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Essays on representations of p-adic groups

The Jacquet module

Bill Casselman
University of British Columbia
cass@math.ubc.ca

In this chapter, we shall associate to every smooth representation π and parabolic subgroup P of G an admissible representation of M_P. These representations turn out to control much of the structure of admissible representations induced from parabolic subgroups, and also to describe the behaviour at infinity on G of the matrix coefficients of π when it is admissible. The origin of most of the results in this section is a lecture of Jacquet's presented at a conference in Montecatini.

1. The Jacquet module

[unipotent-large] **Lemma 1.1.** *If N is a $\mathfrak{p}$-adic unipotent group, it possesses arbitrarily large compact open subgroups.*

Proof. It is certainly true for the group of unipotent upper triangular matrices in GL_n. Here, if a is the diagonal matrix with $a_{i,i} = \varpi^i$ then conjugation by powers of a will scale any given compact open subgroup to an arbitrarily large one. But any unipotent group can be embedded as a closed subgroup in one of these. ▯

Fix the parabolic subgroup $P = MN$. If (π, V) is any smooth representation of N, define $V(N)$ to be the subspace of V generated by vectors of the form

$$\pi(n)v - v$$

as n ranges over N. The group N acts trivially on the quotient

$$V_N = V/V(N)$$

It is universal with respect to this property:

[universality] **Proposition 1.2.** *The projection from V to V_N induces for every smooth R-representation (σ, U) on which N acts trivially an isomorphism*

$$\mathrm{Hom}_N(V, U) \cong \mathrm{Hom}_R(V_N, U) \ .$$

[union-vu] **Lemma 1.3.** *The subspace $V(N)$ is also the union of the subspaces $V(U)$ as U varies over the compact open subgroups of N.*

♣ [unipotent-large] *Proof.* Immediately from Lemma 1.1. ▯

[jacquet-exact] **Proposition 1.4.** *If*

$$0 \to U \to V \to W \to 0$$

is an exact sequence of smooth representations of N, then the sequence

$$0 \to U_N \to V_N \to W_N \to 0$$

is also exact.

Proof. That the sequence

$$U_N \to V_N \to W_N \to 0$$

is exact follows immediately from the definition of $V(N)$. The only non-trivial point is the injectivity of $U_N \to V_N$. If u in U lies in $V(N)$ then it lies in $V(S)$ for some compact open subgroup S of N.

♣ **[projection]** According to Lemma 2.1, the space V has a canonical decomposition

$$V = V^S \oplus V(S) ,$$

and v lies in $V(S)$ if and only if

$$\int_S \pi(s)v\, ds = 0$$

But this last equation holds in U as well, since U is stable under S, so v must lie in $U(S)$. ▯

♣ **[frobenius]** If (π, V) is a smooth representation of G and σ a smooth representation of M, then ♣tells us that

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}(\sigma \,|\, P, G)) \cong \mathrm{Hom}_P(\pi, \sigma\delta_P^{-1/2}) .$$

Since σ is trivial on N, any P-map from V to U factors through V_N. The space $V(N)$ is stable under P, and there is hence a natural representation of M on V_N. The **Jacquet module** of π is this representation twisted by the character $\delta_P^{-1/2}$. This is designed exactly to allow the simplest formulation of this:

[jacquet-frobenius] **Proposition 1.5.** *If (π, V) is any smooth representation of G and (σ, U) one of M then evaluation at 1 induces an isomorphism*

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}(\sigma \,|\, P, G)\big) \cong \mathrm{Hom}_M(\pi_N, \sigma)$$

2. Admissibility of the Jacquet module

Now fix an admissible representation (π, V) of G. Let P, $\overline{P}$ be an opposing pair of parabolic subgroups, K_0 to be a compact open subgroup possessing an Iwahori factorization $K_0 = N_0 M_0 \overline{N}_0$ with respect to this pair. For each a in A_P^{--} let T_a be the smooth distribution

$$\left(\frac{1}{\mathrm{meas}\, K_0}\right) \mathfrak{char}_{K_0 a K_0}\, dx$$

on G. For any smooth representation (π, V) and v in V^{K_0} let τ_a be the restriction of $\pi(T_a)$ to V^{K_0}. Thus for v in V^{K_0}

$$\begin{aligned}\tau_a(v) &= \pi(T_a)\, v \\ &= \sum_{K_0 a K_0/K_0} \pi(g)\, v \\ &= \sum_{K_0/K_0 \cap a K_0 a^{-1}} \pi(k)\pi(a)\, v .\end{aligned}$$

This is valid since the isotropy subgroup of a in the action of K_0 acting on $K_0 a K_0/K_0$ is $aK_0a^{-1} \cap K_0$, hence

$$k \mapsto kaK_0$$

is a bijection of $K_0/K_0 \cap aK_0a^{-1}$ with K_0aK_0/K_0.

[projection] **Lemma 2.1.** *If v lies in V^{K0} with image u in V_N, then the image of $\tau_a v$ in V_N is equal to $\delta_P^{-1/2}(a)\pi_N(a)u$.*

Proof. Since $K_0 = N_0 M_0 \overline{N}_0$, $aK_0a^{-1} = (aN_0a^{-1})M_0(aN_0a^{-1})$. Since $\overline{N} \subseteq a\overline{N}a^{-1}$, the inclusion of N_0/aN_0a^{-1} into $K_0/(aK_0a^{-1} \cap K_0)$ is in turn a bijection. Since the index of aN_0a^{-1} in N_0 or, equivalently, that of N_0 in $a^{-1}N_0a$ is $\delta_P^{-1}(a)$:

$$\begin{aligned} \tau_a(v) &= \sum_{K_0/aK_0a^{-1}\cap K_0} \pi(k)\pi(a)v \\ &= \sum_{N_0/aN_0a^{-1}} \pi(n)\pi(a)v \\ &= \pi(a) \sum_{a^{-1}N_0a/N_0} \pi(n)v\ . \end{aligned}$$

Since $\pi(n)v$ and v have the same image in V_N, this concludes the proof. ▯

[tab] **Lemma 2.2.** *For every a, b in A_P^{--},*

$$\tau_{ab} = \tau_a \tau_b$$

Proof. We have

$$\begin{aligned} T_aT_b &= \sum_{N_0/aN_0a^{-1}} \sum_{N_0/bN_0b^{-1}} \pi(n_1\pi(a)\pi(n_2)\pi(b)v \\ &= \sum_{N_0/aN_0a^{-1}} \sum_{N_0/bN_0b^{-1}} \pi(n_1)\pi(an_2a^{-1})\pi(ab)v \\ &= \sum_{N_0/abN_0b^{-1}a^{-1}} \pi(n)\pi(ab)v \\ &= T_{ab} \end{aligned}$$

since as n_1 ranges over representatives of N_0/aN_0a^{-1} and and n_2 over representatives of N_0/bN_0b^{-1}, the products $n_1\,an_2a^{-1}$ range over representatives of $N_0/abN_0b^{-1}a^{-1}$. ▯

[kernel-ta] **Lemma 2.3.** *For any a in A_P^{--} the subspace of V^{K_0} on which τ_a acts nilpotently coincides with $V^{K_0} \cap V(N)$.*

Proof. Since R is Noetherian and V^{K_0} finitely generated, the increasing sequence

$$\ker(\tau_a) \subseteq \ker(\tau_{a^2}) \subseteq \ker(\tau_{a^3}) \subseteq \ldots$$

is eventually stationary. It must be shown that it is the same as $V^{K_0} \cap V(N)$.

Choose n large enough so that $V^{K_0} \cap V(N) = V^{K_0} \cap V(a^{-n}N_0a^n)$. Let $b = a^n$. Since

$$\tau_b v = \pi(b) \sum_{b^{-1}N_0b/N_0} \pi(n)v\ ,$$

and $\tau_b v = 0$ if and only if $\sum_{b^{-1}N_0b/N_0} \pi(n)v = 0$, and again if and only if v lies in $V(N)$. ▯

The canonical map from V to V_N takes V^{K_0} to $V_N^{M_0}$. The kernel of this map is $V \cap V(N)$, which by
♣ [kernel-ta] Lemma 2.3 is equal to the kernel of τ_{a^n} for large n.

[stable] **Lemma 2.4.** *The image of τ_{a^n} in V^{K_0} is independent of n if n is large enough. The map τ_a is invertible on it. The intersection of it with $V(N)$ is trivial.*

♣ [kernel-ta] *Proof.* Choose n so large that $\ker(\tau_{a^n}) = \ker(\tau_{a^m})$ for all $m \geq n$. By Lemma 2.3 this kernel coincides with $V^{K_0} \cap V(N)$. Let U be the image of τ_{a^n}. If $u = \tau_{a^n}v$ and $\tau_{a^n}v = 0$ then $\tau_{a^{2n}}v = 0$, which means by assumption that in fact $u = \tau_{a^n}v = 0$. Therefore the intersection of U with $V(N)$ is trivial, the projection from V to V_N is injective on U, and τ_a is also injective on it. If $\mathfrak{m}$ is a maximal ideal of R, this remains true for $U/\mathfrak{m}U$, and therefore by ♣ [noetherian] τ_a is invertible on U. This implies that U is independent of the choice of n. ▯

Let $V_N^{K_0}$ be this common image of the τ_{a^n} for large n. The point is that it splits the canonical projection from V^{K_0} to $V_N^{M_0}$, which turns out to be a surjection.

[jacquetdecomp] **Proposition 2.5.** *The canonical projection from $V_N^{K_0}$ to $V_N^{M_0}$ is an isomorphism.*

Proof. Suppose given u in $V_N^{M_0}$. Since M_0 is compact, we can find v in V^{M_0} whose image in V_N is u. Suppose that v is fixed also by $\overline{N}_*$ for some small $\overline{N}_*$. If we choose b in A_P^{--} such that $b\overline{N}_0 b^{-1} \subseteq \overline{N}_*$, then $v_* = \delta^{1/2}(b)\pi(b)v$ is fixed by $M_0\overline{N}_0$. Because $K_0 = N_0 M_0 \overline{N}_0$, the average of $\pi(n)v_*$ over N_0 is the same as the average of $\pi(k)v_*$ over K_0. This average lies in V^{K_0} and has image $\pi_N(b)u$ in V_N. But then $\tau_a v_*$ has image $\delta^{1/2}(a)\pi_N(ab)u$ in V_N and also lies in $V_N^{K_0}$. Since τ_{ab} acts invertibly on $V_N^{K_0}$, we can find v_{**} in $V_N^{K_0}$ such that $\tau_{ab}v_{**} = \tau_a\tau_b v_{**} = \tau_a v_*$, and whose image in V_N is u. ▯

As a consequence:

[jacquet-admissible] **Theorem 2.6.** *If (π, V) is an admissible representation of G then (π_N, V_N) is an admissible representation of M.*

Thus whenever K_0 is a subgroup possessing an Iwahori factorization with respect to P, we have a canonical subspace of V^{K_0} projecting isomorphically onto V^{M_0}. For a given M_0 there may be many different K_0 suitable; how does the space $V_N^{K_0}$ vary with K_0?

[coherence] **Lemma 2.7.** *Let $K_1 \subseteq K_0$ be two compact open subgroups of G possessing an Iwahori factorization with respect to P. If v_1 in $V_N^{K_1}$ and v_0 in $V_N^{K_0}$ have the same image in V_N, then $\pi(\mu_{K_0})v_1 = v_0$.*

3. The canonical pairing

Continue to let K_0 be a compact open subgroup of G possessing an Iwahori factorization $\overline{N}_0 M_0 N_0$ with respect to the parabolic subgroup P, (π, V) an admissible representation of G. Let N_* be a compact open subgroup of N such that $V^{K_0} \cap V(N) \subseteq V(N_*)$.

[annihilation] **Lemma 3.1.** *For v in $V_N^{K_0}$, $\widetilde{v}$ in $\widetilde{V}^{K_0} \cap \widetilde{V}(\overline{N})$, $\langle v, \widetilde{v}\rangle = 0$.*

Proof. This follows easily from the fact that $v = \pi(T_{a^n})u$ for some a in A_P^{--}, u in V^{K_0}, and large n, while $\pi(T_{a^n})\widetilde{v} =$ for large n. ▯

[asymptotic-pairing] **Theorem 3.2.** *If $(\pi, V$ is an admissible representation of G, then there exists a unique pairing between V_N and $\widetilde{V}_{\overline{N}}$ with the property that whenever v has image u in V_N and $\widetilde{v}$ has image $\widetilde{u}$ in $\widetilde{V}_{\overline{N}}$, then for all a in A_P^{--} near enough to 0*

$$\langle \pi(a)v, \widetilde{v}\rangle = \delta_P^{1/2}(a)\langle \pi_N(a)u, \widetilde{u}\rangle \ .$$

Similarly with the roles of V and $\widetilde{V}$ reversed. If the coefficient ring is a field, this pairing gives rise to an isomorphism of $(\widetilde{\pi}_N, \widetilde{V}_{\overline{N}})$ with the contragredient of the representation (π_N, V_N).

Proof. Let u in V_N and $\widetilde{u}$ in $\widetilde{V}_{\overline{N}}$ be given. Suppose that u and $\widetilde{u}$ are both fixed by elements of M_0. Let v be a vector in $V_N^{K_0}$ with image u, and similarly for $\widetilde{v}$ and $\widetilde{u}$. Define the pairing by the formula

$$\langle u, \widetilde{u}\rangle_{\text{can}} = \langle \widetilde{v}, v\rangle \ .$$

♣♣ [annihilation] [coherence] It follows from Lemma 3.1 and Lemma 2.7 that this definition depends only on u and $\widetilde{u}$, and not on the choices of v and $\widetilde{v}$. That

$$\langle \pi(a)v, \widetilde{v}\rangle = \delta_P^{1/2}(a)\langle \pi_N(a)u, \widetilde{u}\rangle_{\text{can}}$$

also follows from Lemma 3.1 and Lemma 2.7. That this property characterizes the pairing follows from the invertibility of τ_a on $V_N^{K_0}$. □

This pairing is called the **canonical pairing**.