

Mathematics 446 — third assignment — solutions

Exercise 1. Locate through the UBC library and JSTOR the article ‘Of the theory of circulating decimal fractions’ by John Robertson in the 1768 volume of *Philosophical Transactions*. Summarize what it says in 300 words or less, including some calculation examples.

The fundamental mathematical point of Robertson’s article is that

$$\frac{1}{9999\dots 9} = 0.000\dots 1\ 000\dots 1\ 000\dots$$

where the denominator on the left has k digits, as does the repeating part $000\dots 1$ on the right. For example

$$\frac{1}{999} = 0.001\ 001\ 001\ 001\dots = 0.\dot{0}0\dot{1}.$$

It is an easy thing to prove, since

$$9999\dots 9 \cdot 0.000\dots 1\ 000\dots 1, 000\dots = 0.99999999\dots,$$

which because of the peculiarities of the decimal system is equal to 1.

From this it follows that if D has k digits (maybe padded at the left with zeroes) then

$$\frac{D}{9999\dots 9} = 0.DDDDD\dots,$$

which is really the main observation of the article. For example

$$\frac{231}{999} = 0.231231231\dots = 0.\dot{2}3\dot{1}.$$

It can be used backwards, too, to tell us how to write any repeating decimal as a fraction. Thus

$$23.34567676767\dots = 23.345 + 0.00067676767\dots = 23.345 + \frac{0.676767\dots}{1000} = 23.345 + \frac{1}{1000} \frac{67}{99}.$$

It is important to realize that Robertson says nothing about why an arbitrary fraction p/q can be expressed as a repeating decimal fraction, nor about the nature of the repetition. I am not sure of the history of this fact, nor when it was first proven. It is not an entirely trivial question since it relates to the equation $n^p = n$ modulo a prime p , which was first observed by Fermat, and also more complicated congruences modulo composite numbers. The earliest complete treatment I am aware of is in *Disquisitiones Arithmeticae* by C. F. Gauss, where it is one of the most elementary topics dealt with.

Exercise 2. Write down and prove by mathematical induction the formula for the finite geometric sum

$$1 + q + q^2 + \dots + q^n.$$

The formula is

$$1 + q + q^2 + \dots + q^n = \frac{q^{n+1} - 1}{q - 1}.$$

For $n = 0$ it is obvious. Assume true for n . Then

$$1 + q + \dots + q^{n+1} = \frac{q^{n+1} - 1}{q - 1} + q^{n+1} = \frac{q^{n+2} - q^{n+1} + q^{n+1} - 1}{q - 1} = \frac{q^{n+2} - 1}{q - 1}.$$

Exercise 3. Prove by mathematical induction the binomial theorem

$$(x+1)^n = x^n + nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} + \cdots + nx + 1$$

where n is an arbitrary positive integer and the coefficient of x^r is

$$c_{n,r} = \frac{n(n-1)\cdots(n-(r-1))}{1\cdot 2\cdot 3\cdots r}$$

Clear for $n=0$. Assume true for n . Then

$$(1+x)^{n+1} = (1+x)^n(1+x) = \left(\sum_{r=0}^n c_{n,r}x^r\right)(1+x)$$

The new coefficient of x^{n+1} is 1, that of x^0 is 1, and for $1 \leq r \leq n$ it is

$$C_{n,r} + C_{n,r-1}$$

It must be shown that

$$C_{n,r} + C_{n,r-1} = C_{n+1,r}.$$

I leave this as an exercise.

Exercise 4. Find a such that

$$a \equiv 14 \text{ modulo } 71$$

$$a \equiv 17 \text{ modulo } 91$$

We must solve

$$14 + A \cdot 71 = 17 \text{ modulo } 91$$

or

$$A \cdot 71 = 3 \text{ modulo } 91.$$

We now must find X such that $XA = 1$ modulo 91. We have

$$91 = 1 \cdot 71 + 20$$

$$71 = 3 \cdot 20 + 11$$

$$20 = 1 \cdot 11 + 9$$

$$11 = 1 \cdot 9 + 2$$

$$9 = 4 \cdot 2 + 1$$

$$2 = 2 \cdot 1$$

If we use the matrix method, we have to calculate

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 32 & -41 \\ -71 & 91 \end{bmatrix}$$

and read k and ℓ from the top row to be 32, -41. More explicitly, the matrices we get in succession are

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}, \begin{bmatrix} 4 & -5 \\ -7 & 9 \end{bmatrix}, \begin{bmatrix} -7 & 9 \\ 32 & -41 \end{bmatrix}, \begin{bmatrix} 32 & -41 \\ -71 & 91 \end{bmatrix}$$

If we use the substitution method, we go backwards.

So to solve for A we set $A = (-41)(3) = -123 = 59$ modulo 91. We have $a = 14 + 59 \cdot 71 = 4203$. It checks.

Exercise 5. Write down a **complete** proof that $\sqrt{3}$ is not a fraction.

Suppose $3 = p^2/q^2$ with p, q relatively prime, or $3q^2 = p^2$. Then 3 divides p^2 , so 3 divides p , $p^2 = 9r^2$. Then $3q^2 = 9r^2$, $q^2 = 3r^2$, so $3|q$. Contradiction.

Exercise 6. Write an essay of 100 words explaining why the Chinese Remainder Theorem is called what it is.

From <http://www.andrews.edu/~calkins/math/biograph/199899/topcrt.htm>:

“We have a number of things, but we do not know exactly how many. If we count them by threes we have two left over. If we count them by fives we have three left over. If we count them by sevens we have two left over. How many things are there?”

This is the first known work involving the Chinese Remainder Theorem (CRT). It is from the book “Sun Tzu Suan Ching” (“Master Sun’s Mathematical Manual”) written by Sun Zi (also called Master Sun). Sun Zi was probably a Buddhist scholar or monk. This is the only problem in the book relating to the CRT, so we don’t know if he made a general method to solve these types of problems. There is some dispute about when problem 26 (up above) was written. Some experts say “Sun Tzu Suan Ching” was written in the late 3rd century BC, others argue it was created in the 4th century.

Exercise 7. Find all the powers of 60 modulo 37. Use this to find the repeating fraction for $1/37$ in base 60.

The powers of 60 modulo 13 are

$$1, 23, 11, 31, 10, 8, 36, 14, 26, 6, 27, 29, 1.$$

So $60^{12} = 1$ modulo 37, and the repeat length of 37 in base 60 is 12. What are the repeating digits? We could recover them from a calculation of

$$\frac{60^{12} - 1}{37}$$

but this is an awfully large number, too large for most calculators to handle exactly. We can use a trick, however. Write

$$\frac{60^{12} - 1}{37} = \frac{(60^6 - 1)(60^6 + 1)}{37}.$$

We know that $60^6 \neq 1$ modulo 37, so 37 must divide the second factor. In fact

$$(60^6 - 1)/37 = 1 : 37 : 17 : 50 : 16 : 13$$

But if we multiply this by $60^6 - 1$ we get

$$1 : 37 : 17 : 50 : 16 : 13 : 0 : 0 : 0 : 0 : 0 : 0 - 1 : 37 : 17 : 50 : 16 : 13$$

which we can calculate by hand. The final answer is

$$0.1 : 37 : 17 : 50 : 16 : 12 : 58 : 22 : 42 : 9 : 43 : 47 : \dots$$

Exercise 8. What is the length of the repeating fraction $1/91$ in base 60?

$91 = 7 \cdot 13$. We have $60^3 = 1$ modulo 7, and $60^4 = 1$ modulo 13. The repeat for 91 is therefore 12.

Exercise 9. Read Book VII. Proposition 1 of Euclid’s Elements (on line at Joyce’s site). Restate the Proposition and rewrite the proof in your own words, using modern algebraic notation.

Exercise 10. Find the continued fraction expansion of (a) $\sqrt{5}$; (b) $\sqrt{3}$; (c) $\sqrt{19}$.

(a)

$$\begin{array}{ccc} x & q & r \\ \sqrt{5} & 2 & \sqrt{5} - 2 \\ \sqrt{5} + 2 & 4 & \sqrt{5} - 2 \end{array}$$

So 2, 4, 4

$$\begin{array}{ccc} \sqrt{3} & 1 & \sqrt{3} - 1 \\ \frac{\sqrt{3} + 1}{2} & 1 & \frac{\sqrt{3} - 1}{2} \\ \sqrt{3} + 1 & 2 & \sqrt{3} - 1 \end{array}$$

So 1, 1, 2, 1, 2, . . .

Last: 4, 2, 1, 3, 1, 3, 8, 2, 1, . . .