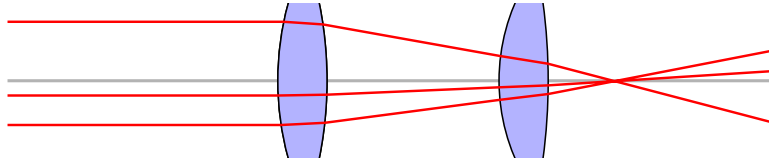


Lens systems

In this note I'll look at how rays of light travel through a system of lenses arranged symmetrically around a linear x -axis. I'll look only at rays contained in a plane through the axis, so in effect I'll be working in 2D.



Real ray tracing

Here, a light ray will be basically a sequence of straight line segments. Each segment is determined by two pieces of data: • an initial position $P = [x, y]$ and • a direction vector $v = [v_x, v_y]$ of length 1. The ray segment is considered to start at P and go in the direction of v until something happens to it, when P and v change. In real life, as with rays from the Sun penetrating the atmosphere at sunset, both position and direction will change continuously, but in the model systems we are looking at here, changes will occur at discrete points of the path. The lens system will be a collection of surfaces which the rays hit and get refracted. Ray tracing amounts to calculating these points and the changes that occur.

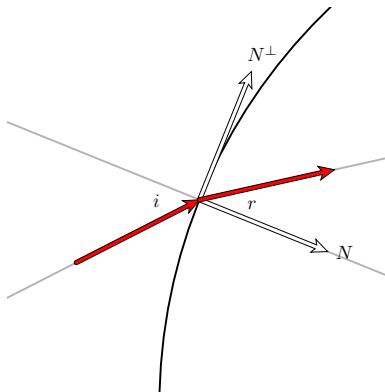
The most interesting change occurs in crossing the surface of a lens from one side to the other. Position remains the same, but the ray is **refracted**, which means it changes direction. Its effect is to rotate the direction vector negatively by $i - r$, where i is the angle of incidence and r the angle of refraction. The angle i is that between v and the normal vector N , oriented so v and N point roughly in the same direction. If n is the index of refraction in a substance, then Snell's Law can be written

$$n_r \sin r = n_i \sin i$$

and solved for r :

$$r = \arcsin(n_i \sin i / n_r)$$

In some circumstances, the ratio $n_i \sin i / n_r$ will have magnitude greater than 1, in which case the ray is reflected.



This is one of the principal calculations in ray-tracing: we are given a point P on the surface $f(x, y) = 0$ and a ray hitting the surface at P . We want to calculate the direction of the ray when it leaves P . We calculate the vector N to be one of the gradients $\pm[\partial f / \partial x, \partial f / \partial y]$ at P , chosen so $N \cdot v \geq 0$. We calculate i according to the recipe $\cos i = v \cdot N$, $\sin i = v \cdot N^\perp$. Here N^\perp is the vector obtained by rotating N through 90° . Then we calculate r by Snell's Law.

The other calculation involved in ray tracing is to figure out where a ray hits a lens surface. If the surface is arbitrary, this can be quite complicated. But I am assuming that the surface is a half-circle with centre on the lens axis. I am also following convention that $R > 0$ if the centre of the half-circle is in the positive

direction from the surface, $R < 0$ otherwise. We must find the points of intersection of the line along the ray with the whole circle, and then exclude those points where the ray itself does not hit, as well as those on the wrong half-circle.

We are first to find t such that $P_* = (x_*, y_*) = P + tv$ lies on the circle. This means solving the quadratic equation

$$(x + tv_x - c_x)^2 + (y + tv_y)^2 = R^2, \quad t^2 + 2tv_x(x - c_x) + ((x - c_x)^2 + y^2 - R^2) = 0$$

if the circle has centre $(c_x, 0)$ and radius R . We must first look only at those $t > 0$. Then among those we must select the one where x_* is greater or less than c_x , depending on the sign of R .

Linear systems

It happens that in most practical systems most rays lie close to the lens axis, both in position and direction. This allows one to trace rays through **linear approximation**.

The basic problem we face in any ray tracing is this: we choose two **reference planes** transecting the axis perpendicularly. If we are given a point P on the first plane and a direction v it starts with, what is the position P_* and the direction v_* when the ray crosses the second plane? Using the techniques in the previous section, we can compute these data explicitly whenever we are given a particular P and v , but there is no reasonable formula for them. In this section, however, we shall find a relatively simple first order approximation that is quite useful.

I recall that a ray is determined by a point $P = (x, y)$ and a direction $v = [\cos t, \sin t]$ where y and t are small, and we start out with x the coordinate of the first plane. Mostly, we fix x and ask how the image point P_* and direction v_* —more precisely y_* and t_* —depend on y and t .

We are now going to assume ourselves in what I call the **linear world**, where quantities of the order of the square of either y or t are to be ignored. So the vector v may be taken to be $[1, t]$, since $\cos t = 1 - t^2/2 + \dots$ and $\sin t = t - t^3/6 + \dots$. We shall just take (y, t) as the ray coordinates, and see how they change as one follows the ray.

A ray is made up of straight segments, and point in between where they bend. (1) In a straight segment the ray is **translated** across a homogeneous region. If s is the distance across this region, then the direction doesn't change but (x, y) changes to $(x + s, y + s \tan t)$. In linear approximation $\tan t$ and t are the same (if t is measured in radians). So in our linear world (y, t) changes to $(y + st, t)$. (2) When the ray crosses a lens surface, it is refracted. Here P doesn't change, but v rotates by $-(i - r)$. Making linear approximations, t changes to $t - (i - r)$ where (in this regime of linear approximations)

$$\begin{aligned} i &= t + (y/R) \\ n_r r &= n_i i \quad (\text{linear form of Snell's Law, since now } \sin i = i) \\ r &= (n_i/n_r) i \\ t &= t - (t + y/R)(1 - n_i/n_r) \\ &= (n_i/n_r)t - (y/R)(1 - n_i/n_r) \end{aligned}$$

(Exercise: explain all these.)

These formulae are complicated, but they simplify if we assign a new coordinate instead of t . Let $\Theta = nt$, where n is the current index of refraction. Then for example, Snell's Law asserts simply that Θ doesn't change as we cross a lens surface perpendicular to the lens axis. In refraction Θ changes to

$$\Theta - (n_r - n_i)/R$$

while in translation y changes to

$$y + (s/n)\Theta.$$

So refraction corresponds to transformation of (y, Θ) by the matrix

$$\begin{bmatrix} 1 & \\ -(n_r - n_i)/R & 1 \end{bmatrix}$$

and translation to

$$\begin{bmatrix} 1 & s/n \\ & 1 \end{bmatrix}$$

A lens system amounts to a sequence of such transitions, and the matrix of a system is the product of the matrices of its components. One thing to keep in mind is that a lens bending backwards (with centre to left of surface) has a **negative** radius R . This may seem a bit strange, but it simplifies calculations enormously. In particular, the formulas above remain valid if the conventions regard R are followed, i means 'left', and r means 'right'.

Compound lenses

In traversing a system of lenses, we just multiply the matrices of each component together. This product is called the **transfer matrix** for the two planes one is looking at.

Properties of lens systems

If a system of rays come into a system from ∞ at the left, they will all meet somewhere on the axis. This is called the **focal point**. If we choose the first plane to the left of the system and the other to be the focal plane, then the transfer matrix must have the property that its upper left entry vanishes. This allows you to calculate where it must be. For a single lens with radii R_1 and R_2 distance d apart, what is it?

Two plane are said to be **conjugate** if all rays leaving a point on the first plane meet at a single point on the second. What condition does this amount to on the transfer matrix? The system will in effect magnify objects. What is the magnification factor? For the same lens as in the previous paragraph, what is the equation relating two conjugate lanes?

A system is called **telescopic** if a system of parallel rays coming from the left (any direction) remain a parallel after they have passed through. What condition on the transfer matrix? What is the magnification here?

A **thin lens** is an ideal pair of spherical surfaces at distance 0 apart from each other. If the radii are R_1 and R_2 what is the focal distance?

Show that if two thin lenses of focal lengths f_1 and f_2 are placed at distance $f_1 + f_2$ the system is telescopic.