3-Dimensional Geometry:

Solid Angles and Girard’s Theorem

Math 308 - Final Project
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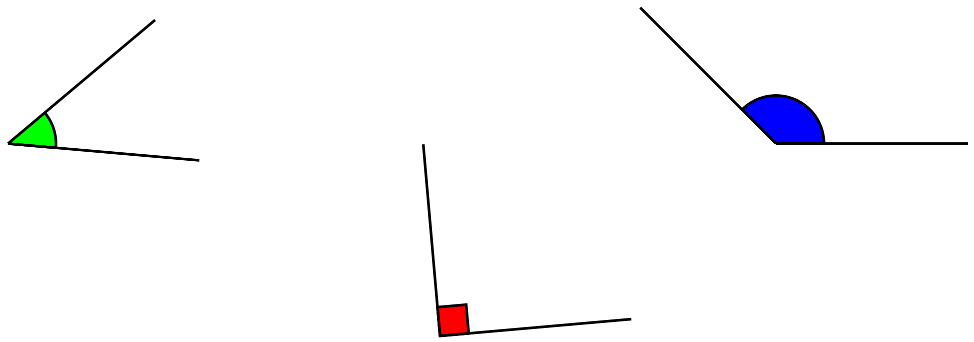
Novice mathematicians are often introduced to Geometry in a plane, learning theorems and properties of this 2-dimensional space. However, the world of solids, the 3-D world, is often neglected. Many of the objects and concepts found in the plane have analogues in 3-dimensional space.

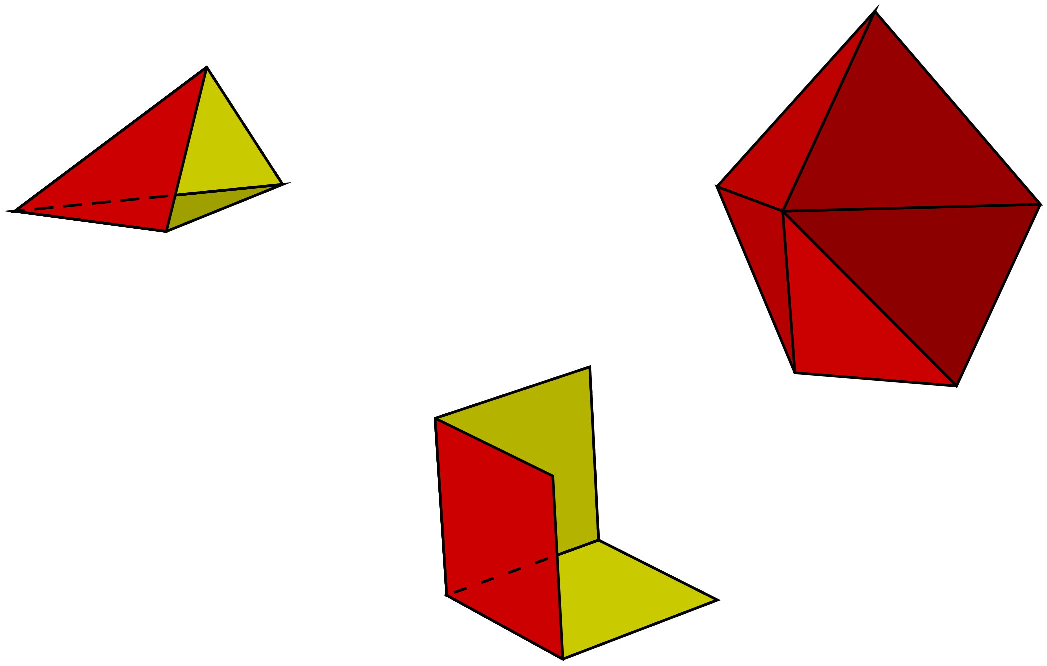
In 2 dimensions - The Plane Angle:

A plane angle is the region contained by two intersecting non-collinear lines meeting at a point.



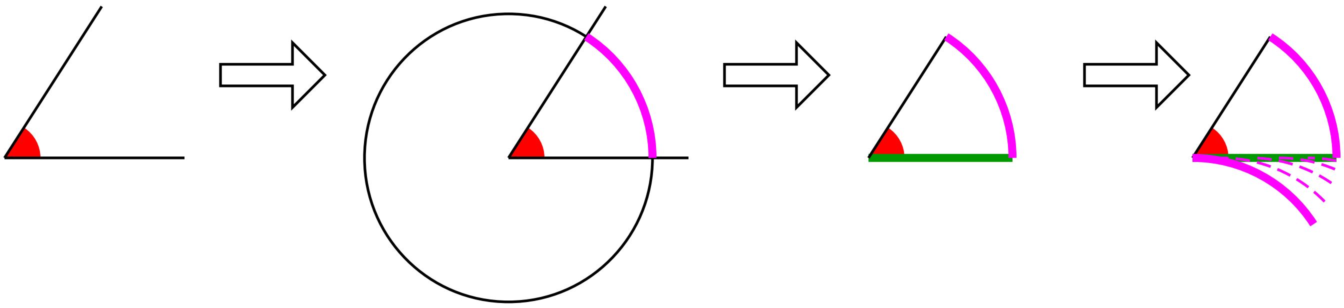
In 3 dimensions - The Solid Angle:

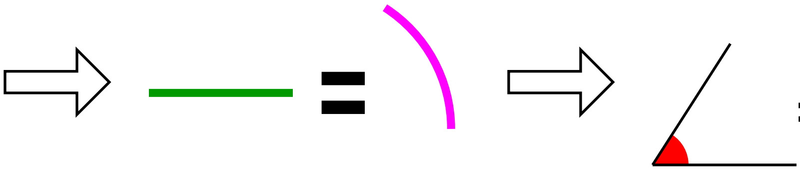
Similarly, a solid angle is the region contained by three or more non-coplanar planes meeting at a point.



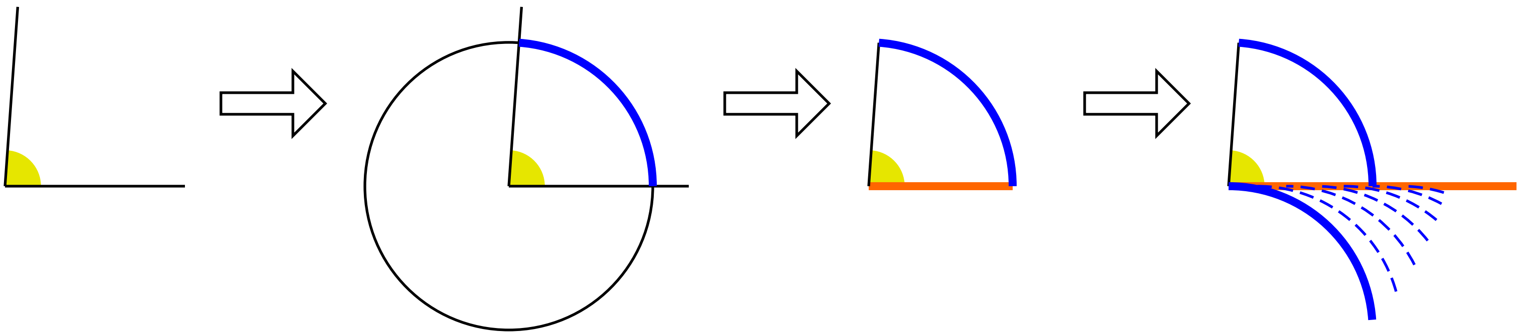
Plane Angles Review - Measurement:

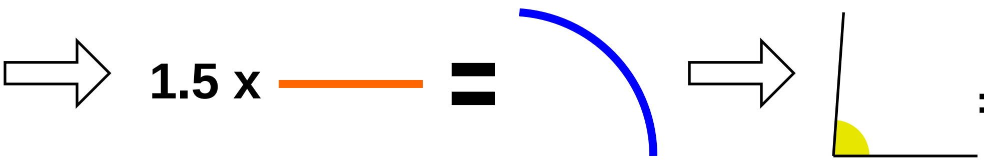
A plane angle is measured using the length of the subtending arc on a circle of radius one unit, centred at the angle's vertex. This ratio of arc length to radius is called the radian.





= 1 radian

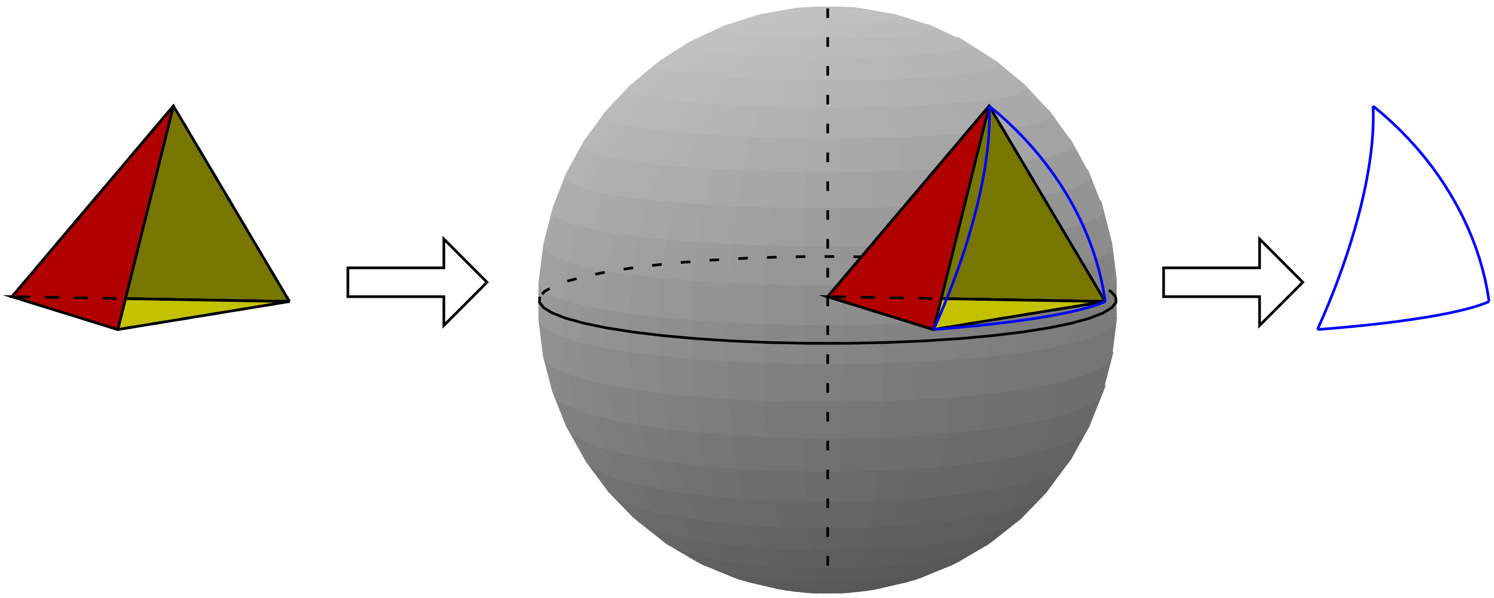




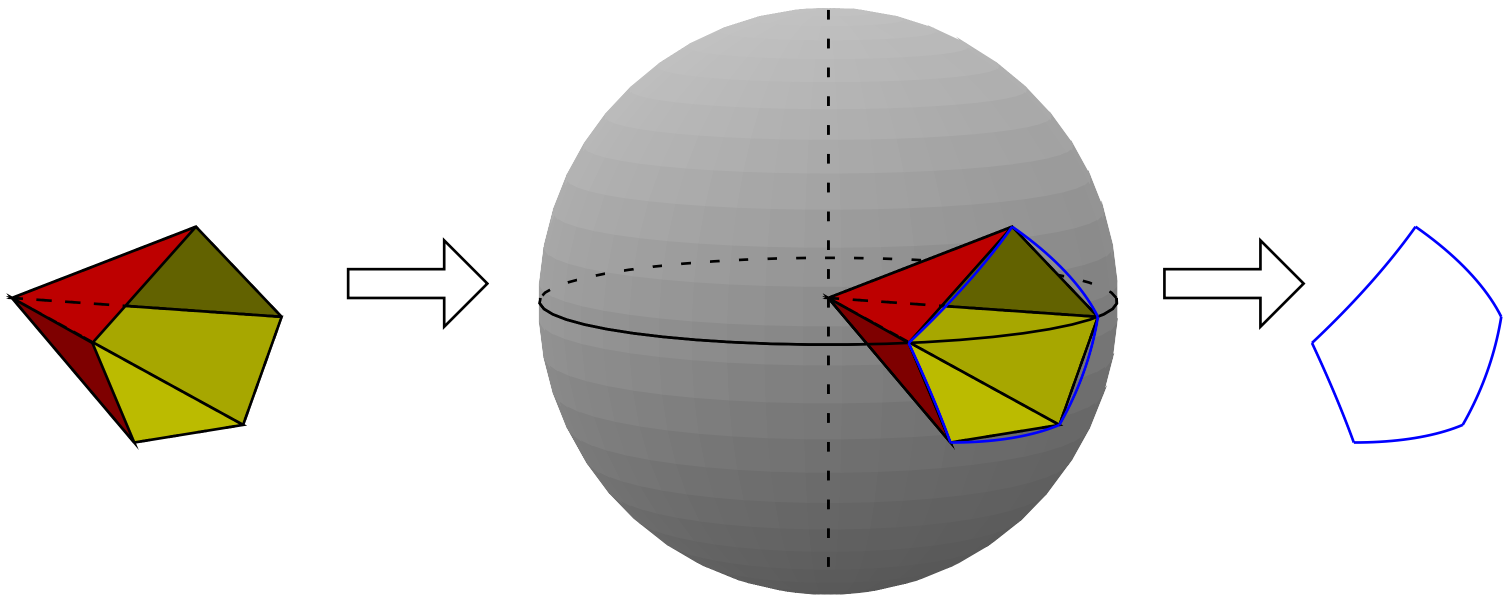
= 1.5 radians

Solid Angles - Measurement:

A solid angle is measured using the area of the subtending spherical polygon on a sphere of radius one unit, centred at the angle's vertex. A spherical polygon is a closed geometric figure on the surface of a sphere which is formed by arcs of great circles. The ratio of spherical surface area to radius squared is called the steradian. Note that while equal plane angles are similar, there are an infinite number of distinct spherical polygons with a given area and therefore an infinite number of distinct solid angles that are equal.



But how do we measure the area of a polygon on the surface of a sphere?



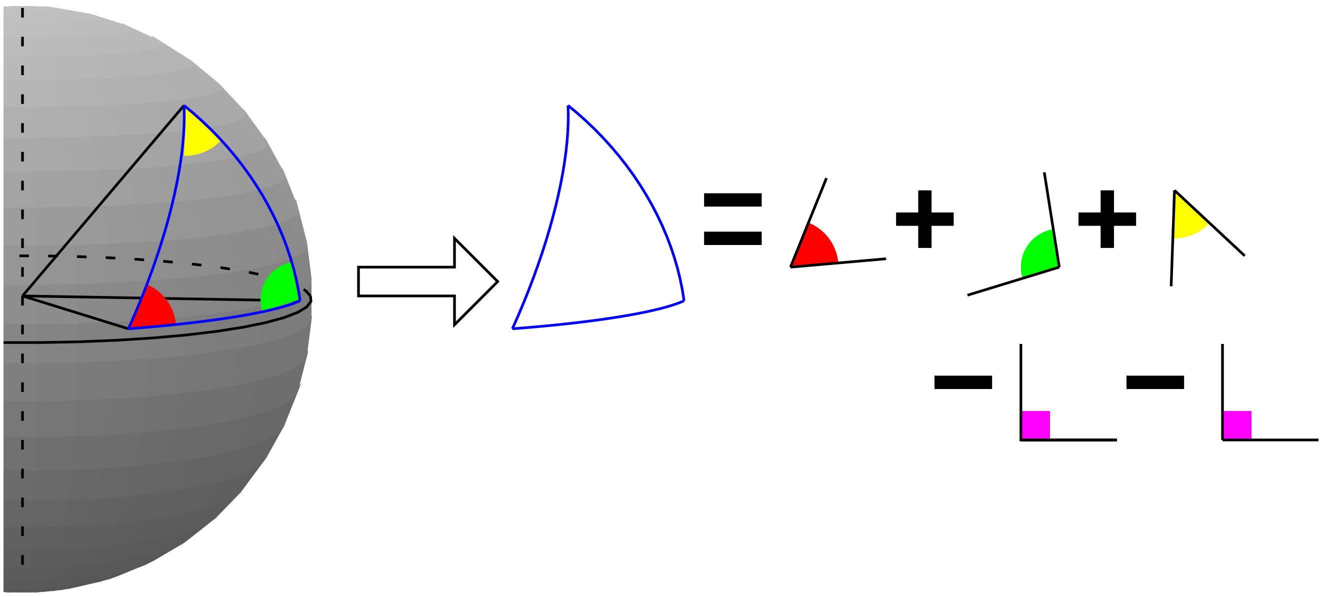
Solid Angles -

Girard’s Theorem:

"The area of a spherical triangle is equal to the sum of its angles less 2 right angles."

$$\text{area of triangle*} = \left(\sum \text{angles of triangle}\right) - \pi$$

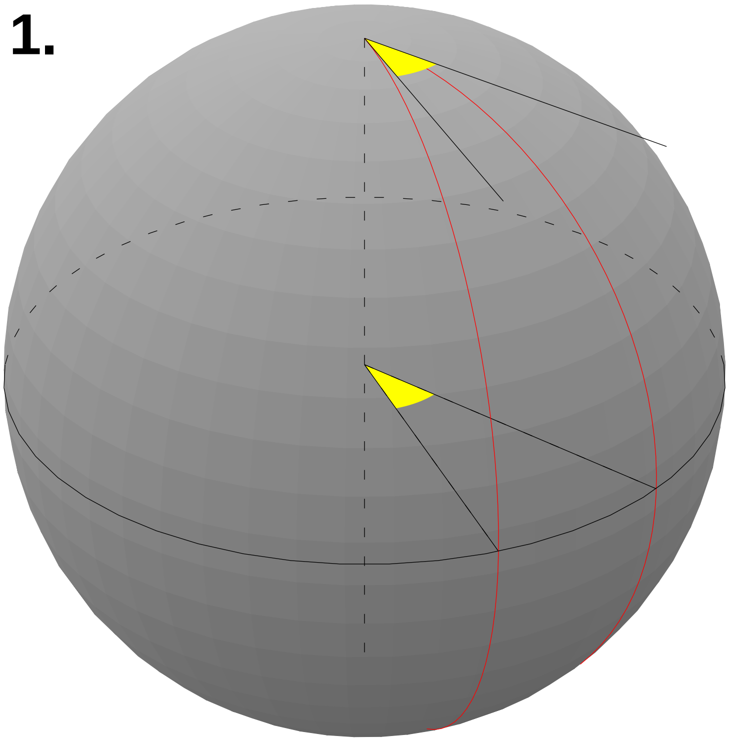
*** area is measured in (radius = 1 unit) squared, otherwise multiply area by r^2.**



Girard’s Theorem - Proof:

Facts: - The surface area of a sphere is 8 right angles, or 4π.
- The angle between arcs of two great circles on the surface of a sphere is defined to be the angle between the tangents to the curves.

1.



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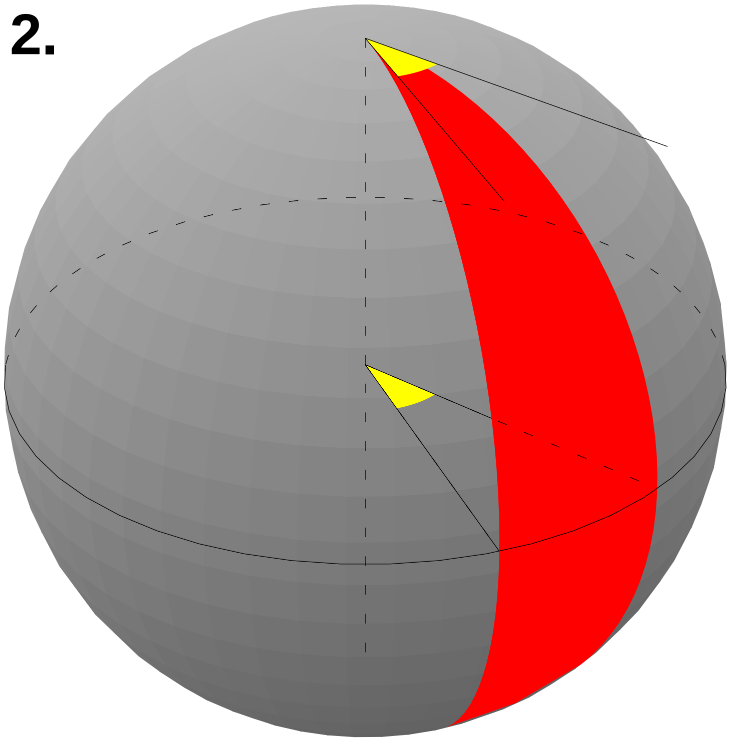

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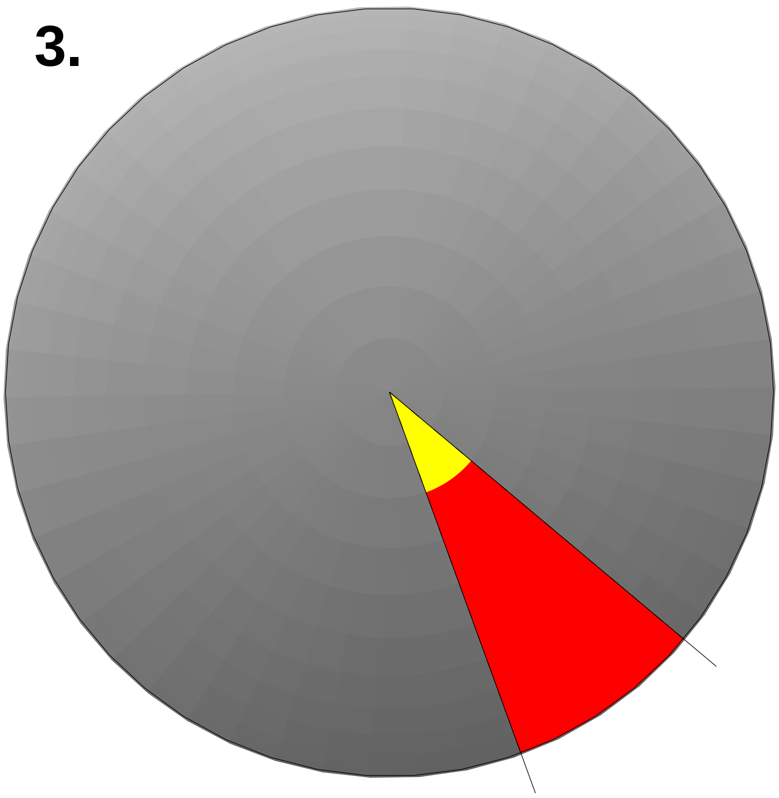
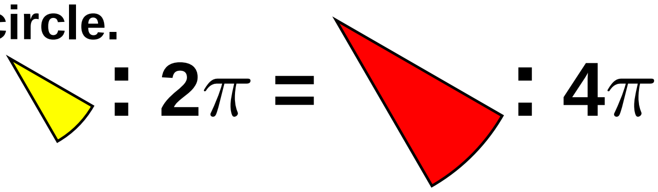
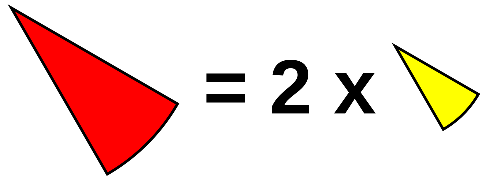


From Figs. 1-3, one can see that the proportion of the area of the lune (the red section) to the overall surface area is equal to the proportion of the angle to the full circle.

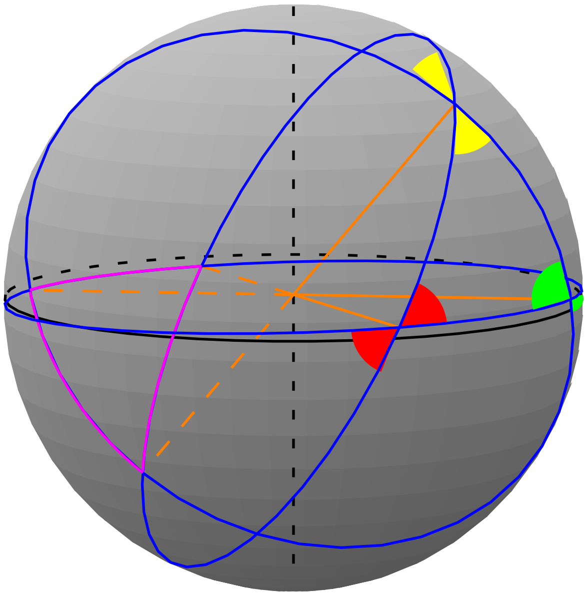


i.e.

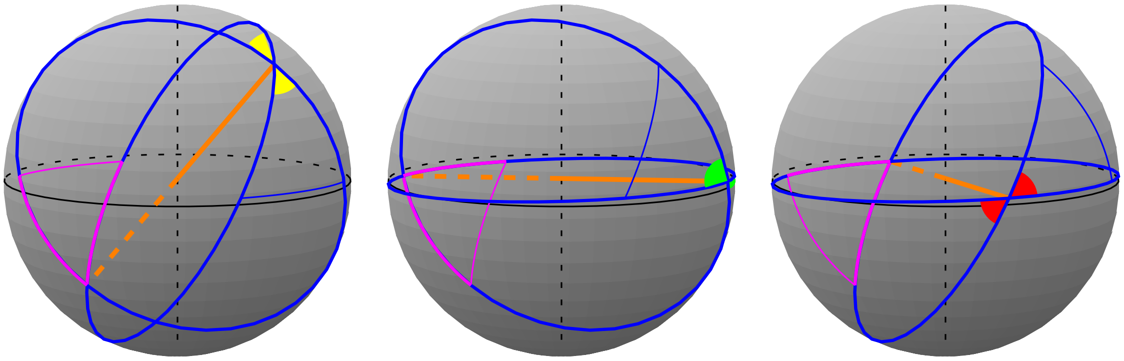


Hence a lune’s area is twice its defining angle.

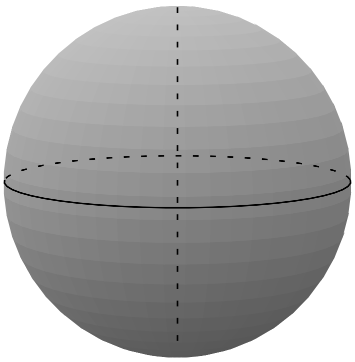
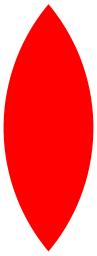
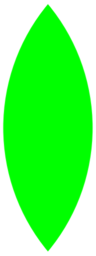
Girard’s Theorem - Proof (Continued):

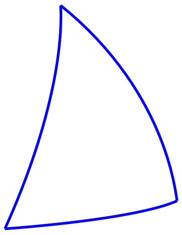


By extending the arcs to complete the great circles, one can see that 3 pairs of lunes are defined on the sphere’s surface. Also defined is the antipodal triangle (purple) on the opposite surface of the sphere, which is an exact image of the original triangle.



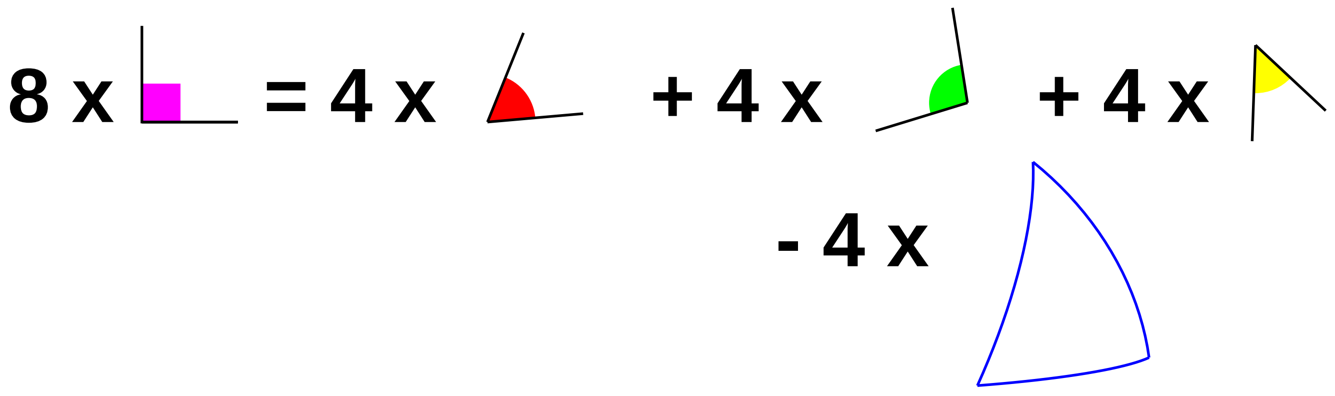
One can see that these 6 lunes cover the entire sphere and only overlap on the 2 triangles. Thus:

 **= 2 x**  **+ 2 x**  **+ 2 x**

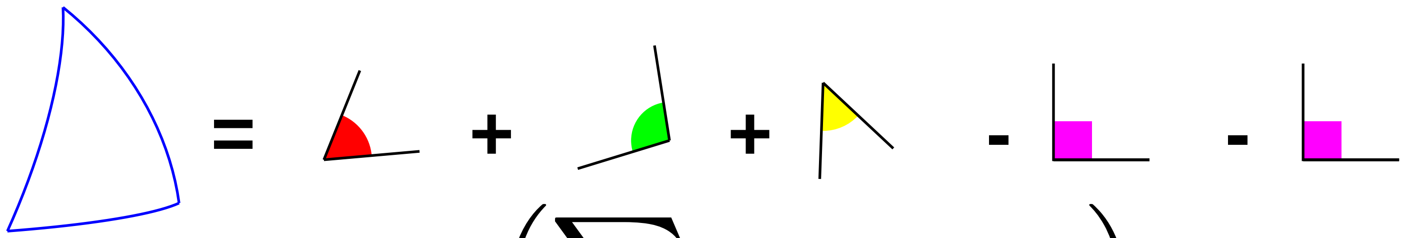
- 4 x 

Girard's Theorem - Proof (Continued):

By substituting in the area formulas we know, we get:



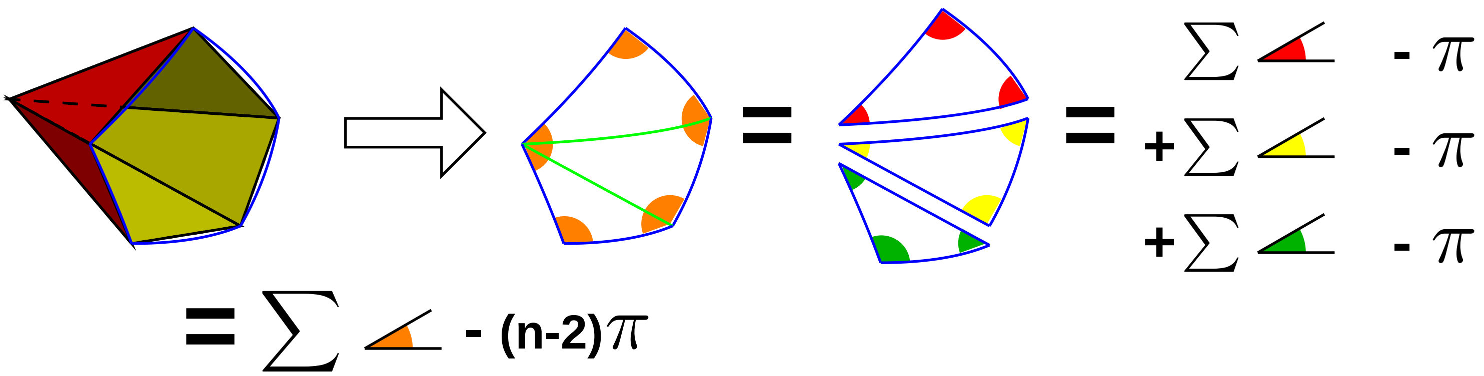
i.e.



$$\text{area of triangle} = \left(\sum \text{angles of triangle}\right) - \pi$$

Q.E.D.

Generalization: Just like planar polygons, the area of an n-sided spherical polygon can be calculated by dividing it into (n-2) triangles and adding up their individual areas.



Thus:

$$\text{area of n-sided spherical polygon} = \left(\sum \text{angles}\right) - (n-2)\pi$$

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