Partial solutions to the second assignment

1. The Egyptians always expressed fractions as sums of unit fractions $1/N$*. This raises some mildly interesting mathematical questions.*

(a) Is it true that every fraction f *between* 0 *and* 1 *can be expressed as a sum of distinct unit fractions? If so, prove it. If not, give an example, and explain which can be expressed in this way (with poroofs).*

(b) Can f *have an infinite number of such expressions? An infinite number with a given number of terms?*

(c) Find all such expressions for $2/45$ *involving two terms; three terms.*

(d) Find all such expressions for $2/47$ *involving two or three terms. For* $2/53$*.*

The simplest case is to show that $2/q$ with q odd can be expressed as a sum of distinct unit fractions. This is easy:

$$\frac{2}{2n-1} = \frac{1}{n} + \frac{1}{n(2n-1)} .$$

But then any fraction can be reduced to thsi by a recursion process:

$$\frac{p}{q} = \frac{p-1}{q} + \frac{1}{q} \dots$$

so that any time a numerator greater tha 2 pops up in this process we can reduce it by 1. (**Warning**: This is not a complete proof, but only a suggestion for one.

(b) There can certainly be more than one expression like this, and an infinite number in total, but only a finite number of given degree.

Why is that? Try showing that there are only a finite number of expressions like this of two terms. Of three. Of four.

1. Find the base 60 *expressions for (a)* 180*, (b)* 456*, (c)* $5,000$*, and (d)* $314,678$*.*

$$3:0,\ 7:36,\ 1:23:20,\ 1:27:24:38 .$$

1. Write in detail a proof that if B *is an integer larger than* 1*, every positive integer* n *can be expressed uniquely as a sum*

$$n = n_0 + n_1 B + n_2 B^2 + \cdots + n_k B^k$$

with $0 \le n_i < B$*,* $n_k > 0$*. Write down an explicit algorithm for finding the* n_i*.*

Use the second form of induction on the number. Existence for 1 is clear. Next uniqueness for 1.

$$n = n_0 + n_1 B + n_2 B^2 + \cdots + n_k B^k$$

with $n_k > 0$ then $n > B^k$. If

$$1 = n_0 + n_1 B + n_2 B^2 + \cdots + n_k B^k$$

this implies $k = 0$, and then that $n_0 = 1$.

Existence in general. Assume true for $m < n$. If we divide n by B we get

$$n = qB + r, \quad 0 \le r < B$$

and then we can set $n_0 = r$, and apply induction to q to get an expression for n. Uniqueness is similar. The term n_0 has to be the remainder upon division by B, and the remaining expression is B times that for q. This gives the algorithm as well, by means of successive division by B.

1. Find the infinite sexagesimal expansion for $1/3$*,* $1/5$*,* $1/11$*,* $1/13$*.*

$$
\begin{aligned}
1/3 &= 0.20 : 0 : 0 : \ldots \\
1/5 &= 0.12 : 0 : 0 : \ldots \\
1/11 &= 0.5 : 27 : 16 : 21 : 49 : 5 : 27 : \ldots \\
1/13 &= 0.4 : 36 : 55 : 23 : 4 : 36 : \ldots
\end{aligned}
$$

1. Find the first 8 *'digits' of* $\sqrt{2}$ *in base* 60*.*

Postponed.

1. Read the selection by Newman. Tell me what the problem being solved on the two-page spread is, and what and where the solution is on those pages.

He is solving

$$(3 + 1/3 + 1/5)x = 1$$

which gives $x = 15/53$ or in Egyptian fashion

$$1/4 + 1/106 + 1/53 + 1/212\,.$$

These are ticked of at the lower left of the page, so you have a check on your guess.