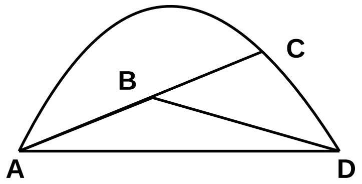
Mathematics 446 — partial solutions to the fifth assignment

1. Finish the proof that if $A < K$ then for large n the circumscribed polygons of 2^n sides are less than K. Your goal here is just to make Archimedes' argument more explicit. Draw some nice pictures.

I'll use Euclid's proof of his XII.2 as the basic example of how the method of exhaustion works.

2. Finish the proof of Archimedes claim about two concave polygonal paths from P to Q. Be careful about what you mean by 'concave' and its consequences. I'll give you this hint: use induction on the number of segments in the bottom curve.

The case of 2 lower segm,enst should explain all.



We have

$$
\begin{aligned}
AB + BC &< \mathrm{arc}\, AC \\
BD &< BC + \mathrm{arc}\, CD \\
AB + BC + BD &< \mathrm{arc}\, AC + BC + \mathrm{arc}\, CD \\
AB + BD &< \mathrm{arc}\, AC + \mathrm{arc}\, CD
\end{aligned}
$$

3. Translate the table from Ptolemy. The first column is made up of angles, the seconmd is proportional to the **chord** *of the angle. What is the constant of proportion, and what is the third column?*

The first column is angles at a spacing of half a degree, the next is 60 times the chord, and the third is proportional to the differences in column two.

4. Read the proof of Euclid XII.2 and write it your own words, being careful about the use of XII.1.

Proposition.The ratio of the area of two circles is the square of the ratio of their diameters.

The starting point of the proof is the XII.1, which says that the analogous fact is true for similar convex polygons, and this in turn is reduced to the case of triangles by decomposing a convex polygon into triangles. From now on we'll just assume that without proof.

Now consider two circles C_d and C_D of diameters d and D, areas A_d and A_D.

We want to shwo the ratio A_d/A_D is the same as the ration d^2/D^2. If it is not, the either $A_d/A_D < d^2/D^2$ or $A_d/A_D > d^2/D^2$. WE shall show that each of these leads to a contradiction.

Suppose $A_d/A_D < d^2/D^2$. Then

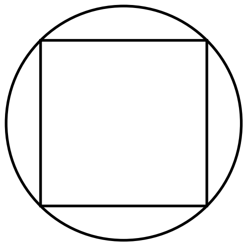
$$A_d < (d^2/D^2)A_D$$

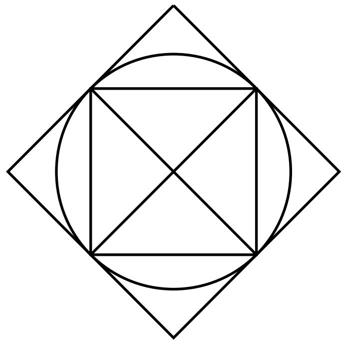
or

$$A_d = (d^2/D^2)S$$

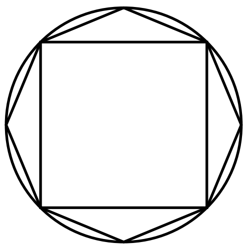
with $S < A_D$.

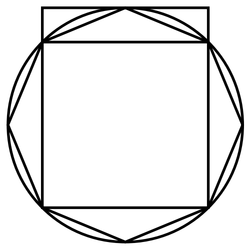
Inscribe a square in the circle C_D. Its area is greater than half A_D.





Subdivide the inscribed square to make an octagon, and keep on subdividing that in turn. In each subdvision the excess of the area of the circle over that of the inscribed polygon is reduced by more than one half.





Therefore sooner or later this excess will be less than the excess of A_D over S, and then the area of the inscribed polygon will have area $> S$.

At this point we have

$$\begin{aligned}
A_d &= \frac{d^2}{D^2}\, S \\
S &< I_D < A_D \\
\frac{A_d}{S} &= \frac{d^2}{D^2} \\
I_d = \frac{d^2}{D^2}\, I_D & \\
\frac{I_d}{I_D} &= \frac{d^2}{D^2} \\
&= \frac{A_d}{S} \\
\frac{I_d}{A_d} &= \frac{I_D}{S} \\
\frac{I_d}{A_d} &< 1 \\
\frac{I_D}{S} &< 1 \\
I_D &< S
\end{aligned}$$

a contradiction. So it is not true that $A_d/A_D < d^2/D^2$.

Suppose that $A_d/A_D > d^2/D^2$. Then $A_D/A_D < D^2/d^2$. But the case already dealt with shows that this can't be true.

Euclid's proof is rather subtle in the way he avoids some nasty traps in logic. Particularly nice is the wy he reduces the second half of this argument to the first half by appealing to a basic fact about ratios. Another and more direct proof of the second poart could have been done the way Archimedes does, using circumscribed polygons.

5. I didn't talk enough about it in class, but this week I will. Do the question from last time:

How many sides of the hexagon would you need to get the difference between inner and outer perimeters to be less than 10^{-8}*?* 10^{-16}*?*

Done sufficiently (but not completely) in the solutions for the last assignment.