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Essays on representations of p-adic groups

Analysis on profinite groups

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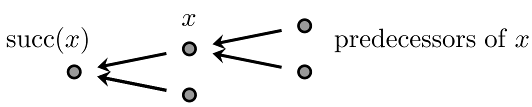
All $\mathfrak{p}$-adic groups have a topology in which every point has a countable basis of compact open neighbourhoods. Analysis on such spaces is essentially algebra, and in particular the theory of Haar measures is elementary. In order to emphasize this, I shall discuss such spaces in purely combinatorial terms.

1. Trees and topologies

A p-adic integer can be identified with a sequence (x_n) of compatible integers in the finite rings $\mathbb{Z}/p^n$. We can make a graph with directed edges out of these data: there is one node in the graph for each pair (n, x) with $n > 0$ and x in $\mathbb{Z}/p^n$, and an edge from (n, x) to $(n-1, y)$ if $n \geq 2$ and $y \equiv x \bmod p^{n-1}$. A p-adic integer then amounts to a one-way path of nodes in this graph, coming from infinity and terminating at one of the initial nodes in $\mathbb{Z}/p$. These observations should motivate the following discussion.

A **rooted tree** consists of (1) a set of **nodes**, (2) a designated **root node**, and (3) for every node other than the root an assignment of **immediate successor** node $\mathrm{succ}(x)$, satisfying the condition that from any node x there exists a unique sequence $x_n = x, x_{n-1}, \ldots, x_0$ with each x_{i-1} the successor of x_i and x_0 the root node. A **successor** is defined inductively by the condition that it be either an immediate successor or a successor of an immediate successor.

A node x is an **immediate predecessor** of another node y if $y = \mathrm{succ}(x)$, and a predecessor if linked by a chain of immediate predecessors.



A rooted tree is said to be **locally finite** if the number of immediate predecessors of every node is finite. A **chain** in the tree rooted at the node x is a finite sequence of nodes $x_0 = x, x_1, \ldots, x_n$ where each x_{i+1} is an immediate predecessor of x_i. It is said to have length n. A **branch** of a rooted tree is a sequence (finite or infinite) of nodes (x_i) where x_0 is the root and each x_{i+1} is an immediate predecessor of x_i, satisfying the condition that the sequence stops only at a node with no predecessors.

Locally finite rooted trees possess a recursive structure, since if T is any locally finite rooted tree and x is a node of T then the set of all predecessors of x in T, together with x itself, make up a locally finite tree with root x. Any subset of T that contains along with a node its successor will also be a locally finite tree with the same root as T.

[finite-depth] **Lemma 1.1.** *In any locally finite tree, the number of chains of length n rooted at a given node is finite.*

Proof. That this is true for chains of length 1 is the definition of locally finite. The proof proceeds by induction on chain length.

[konigs-lemma] **Lemma 1.2.** (König's Lemma) Every locally finite rooted tree with an infinite number of nodes has an infinite branch.

Proof. Let ρ be the root of the given tree, assumed to possess an infinite number of nodes. It follows from the previous result that there exist chains rooted at ρ of arbitrary length, and in particular that the set C_0 of all chains rooted at $x_0 = \rho$ is infinite. Since the number of immediate predecessors of x_0 is finite, the subset C_1 of chains in C_0 passing through some one of them, say x_1, is infinite. Similarly there must exist an infinite number among the chains in C_1 whose third nodes agree. By induction, we obtain for each n a sequence of sets of chains

$$C_0 \supseteq C_1 \supseteq \ldots \supseteq C_n \supseteq C_{n+1} \supseteq \ldots$$

where all the chains in C_n agree with each other in their first $n + 1$ nodes, and agree in their first n nodes with the chains in C_{n-1}. By choosing x_n to be the common n-th node of the chains in C_n we assemble an infinite branch. ▯

The reasoning here, although plausible, is specious or at least highly subtle, since disguised in it is a weak form of the Axiom of Choice. König's Lemma is one of those results that hovers on the edge of obviousness, like a dim star one cannot quite focus on directly. The degree to which it is not obvious becomes more apparent when one sees (for example in the enlightening discussion of §2.3.4.3 of [Knuth:1973]) some of its immediate corollaries.

Suppose T to be a locally finite rooted tree with root ρ. For every node x in T define Ω_x to be the set of all branches passing through x. König's Lemma guarantees that this is never empty. Let Ω_T be the set of all branches of T, which is the same as Ω_ρ. We can make a topological space out of Ω_T by defining as basis of open sets the sets Ω_x—if ω is any branch in the tree then the sets Ω_x for each of its nodes x define a basis of neighbourhoods of ω. Two distinct branches must eventually diverge, and therefore it is easy to see that this topology is Hausdorff. The next result shows that this topology is otherwise somewhat special.

[omega-closed] **Proposition 1.3.** *If T is a locally finite rooted tree and x a node in T then Ω_x is closed in Ω_T.*

In other words, any point of Ω_T has a basis of neighbourhoods that are both open and closed.

Proof. Let $x_0 = \rho, x_1, \ldots, x_n = x$ be the chain from ρ to x, and let $X = \{x_i\}$. Let Y be the union of all of the immediate predecessors of the x_i for $i < n$, except for x_{i+1}. In other words, y is in Y if it is an immediate predecessor of some x_i and is not in X. Since T is locally finite, Y is finite. Every branch in T that does not pass through x has to branch off at one of the x_i with $i < n$, and therefore the complement of Ω_x is the union of the Ω_y for y in Y. ▯

[compactness] **Proposition 1.4.** *If T is a locally finite rooted tree then the topological space Ω_T is compact.*

Proof. This amounts to the following assertion:

> *Suppose X to be a set of nodes of T such that the sets Ω_x (x in X) cover Ω_T. Then there exists a finite set of Ω_x (x in X) covering Ω_T.*

The assumption means that every branch in Ω_T lies in some Ω_x with x in X, or equivalently that every branch in Ω_T has a node in X. If X contains the root node, we are immediately through.

Otherwise, let $X_{\min}$ be the set of nodes in X that are minimal—that is to say, x lies in $X_{\min}$ if it lies in X and in the path from the root to x there are no elements of X other than x. It is clear that that the Ω_x for x in $X_{\min}$ cover Ω_T, since every branch has to have a first element in X.

It suffices to show that $X_{\min}$ is finite, since a path from x in X to the root must pass through a node in $X_{\min}$. Let Y be the set of nodes in T that are not in X and none of whose successors are in X. In particular, Y contains the root node of T. Any successor of a node in Y will also be in Y, so that Y itself is a rooted tree and we can apply König's Lemma to it. It is not possible for Y to contain an infinite branch, since any infinite branch in Y would also be a branch in T, and by assumption every branch in T must contain a node in X. König's Lemma tells us that Y must be finite. But since X does not contain the root node, every element of $X_{\min}$ is the predecessor of some node in Y, and since the tree is locally finite $X_{\min}$ must be finite. ▯

Applying this to each of the rooted trees Ω_x:

[totally-disconnected] **Proposition 1.5.** *Every point in the topological space Ω_T possesses a countable basis of compact open neighbourhoods.*

2. Locally profinite spaces

Suppose we are given a sequence of finite sets $X_0, X_1, \ldots$ and for each $n > 0$ a surjection $\pi_n \colon X_n \to X_{n-1}$. We can make a finite union of trees from these data by taking the nodes of our graph to be the points of the X_n and defining the successor of x in X_n to be $\pi_n(x)$. The branches of this tree are the infinite sequences (x_n) where $\pi_n(x_n) = x_{n-1}$, and for a given point x in X_n the set Ω_x consists of all sequences with $x_n = x$. The topological space we get in this way is called the **projective limit** of the given sequence of finite sets. ♣ [compactness] According to Proposition 1.4 it possesses a natural Hausdorff topology with respect to which it is compact.

An arbitrary topological space is said to be **profinite** if it homeomorphic to the projective limit of a sequence of finite sets. It is said to be **locally profinite** if every point possesses a profinite neighbourhood.

This terminology is not quite in agreement with common usage, since I am restricting limits to sequences of spaces. The spaces I use here might more strictly be called **sequentially profinite**.

3. $\mathfrak{p}$-adic fields

The ring of p-adic integers $\mathbb{Z}_p$ is defined to be the projective limit of the finite rings $\mathbb{Z}/p^n$. The ordinary integers $\mathbb{Z}$ may be embedded in $\mathbb{Z}_p$, since m may be identified with the sequence $(m \bmod p^n)$. If q is an integer relatively prime to p then for every $n > 0$ there exists a multiplicative inverse of q modulo p^n, so that all rational numbers m/q with q prime to p may also be identified with elements of $\mathbb{Z}_p$. More generally, the p-adic integers with multiplicative inverses in $\mathbb{Z}_p$ are precisely those whose image in $\mathbb{Z}/p$ does not vanish. A p-adic rational number other than 0 can be identified with a unique expression m/p^k where m is a unit in $\mathbb{Z}_p$. These make up the field $\mathbb{Q}_p$.

Generalizing this construction, a **$\mathfrak{p}$-adic field** is defined to be a field $\mathfrak{k}$ containing a ring $\mathfrak{o}$ and an ideal $\mathfrak{p}$ of $\mathfrak{o}$ satisfying these conditions:

(a) $\mathfrak{o}/\mathfrak{p}$ is a finite field, say of q elements;
(b) the canonical projections from $\mathfrak{o}$ to $\mathfrak{o}/\mathfrak{p}^n$ and from $\mathfrak{o}/\mathfrak{p}^{n+1}$ to $\mathfrak{o}/\mathfrak{p}^n$ identify the ring $\mathfrak{o}$ with the projective limit of the quotients $\mathfrak{o}/\mathfrak{p}^n$;
(c) if ϖ lies in $\mathfrak{p} - \mathfrak{p}^2$, then every non-zero element of $\mathfrak{k}$ may be written as $u\varpi^n$ where u lies in $\mathfrak{o} - \mathfrak{p}$.

If x is an element of $\mathfrak{o} - \mathfrak{p}$ there exists an element y of $\mathfrak{o} - \mathfrak{p}$ such that $xy \equiv 1$ modulo $\mathfrak{p}$. If then $m = xy - 1$ the series

$$u = 1 - m + m^2 - m^3 + \cdots$$

converges to an element of $\mathfrak{o}$ because of condition (b), since modulo any power of $\mathfrak{p}$ the series terminates. The limit will be a multiplicative inverse of xy, so that yu will be a multiplicative inverse of x. Hence every element of $\mathfrak{o} - \mathfrak{p}$ is a unit of $\mathfrak{o}$. The ideal $\mathfrak{p}$ is the only prime ideal of $\mathfrak{o}$ other than (0), and every non-zero ideal of $\mathfrak{o}$ is a power of $\mathfrak{p}$. If ϖ lies in $\mathfrak{p} - \mathfrak{p}^2$ then multiplication by ϖ^n induces a bijection of $\mathfrak{o}/\mathfrak{p}$ with $\mathfrak{p}^n/\mathfrak{p}^{n+1}$, so that $\mathfrak{o}/\mathfrak{p}^n$ is a finite ring of cardinality q^n.

The field $\mathbb{Q}_p$ of p-adic integers is a $\mathfrak{p}$-adic field with $\mathfrak{o} = \mathbb{Z}_p$ and $\mathfrak{p} = (p)$. Every finite algebraic extension of a $\mathfrak{p}$-adic field is a $\mathfrak{p}$-adic field. The completion of any algebraic number field of finite degree with respect to any non-zero prime ideal of its ring of algebraic integers is a $\mathfrak{p}$-adic field, and is a finite extension of $\mathbb{Q}_p$. The field of power series in x with coefficients in $\mathbb{F}_q$ and a finite number of negative powers of x

$$c_{-n}x^{-n} + c_{-(n-1)}x^{-(n-1)} + \cdots$$

make up the quotient field $\mathbb{F}_q((x))$ of the ring of formal power series $\mathbb{F}_q[[x]]$. It is a $\mathfrak{p}$-adic field with $\mathfrak{p} = (x)$. The completion of any $\mathbb{F}_q$-rational local ring on an algebraic curve over $\mathbb{F}_q$ is isomorphic to it. Conversely, any $\mathfrak{p}$-adic field is a finite algebraic extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((x))$.

Throughout this book we will work with a fixed $\mathfrak{p}$-adic field $(\mathfrak{k}, \mathfrak{o}, \mathfrak{p})$, where $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$. If $x = u\varpi^n$ with u a unit in $\mathfrak{o}$ then its norm $|x| = |x|_\mathfrak{p}$ is defined to be

$$|x| = q^{-n} .$$

Thus when $n \geq 0$ the index of the ideal (x) in $\mathfrak{o}$ is $|\ x\ |^{-1}$.

The ring $\mathfrak{o}$ is a profinite space. The ideals $\mathfrak{p}^n$ form a basis of neighbourhoods of 0. Any closed or open subspace of a finite dimensional vector space over $\mathfrak{k}$ will also be a locally profinite space.

4. Smooth functions on a locally profinite space

Fix a coefficient field $\mathbb{D}$ of characteristic 0. Much later it will be assumed to be algebraically closed, and occasionally $\mathbb{C}$, but for the moment it might even be just $\mathbb{Q}$.

If X is a locally profinite space and V a vector space over $\mathbb{D}$, then the space $C^\infty(X, V)$ is that of all locally constant functions on X with values in V, and $C_c^\infty(X, V)$ is the subspace of functions with compact support. Let $C^\infty(G)$ and $C_c(G)$ be these spaces with $V = \mathbb{D}$, and let $\mathcal{D}(G)$ be the linear dual of $C_c(G)$, a space of **distributions**.

For any open set U of X let $\mathrm{TR}/_U$ be its characteristic function. Every function in $C_c^\infty(X, V)$ is a finite sum of functions $\chi_U \cdot v$ with v in V.

[excision] **Proposition 4.1.** *(Excision Lemma) If Y is a closed subset of the locally profinite space X, then the natural maps make an exact sequence*

$$0 \to C_c^\infty(X - Y, V) \to C_c^\infty(X, V) \to C_c^\infty(Y, V) \to 0 .$$

Proof. The only interesting point is the final surjectivity. Suppose f to be in $C_c^\infty(Y)$. Because of ♣ [totally-disconnected] Proposition 1.5, we can find a covering $\{U_i\}$ of the support of f by compact open subsets U_i of X with the property that f is constant on each $U_i \cap Y$. The union of any of the U_i is compact, hence closed in X, so that the sets

$$U_{*,i} = U_i - \bigcup_{j<i} U_j$$

are also open. The mutually disjoint sets $U_{*,i}$ also cover the support of f, and on each $U_{*,i} \cap Y$ the function f takes a constant value v_i. The linear combination $\sum \mathrm{TR}/_{U_{*,i}} \cdot v_i$ then lies in $C_c^\infty(X, V)$ and has image f in $C_c^\infty(Y, V)$. ▯

5. Profinite groups

A (sequentially) **profinite group** is defined to be the projective limit of a sequence of surjective homomorphisms of finite groups. By definition it possesses as basis of neighbourhoods of the identity a sequence of normal subgroups. A **locally profinite group** is a locally profinite space with a group structure that is continuous in the locally profinite topology and that in addition possesses a profinite subgroup as a neighbourhood of the identity. A Hausdorff locally compact group is locally profinite if and only if there exists a countable set of compact open subgroups forming a basis of neighbourhoods of the identity.

In $\mathrm{GL}_n(\mathfrak{k})$ the subgroup $\mathrm{GL}_n(\mathfrak{o})$ of invertible matrices with coefficients in $\mathfrak{o}$ is a compact open subgroup, with the congruence subgroups

$$\mathrm{GL}_n(\mathfrak{p}^m) = \big\{ g \in \mathrm{GL}_n(\mathfrak{o}) \,\big|\, g \equiv I \ (\mathfrak{p}^m) \big\}$$

forming a basis of neighbourhoods of the identity. Hence the group of $\mathfrak{k}$-rational points on any closed subgroup of $\mathrm{GL}_n(\mathfrak{k})$, and in particular the group of $\mathfrak{k}$-rational points on any affine algebraic group defined over $\mathfrak{k}$, is a locally profinite group.

[k-stable] **Proposition 5.1.** *Suppose X to be any locally profinite space on which the locally profinite group G acts continuously. Then for every compact open set Ω in X there exists a compact open subgroup K of G such that $K\Omega = \Omega$.*

Proof. Because the action of G is continuous, left multiplication in G is continuous, for any point x of Ω there exists a compact subgroup K and a neighbourhood U of x such that $KU \subseteq \Omega$. Since Ω is compact, Ω will be covered by a finite number of these, say by the $K_i U_i$. Then Ω will be stable with respect to the intersection of the K_i. ▯

If G is a locally profinite group, then it acts by the right- and left-regular representations on $C^\infty(G)$, $C_c^\infty(G)$, and $\mathcal{D}(G)$ according to the recipes

$$\begin{aligned} L_g f(x) &= f(g^{-1}x) \\ R_g f(x) &= f(xg) \\ \langle L_g \Phi, f \rangle &= \langle \Phi, L_{g^{-1}} f \rangle \\ \langle R_g \Phi, f \rangle &= \langle \Phi, R_{g^{-1}} f \rangle \ . \end{aligned}$$

We have

$$L_{g_1 g_2} = L_{g_1} L_{g_2}, \quad R_{g_1 g_2} = R_{g_1} R_{g_2} \ .$$

A **smooth** function is one that is locally constant, and a **smooth distribution** is one that is locally right-invariant under some open subgroup of G. **Uniform** smoothness means global right-invariance under some fixed open subgroup.

[right-left] **Lemma 5.2.** *If G is a locally profinite group and f a function on G with values in V with compact support, the following are equivalent:*

(a) *the function f lies in $C_c^\infty(G, V)$;*
(b) *there exists a compact open subgroup K of G such that $L_k f = f$ for all k in K;*
(c) *there exists a compact open subgroup K of G such that $R_k f = f$ for all k in K.*

This is straightforward. As a consequence, the notion of smoothness doesn't depend on whether left or right multiplication is used in the definition.

The theory of Haar measures on locally profinite groups is very simple. In essence, as will be explained in a moment, integrals are always sums—even though occasionally infinite.

[haar] **Proposition 5.3.** *Let G be a locally profinite group. Given a compact open subgroup K and a constant c_K in $\mathbb{Q}^\times$, there exists a unique right G-invariant distribution μ on G such that*

$$\langle \mu, \mathrm{TR}/_K \rangle = c_K \; .$$

Proof. Suppose that the distribution μ is known to exist. If K_* is a compact open subgroup contained in K then

$$\langle \mu, \mathrm{TR}/_{K_*} \rangle = [K\colon K_*]^{-1} c_K$$

since K is the disjoint union of the $K_* x$ as x runs over representatives of $K_* \backslash K$, and $\langle \mu, \mathrm{TR}/_{K_* x} \rangle = \langle \mu, \mathrm{TR}/_{K_*} \rangle$. If K_* is an arbitrary compact open subgroup, then

$$\langle \mu, \mathrm{TR}/_{K_*} \rangle = \frac{[K_*\colon K \cap K_*]}{[K\colon K \cap K_*]}\, c_K \; .$$

But knowing $\langle \mu, \mathrm{TR}/_{K_*} \rangle$ for all compact open subgroups K_*, together with right G-invariance, determines $\langle \mu, f \rangle$ for any smooth function f since f is a linear combination of $\mathrm{TR}/_{K_* x}$ for some one K_* and a finite set of x in G. This argument when run backwards gives the recipe for constructing μ. ▯

I shall call such any $\mathbb{Q}$-distribution on G with positive c_K that is right G-invariant a (rational) **right Haar measure** on G, and if $d_r x$ is one write

$$\mathrm{meas}(U, d_r x) = \langle d_r x, \mathrm{TR}/_U \rangle$$
$$\int_G f(x)\, d_r x = \langle d_r x, f \rangle \; .$$

If $d_r x$ is any right Haar measure on G then any left translation $L_x\, d_r x$ is also a right Haar measure, and must be therefore a scalar multiple of $d_r x$. In other words for each g in G there exists a scalar $\delta_G(g)$ such that

$$\int_{gU} d_r x = \delta_G(g) \int_U d_r x$$

for all compact open subsets U of G, or in brief

$$d_r gx = \delta_G(g)\, d_r x \, .$$

The constant $\delta_G(g)$ is independent of the right Haar measure chosen, since all others are just scalar multiples of it. It depends multiplicatively on g:

$$\delta_G(g_1 g_2) = \delta_G(g_1)\delta_G(g_2) \; .$$

This multiplicative character of G with values in the positive rational numbers is called the **modulus character** of G. If $d_r x$ is a right Haar measure on G then $\delta_G(x)^{-1} d_r x$ is a left Haar measure.

The modulus character also clearly characterizes how conjugation affects measures, since

$$\mathrm{meas}(g\,U g^{-1}) = \delta_G(g)\, \mathrm{meas}(U) \; .$$

The group is called **unimodular** if δ_G is trivial. If K is a compact subgroup of G then the image of any character of K with values in the positive rational numbers has to be a torsion group, hence trivial. Therefore every compact group is unimodular.

I have said that, in practice, integration on G amounts to summation. Let's make this explicit. Suppose that V is a vector space over $\mathbb{Q}$ and that f lies in $C_c^\infty(G,V)$ and is right-invariant under the compact open subgroup K. Then

$$\begin{aligned}\int_G f(x)\,d_rx &= \sum_{G/K}\int_{gK} f(x)\,d_rx \\ &= \sum_{G/K} f(gK)\int_{gK} d_rx \\ &= \sum_{G/K} f(gK)\,\mathrm{meas}(gK) \\ &= \mathrm{meas}(K)\sum_{G/K} f(gK)\delta_G(g)\,.\end{aligned}$$

Assume for the moment a right-invariant Haar measure dg chosen on G. If φ is a smooth function on G with values in V then the formula

$$\langle D_\varphi, f\rangle = \int_G \varphi(x)f(x)\,dx \quad \big(f\in C_c^\infty(G,\mathbb{Q})\big)$$

defines a distribution D_φ on G with values in V, and

$$R_gD_\varphi = D_{R_g\varphi}, \quad L_gD_\varphi = \delta_G(g)^{-1}D_{L_g\varphi}\,.$$

If φ is right-invariant under an open group K, then D_φ will also be right-invariant under K.

Conversely, suppose D to be a smooth distribution, which will be locally right-invariant by some compact open subgroup K. We can associate to it a function value at g in G by the formula

$$\varphi(g) = \frac{\langle D, \mathrm{TR}/_{gK}\rangle}{\mathrm{meas}(gK)}\,.$$

Local right-invariance of D implies immediately that

$$\langle D, \mathrm{TR}/_{gK}\rangle = \langle D, \mathrm{TR}/_{gK_*}\rangle\left(\frac{\mathrm{meas}(gK)}{\mathrm{meas}(gK_*)}\right)\,.$$

for any compact open subgroup K_* of K, which means that the definition of $\varphi(g)$ is independent of the choice of K with respect to which D is right-invariant. It is also straightforward to see that D is then the same as D_φ. We have proven:

[smooth-distribution] **Proposition 5.4.** *Suppose V to be a vector space over $\mathbb{Q}$. Given a right-invariant Haar measure on G, the correspondance $\varphi\mapsto D_\varphi$ is a right-G-covariant isomorphism between the space of smooth functions with values in V and that of V-valued smooth distributions on G.*

There is one large class of distributions we shall often use. Suppose H to be any compact subgroup of G (not necessarily open). Then associated to H is the distribution μ_H (not necessarily smooth) defined by the formula

$$\langle \mu_H, f\rangle = \frac{1}{[H\colon H\cap K]}\sum_{H/H\cap K} f(h)$$

if f in $C_c^\infty(G)$ is right-invariant under K. In effect it evaluates the average value of f on H.

6. Quotients

If G is a locally profinite group and H a closed subgroup, the quotient $H\backslash G$ is also a locally profinite space. It can be covered by translates of the compact quotients $(H \cap K)\backslash K$, where K is a compact open subgroup of G. Such a quotient may be identified with the projective limit of finite quotients $(H \cap K)K_n\backslash K$ where K_n is a sequence of open normal subgroups of K.

[sections] **Lemma 6.1.** *There exists a continuous section of the canonical projection $G \longrightarrow H\backslash G$ over all of $H\backslash G$.*

Proof. It suffices to prove this when G is compact. Let H_n be a shrinking sequence of compact open subgroups of H. The first claim is that for each n there exist continuous sections of the canonical projections

$$H_n\backslash G \longrightarrow H\backslash G\ .$$

This is because if K is a compact open subgroup of G small enough so that $K \cap H \subseteq H_n$ then this projection is a bijection on K and any of its G-translates.

We now look at a tree whose nodes are continuous sections

$$s_n \colon H\backslash G \longrightarrow H_n\backslash G$$

and where the successor of a section is its composition with the canonical projection from $H_{n+1}\backslash G$ to $H_n\backslash G$. Since H_{n+1} has finite index in H_n, such systems form a locally finite tree. By König's Lemma
♣ [konigs-lemma] (Lemma 1.2) there exists an infinite branch in this tree, hence a sequence of compatible continuous sections $s_n \colon H\backslash G \to H_n\backslash G$, hence a map from $H\backslash G$ to the projective limit of the $H_n\backslash G$, which can be canonically identified with G. □

[h-projection] **Lemma 6.2.** *If H be a closed subgroup of G and $d_r h$ be a right Haar measure on H, then the map taking f to $\overline{f}$ where*

$$\overline{f}(g) = \int_H f(hg)\, d_r h$$

is a surjection from $C_c^\infty(G)$ to $C_c^\infty(H\backslash G)$, and commutes with the right action of G.

Proof. Every function in $C_c^\infty(H\backslash G)$ is a linear combination of the functions $\mathrm{TR}/_{HxK}$ for some x in G and compact open subgroup K. This is the image under the map given of $\mathrm{meas}(H \cap xKx^{-1}, d_r h)^{-1}\mathrm{TR}/_{xK}$. □

If H and G are both unimodular then invariant Haar measures on G and H will determine a G-invariant measure on the quotient $H\backslash G$ (as we'll see in a moment). But in the situation we'll be mostly concerned with, G will be unimodular and H not, and one has to be more careful.

An indirect approach seems best. Suppose G and H to be arbitrary locally profinite groups, and suppose that we are given right Haar measures on both. If F is a smooth linear functional on $C_c^\infty(H\backslash G)$, then we can define a smooth distribution D_F on G according to the formula

$$\langle D_F, f\rangle = \langle F, \overline{f}\rangle\ .$$

The map taking F to D_F is injective, since integration defines a surjective map from $C_c(G)$ to $C_c(H\backslash G)$. Since

$$\overline{L_h f}(g) = \int_H f(h^{-1}xg)\, d_r x = \delta_H(h)\overline{f}(g)$$

we must have

$$L_h D_F = \delta_H^{-1}(h) D_F$$

for all h in H. Since D_F is smooth it is associated to a smooth function φ on G, and this condition on D translates to the condition on φ that

$$\delta_G(h)^{-1} L_h\varphi = \delta_H(h)^{-1}\varphi, \quad \varphi(hg) = \delta_H(h)\delta_G(h)^{-1}\varphi(g)\ .$$

for all h in H, g in G. In fact:

[h-g-distributions] **Proposition 6.3.** *Given right-invariant Haar measures on H and G, the space of smooth distributions on $H\backslash G$ may be identified with the space of smooth functions φ on G such that*

$$\varphi(hg) = \delta_H(h)\delta_G(h)^{-1}\varphi(g) \; .$$

Proof. It remains to be shown that if φ satisfies this condition then

$$\int_G \varphi(x) f(x) \, d_r x$$

depends only on $\overline{f}$. Suppose K to be a compact open subgroup of G such that f is right-invariant under K and that φ is right-invariant under K on the support of f. Then

$$\int_G \varphi(x) f(x) \, d_r x = \sum_{H\backslash G/K} \int_{HgK} \varphi(x) f(x) \, d_r x$$

and since $h \mapsto hgK$ is a bijection between $H/H \cap gKg^{-1}$ and HgK/K

$$\begin{aligned} \int_{HgK} \varphi(x) f(x) \, d_r x &= \sum_{H/H\cap gKg^{-1}} \varphi(hg) f(hg) \operatorname{meas}(hgK) \\ &= \sum_{H/H\cap gKg^{-1}} \delta_H(h)\delta_G^{-1}(h)\varphi(g) f(hg)\delta_G(h) \operatorname{meas}(gK) \\ &= \operatorname{meas}(gK)\varphi(g) \sum_{H/H\cap gKg^{-1}} \delta_H(h) f(hg) \\ &= \varphi(g) \left[\frac{\operatorname{meas}_G(gK)}{\operatorname{meas}_H(H \cap gKg^{-1})} \right] \int_H f(hg) d_r h \end{aligned}$$

which concludes the proof. ▯

Define $\Omega^\infty(H\backslash G)$ to be the space of smooth distributions on $H\backslash G$, which is also the space of smooth **one-densities** on $H\backslash G$. Define $\Omega_c^\infty(H\backslash G)$ to be the subspace of distributions in $\Omega^\infty(H\backslash G)$ that are of compact support. Such functions can be paired with the constant function 1 on $H\backslash G$. For any $\mathbb{Q}$ vector space V, define similarly $\Omega^\infty(H\backslash G, V) = \Omega^\infty(H\backslash G) \otimes V$. As a consequence:

[one-densities] **Corollary 6.4.** *Let V be a vector space over $\mathbb{Q}$. With notation as in the Proposition, there exists a unique right G-invariant linear map $d_r\overline{x}$ from $\Omega_c^\infty(H\backslash G, V)$ to V, which will be denoted simply by integration, such that*

$$\int_G f(g) \, d_r x = \int_{H\backslash G} d_r \overline{x} \int_H \delta_H^{-1}(h)\delta_G(h) f(hx) \, d_r h \; .$$

for any f in $C_c^\infty(G)$.