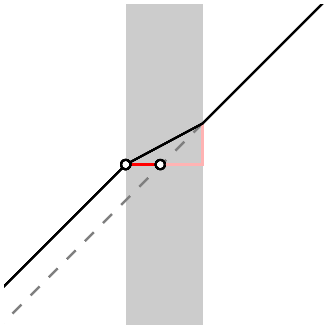
Mathematics 309 — Spring 2004 — Fifth homework solutions

1. *When you look at an object through a piece of flat glass, where does it* **seem** *to be (in terms of the thickness of the glass and its index of refraction n)?*

If a light ray enters a plate of glass and exist the other side, the entering and exiting rays are parallel.



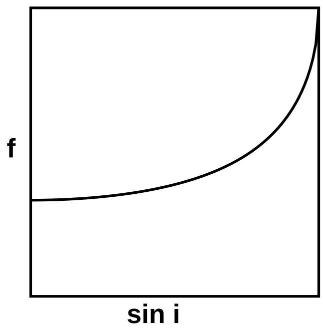
The effect of the glass is to shift the ray forward a bit, the segment in the figure. How much exactly? A bit of geometry shows the distance to be

$$d - d\left(\frac{\tan r}{\tan i}\right)$$

which can also be written as df where

$$\begin{aligned} f &= 1 - \frac{\tan r}{\tan i} \\ &= 1 - \frac{\sin r}{\sin i}\sqrt{\frac{1-\sin^2 i}{1-\sin^2 r}} \\ &= 1 - \frac{1}{n}\sqrt{\frac{1-\sin^2 i}{1-\sin^2 i/n^2}} \\ &= 1 - \frac{1}{n}\sqrt{\frac{1-s^2}{1-s^2/n^2}} \quad (s = \sin i) \end{aligned}$$

The fraction f varies between $1 - 1/n$ and 1 as i goes from 0 to $90°$ and remains close to $1/n$ for much of the range, as shown by this graph:



Even without plotting the graph explicitly you can estimate what its shape is, since for small s

$$\frac{1-s^2}{1-s^2/n^2} \sim (1-s^2)(1+s^2/n^2) = 1 - s^2(1-1/n^2) + s^4/n^2 \sim 1 - s^2(1-1/n^2)$$

and

$$\sqrt{\frac{1-s^2}{1-s^2/n^2}} \sim \sqrt{1-s^2(1-1/n^2)} \sim 1 - s^2(1-1/n^2)/2 \ ,$$

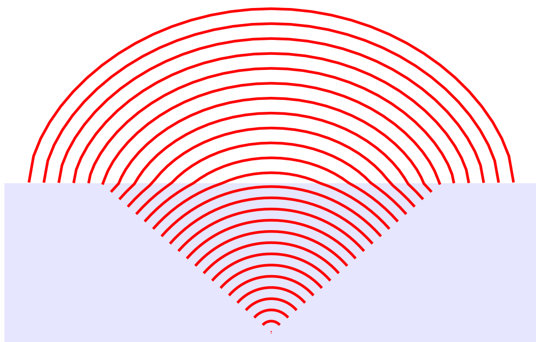
which means the near s its graph is essentially the parabola

$$f = (1-1/n) + s^2(1/n)(1-1/n^2)$$

at its minimum, which guarantees its flatness. This means that the magnitude of the shift perceived is effectively constant over that range. So the answer in this range is pretty simple—the object is shifted forward a distance $d - d/n$, which means roughly that the **effective thickness** of the plate glass is d/n.

2. *Place a red object at a depth of one meter under water. Draw on a single page the wave front of light rays at an optical distance of 1 m, 1.5 m, 2 m., both exact and according to the linear theory. Place the object at the bottom of a page, 5 cm = 1 m. Find the exact equation of the wave front surface outside the water at optical distance* d.

The following figure shows the wave fronts at optical distance $0, 0.1, \ldots, 2.5$.



There is one uniform way to attack all problems of this kind. including the next two as well as this one: parametrizing a light ray by optical length σ from the starting point. This parametrization will in all the cases we are looking at give a formula that depends on cases. The ray breaks up into one or more straight line segments, on each of which the formula is simple.

The most difficult part of the problem is one we have already dealt with, to figure out exactly how the ray breaks up, and what its direction is on each segment.

Here, the first problem is to figure out the optical distance to the point where the ray hits the water surface. A ray starting at $P_0 = (-1, 0)$ is determined by its direction, which is also determined by the angle of refraction. If the refracted angle is r, the horizontal coordinate of this point is $x = \tan(r)$, and the true length of the path to that point is $s_1 = \sqrt{1+x^2}$. That means optical length $\sigma_1 = sn$. Let

$$P_1 = (x, 0) \ .$$

If we define the initial **reduced velocity**

$$v_0 = \frac{[\sin r, \cos r]}{n}$$

then up to optical distance σ_1 the ray has direction v_1 and the point at optical distance σ is

$$P_0 + \sigma\, v_0 \ .$$

Outside the pool the velocity vector is

$$v_1 = [\sin i, \cos i] \; .$$

Here i is the incidence angle, $i = \operatorname{asin}(n \sin r)$. The paths that exit the pool are those with $|r| \le r_{\max}$, where $r_{\max} = \operatorname{asin}(1/n)$. The others reflect. The position optical distance $\sigma \ge \sigma_1$ is therefore

$$P_1 + (\sigma - \sigma_1) v_1 \; .$$

In summary:

$$P(\sigma) = \begin{cases} P_0 + \sigma\, v_0 & 0 \le \sigma < \sigma_1 \\ P_1 + (\sigma - \sigma_1) v_1 & \sigma_1 \le \sigma \end{cases}$$

Once we know how to calculate $P(\sigma)$ for each ray, we just join together a number of them, all with the same value of σ, to sketch the wave-front at distance σ.

In summary: the basic technique to use in plotting rays and wave fronts both is to start with P_0, v_0, and then the calculate σ_1, P_1, v_1, etc. Here the P_i are the breakpoints of the ray, the v_i are the reduced velocities, and σ_i is the optical distance from P_0 to P_i.

3. *An object is placed at* $x = -7$ *in front of a hemispherical lens of radius* 3 *whose centre is* $(0, 0)$*. Draw the exact and linear wave fronts at optical distance* 4*,* 5.5*,* 7*,* 8.5 *from the object by plotting points on the* 11 *rays at angles* $\pm i/100$ *radians for* $0 \le i \le 10$.

We use the same technique here. The first step is to figure out the points, optical lengths, and reduced velocities P_0, v_0, σ_1, P_1, etc.

The ray starts at point $P_0 = (-7, 0)$. with a velocity $v_0 = [\cos\theta_0, \sin\theta_0]$, where θ_0 here varies over a small range.

The distance to the first hit is the usual formula

$$s_1 = \sigma_1 = -v_0 \bullet \Delta P - \sqrt{(v_0 \bullet \Delta P)^2 - \|\Delta P\|^2 + R^2}$$

where $\Delta P = [-7, 0]$, $R = 3$. Then

$$P_1 = P_0 + s_1 v_0 \; .$$

Let this point be $P_1 = (-R\cos\alpha, R\sin\alpha)$.

At P_1 the ray bends down by $i - r$, where (as explained earlier in the course)

$$i = \theta_0 + \alpha, \quad r = \operatorname{asin}(\sin i/n) \; .$$

So we set

$$\theta_1 = \theta_0 - (i - r), \quad v_1 = [\cos\theta_1, \sin\theta_1]/n$$

(reduced velocity).

Finally, let σ_2 be the (total) optical distance to the far surface of the lens, hitting at P_2, and θ_2 be the new direction exiting. We find σ_2 by solving

$$P_1 + (\sigma_2 - \sigma_1) v_1 = (0, y), \quad \sigma_2 = \sigma_1 - x_1/v_{1,x} \; .$$

This gives

$$P_2 = (0, y_1 + (\sigma_2 - \sigma_1) v_{1,y}) \; .$$

The ray again refracts, changing θ to i, so we set

$$r = \theta_1, \; \theta_2 = i = \operatorname{asin}(n \sin r)$$

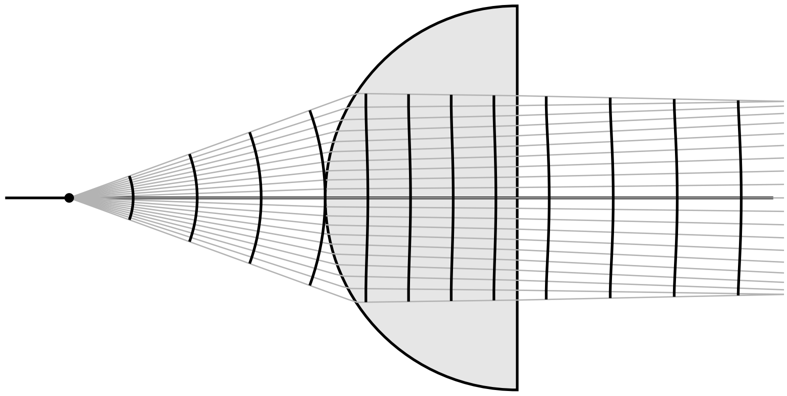
and

$$v_2 = [\cos\theta_2, \sin\theta_2] \; .$$

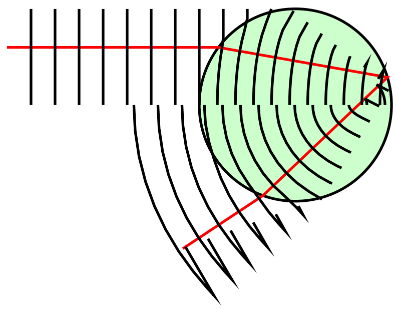
We have

$$P(\sigma) = \begin{cases} P_0 + \sigma v_0 & \text{if } \sigma \le \sigma_1 \\ P_1 + (\sigma - \sigma_1)v_1 & \text{if } \sigma_1 < \sigma \le \sigma_2 \\ P_2 + (\sigma - \sigma_2)v_2 & \text{otherwise} \end{cases}$$

In drawing, I did this for the range $\pm 20^\circ$ instead.



4. *Horizontal light rays enter a drop of water of radius* 1*. Draw the wave front at an optical distance of* 0.4 *from the point furthest left on the drop. Do this by locating points on* 10 *rays at heights* $y = 0, 0.1, \ldots 0.9$*.*



Here for the ray starting at height y

$$\begin{aligned} x_0 &= -1 \\ y_0 &= y \\ P_0 &= (x_0, y_0) \\ v_0 &= [1, 0] \end{aligned}$$

$$\begin{aligned} x_1 &= -\sqrt{1 - y^2} \\ y_1 &= y \\ \sigma_1 &= x_1 - x_0 \\ P_1 &= (x_1, y_1) \\ i &= \operatorname{asin}(y) \\ r &= \operatorname{asin}(y/n) \\ \alpha_1 &= \pi - i \\ \theta_1 &= -(i - r) \\ v_1 &= [\cos\theta_1, \sin\theta_1]/n \end{aligned}$$

$$
\begin{aligned}
\alpha_2 &= \alpha_1 - (\pi - 2r) \\
x_2 &= \cos(\alpha_2) \\
y_2 &= \sin(\alpha_2) \\
\sigma_2 &= \sigma_1 + 2n\cos(r) \\
P_2 &= (x_2, y_2) \\
\theta_2 &= \theta_1 - (\pi - 2r) \\
v_2 &= [\cos\theta_2, \sin\theta_2]/n
\end{aligned}
$$

$$
\begin{aligned}
\alpha_3 &= \alpha_2 - (\pi - 2r) \\
x_3 &= \cos(\alpha_3) \\
y_3 &= \sin(\alpha_3) \\
\sigma_3 &= \sigma_2 + 2n\cos(r) \\
P_3 &= (x_3, y_3) \\
\theta_3 &= \theta_2 - (i - r) \\
v_3 &= [\cos\theta_3, \sin\theta_3]
\end{aligned}
$$

$$
P(\sigma) = \begin{cases} P_0 + \sigma v_0 & 0 \le \sigma < \sigma_1 \\ P_1 + \sigma v_1 & \sigma_1 \le \sigma < \sigma_2 \\ P_2 + \sigma v_2 & \sigma_2 \le \sigma < \sigma_3 \\ P_3 + \sigma v_3 & \sigma_3 \le \sigma \end{cases}
$$