

Euclid's theory of ratios

The most important thing to know about Euclid's theory of ratios is that in some sense there is none. That is to say, Euclid never anywhere says exactly what a ratio is. The reason, roughly, is that the way in which the Greeks of his time dealt with real numbers was very primitive—far more primitive, apparently, than how the Babylonians of a thousand and some years earlier dealt with them. In particular, they didn't have anything like decimal notation, neither in base 10 nor in any other base, and their notions of limits and convergence were weak. Even their notion of common fractions seems not to have been very well developed.

Because of this lack, statements in Euclid about ratios have to be worded so as to avoid the notion of a real number other than integers. For example, we would say that the ratio of the diagonal to the side of a square was $\sqrt{2}$. For the Greeks, the way to say this was that the square on the diagonal has area equal to twice the areas of the square on a side. They did know something about this ratio, because they knew that the side and the diagonal of a square were not commensurable. We would say that $\sqrt{2}$ was not a rational number, but although the assertion of incommensurability was equivalent to this, they did not phrase it the way we would. One way to say that $\sqrt{2}$ is not a rational number is to say that there is no rational number whose square is 2, and Euclid also proves this (in Book X).

Nonetheless, geometry in Euclid does deal with ratios, although in a somewhat indirect manner. The basic tool in Euclid's treatment of ratios is the paradoxical fact that although he does not know what a ratio is, exactly, he does have an extremely useful if subtle criterion for telling when two ratios are the same. This definition and its direct consequences are the content of Book V, one of the two most difficult of all *The Elements*, and its geometric consequences are worked out mostly in Book VI. One curious feature of Book V is that it is not about geometry *per se* but instead about a general theory of magnitudes. Archimedes, for example, used it to talk about weights.

Let me give two examples of what Euclid's theory can do. Proposition VI.1 asserts (as a special case)

- *Suppose a and b to be two line segments, and suppose that we erect on each a rectangle of the same height h . Then the ratio of the areas A and B of these rectangles is that same as the ratio of the lengths of the segments a and b .*

and VI.4 asserts that

- *Suppose that two triangles hve the same angles. Then the ratios of corresponding sides are equal.*

In both of these propositions, Euclid gets by with what he knows how to deal with, the **equality of ratios** rather than saying exactly what the ratios are. They are typical.

Before I introduce Euclid's definition, let me discuss why it looks so complicated. Take an easy case—suppose that, in our terms, the ratio of b to a is 3 to 2. That means that $a = 2d$ and $b = 3d$ for some smaller segment d . Then we can show immediately that $A = 2D$ and $B = 3D$ where D is the rectangle of height h erected on D . So of course the theorem is true. In essence, when a and b are commensurable the theorem is straightforward, because in effect we can show that the ratio of a to b and also of A to B is the same as a ratio of positive integers. But if a and b are not commensurable, it is difficult to see how to proceed. Euclid's proof depends on the following idea, as we would phrase it: *a real number x is completely determined by the two sets (a) L_x , the rational numbers less than x and (b) R_x , the rational numbers greater than x .* In modern terms this is very close in fact to how we are accustomed to think of areal number as being defined by its decimal expansion, since the decimal expansion amounts to a sequence of finite decimal fractions snuggling up to the number from below.

Therefore, if x and y are two real numbers, in order to test whether $x = y$ it suffices to check whether $L_x = L_y$ and $R_x = R_y$. If $x = a/b$ and $y = A/B$, then $m/n > a/b$ if and only if $mb > na$, and $m/n < a/b$ if and only if $mb < na$. This leads directly to Euclid's definition, modernized somewhat:

- Magnitudes a and b , A and B , are said to be in the same ratio when for any positive integers m and n the magnitudes (i) if $ma > nb$ then $mA > nB$; (ii) if $ma < nb$ then $mA < nB$.

We write $a:b = A:B$ in these circumstances.

The definition of equality of ratios in Book V is one of three basic tools that Euclid has to deal with what we would think of as real numbers. Another one is something called **Archimedes' axiom**, although it is used in Euclid. First of all, in Book V Euclid calls two magnitudes a and b comparable if there exists $n > 0$ such that $na > b$, and also some n such that $nb > a$. Later on (in Book X) he assumes implicitly that two different line segments are comparable, and then uses this axiom to say something about what happens when an interval is halved repeatedly.

A third tool in Euclid's kit is an argument called the **method of exhaustion**. We'll see some examples later on.

Let me finish here with proving Proposition VI.1 in the case of rectangles. The point here is very simple: to every segment a corresponds a unique rectangle, that I'll write as $h \times a$, with that segment as base, and a fixed height h . This construction satisfies two simple rules: (1) If $a > b$ then $h \times a > h \times b$, and (2) $h \times ma = m(h \times a)$. These imply immediately that the criterion of Definition V.5 is satisfied. It must be shown that if $ma > nb$ then $m(h \times a) > n(h \times b)$, and similarly when $ma < nb$. But if $ma > nb$ then $h \times ma > h \times nb$ by (1), and then $m(h \times a) > n(h \times b)$ by (2).