

Aberrations: Curvature of Field¹

JEREMY D. A. JANZEN

ABSTRACT. A definition and introduction of third order aberrations will be given. Curvature of field will be explained in more detail which includes a look at how the Seidel sums explain this aberration.

1 Introduction

An aberration is a deviation from the ideal convergence of rays in a lens system. In the linear theory developed in class we could show that any point on the right conjugate plane would focus to a single point on the left conjugate plane for any angle as long as the angle was small and the y value was close to the axis.

We required the distance to the y axis and the angle to be small in the linear theory because we didn't use Snell's law as stated as

$$n_i \sin i = n_r \sin r, \tag{1}$$

we took only the first term of the Taylor expansion for $\sin \theta$,

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \dots \tag{2}$$

which gave us Snell's law as

$$n_i i = n_r r. \tag{3}$$

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In this talk we will be considering third order aberrations. These are aberrations which come about when the third order term in the Taylor expansion becomes important in the approximation due to a larger incidence angle. By taking the first two terms of the Taylor expansion for $\sin \theta$, we have Snell's law written as

$$n_i \left(i - \frac{i^3}{6} \right) = n_r \left(r - \frac{r^3}{6} \right). \quad (4)$$

2 Deviation from the ideal spherical surface

One way of looking at aberrations is to consider the optical distance. In the linear theory all the rays from a point in the object plane are mapped to a single point in the image plane. This implies that the optical distance traveled by all the rays is the same. The wave front surfaces will be spherical coming from the point in the object plane and after exiting the lens system will also be spherical around the point in the image plane to which they all focus. However, when we compute the exact surfaces, they deviate from the spherical ideal. This is due to the fact that the optical distances from the object point to the image point are not the same for all rays traveling through the system according to the exact theory.

The amount of deviation Δ is a function of the distance from the optical axis and is parameterized by $\Delta = f(C_{22}, C_{12}, C_{11}, C_{20}, C_{10})$. Each parameter represents one of the five third order aberrations which respectively are

1. Spherical
2. Coma
3. Astigmatism
4. Curvature of Field
5. Distortion

3 Overview of the five aberrations

3.1 Spherical aberration

Rays coming from a point source on the axis entering the lens further from the optical axis focus closer to the lens than rays which enter close to the optical axis. See Figure 1.

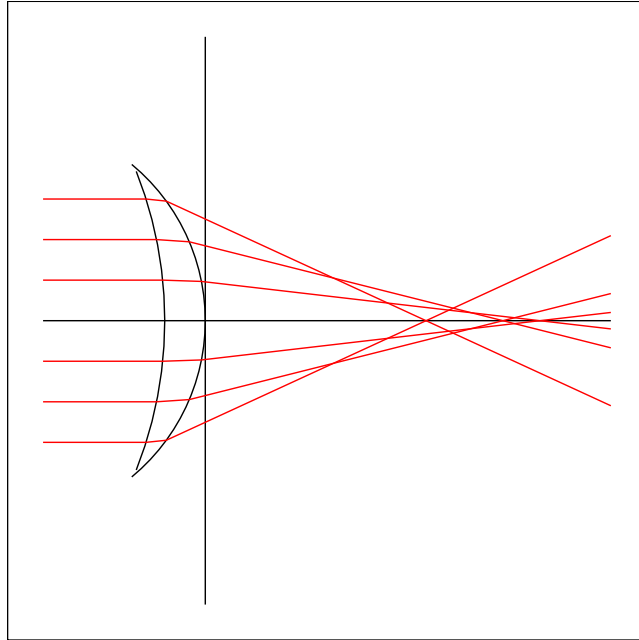


Figure 1: Spherical aberration which shows that the marginal rays focus closer to the lens than the central rays.

3.2 Coma

Rays coming from a point source off the axis focus at varying heights on the image plane depending on where they enter the lens. See Figure 2.

3.3 Astigmatism

Rays coming from a point source off the axis focus in two lines on different surfaces due to the elliptical shape of the lens for a cone of rays coming from an angle to the lens.

3.4 Curvature of field

Rays coming from off axis points encounter a lens with a shorter focal length than point closer to the axis which gives rise to curvature of the focal plane. See Figure 4 and Figure 5.

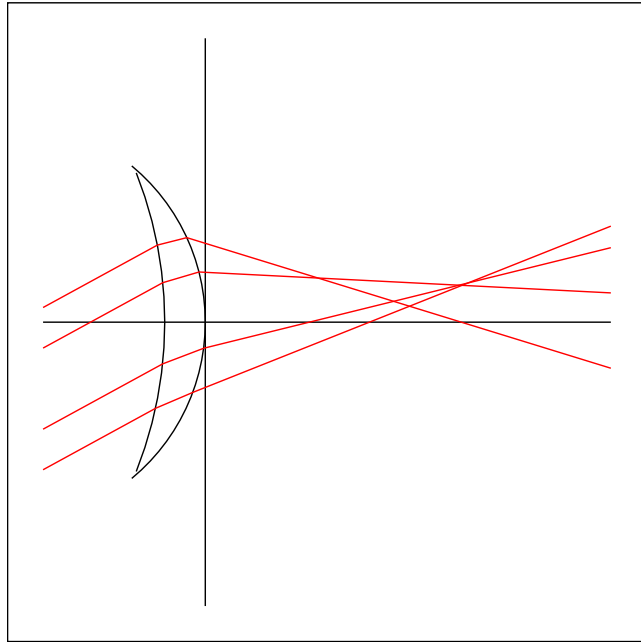


Figure 2: This figure shows coma as well as spherical aberration. The marginal rays focus lower due to coma and closer due to spherical aberration. If only coma was involved the marginal rays would focus on the same plane but at a lower point

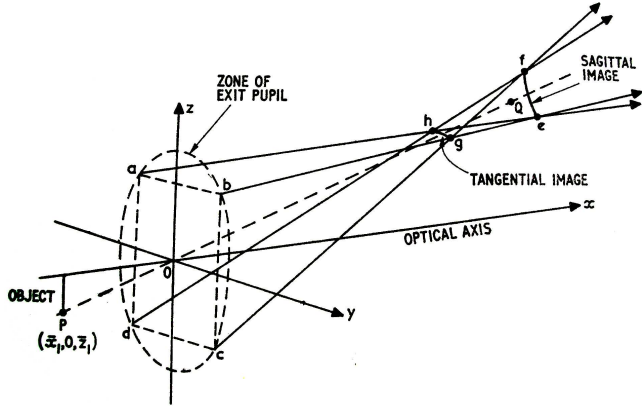


Figure 3: This figure shows astigmatism.

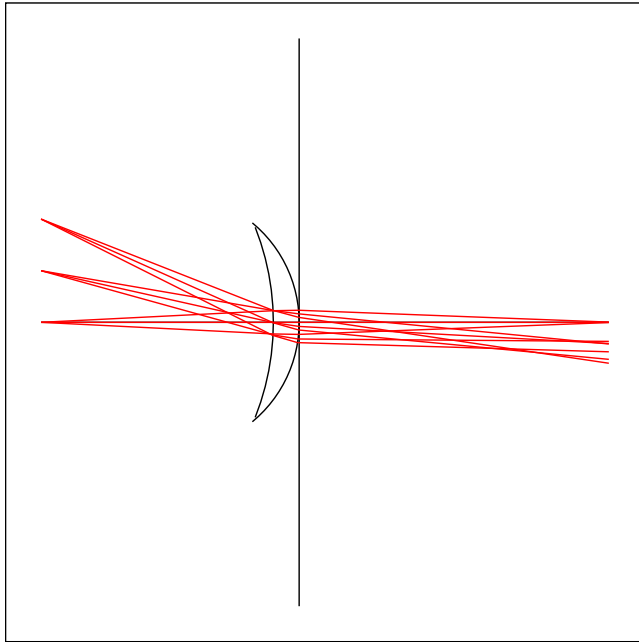


Figure 4: This figure shows how rays coming from different points will focus at different distances from the lens.

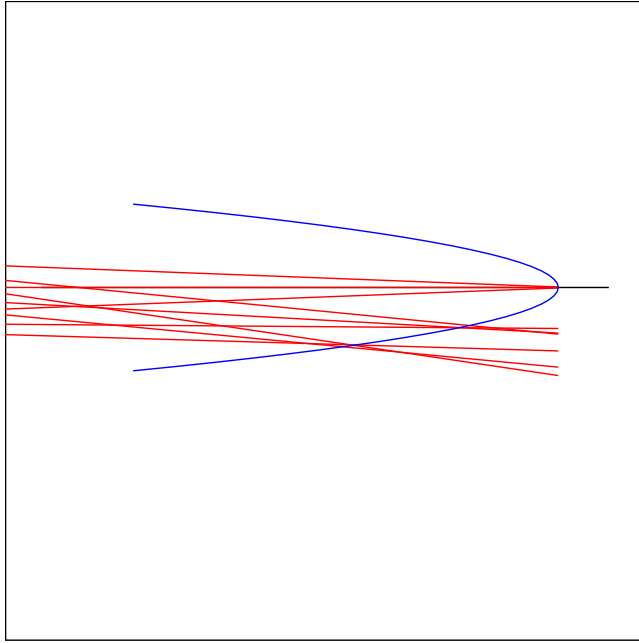


Figure 5: This figure shows an approximation to the curved field of focus by an parabola.

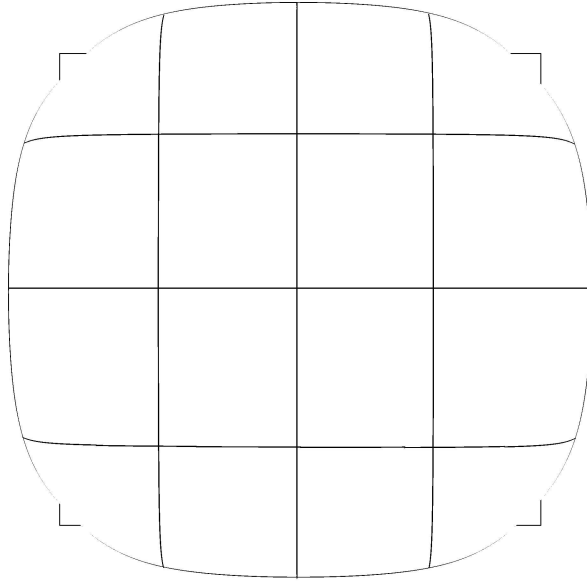


Figure 6: This figure shows barrel distortion of a grid.

3.5 Distortion

The displacement of rays coming from off axis points which causes variance in the magnification of the image.

4 The Mathematics

We will now describe the wavefronts as they exit the lens system. Our coordinate system will have axes x' , y' , and z' . at the image plane. See Figure 7. Define a reference plane to be (x'_1, y'_1) and set it to be the exit pupil. Let P' be the image point of P on the image plane and Q the point on the reference plane on the optical axis. The equation of a sphere with P' at the center and radius $P'Q$ is

$$(x'_1 - x'_p)^2 + (y'_1 - y'_p)^2 + z'_1 = r^2 \quad (5)$$

Let Δ be the function of deviation. Introducing Δ into (5) gives

$$(x'_1 - x'_p)^2 + (y'_1 - y'_p)^2 + z'_1 = (r + \Delta)^2. \quad (6)$$

We can rewrite $(r + \Delta)^2$ as $r^2 + 2r\Delta + \Delta^2$. We can neglect Δ^2 by requiring Δ to be small. Finally we have

$$(x'_1 - x'_p)^2 + (y'_1 - y'_p)^2 + z'_1 - r^2 - 2r\Delta = 0. \quad (7)$$

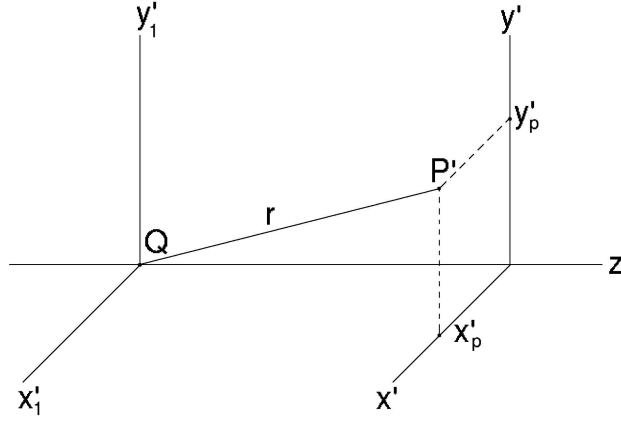


Figure 7: The image plane and the reference plane are shown.

We now have an equation for our wavefront but we still need to find Δ . Δ will be a function of the position of P_i and of the position of a point on the wavefront, that is it depends on x'_p, y'_p, x'_1 and y'_1 . Because we are assuming rotational symmetry, Δ will only depend on the combinations of these variables which do not vary when we rotate around the optical axis. Therefore, convert to polar coordinates in the reference plane and image plane. That gives

$$\begin{aligned} x'_p &= \rho \sin \alpha \\ y'_p &= \rho \cos \alpha \\ x'_1 &= r_1 \sin \beta \\ y'_1 &= r_1 \cos \beta \end{aligned}$$

By an argument of symmetry we have Δ as a function of only

$$2u'_1 = x'_p{}^2 + y'_p{}^2 = \rho^2 \quad (8)$$

$$\begin{aligned} u'_2 &= x'_p x'_1 + y'_p y'_1 \\ &= r_1 \rho (\sin \alpha \sin \beta + \cos \alpha \cos \beta) \end{aligned} \quad (9)$$

$$\begin{aligned} &= r_1 \rho \cos(\beta - \alpha) \\ 2u'_3 &= x'_1{}^2 + y'_1{}^2 = r_1^2 \end{aligned} \quad (10)$$

Since Δ is complicated we will expand Δ in a power series up to the quadratic terms. That gives us

$$\begin{aligned} \Delta &= a_0 + b_0 u'_1 + b_1 u'_2 + b_2 u'_3 \\ &+ \frac{1}{2} c_0 u'_1{}^2 + c_1 u'_1 u'_2 + c_2 u'_1 u'_3 \\ &+ \frac{1}{2} c_3 u'_2{}^2 + c_4 u'_2 u'_3 + \frac{1}{2} c_5 u'_3{}^2 + \dots \end{aligned} \quad (11)$$

Now we will need to find coefficients a_i , b_i and c_i which will be dependent on the particular optical system.

We now would like to find the rays, which we know are normal to the wavefront. Recall the general formula for finding the normal to a point (x'_1, y'_1, z'_1) for a surface $f(x'_1, y'_1, z'_1)$,

$$\frac{x'_1 - x'}{\frac{\partial f}{\partial x'_1}} = \frac{y'_1 - y'}{\frac{\partial f}{\partial y'_1}} = \frac{z'_1 - z'}{\frac{\partial f}{\partial z'_1}}$$

We apply this to equation (7) to get two equations

$$x' = x'_p + r \frac{\partial \Delta}{\partial x'_1} + \frac{z'}{z'_1} x'_1 - \frac{z'}{z'_1} x'_p - \frac{z'}{z'_1} r \frac{\partial \Delta}{\partial x'_1} \quad (12)$$

$$y' = y'_p + r \frac{\partial \Delta}{\partial y'_1} + \frac{z'}{z'_1} y'_1 - \frac{z'}{z'_1} y'_p - \frac{z'}{z'_1} r \frac{\partial \Delta}{\partial y'_1} \quad (13)$$

Since we have only expanded the power series for Δ to the quadratic terms we require that x'_p , y'_p , $\partial \Delta / \partial x'$, and $\partial \Delta / \partial y'$ are small. We will be considering points near P_i so z' is small and z'_1 will be close to r . By dropping terms with a small coefficient and replacing z'_1 with r we have

$$x' = x'_p + z' \frac{x'_1}{r} + r \frac{\partial \Delta}{\partial x'_1} \quad (14)$$

$$y' = y'_p + z' \frac{y'_1}{r} + r \frac{\partial \Delta}{\partial y'_1}$$

When we are in the paraxial image plane, $z' = 0$ so the term with z' in equation (14) will disappear which gives

$$\begin{aligned} x' &= x'_p + r \frac{\partial \Delta}{\partial x'_1} \\ y' &= y'_p + r \frac{\partial \Delta}{\partial y'_1}. \end{aligned} \quad (15)$$

We now want to write out Δ . The partial derivatives of each u'_i are

$$\frac{\partial u'_1}{\partial x'_1} = 0$$

$$\frac{\partial u'_1}{\partial y'_1} = 0$$

$$\frac{\partial u'_2}{\partial x'_1} = x'_p$$

$$\frac{\partial u'_2}{\partial y'_1} = y'_p$$

$$\frac{\partial u'_3}{\partial x'_1} = x'_1$$

$$\frac{\partial u'_3}{\partial y'_1} = y'_1$$

Now we can find the partial derivatives of Δ .

$$\begin{aligned}\frac{\partial \Delta}{\partial x'_1} &= b_1 x'_p + b_2 x'_1 + c_1 u'_1 x'_p + c_2 u'_1 x'_1 \\ &+ c_3 u'_2 x'_1 + c_4 u'_3 x'_p + c_5 u'_3 x'_1\end{aligned}\quad (16)$$

$$\begin{aligned}\frac{\partial \Delta}{\partial y'_1} &= b_1 y'_p + b_2 y'_1 + c_1 u'_1 y'_p + c_2 u'_1 y'_1 \\ &+ c_3 u'_2 y'_1 + c_4 u'_3 y'_p + c_5 u'_3 y'_1\end{aligned}\quad (17)$$

The coefficients b_i can be determined by realizing that when we have a spherical wavefront when $\Delta = 0$ so we have

$$\begin{aligned}x' &= x'_p + z' \frac{x'_1}{r} \\ y' &= y'_p + z' \frac{y'_1}{r}\end{aligned}\quad (18)$$

and then by seeing that equation (15) with equations (16) and (17) substituted in should equal equation (18). This then requires that $b_1 = 0$ and $b_2 = z'/r^2$.

We now would like to write the aberration equations with respect to the variables x and y of the point P in the object plane and x_1 and y_1 in the entrance pupil rather than the exit pupil. If we assume the linear theory we can multiply x and y by a constant μ which is the magnification and we can multiply x_1 and y_1 by a constant μ_1 which is the magnification from one side of the lens system to the other, from entrance pupil to exit pupil. So our equation will now be

$$\begin{aligned}x' &= \mu x + C_{10} u_1 x + C_{20} u_1 x_1 + C_{11} u_2 x \\ &+ C_{21} u_2 x_1 + C_{12} u_3 x + C_{22} u_3 x_1\end{aligned}\quad (19)$$

$$\begin{aligned}y' &= \mu y + C_{10} u_1 y + C_{20} u_1 y_1 + C_{11} u_2 y \\ &+ C_{21} u_2 y_1 + C_{12} u_3 y + C_{22} u_3 y_1\end{aligned}\quad (20)$$

where

$$\begin{aligned}C_{10} &= r c_1 \mu^3 \\ C_{20} &= r c_2 \mu^2 \mu_1 \\ C_{11} &= r c_3 \mu^2 \mu_1 \\ C_{21} = C_{12} &= r c_4 \mu \mu_1^2 \\ C_{22} &= r c_5 \mu_1^3\end{aligned}$$

When given a lens system, the values of $C_{i,j}$ can be determined and tell us what aberrations are present. Calculating these is outside the scope of this talk, however we can get an understanding of Field Curvature using this mathematical foundation.

5 Curvature of field

Curvature of field is closely related to astigmatism so we will consider them together. Assume that all the coefficients $C_{i,j}$ are zero except C_{20} and C_{11} which are responsible for field curvature and astigmatism. Take a point in the object plane on the x axis so $y = 0$. We have

$$\begin{aligned}x' - \mu x &= \mu_1 \frac{x_1}{r} z' + C_{20} u_1 x_1 + C_{11} u_2 x \\y' - \mu y &= \mu_1 \frac{y_1}{r} z' + C_{20} u_1 y_1\end{aligned}\quad (21)$$

Let $\Delta x = x' - \mu x$ and $\Delta y = y' - \mu y$ these are the differences between the x and y coordinates of the image point P' and the coordinate of the ray. Solve for Δx and Δy and change to polar coordinates.

$$\Delta x' = \mu_1 \frac{z'}{r} r_1 \sin \phi + \frac{1}{2} C_{20} r \rho^2 \sin \phi + C_{11} r \rho^2 \sin \phi \quad (22)$$

$$\Delta y' = \mu_1 \frac{z'}{r} r_1 \cos \phi + \frac{1}{2} C_{20} r \rho^2 \cos \phi$$

$$\Delta x' = r_1 \left(\mu_1 \frac{z'}{r} + \rho^2 \left(\frac{1}{2} C_{20} + C_{11} \right) \right) \sin \phi \quad (23)$$

$$\Delta y' = r_1 \left(\mu_1 \frac{z'}{r} + \frac{1}{2} \rho^2 C_{20} \right) \cos \phi$$

If we square both equations in (23) and add them together after solving for \sin^2 and \cos^2 we get

$$\begin{aligned}1 &= \frac{\Delta x'^2}{r_1^2 \left(\mu_1 \frac{z'}{r} + \rho^2 \left(\frac{1}{2} C_{20} + C_{11} \right) \right)^2} \\ &+ \frac{\Delta y'^2}{r_1^2 \left(\mu_1 \frac{z'}{r} + \frac{1}{2} \rho^2 C_{20} \right)^2}\end{aligned}\quad (24)$$

The denominators are the lengths of the axes squared. Let

$$A = r_1 \left(\mu_1 \frac{z'}{r} + \rho^2 \left(\frac{1}{2} C_{20} + C_{11} \right) \right) \quad (25)$$

$$B = r_1 \left(\mu_1 \frac{z'}{r} + \frac{1}{2} \rho^2 C_{20} \right)$$

When $A = 0$ we have the tangential focal line which is horizontal

$$z'_{\text{tangential}} = -\frac{1}{2} \frac{r}{\mu_1} \rho^2 (C_{20} + 2C_{11}) \quad (26)$$

and when $B = 0$ we have the meridional focal line which is vertical

$$z'_{\text{meridional}} = -\frac{1}{2} \frac{r}{\mu_1} \rho^2 C_{20} \quad (27)$$

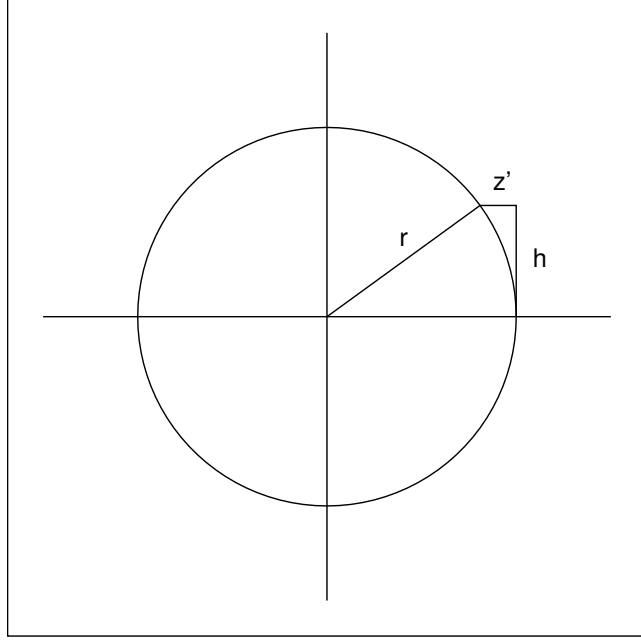


Figure 8: Illustration for curvature of field.

Now, C_{20} and C_{11} are also related to the curvature of field. The equation for a triangle inside a circle (see Figure 8) with the radius as its hypotenuse is

$$h^2 + (z' - r)^2 = r^2$$

We can rewrite this as

$$h^2 + z'^2 - 2z'r + r^2 = r^2$$

We can approximate and solve for z' by taking z'^2 to be zero to get

$$z' = \frac{h^2}{2r}$$

We can see that the expressions for $z'_{tangential}$ and $z'_{meridional}$ can be equated to

$$z' = \frac{\mu^2 \rho^2}{2r'}$$

which when solved for curvature, the inverse of the radius, we have

$$R'_{meridional} = \frac{1}{r'_{meridional}} = -\frac{r}{\mu^2 \mu_1^2} C_{20} \quad (28)$$

$$R'_{tangential} = \frac{1}{r'_{tangential}} = -\frac{r}{\mu^2 \mu_1^2} (C_{20} + 2C_{11}) \quad (29)$$

This shows that the curvature is directly proportional to C_{20} and that the difference in curvature between the tangential plane and meridional plane is directly proportional to C_{11} . Similar demonstrations can be done for the other aberrations as well to see how $C_{i,j}$ change the image from the ideal.

References

- [1] Blaker Warren J. *Geometric Optics: The Matrix Theory*. Marcel Dekker, Inc, New York, 1971.
- [2] Brouwer Willem *Matrix Methods in Optical Instrument Design*. W. A. Benjamin, Inc, New York, 1964.
- [3] Meyer-Arendt Jurgen R. *Introduction to Classical and Modern Optics* Fourth Edition. Prentice Hall, New Jersey, 1995.
- [4] Tenquist D. W. and Whittle R. M. and Yarwood J. *University Optics*. Ilife Books Ltd, London, 1969.
- [5] Welford W. J. *Aberrations of Optical Systems*. Adam Hilger Ltd, Bristol, 1986.