Plotting planets

Tracking planets in space involves working with several coordinate systems. The final step is transferring to an Earth-based coordinate system, but that story will be told elsewhere.

The planetary coordinate system

The coordinate system simplest for plotting a planet's motion is one determined by its own orbit:

- the Sun (assumed to be fixed) is at the origin;
- the positive x-axis passes through perihelion;
- the motion is counter-clockwise.

In this system, what you need to plot the planet are these data:

- the semi-major axis a;
- the eccentricity e;
- a moment T at which the planet passed through perihelion.

These are three out of six of the planet's **elements**, which specify everything needed to plot it.

From these we can calculate f, the distance from the centre of the orbit to the Sun; b, the semi-minor axis; and the period P. We shall use here the A.U. (astronomical unit) as the unit of length and the Earth year as unit of time.

$$b = a\sqrt{1 - e^2}$$
$$f = ae$$
$$P = a^{3/2} \quad \text{(by Kepler's third law)}$$

To calculate where the planet is at time t:

- calculate the **mean anomaly** $M = 2\pi(t - T)/P$;
- find the **eccentric anomaly** E by solving Kepler's equation $M = E - e\sin E$;
- the coordinates with respect to the orbit's centre are then $(a\cos E, b\sin E)$;
- shift the origin to the focus to find the position to be $(a\cos E - f, b\sin E)$.

The mean anomaly M is the angular coordinate for an imaginary **mean** planet with a circular orbit of radius equal to a, whereas the eccentric anomaly is an angular coordinate for n eccentric orbit. This terminology is itself eccentric, but time-honoured. Often the value of M at a given moment is specified instead of giving T directly. If $M = M_0$ at time T_0 then we solve

$$M_0 = 2\pi\left(\frac{T_0 - T}{P}\right), \quad T = T_0 - \left(\frac{M_0}{2\pi}\right)P$$

to get T. Notice how units remain consistent here.

Both anomalies are angles. If angles are given in degrees, as they usually are in astronomical tables, be sure to convert to radians for calculations.

Coordinates in the solar system

Normally, coordinates in the solar system are chosen to make calculations of gravitational motion in the system as well as observation from Earth as simple as possible. This gives us the **ecliptic coordinate system**.

The origin is at the centre of a fixed Sun (in these notes the motion of the Sun will be ignored). The (x, y) plane is chosen to be that in which the Earth's orbit lies. The positive z-direction is that in which the Pole star lies. It remains to specify the direction of the positive x-axis. This is determined by the spin of the Earth! For an observer

living on the Equator, in the winter the Sun sets to the south of West, and in the summer it sets to the north of it. Also in the winter, days are shorter than nights, and vice-versa in the summer. There is a moment in the spring when it sets exactly at due West, and when days and nights are of equal length, called the **vernal equinox**. The direction of the Sun at that moment is chosen to be the direction of the positive x-axis in the ecliptic coordinate system.

The **longitude** of an object in the solar system is the angular coordinate of its projection onto the ecliptic plane. Thus the Earth's position has longitude 0 at the **autumnal equinox**, and longitude π or $180°$ at the vernal equinox.

The unit of length is invariably the **astronomical unit** (A.U.), essentially the semi-major axis of the Earth's orbit, and the unit of time is the Earth year (again, subtleties involved in this definition due to complexities in the Earth's motion will be ignored).

Incidentally, the Earth approaches closest to the Sun when its longitude is $102.9°$, in mid-winter in the northern hemisphere. Seasonal change of climate is almost entirely due to the amount of radiation received from the Sun, which varies during the year because the Earth spins around axis which is tilted about $23°$ with respect to the ecliptic plane.

Orbital elements

The orbits of all planets except Earth lie in planes different from the ecliptic. In order to plot a planet's motion, planetary coordinates must be converted to ecliptic coordinates. How to do this is determined by the remaining three of the planet's elements.

As the planet moves around its orbit, there will be a unique moment when it cuts through the ecliptic plane from below. The point where this occurs is called the planet's **ascending node**, and its ecliptic longitude is called Ω.

The angle between the planet's orbit and the ecliptic plane is called its **inclination** i. It can be anywhere in the range $0°$ to $180°$. If it is less than $90°$ then the motion is **direct** (in the same direction as the Earth's motion), otherwise **retrograde**. None of the major planets have retrograde motion, but many comets do.

The angle between the ascending node and the perihelion point of the orbit is called the **argument of perihelion** ω. This measures the proportion of its orbit the planet traverses after it crosses the ecliptic to its perihelion. Sometimes this is replaced by a fictional quantity $\Pi = \Omega + \omega$, which has no direct geometric meaning. This also is irrational but time-honoured.

The phase of the planet's motion is determined by specifying a time T at which it passed perihelion. As I have mentioned, this is also determined by the mean anomaly M_0 (an angle in the planetary coordinate system) at a fixed moment T_0. In astronomical practice, it is usually specified by the **mean longitude** at a fixed moment T_0.

Mean longitude is a peculiar quantity, even further removed from geometrical significance than the mean anomaly. It is equal to the sum of the already odd quantity Π referred to above and the mean anomaly:

$$L(t) = \Pi + M(t) \,.$$

So if we fix a time T_0 and set L_0 equal to L at that moment, then we can fix a time of perihelion passage T by solving the equation

$$L_0 = \Pi + 2\pi \left(\frac{T_0 - T)}{P} \right) \,.$$

Of course if angles are measured in degrees, 2π is replaced by $360°$.

In the table below, mean longitude is that at midnight Greenwich Mean Time in between December 31, 1999 and January 1, 2000.

Planet	a	e	Ω	Π	L	i
Mercury	0.387	0.206	48.3°	77.46°	252.3°	7.00°
Venus	0.723	0.007	76.7°	131.6°	182.0°	3.39°
Earth	1.000	0.017	–	102.9°	100.5°	0.00°
Mars	1.524	0.093	49.6°	336.1°	355.4°	1.85°
Jupiter	5.203	0.048	100.4°	14.3°	34.4°	1.30°
Saturn	9.555	0.056	113.7°	93.1°	50.1°	2.49°
Uranus	19.22	0.046	74.0°	173.0°	314.1°	0.77°
Neptune	30.11	0.009	131.8°	48.1°	304.3°	1.77°

For the Earth, Ω and ω are not well-determined. What is important is only Π. But to allow dealing with Earth by procedures good for all the other planets, Ω is usually arbitrarily set to be $0°$.

Plotting in 3D

How can you use the data to plot the motion of a planet? We have to figure out how to transform from planetary coordinates to ecliptic ones. In order to do this, we need to know for each planet two vectors: the unit vector u pointing from the Sun towards perihelion, and the vector w pointing up perpendicularly to the orbit. These two determine the unit vector v pointing along the positive y-axis:

$$v = w \times u .$$

These three vectors u, v, and w determine make up a **frame** for the planetary coordinate system. A point with planetary coordinates x, y, z is equal to the vector sum

$$xu + yv + zw .$$

How do we find u and w?

Let α be the unit vector pointing to the ascending node of the orbit. This is $(1, 0, 0)$ rotated in the ecliptic plane by Ω:

$$\alpha = (\cos\Omega, \sin\Omega, 0) .$$

The orbital axis w of the planet is the z-axis $(0, 0, 1)$ rotated around the axis through α by the inclination i. The vector u is α rotated around w by ω.

So to calculate w and then u we need to understand 3D rotations.

3D rotations

In 3D rotations take place around axes. The convention is that the direction of positive rotation is according to the **right hand rule**.

Problem: Given

- an axis α (that is to say a vector in 3D);
- an angle θ; and
- a vector v.

What is v rotated by θ around α?

Normalize α to have length 1—replace it by $\alpha/\|\alpha\|$. Express $v = v_0 + v_\perp$ where v_0 is the projection of v onto the axis:

$$v_0 = (v \bullet \alpha)\alpha, \quad v_\perp = v - v_0 .$$

Let v_* be $v_\perp$ rotated by $\pi/2$ in the plane perpendicular to α:

$$v_* = \alpha \times v_\perp \quad (\text{cross product}) .$$

Then the rotated vector is

$$v_0 + (\cos\theta)v_\perp + (\sin\theta)v_* \ .$$

Recall that

$$(x_1, y_1, z_1) \bullet (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

and

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = \left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} , \ - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} , \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \right) \ .$$