Mathematics 446 — partial solutions to the third assignment

1. Redo the questions from this week about unit fractions, to the extent discussed below. First explain in detail why any fraction $2/N$ only has a at most finite number of expressions as the sum of two unit fractions. Explicitly, if

$$\frac{2}{N} = \frac{1}{a} + \frac{1}{b}$$

show lower and upper bounds on a and b. What we did in class wasn't quite accurate, but the same basic idea works: try $a = n = (N+1)/2, n+1, \ldots, N$, and then show that you need not look further. Do this by first swapping a and b if necessary to get $b > a$ and then show that $a < N$.

Next do the same for three terms by showing bounds for $a < b < c$ with

$$\frac{2}{N} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \,.$$

In class I was also little sloppy on this last part, but you can still find some simple bounds by being careful. Again $(N+1)/2 < a$ is easy to see. And you can also find an easy upper bound on a. This means that only a finite number of a are possible. But then for each a only a finite number of b and c are possible.

Suppose we want to list all sums

$$\frac{2}{N} = \frac{1}{a} + \frac{1}{b}$$

where N is odd. Then we must have $1/a < 2/N$ or $a > N/2$. If $N = 2n-1$ then the smallest possibility si $a = n$, and we have seen that this works, with $b = n(2n-1)$. As a increase, b decreases. But we must also have $b > N/2$, so there are a finite number of possibilities for sure.

In fact, as a increases and b decreases the two will pass each other by, and at that point we shall start to duplicate our list with a and b swapped.

Now look at three terms:

$$\frac{2}{N} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \,.$$

with $a < b < c$. Again $a \ge n$ if $N = 2n-1$. But if a gets large, b and c have to get large also, and the sum on the right has to get small, smaller than $2/N$. Explicitly

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{3}{a}$$

so if $3/a < 2/N$ or $a > 3N/2$ there is no solution.

2. Read the paper 'Ancient Babylonian algorithms' by Donald Knuth, in Communications of the Association for Computing Machinery volume **15***, issue 7 (1972), pages 671–677. Also the correction to it in the letter on page 108 of volume* **19***, issue 2 (1976). These are available elctronically through UBC elinks. The correction is in the letter section that starts on page 105 of the issue, and you might have to download the whole section.*

Translate line by line into algebra and English the procedure exhibited on page 672 and also the first, second, and fourth on page 673. Explain Knuth's remark at bottom right on page 673 that "the stated parameters 1 and 5 cannot possibly correspond . . . " Your primary goal is to make it clear to a modern reader what the Babylonian text is talking about. Do not deviate more than necessary from the Babylonian text.

p. 672:

We are given a volume of dimensions ℓ, w, h. We have $h = 3:20$ amd the volume is $27:46:40$. Also $\ell = w + 50$. We want to find ℓ and h. So we start out with

$$\ell w h = 27:46:40, \quad h = 3;20, \quad \ell = w + 50 \,.$$

We divide the volume by h, giving

$$\ell w = 8:20 \,.$$

So we now want to solve

$$\ell - w = 50 \quad \ell w = 8:20\,.$$

Here we fix the decimal point so $50 = 50/60 =$ say a and $8:20 = 8 + 1/3 = b$. So this becomes in algebraic terms

$$\ell - w = a, \ell w = \text{ say } b$$

Using algebra, we are setting $w = b/\ell$ and solving

$$\ell - b/\ell = a, \quad \ell^2 - a\ell - b = 0\,.$$

The tablet tells us, take half of a and square it, getting $(a/2)^2$. Add b, making $(a/2)^2 + b$. Take the square root of that, making $\sqrt{(a/2)^2 + b}$. Add $a/2$ to this, also subtract it. This gives both ℓ and w.

3. Transliterate the numbers in rows 5–10 (not counting the head) of the tablet **thureau-dangin.pdf** *available on the course web site—first into sexagesimal form, then into decimal. The article by Knuth lists some of the first few rows from that tablet, to help you see what's going on.*

Knuth gives you the first four:

1						1							
1	0	16	53	53	20	59	43	10	50	52	48		
1	0	40	53	20		59	19	34	13	7	30		
1	0	45				59	15	33	20				
1	1	2	6	33	45	58	58	56	33	45			
1	1	26	24			58	35	37	30				
1	1	30	33	45		58	31	33	35	18	31	64	

etc. Ambiguous lines like the sixth can be decided by checking inverses.

4. Do the problem from the last assignment about deciphering the two-page spread from the paper by James R. Newman, if you have not already done it. Keeo in mind that although it is not easy to figure out what the problem is, the solution is exhibited somewhere on the page, so you have a good check on things. It ought to be an interesting exercise to figure out what almost every part of the spread means. Whatever your solution, justify it in your own words.