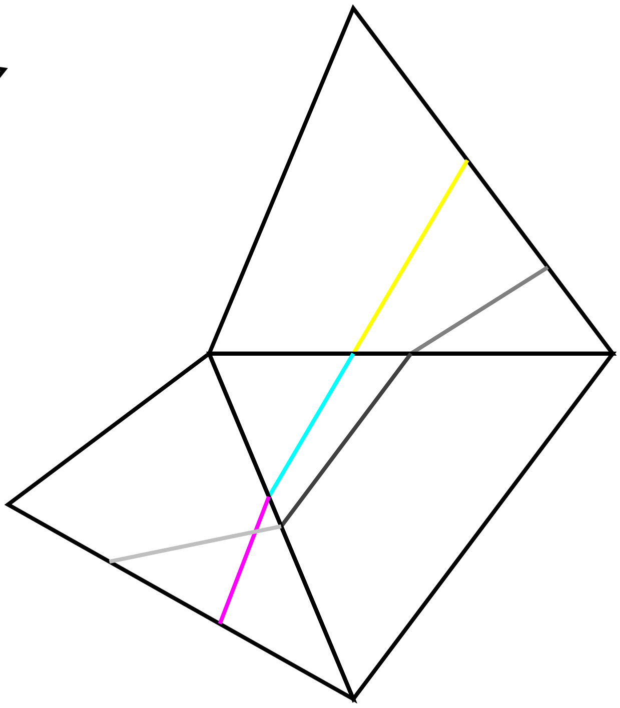


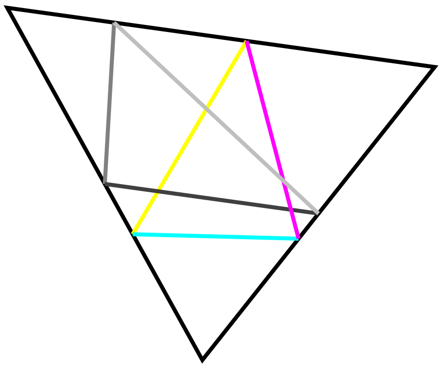


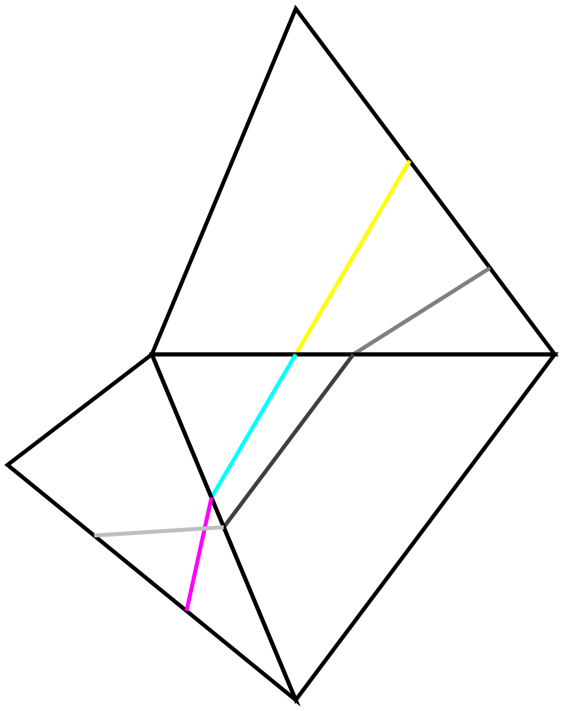


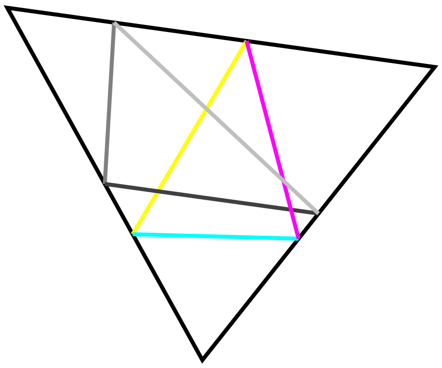


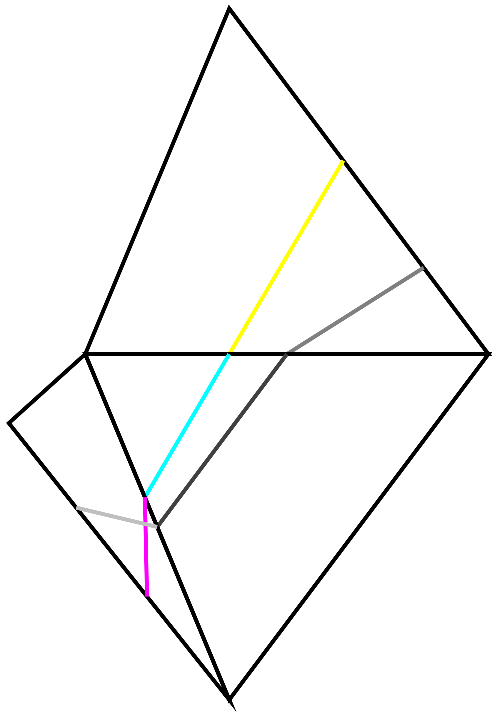


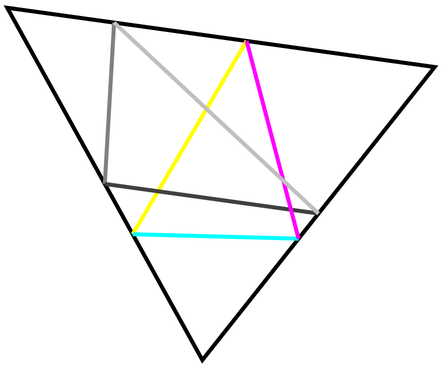


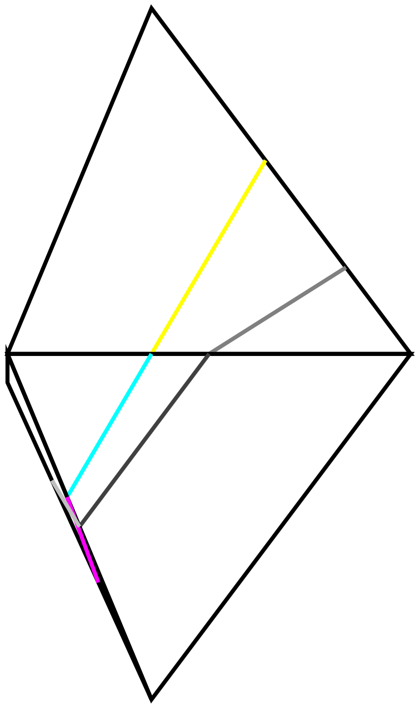


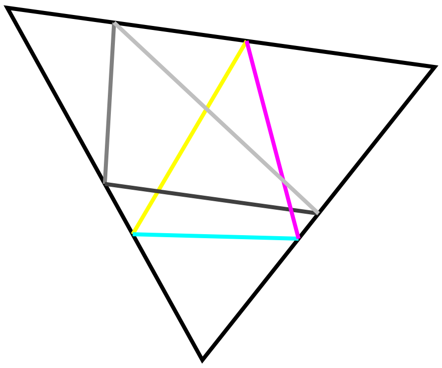


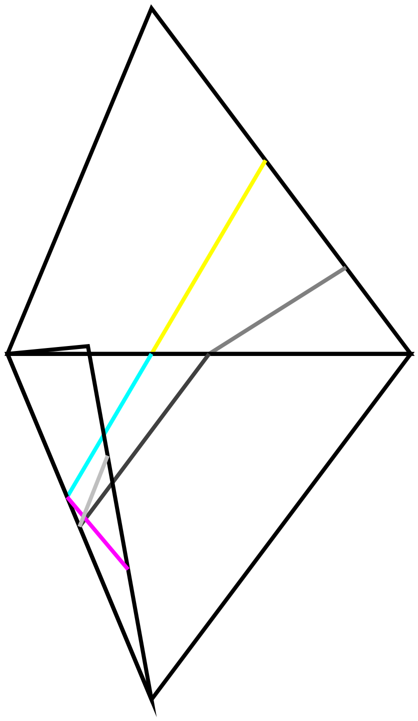


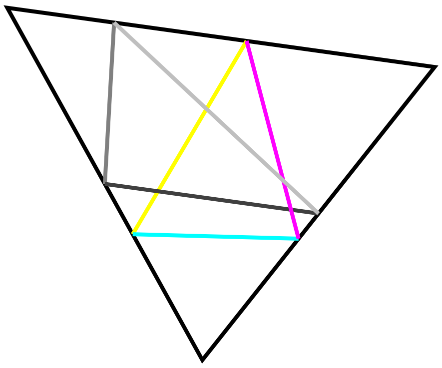


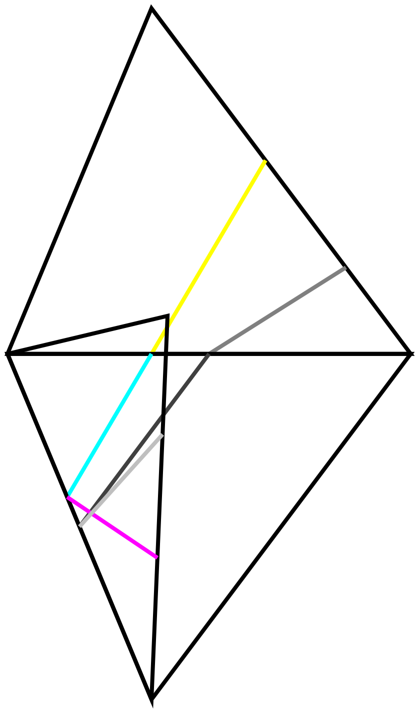


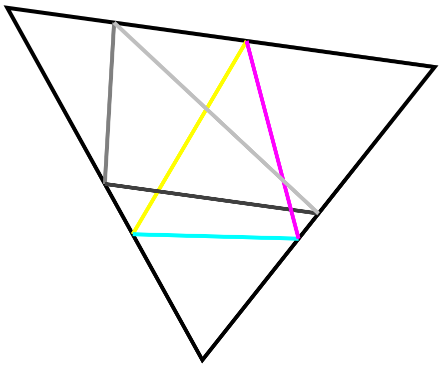


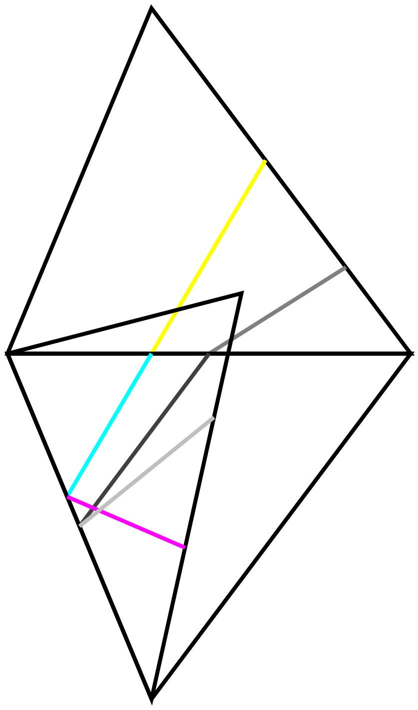


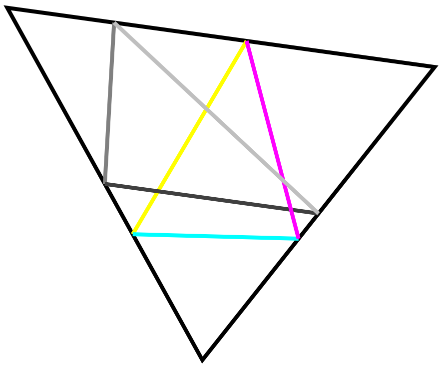


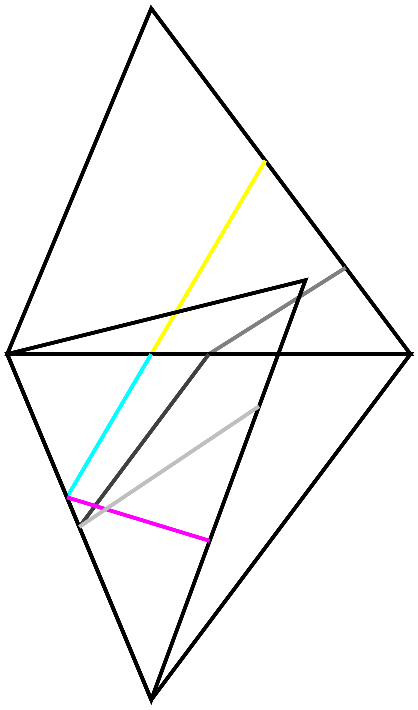


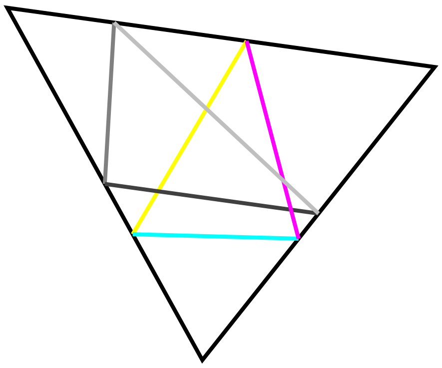


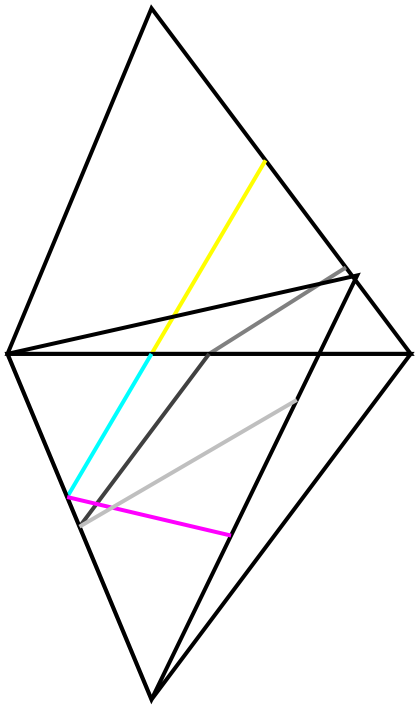


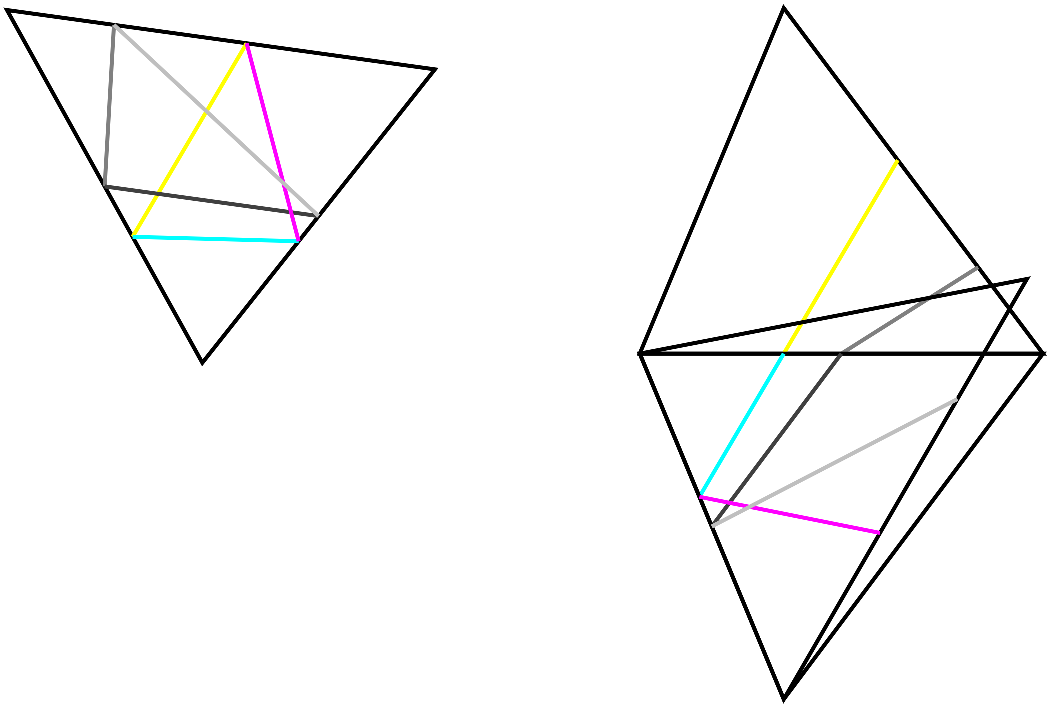


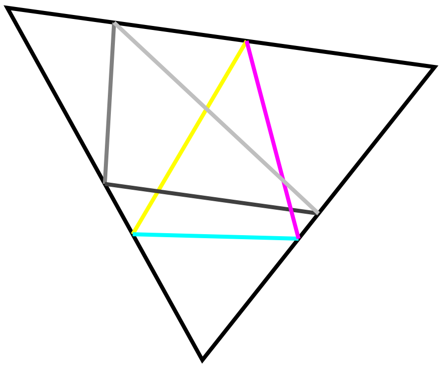


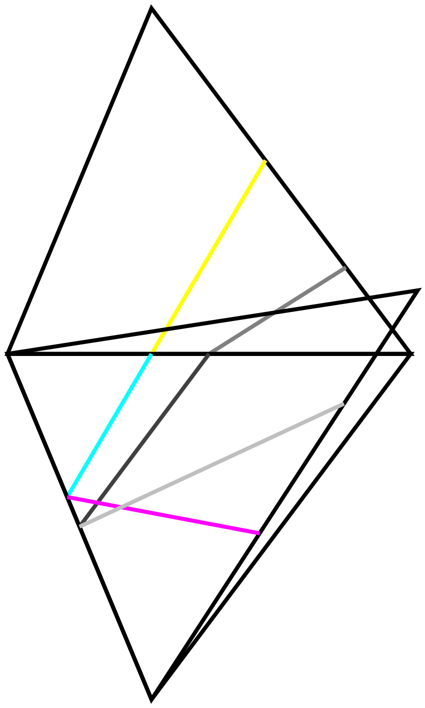


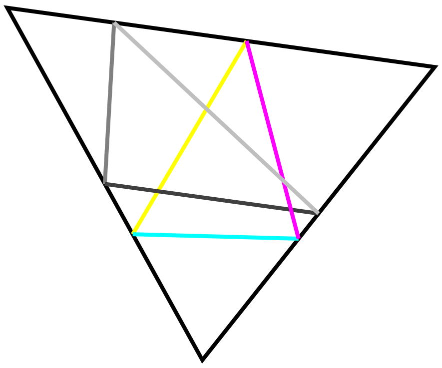


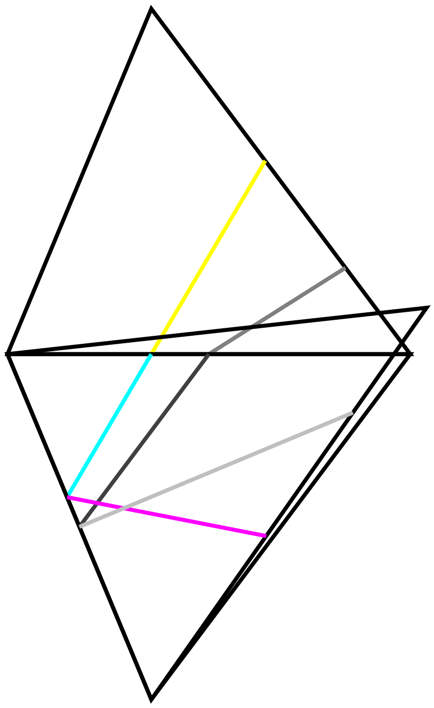


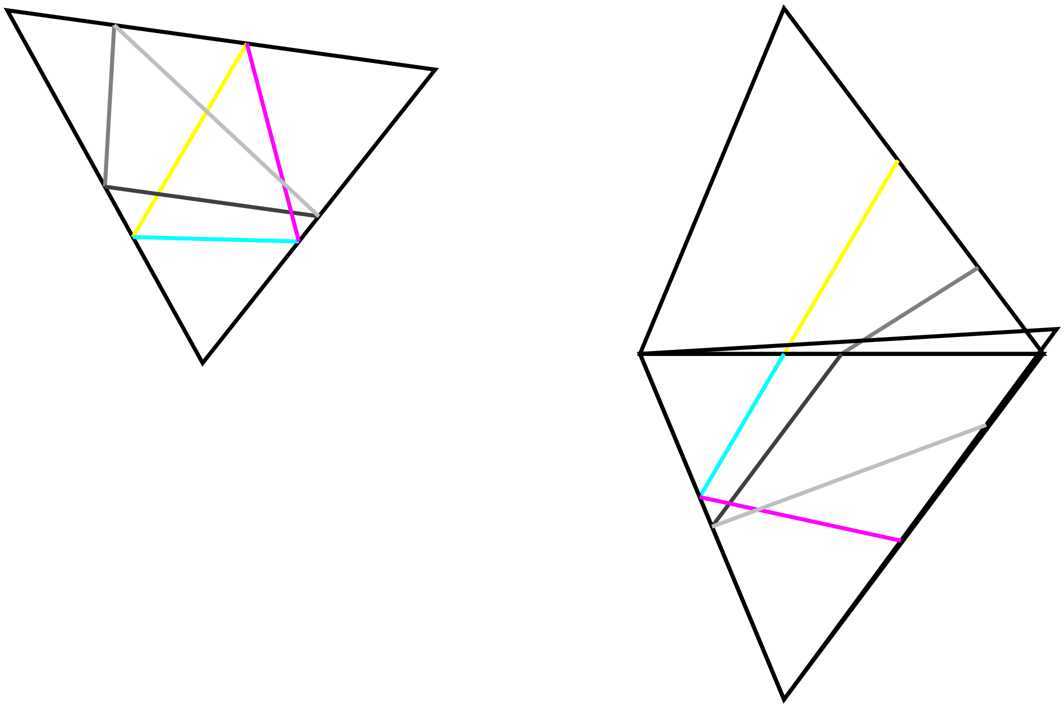


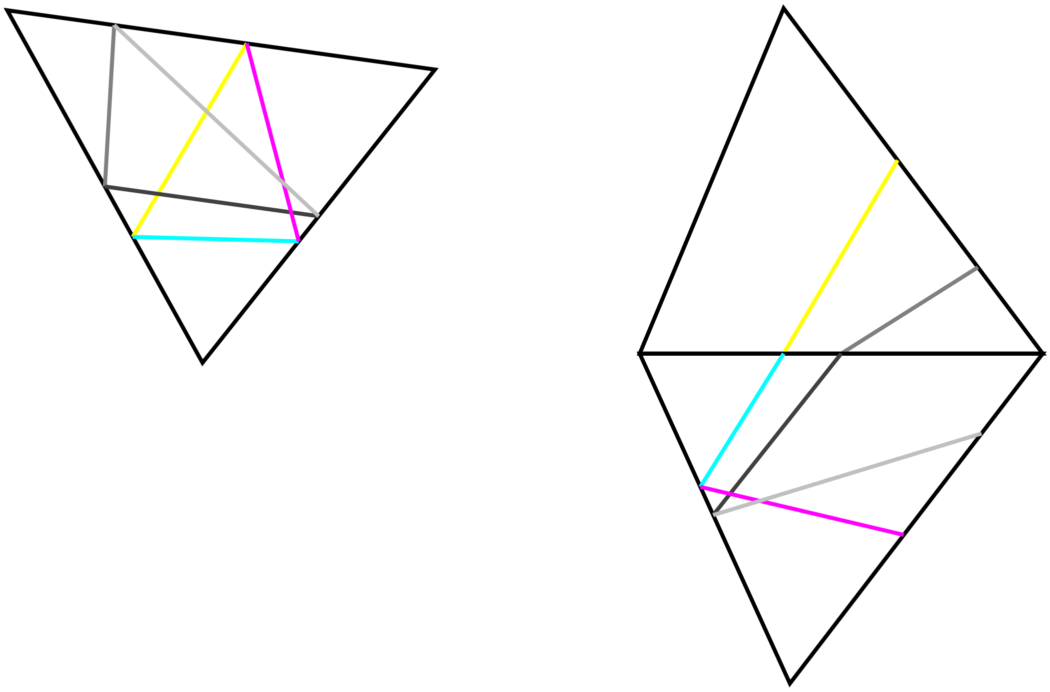


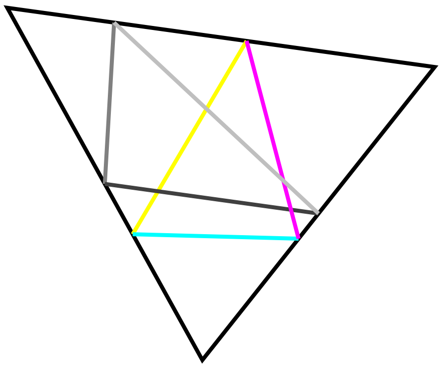


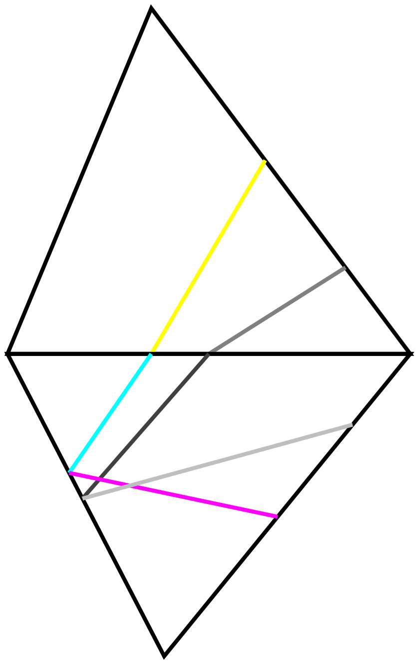


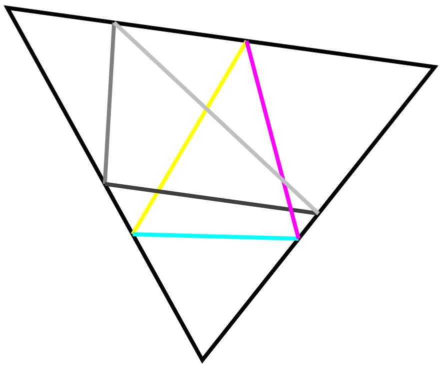


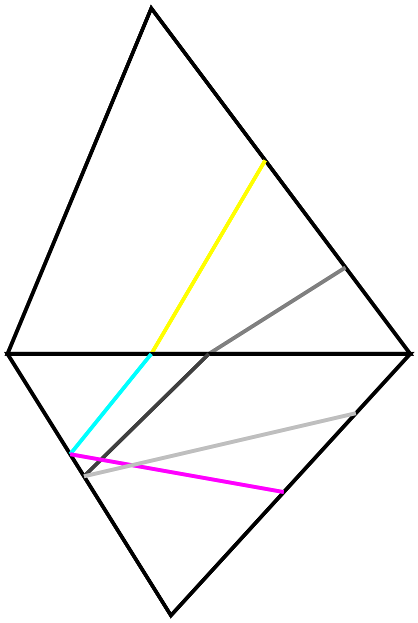


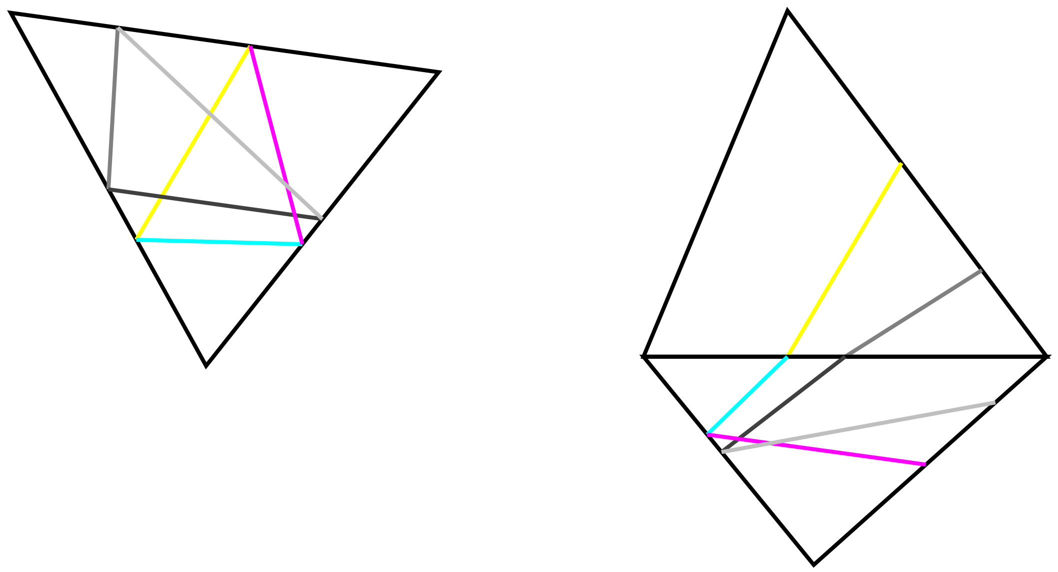


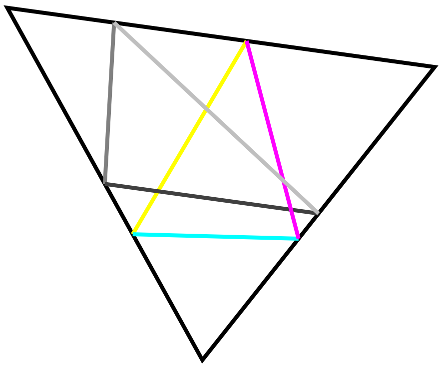


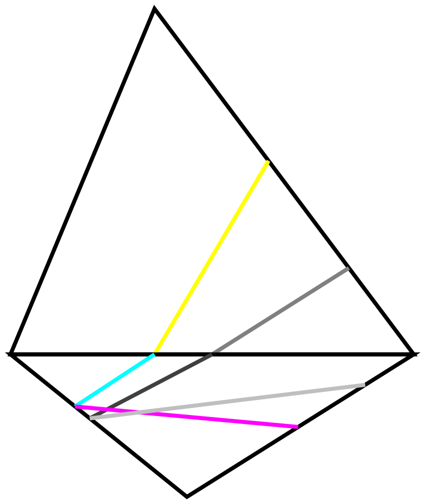


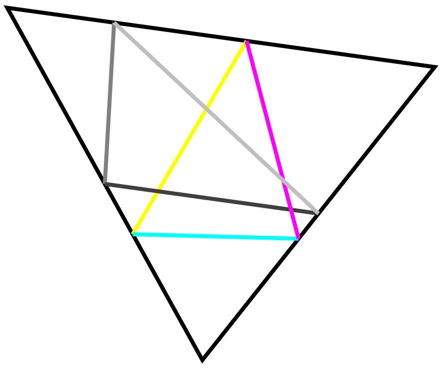


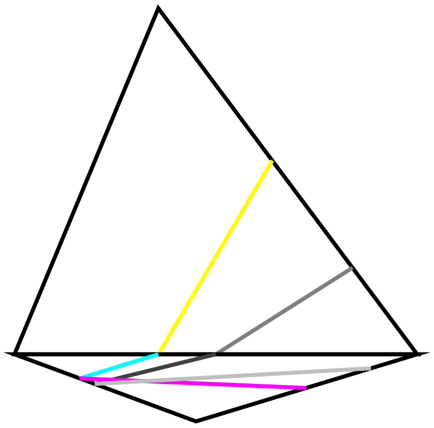


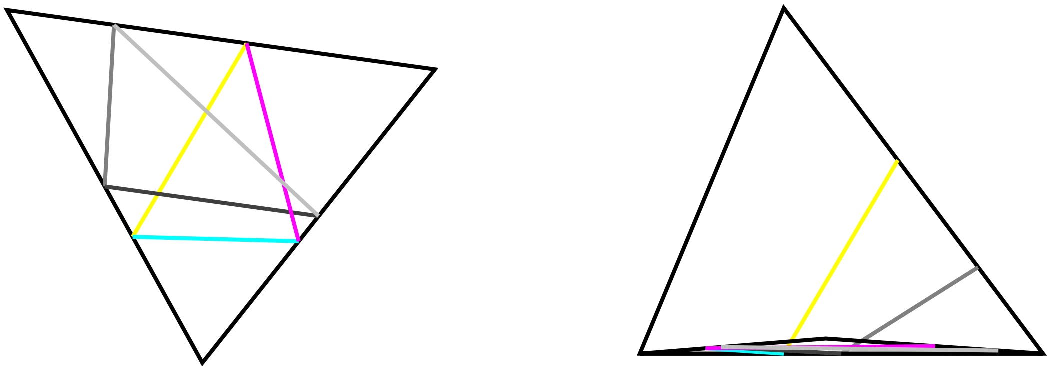


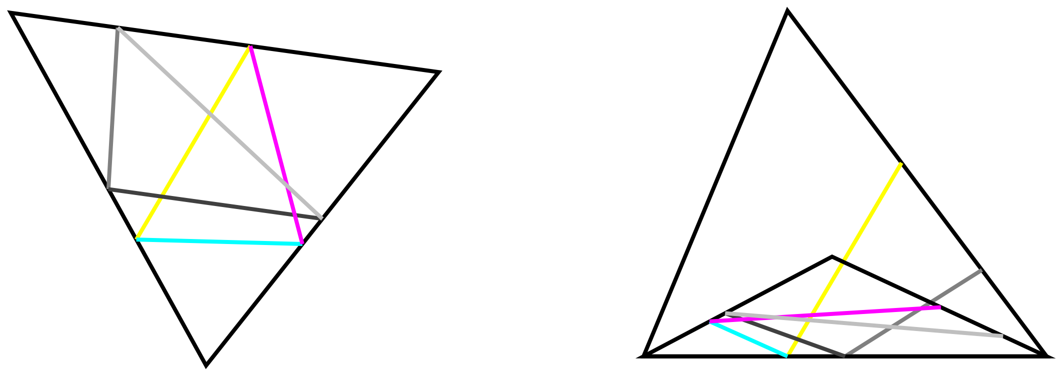


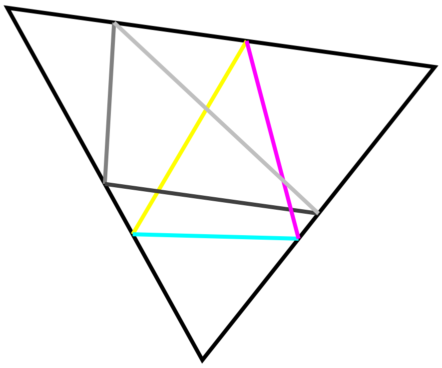


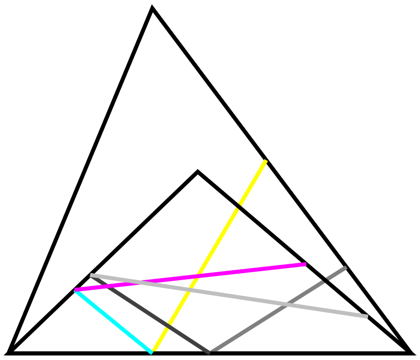


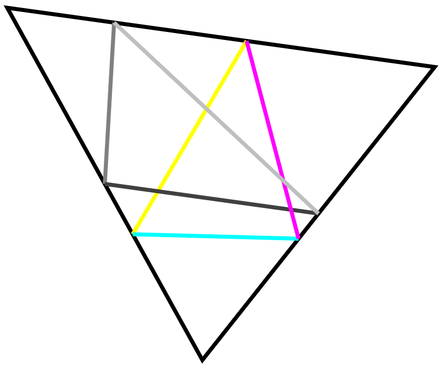


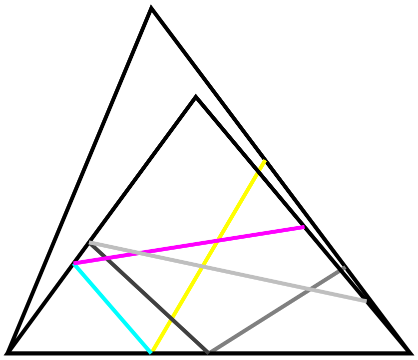


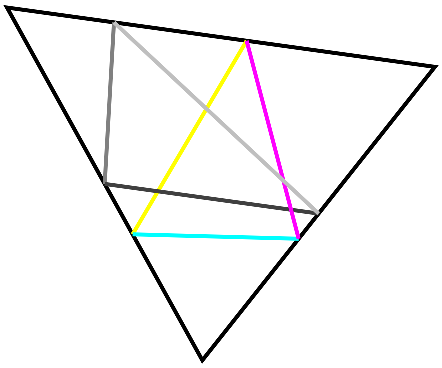


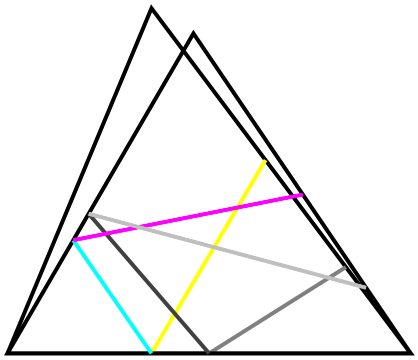


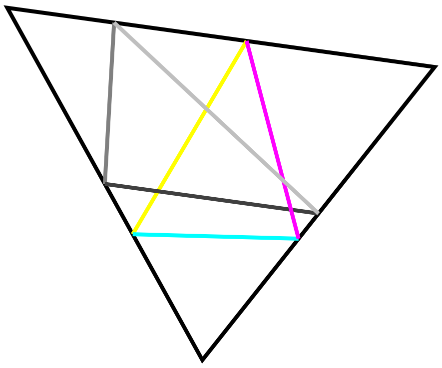


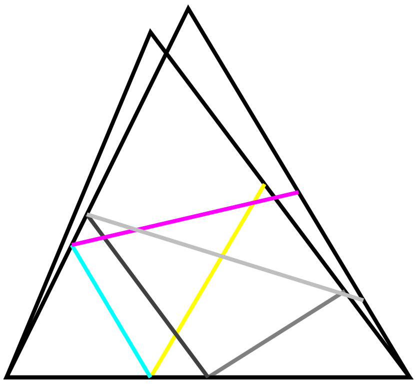


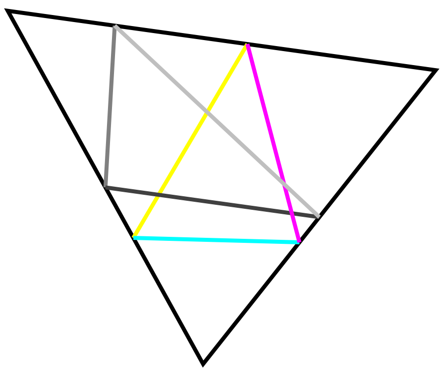


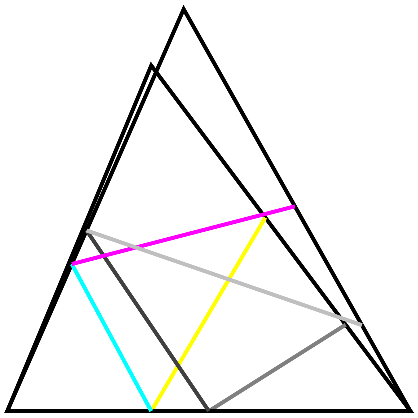


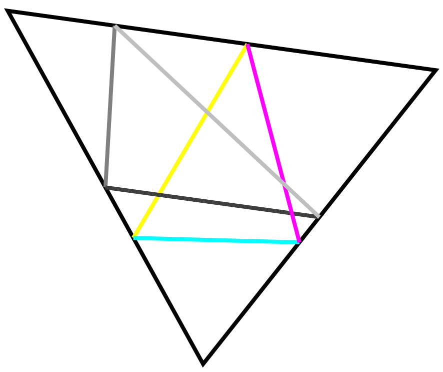


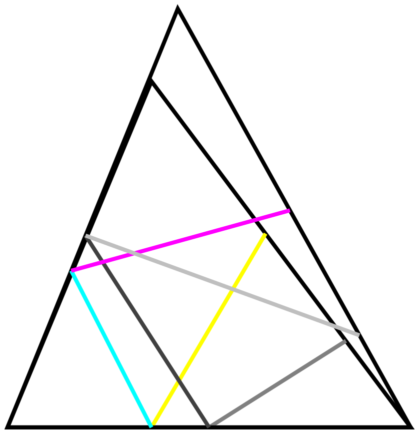


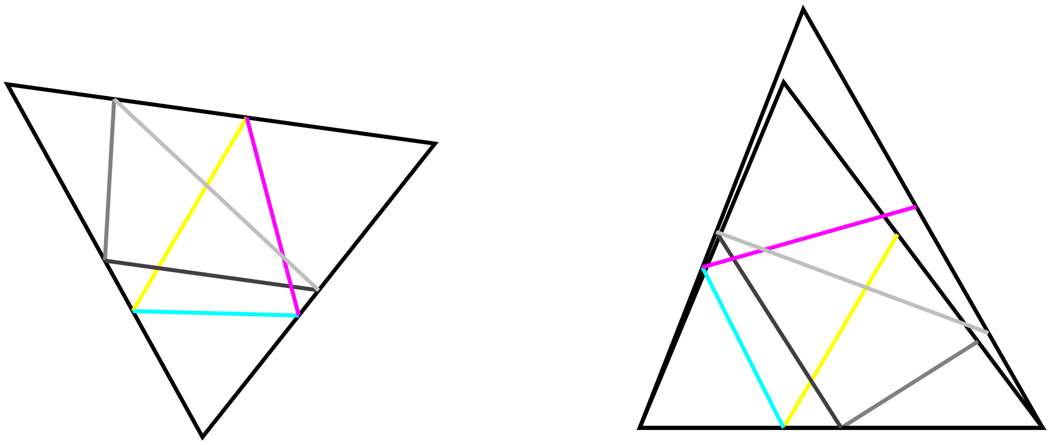


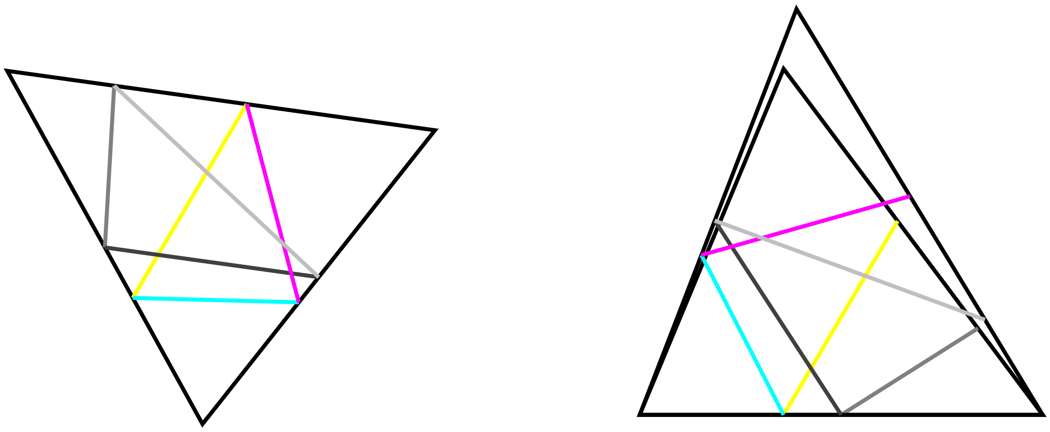


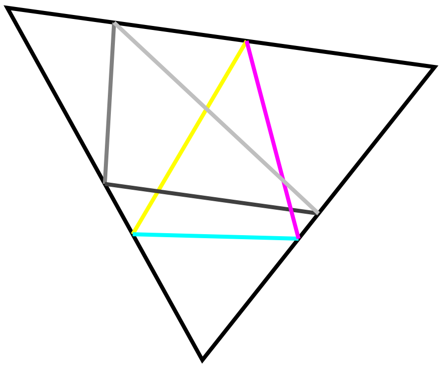


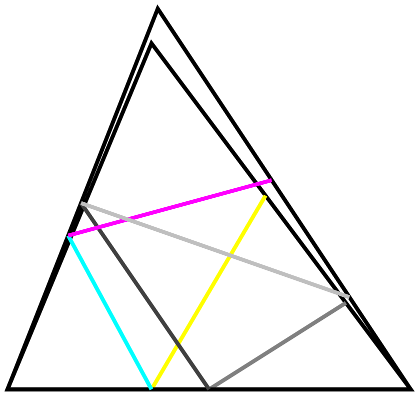


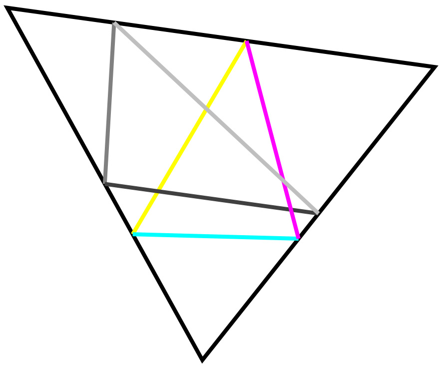


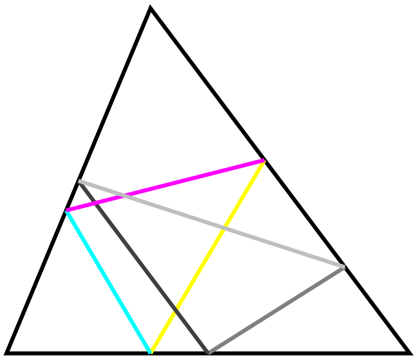






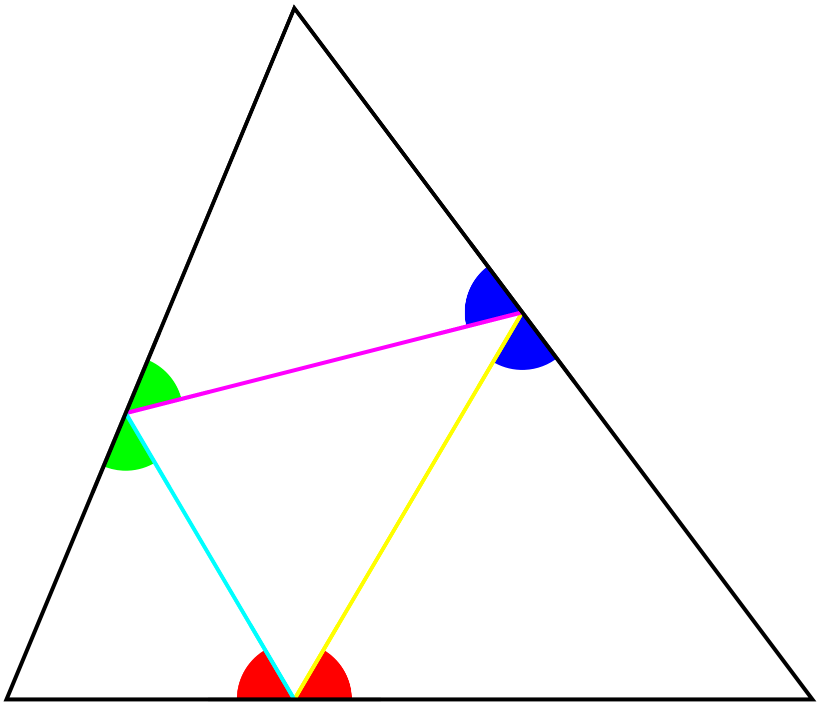






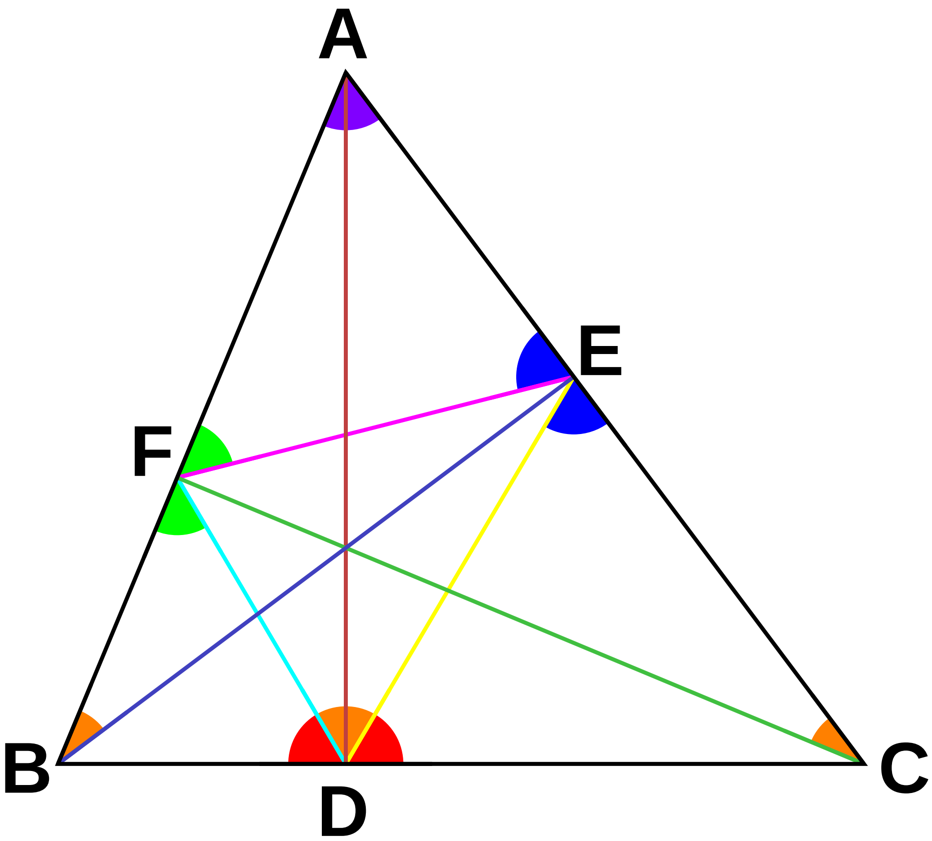
There.

Now that we have proven that this triangle is the one with the smallest perimeter, let's learn how to find it by examining its properties.

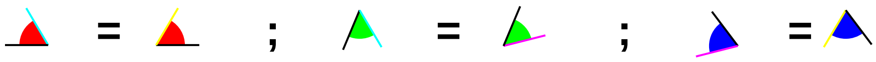


Remember the angles of the same colors are equal.

Connect the lines.

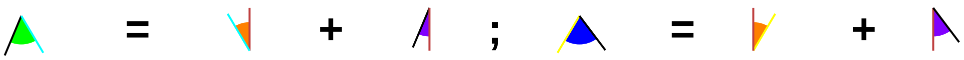


**Note the three lines conveniently come together at a single point.
We'll see why later.**

We know 

AND  **= 180 degrees**

that's equal to  **= 180 degrees**

Moreover, 

Therefore, 

= 

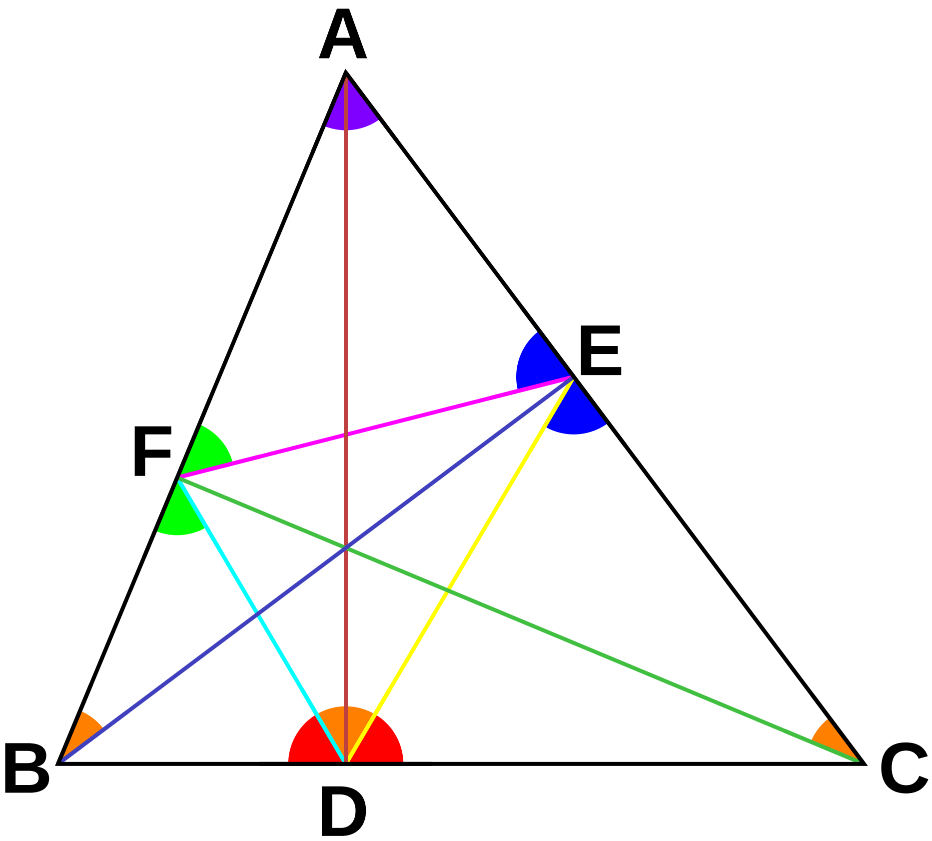
After cancelling out the same angles from the both sides:

Therefore, 

As a result, AFDC is an inscribed quadrilateral.

Similarly, ABDE and BCEF are also inscribed quadrilaterals

(The proof of this can be derived easily from Euclid's propositions on cilcles.)



Since AFDC is an inscribed quadrilateral,

 = 

(They share the same chord AF)

Similarly,

 = 

 = 

Therefore,

 = 

Therefore,  =  = 90 degrees

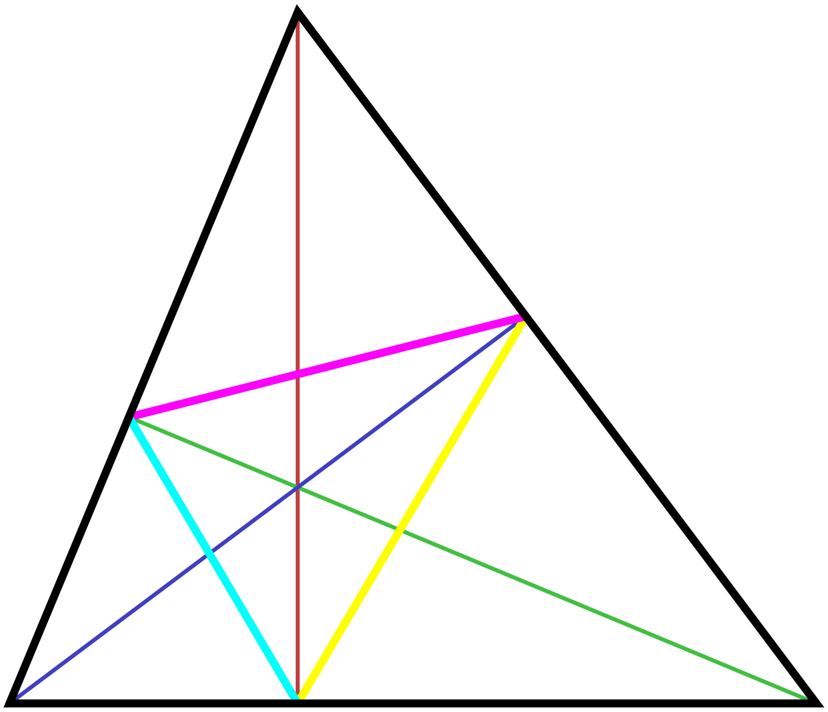
Therefore, AD is a height of the triangle!

Similarly, BE and CF are also heights of the triangle.

(This explains the coming together of the three points.)

So now we know the three vertices of the small triangle are the footpoint of the three altitudes of the acute-angle triangle.

Conclusion:



In a given acute-angle triangle, to find the inscribed triangle of a minimum perimeter, find the footpoints of the three altitudes (heights) and connect them to make them the vertices of an inscribed triangle.