Mathematics 446 — sixth assignment solutions

Exercise 1. *Use the binomial theorem to calculate $\sqrt[3]{999}$ correct to 17 decimals after the decimal point.*

$$(1-1/1000)^{1/3} = 1 - 1/3\cdot 10^{-3} + \frac{(1/3)(1/3-1)}{2!}10^{-6} - \frac{(1/3)(1/3-1)(1/3-2)}{3!}\cdot 10^{-9} + \frac{(1/3)(1/3-1)(1/3-2)(1/3-3)}{4!}\cdot 10^{-12} \pm\cdots$$

which is also

$$\begin{aligned}
&1 - 1/3\cdot 10^{-3} - \frac{(1/3)(1-1/3)}{2!}10^{-6} - \frac{(1/3)(1-1/3)(2-1/3)}{3!}\cdot 10^{-9} \\
&\quad - \frac{(1/3)(1-1/3)(2-1/3)(3-1/3)}{4!}\cdot 10^{-12} \pm\cdots \\
&= 1 - 1/3\cdot 10^{-3} - \frac{(1/3)(2/3)}{2!}10^{-6} - \frac{(1/3)(2/3)(5/3)}{3!}\cdot 10^{-9} - \frac{(1/3)(2/3)(5/3)(8/3)}{4!}\cdot 10^{-12} \\
&\quad - \frac{(1/3)(2/3)(5/3)(8/3)(11/3)}{5!}\cdot 10^{-15} - \frac{(1/3)(2/3)(5/3)(8/3)(11/3)(14/3)}{6!}\cdot 10^{-18} - \cdots \\
&= 1 - \frac{1}{3}\cdot 10^{-3} - \frac{1}{9}\cdot 10^{-6} - \frac{5}{81}\cdot 10^{-9} - \frac{10}{243}\cdot 10^{-12} - \frac{22}{729}\cdot 10^{-15} - \frac{154}{6561}\cdot 10^{-18} - \cdots
\end{aligned}$$

The sum of negative terms is the sum of the rows

0.0	0	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
0.0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
0.0	0	0	0	0	0	0	0	0	0	6	1	7	2	8	3	9	5	1
0.0	0	0	0	0	0	0	0	0	0	0	0	0	4	1	1	5	2	3
0.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	2
0.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

which you can add by hand. Note that nowhere do you need to do any fancy calculation with lots of digits, since as the fractions get more complicated the number of digits needed goes down.

Exercise 2. *Assuming you were using the series*

$$\int_0^x \frac{dt}{1+t^2} = \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} \pm\cdots$$

with $x = 1/\sqrt{3}$, how many terms would you have to use to get π to 20 decimals?

Because this is an alternating series, the error is bounded by the first term left out. Therefore we just have to find n such that $x^n/(2n+1) < 10^{-20}$. This happens first around $n = 38$.

Exercise 3. *Use the Taylor series for e to compute it to 17 decimals' accuracy.*

0	1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0.1	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
4	0.0	4	1	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
5	0.0	0	8	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
6	0.0	0	1	3	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	0.0	0	0	1	9	8	4	1	2	6	9	8	4	1	2	6	9	8	4
8	0.0	0	0	0	2	4	8	0	1	5	8	7	3	0	1	5	8	7	3
9	0.0	0	0	0	0	2	7	5	5	7	3	1	9	2	2	3	9	8	6
10	0.0	0	0	0	0	0	2	7	5	5	7	3	1	9	2	2	3	9	9
11	0.0	0	0	0	0	0	0	2	5	0	5	2	1	0	8	3	8	5	4
12	0.0	0	0	0	0	0	0	0	2	0	8	7	6	7	5	6	9	8	7
13	0.0	0	0	0	0	0	0	0	0	1	6	0	5	9	0	4	3	8	4
14	0.0	0	0	0	0	0	0	0	0	0	1	1	4	7	7	4	5	5	9
15	0.0	0	0	0	0	0	0	0	0	0	0	0	7	6	4	7	1	6	4
15	0.0	0	0	0	0	0	0	0	0	0	0	0	0	7	6	4	7	1	3
16	0.0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	7	7	9	5
17	0.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	8	1	1
18	0.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	5	6
19	0.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8
20	0.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Exercise 4. *Read Dedekind's essay. (a) The sqequence x_n converges to x if for any $\epsilon > 0$ there exists N such that $|x - x_n| < \epsilon$ if $n \geq N$. (b) Prove that if x_n is an increasing sequence of numbers that is bounded, then it converges to a number x. (c) Prove that if x_n is a sequence of terms which alternate in sign and decrease in value, then the series*

$$x_1 + x_2 + \cdots + x_n + \cdots$$

converges to a real number.

(b) Each x_n is a cut (A_n, B_n), by definition of a real number. Since the x_n are increasing, $A_n \subseteq A_{n+1}$. Define A to be the union of all the A_n, and let B be the complement of A, or the intersection of all the B_n. Because the sequence is bounded, the set B is not empty. It is easy to see that (A, B) is a cut: (a) If a is a rational number in A and b one in B, than a lies in some A_n, while b lies in B_n. Therefore $a < b$. Therefore B contains an infinity ofrational numbers. Hence (A, B) is a cut. It is possible that B has a least element, in which case we move it to A. Then (A, B) by definition defines a real number, call it x. Proving that the x_n converge to x according to the definition in (a) is easy. We shall show that given $\varepsilon > 0$ there exists N such that $x - x_n < \varepsilon$ for $n \geq N$. Well $x - \varepsilon < x$, and so by one of Dedekind's results there exists a rational r with $x - \varepsilon < r < x$. Then $r \in A$, hence lies in some A_N, then in all A_n with $n \geq N$ since then $A_N \subseteq A_n$. But then for $n \geq N$ we have $A_{x-\varepsilon} \subseteq A_N \subseteq A_n \subseteq A_x$, so $x - \varepsilon < x_n < x$.

(c) Suppose the terms start out positive. Then the sums

$$(x_1 - x_2),\ (x_1 - x_2) + (x_3 - x_4),\ \ldots$$

are an increasing sequence bounded by x_1. Apply (b).

Exercise 5. *(a) Prove that the sequence*

$$x_n = (1 + 1/n)^n$$

is increasing and bounded and therefore converges to some real number.

We have

$$\begin{aligned}(1+1/n)^n &= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\left(\frac{1}{n}\right)^2 + \cdots + \frac{n!}{n!}\cdot\left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!}\cdot(1-1/n) + \frac{1}{3!}\cdot(1-1/n)(1-2/n) + \cdots + (1/n)^n + \text{ a lot of zeroes}\end{aligned}$$

The k-th term is less than $1/k!$ so the entire series is less than that for e, which we know by comparison to a geometric series to be less than 3. Furthermore, as n increases each term increases, so the entire sum is as well.

Exercise 6. *(a) Define the product of two real numbers in Dedekind's sense. (b) Using Dedekind's definitions, prove that* $\sqrt{2}\sqrt{3} = \sqrt{6}$.

(a) There are several cases, according to sign. The technical difficulty is that negative times negative is positive. The only important case is that where both are positive. So suppose $x = (A_x, B_x)$, $y = (A_y, B_y)$ are both positive. Let A_x^+ be the positive numbers in A_x, A_x^+ those in A_y. Define A_{xy}^+ to be the set of rs with r in A_x^+, s in A_y^+, and let A_{xy} be the set of all numbers either in A_{xy}^+ or less than a number if A_{xy}^+. Let B_{xy} be the complement of A_{xy}, with a least element swapped over to A_{xy} if necessary.

(b) Dedekind defines $\sqrt{N}$ to be $A_{\sqrt{N}}^+$ together with all less than something in $A_{\sqrt{N}}^+$ if $A_{\sqrt{N}}^+$ is the set of all $r > 0$ with $r^2 < N$. To show $\sqrt{2}\sqrt{3} = \sqrt{6}$ amounts to showing that $A_{\sqrt{2}}^+ A_{\sqrt{3}}^+ = A_{\sqrt{6}}^+$.

If $r^2 < 2$ and $s^2 < 3$ then $(rs)^2 < 6$, so one inclusion is OK. Conversely, it must be shown that if $t^2 < 6$ then we can write $t = rs$ with $r^2 < 2$ and $s^2 < 3$. I leave this as an exercise.