Mathematics 446 — fifth assignment solutions

Exercise 1.

Exercise 2.

Exercise 3. *Suppose given a series of numbers X_n such that $X_n = cX_{n-1} + dX_{n-2}$. Show that there exist numbers a and b such that $X_n = a\alpha^n + b\beta^n$ where α and β are the roots of*

$$x^2 = cx + d \, .$$

Explain how to determine a and b explicitly.

We have

$$\begin{aligned} X_0 &= a + b \\ X_1 &= a\alpha + b\beta \end{aligned}$$

which we can solve to get

$$a = \frac{X_1 - X_0\beta}{\alpha - \beta}, \quad b = \frac{X_1 - X_0\alpha}{\beta - \alpha} \, .$$

Now we want to prove that for all n

$$X_n = a\alpha^n + b\beta^n \, .$$

The proof goes by strong induction, with starting values $n = 0$ and $n = 1$. Then for $n > 2$

$$\begin{aligned} X_n &= cX_{n-1} + dX_{n-2} \\ &= c(a\alpha^{n-1} + b\beta^{n-1}) + d(a\alpha^{n-2} + b\beta^{n-2}) \\ &= a\alpha^{n-2}(c\alpha + d) + b\beta^{n-2}(c\beta + d) \\ &= a\alpha^{n-2}\alpha^2 + b\beta^{n-2}\beta^2 \\ &= a\alpha^n + b\beta^n \, . \end{aligned}$$

Exercise 4.

Exercise 5. *(a) Write down the explicit formula for the length of a regular polygon of $2n$ sides, given that for n sides. (b) Give an estimate as accurate as you can of how many subdivisions of the square you would have to make to calculate the area of a circle, hence π, correctly to 10 decimals. You will want to use the previous exercise on parabolas. Explain.*

See the notes on π.