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Essays on representations of p-adic groups

The Bruhat filtration

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If P and Q are parabolic subgroups of G then $P\backslash G/Q$ is finite. In this chapter I will

- construct a filtration of the restriction to Q of a representation of G induced from P;
- describe the associated graded representation;
- describe also the graded Jacquet module.

I begin with some abstract considerations.

Contents

1. Smoothly induced representations

In this section suppose P and G to be any locally profinite groups, and p the canonical projection from G onto the quotient $P\backslash G$. Let X be a locally closed P-stable subspace of G of the form $p^{-1}(Y)$, where Y is a locally closed subset of $P\backslash G$. There exist global continuous sections of p, and hence of the restriction of p to X.

Let (σ, U) be a smooth representation of P. Define $|_c(\sigma\,|\,P, X)$ to be the space of smooth functions $f\colon X \to U$ with compact support modulo P such that

$$f(px) = \sigma(p)f(x)$$

for all x in X, p in P. The condition of compact support means that for every f in the space there exists a compact subset Ω of X with the support of f contained in $P\Omega$.

For any $f \in C_c^\infty(G, U)$ and x in X, the function $f(px)$ lies in $C_c^\infty(P, U)$. Therefore we can integrate to define a new function on X:

$$\Pi_\sigma f(g) = \int_P \sigma^{-1}(p) f(pg)\, d_r p$$

[projections] **Proposition 1.1.** *The map Π_σ takes $C_c(X, U)$ to $|_c(\sigma\,|\,P, X)$ and is surjective.*

Proof. If f has support in Ω then $\pi_\sigma f$ has support in $P\Omega$, so the support of $\Pi_\sigma f$ is certainly compact modulo P.

Now suppose $f\, C_c^\infty(X, U)$. It will have support on some compact subset Ω of X, which we may as well assume to be of the form $K \times S$, where K is a compact open subgroup of G and S is the section of an open set in the quotient $Y = P\backslash X$. In showing that $\Pi_\sigma f$ lies in $\mathrm{Ind}_c(\sigma\,|\,P, G)$ we may assume that f is constant on Ω, say $f(x) = u$ for x in Ω. But then if $F(x) = \Pi_\sigma f(x)$ we have for $x = p_0 s_0$ in Ω with p_0 in K, s_0 in S:

$$\begin{aligned} F(x) &= \int_P \sigma^{-1}(p) f(px)\, dx \\ &= \int_{p\,|\,px\in\Omega} \sigma^{-1}(p) u\, dx \\ &= \int_{p\,|\,pp_0s_0\in KS} \sigma^{-1}(p) u\, dx \\ &= \int_{p\,|\,p\in K} \sigma^{-1}(p) u\, dx \end{aligned}$$

which is independent of x. Therefore Π_σ is a map from $C_c(X, U)$ to $|_c(\sigma\,|\,P, X)$.

The same calculation shows that Π_σ is surjective, if we choose K small enough to fix u. ▯

[ind-excision] **Corollary 1.2.** *If Y is a P-stable closed subset of X then*

$$0 \to |_c(\sigma\,|\,P, X - Y) \to |_c(\sigma\,|\,P, X) \to |_c(\sigma\,|\,P, Y) \to 0$$

is exact.

♣ [excision] *Proof.* Only the final surjectivity is non-trivial. It follows from • and the previous Proposition. ▯

Much more elementary:

[ind-components] **Lemma 1.3.** *If X is the finite union of disjoint closed P-stable subsets X_i then restriction induces an isomorphism of $|_c(\sigma\,|\,P, X)$ with the direct sum of the subspaces $|_c(\sigma\,|\,P, X_i)$.*

Now let Q and N be closed subgroups of P. Assume that N is normal in P, QN closed in P, and that N has arbitrarily large compact open subgroups. This implies that N is unimodular, since δ must be trivial on each one of these compact groups. Let $\delta = \delta_{Q\cap N\backslash N}$ be the modulus character of Q acting on the quotient $Q \cap N\backslash N$. Since $Q \cap N$ also has arbitrarily large compact open subgroups, $Q \cap N$ is unimodular so that the restriction of δ to the normal subgroup $Q \cap N$ is trivial.

If (σ, U) is a smooth representation of Q, let $u \mapsto \overline{u}$ be the canonical projection onto the Jacquet module $U_{Q\cap N}$. Since any f in $|_c(\sigma\,|\,Q, P)$ has compact support modulo Q, for any p in P the function $R_p f$ restricted to N has compact support modulo $Q \cap N$. For any q in $Q \cap N$, n in N, p in P we have

$$\overline{f(qnp)} = \overline{\sigma(q) f(np)} = \overline{f(np)}\ .$$

Therefore the integral

$$\overline{f}(p) = \int_{Q\cap N\backslash N} \overline{f(np)}\, dn$$

is well defined. Its definition depends only on a choice of measure on $Q \cap N\backslash N$, and is otherwise canonical. The function $\overline{f}(p)$ maps P into $U_{Q\cap N}$.

[abstract-jacquet] **Proposition 1.4.** *If (σ, U) is a smooth representation of Q then the map $f \mapsto \overline{f}$ induces an isomorphism*

$$|_c(\sigma\,|\,Q, P)_N \cong |_c(\sigma_{Q\cap N}\delta_{Q\cap N\backslash N}\,|\,QN/N, P/N)\ .$$

Keep in mind that $QN/N \cong Q/Q \cap N$.

Proof. A change of variable in the integral defining $\overline{f}$ shows that $\overline{f}(np) = \overline{f}(p)$ for all p in P, since N is unimodular. Thus $\overline{f}$ may be identified with a function on P/N.

For q in Q we have

$$\begin{aligned}\overline{f}(qp) &= \int_{Q\cap N\backslash N} \overline{f(nqp)}\,dn \\ &= \int_{Q\cap N\backslash N} \overline{f(q\,q^{-1}nq\,p)}\,dn \\ &= \delta(q)\int_{Q\cap N\backslash N} \overline{\sigma(q)f(np)}\,dn \\ &= \sigma_{Q\cap N}(q)\delta(q)\overline{f}(p)\ .\end{aligned}$$

If f has support on $Q\Omega$ then $\overline{f}$ has support on $QN\Omega$. Therefore $f\mapsto\overline{f}$ is a map from $|_c(\sigma\,|\,Q,P)_N$ to $|_c(\sigma_{Q\cap N}\delta\,|\,QN/N,P/N)$.

It must be shown that for each compact open subgroup K of P the map $f\mapsto\overline{f}$ induces an isomorphism

$$|_c(\sigma\,|\,Q,P)_N^K \cong |_c(\sigma_{Q\cap N}\delta\,|\,QN/N,P/N)^K\ .$$

The subspace $|_c(\sigma\,|\,Q,P)^K$ is the direct sum of the subspaces of functions with support on the double cosets QxK which are right invariant with respect to K. Since $QxK=QxKx^{-1}\,x$, right multiplication by x identifies this with the subspace of functions on $QxKx^{-1}$ invariant under xKx^{-1}. Similarly for the space $|_c(\sigma_{Q\cap N}\delta\,|\,QN/N,P/N)^K$. The map $f\mapsto\overline{f}$ takes functions with support on QxK to those with support on $QNxK$. Therefore we are reduced to showing that $f\mapsto\overline{f}$ induces an isomorphism

$$\begin{aligned}&\{f\in C_c^\infty(QK,U)\,|\,f(qk)=\sigma(q)f(1)\}_N \\ &\qquad\cong\{f\in C^\infty(QNK,U_{Q\cap N})\,|\,f(qnk)=\sigma_{Q\cap N}(q)f(1)\}\ .\end{aligned}$$

The space $\{f\in C_c^\infty(QK,U)\,|\,f(qk)=\sigma(q)f(1)\}$ may be identified with $U^{Q\cap K}$, while the space $\{f\in C^\infty(QNK,U_{Q\cap N})\,|\,f(qnk)=\sigma_{Q\cap N}(q)f(1)\}$ may be identified with $U_{Q\cap N}^{Q\cap K}$. It only remains to show that in terms of these identifications $f\mapsto\overline{f}$ translates to some multiple of the canonical projection from $U^{Q\cap K}$ to $U_{Q\cap N}$. Hence we must calculate $\overline{f}$ when

$$f(p)=\begin{cases}\sigma(q)u & \text{if } p=qk\\ 0 & \text{otherwise}\end{cases}$$

and show that $\overline{f}(1)$ is the canonical projection of $f(1)$. But

$$\begin{aligned}\overline{f}(1) &= \int_{Q\cap N\backslash KQ\cap N} \overline{f(\nu n)}\,dn \\ &= \int_{K\cap Q\cap N\backslash K\cap N} \overline{f(n)}\,dn \\ &= \text{const}\ \overline{f(1)}\ .\end{aligned}$$

2. The Bruhat order

If P and Q are any two parabolic subgroups then G is a finite disjoint union of double cosets PxQ. These double cosets and the closure relations among them can be parametrized in terms of the Weyl group.

Fix a minimal parabolic subgroup $P_\emptyset$, and a maximal split torus $A_\emptyset$ contained in it, and let W be the corresponding Weyl group, Δ the basis of positive roots determined by the choice of $P_\emptyset$.

For any subsets Θ, Ω in Δ, $P_\Theta\backslash G/P_\Omega$ is the disjoint union of cosets $P_\Theta x P_\Omega$ as x ranges over representatives of $W_\Theta\backslash W/W_\Omega$, which we can choose from among the particular representatives $[W_\Theta\backslash W/W_\Omega]$. We define a partial order on the double cosets $P\backslash G/Q$ according to which $PxQ \le PyQ$ if and only if $PxQ \subseteq \overline{PyQ}$. This is called the **Bruhat order**. What does it translate to in terms of W?

We answer this first for $\Theta = \Omega = \emptyset$. For each x in W let $C(x)$ be the double coset $P_\emptyset x P_\emptyset$. For x and y in W, we say that $x \le y$ when y has a reduced expression

$$y = s_1 s_2 \ldots s_n$$

and

$$x = s_{i_1} s_{i_2} \ldots s_{i_r}$$

is a product, in order, of a subsequence of the s_i. In these circumstances

$$\begin{aligned} C(y) &= C(s_1)C(s_2)\ldots C(s_n) \\ \overline{C(y)} &= \overline{C(s_1)}\,\overline{C(s_2)}\ \ldots\ \overline{C(s_n)} \end{aligned}$$

and since

$$\overline{C(s)} = \{1\} \cup C(s)$$

$$C(x) \subseteq \overline{C(y)}\ .$$

[bruhat-closure] **Proposition 2.1.** *The closure of $C(y)$ is the union of all the $C(x)$ for $x \le y$ in W.*

Therefore if x and y lie in $[W_\Theta\backslash W/W_\Omega]$, then $P_\emptyset x P_\emptyset \le P_\emptyset y P_\emptyset$ if and only if $y \le x$ where the order is that whereby $x \le y$ if and only if y has a reduced expression and x is a product $s_{i_1} s_{i_2} \ldots s_{i_k}$ for some sequence $i_1 < i_2 < \ldots \le i_k$.

More generally:

[parabolic-bruhat-order] **Proposition 2.2.** *If x lies in $[W_\Theta\backslash W]$ and y is the element of smallest length in $[W_\Theta x w_{\ell,\Omega}]$, then the closure of $P_\emptyset w_{\ell,\Theta} y P_\emptyset$ is the same as the closure of $P_\Theta x P_\Omega$.*

Proof. ▯

Algorithm to determine closures? Product relations for N_w?

3. The filtration

Throughout the rest of this chapter, fix a parabolic subgroup P of G and a smooth representation (π, V) of M_P. In the first section I will construct for every parabolic subgroup Q of G a filtration of $\mathrm{Ind}(\sigma\,|\,P, G)$ as a representation of Q. In the second I shall describe the Jacquet module of each graded term associated to this filtration.

[pq] **Lemma 3.1.** *If P and Q are both parabolic subgroups of G, then the image of $Q \cap P$ in P/N_P is also one, with unipotent radical equal to the image of $N_Q \cap P$.*

Let $M_{P,Q}$ be the reductive factor of the image of $P \cap Q$ in P/N_P. It is the same as the reductive factor of $P \cap Q$, so that $M_{P,Q}$ and $M_{Q,P}$ are canonically isomorphic.

Describe it explicitly in terms of $\Theta \subseteq \Delta$: $M_{\Theta\cap\Psi}$, unipotent radical $N_{P,Q}$.

Now P and Q are both parabolic subgroups, as is xQx^{-1}. The image of xQx^{-1} in P/N_P is also a parabolic.

Suppose P and Q to be parabolic subgroups of G. Let (σ, U) be a smooth representation of P/N_P, and for the moment let

$$I = \mathrm{Ind}(\sigma\,|\,P, G)\ .$$

If X is any union of double cosets in G, let I_X be $\mathrm{Ind}_c(\sigma\,|\,P, X)$. For example, if X is open in G then I_X is the subspace of function in I with support in X. Let $X_{\min}$ be the union of closed $P \times Q$ cosets in X.
♣ [ind-excision] Then $X - X_{\min}$ is open in X and Corollary 1.2 asserts that

$$0 \to I_{X-X_{\min}} \to I_X \to I_{X_{\min}} \to 0$$

is exact.

Furthermore $I_{X_{\min}}$ is the direct sum of spaces I_Y as Y ranges over the $P \times Q$ cosets in $X_{\min}$.

4. The Jacquet module

Suppose $X = PxQ$ is a single double coset in G. What is the Jacquet module of the representation of Q on $\mathrm{Ind}_c(\sigma\,|\,P, PxQ)$? The image of $xQx^{-1} \cap P$ in P/N_P is a parabolic subgroup R; let S be the image of $x^{-1}Px \cap Q$ in Q/N_Q. Then conjugation by x^{-1} induces an isomorphism of M_R with M_S.

For f in $\mathrm{Ind}_c(\sigma\,|\,P, PxQ)$ we define

$$\overline{f}(q) = \int_{N_Q\cap x^{-1}Px\backslash N_Q} \overline{f(xnq)}\,dn$$

where $u \mapsto \overline{u}$ is the canonical projection from U to U_{N_R}.

[jacquet-induced] **Theorem 4.1.** *The map $f \mapsto \overline{f}$ induces an isomorphism*

$$\mathrm{Ind}_c(\sigma\,|\,P, PxQ)_{N_Q} \cong \mathrm{Ind}(x^{-1}\sigma_{N_R}\,|\,S, M_Q)$$

♣ [abstract-jacquet] *Proof.* This follows from Proposition 1.4, since multiplication by x and restriction to Q give

$$|_c(\sigma\,|\,x^{-1}Px \cap Q, Q) \cong\ |_c(x^{-1}\sigma\,|\,P \cap xQx^{-1}, xQx^{-1})\ .$$

We just have to get the δ factor correct.

In certain circumstances the expressions in this can be calculated explicitly, and then this gives a usable formula for the pairing.