Commensurability

We now know quite a bit about fractions, but they are not really what I am most interested in. I am interested in the myriad of numbers out there that are *not* frcations. The simplest ones are square roots and their relatives, and we'll look at those first.

1. The square root of 2 is not a fraction

If $\sqrt{2}$ were a fraction, we could write it as p/q where the gcd of p and q is 1. If

$$\sqrt{2} = \frac{p}{q}$$

then

$$2 = \frac{p^2}{q^2}, \quad 2q^2 = p^2 \ .$$

Thus 2 divides p^2. By a result in the notes on divisibility, 2 must divide p, we can write $p = 2p_\bullet$, and then get

$$2q^2 = (2p_\bullet)^2 = 4p_\bullet^2, \quad q^2 = 2p_\bullet^2 \ .$$

But we can repeat the argument: 2 must divide q. But this contradicts the initial assumption that p and q are relatively prime.

A similar argument will work for other square roots other than those which are actually integers. The best result that follows from a similar argument is this:

Theorem. *If $r = p/q$ is a rational root of the polynomial equation*

$$A(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where the a_i are integers, then p divides a_0 and q divides a_n.

This implies immediately that the k-th root of N is never a fraction unless N is a perfect k-th power.

Proof. We need first

Lemma. *If r is relatively prime to s then it is relatively prime to r^n.*

Left as exercise.

If p/q is a root of $A(x)$ then

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$$

which can be rewritten

$$a_n p^n = -(a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n) \ .$$

Since q divides the right, it divides ther left. Since it is relatively prime to p^n, it must divide a_n. Similarly, r must divide a_0.

2. The geometric Euclidean algorithm

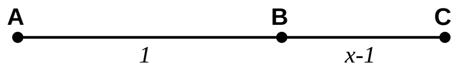
Two line segments are said to be **commensurable** if they are both integral multiples of some common (smaller) segment. For example, if one segment is of unit length and the other is of length $3/2$ then they are both multiples of a segment of length $1/2$, so they are commensurable. In general, two segments are commensurable precisely when their ratio is a fraction, since if one is m times a segment and the other is n times the same segment, then the ratio is m/n.

The Euclidaen algorithm as spelled out in class was applied only to integers, but the same process will produce a common measure of any two given commensurable segments. Say the segments are a and b units long. If d is the common measure of both, then it will be the coomon measure of $a - qb$ if q is an integer. So we find q such that this has length less than b, which is always possible; swap a and b, and continue on until one segment fits into the other an even number of times.

But the converse is also true: two segments are *not* commensurable precisely when their ratio is not a fraction, or in other words when the geometric Euclidean algorithm dosn't stop.

Let's look at a famous example of this.

Suppose a single line segment AC is partioned into smaller segments AB and BC with this property: The ratio of AB to AC is the same as the ratio of BC to AB.



Choose units of length so that x is the length of the whole segment and 1 is that of the larger half. The length of the smaller half is $x - 1$.

Let x be the length of the whole segment, and scale We can see immediately that $1 < x < 2$.

By definition we have an equation

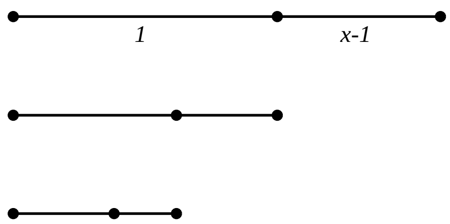
$$\frac{x}{1} = \frac{1}{x-1}$$

which leads to

$$x^2 - x - 1 = 0, \quad x = \frac{1+\sqrt{5}}{2} = 1.61803398\ldots\ .$$

This number is called the **golden ratio**.

Let's apply the Euclidean algorithm to the segments 1 and x. Since $1 < x < 2$, we have the first quotient $q_0 = 1$. The remainder is $r = x - 1$. So now we are looking at the two segments 1 and $x - 1$. But by definition the ratio $x - 1 :: 1$ is the same as $1 :: x$. In other words, in performing one step of the Euclidean algorithm we are just scaling everything by $1/x$. The second quotient q_1 is again 1.



As is the third, fourth, etc. The process never stops, and we see that *the golden ratio is not a rational number.*

In general, if y is any number larger thna 1, can apply we apply the Euclidean algorithm to the intervals of length 1 and x to test whether x is a rational number or not. It will be rational if and only if the process stops. Suppose for convenience that $x > 1$. The first quotient is the largest integer less than or equal to x, the **floor** $\lfloor x \rfloor$ of x.

We get $q_0 = \lfloor x \rfloor$, $r = x - q$ with $0 \le r < 1$. Then we apply the same division to 1 and r, dividing 1 by r and setting $q_1 = \lfloor 1/r \rfloor$. In effect we are setting a new value of x to be $1/r$. So we can describe the process in brief like this to find the succession of quotients:

(1) Start with $x > 1$.

(2) Set $q = \lfloor x \rfloor$, $r = x - q$.

(3) If $r > 0$, set the new value of x to be $1/r$. Loop again to (2). Otherwise stop.

Let's try another example, $x = \sqrt{2}$. Here $1 < x < 2$ since $1 < 2 < 4$, so $q_0 = 1$, $r_0 = \sqrt{2} - 1$. Next

$$x := \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}+1}\frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \ .$$

The $q_1 = 2$, $r_1 = (\sqrt{2}+1) - 2 = \sqrt{2} - 1$ *again*. So we are looping, and the succession of quotients here is 1, 2, 2, ...

These examples are typical:

Theorem. *If N is not a perfect square and $x = \big(a + b\sqrt{N}\big)/c$ with integers a, b, and c then the succession of quotients is always eventually periodic and non-vanishing. Conversely, if the succession of quotients is periodic then x is of this form.*

Let's looka t just one example of how to go backwards here. What number x gives rise to the succession 1, 2, 1, 2, ...? We have

$$\begin{aligned} x &= 1 + r_0 \\ \frac{1}{r_0} &= 2 + r_1 \\ \frac{1}{r_1} &= 1 + \ldots \\ &= x \ . \end{aligned}$$

Therefore

$$\begin{aligned} r_1 &= \frac{1}{x} \\ 2 + \frac{1}{x} &= r_0 \\ &= \frac{1}{x-1} \ . \end{aligned}$$

which leads to the quadratic equation

$$x = 2x(x-1) + (x-1), \quad 2x^2 - 2x - 1 = 0 \ .$$