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Essays on the structure of reductive groups

Root systems

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Suppose G to be a reductive group defined over an algebraically closed field k. There exist in G maximal subgroups which are **tori**, that is to say algebraically isomorphic to a product of copies of $k^\times$. Fix one of them, say A. The adjoint action of A on $\mathfrak{g}$ is the direct sum of eigenspaces. The 0-eigenspace is $\mathfrak{a}$, and the rest are of dimension one. An eigencharacter $\lambda \neq 0$ is called a **root** of the Lie algebra. Let Σ be the set of all roots, a subset of the group

$$X^*(A) = \mathrm{Hom}(A, \mathbb{G}_m)$$

of algebraic characters of A. Associated to each root λ is an eigenspace $\mathfrak{g}_\lambda$ and a corresponding subgroup U_λ of G isomorphic to the additive group $\mathbb{G}_a$. There also exist in G maximal solvable subgroups containing A, called **Borel subgroups**. Such a group B contains a normal unipotent subgroup U such that the quotient B/U is isomorphic to A. Fix one of these Borel subgroups.

The standard example is $G = \mathrm{GL}_n$, with A the group of diagonal matrices, B that of upper triangular matrices, U the subgroup of unipotent matrices in B. If ε_i is the character

$$\begin{bmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ 0 & 0 & t_3 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & t_n \end{bmatrix} \longmapsto t_i \,,$$

the roots are $\varepsilon_i/\varepsilon_j$ for $i \neq j$, written additively $\varepsilon_i - \varepsilon_j$.

The Lie algebra of U is the direct sum of root spaces $\mathfrak{g}_\lambda$. If Σ^+ is the set of roots occurring, which are called **positive roots**, then Σ is the disjoint union of Σ^+ and $\Sigma^- = -\Sigma^+$. There exists in Σ^+ a subset Δ with the property that every $\lambda > 0$ is a unique integral linear combination of elements of Δ with non-negative coefficients. It is called a **basis** of Σ. For each $\lambda > 0$, choose an isomorphism u_λ of $\mathbb{G}_a$ with U_λ. There then exists a unique isomorphism $u_{-\lambda}$ of $\mathbb{G}_a$ with $U_{-\lambda}$ such that

$$u_\lambda(-x)u_{-\lambda}(1/x)u_\lambda(-x)$$

lies in the normalizer of A. There exists a unique homomorphism $\lambda^\vee$ from SL_2 to G taking

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \longmapsto u_\lambda(x), \quad \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \longmapsto u_{-\lambda}(x)\,.$$

The condition on $u_{-\lambda}$ is motivated by the equation

$$\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-xy & -2x+x^2y \\ y & 1-xy \end{bmatrix} \,,$$

which shows that the left hand side will lie in the normalizer of the diagonal matrices in SL_2 if and only if $y = 1/x$. When $y = 1/x$, we have

$$\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/x & 1 \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -x \\ 1/x & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \text{ say } s_\lambda(x)\,.$$

Let $X_*(A)$ be the group of **cocharacters** of A, the group of algebraic homomorphisms from $\mathbb{G}_m$ to A. If we assign A coordinates (x_i) then the characters are all of the form $\prod x_i^{m_i}$ and the cocharacters of the form $t \mapsto (t^{m_i})$, so both groups $X^*(A)$ and $X_*(A)$ are free modules over $\mathbb{Z}$. They are canonically dual to each other. Given $\lambda^\vee$ in $X_*(A)$ and μ in $X_*(A)$, the pairing is defined by the equation

$$\mu\big(\lambda^\vee(x)\big) = x^{\langle \mu, \lambda^\vee \rangle}$$

for x in $k^\times$. The homomorphism $\lambda^\vee$ from SL_2 to G determines also the cocharacter

$$x \longmapsto \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \stackrel{\lambda^\vee}{\longmapsto} A \subset G\,,$$

which I'll also express as $\lambda^\vee$.

The **Weyl group** W of G with respect to A is the quotient of the normalizer $N_G(A)$ by A. For a fixed λ, all the $s_\lambda(x)$ have the same image in W. The group W is generated by the images of the $s_\alpha(1)$ for α in Δ. The Weyl group W acts on $X^*(A)$ by conjugation, and the $s_\lambda(1)$ act as reflections. Elements of W, and in particular the reflections s_λ, take Σ into itself. For $G = \mathrm{GL}_n$, for example, the reflections swap ε_i and ε_j.

As we shall see in the next section, this means that reductive groups give rise to **root systems**. Important properties of these groups follow immediately from properties of their root systems, which are worth studying on their own. This chapter is concerned with the structure of abstract root systems. Applications to the structure of reductive groups will come later.

1. Definitions

I first recall that a **reflection** in a finite-dimensional vector space is a linear transformation that fixes vectors in a hyperplane, and acts on a complementary line as multiplication by -1. Every reflection can be written as

$$v \longmapsto v - \langle f, v \rangle f^\vee$$

for some linear function $f \neq 0$ and vector $f^\vee$ with $\langle f, f^\vee \rangle = 2$. The function f is unique up to non-zero scalar.

I define a **root system** to be

- a quadruple $(V, \Sigma, V^\vee, \Sigma^\vee)$ where V is a finite-dimensional vector space over $\mathbb{R}$, $V^\vee$ its linear dual, Σ a finite subset of $V - \{0\}$, $\Sigma^\vee$ a finite subset of $V^\vee - \{0\}$;
- a bijection $\lambda \mapsto \lambda^\vee$ of Σ with $\Sigma^\vee$

subject to these conditions:

- for each λ in Σ, $\langle \lambda, \lambda^\vee \rangle = 2$;
- for each λ and μ in Σ, $\langle \lambda, \mu^\vee \rangle$ lies in $\mathbb{Z}$;
- for each λ the reflection

$$s_\lambda\colon\ v \longmapsto v - \langle v, \lambda^\vee \rangle \lambda$$

takes Σ to itself. Similarly the reflection

$$s_{\lambda^\vee}\colon\ v \longmapsto v - \langle \lambda, v \rangle \lambda^\vee$$

in $V^\vee$ preserves $\Sigma^\vee$.

Sometimes the extra condition that Σ span V is imposed, but often in the subject one is interested in subsets of Σ which again give rise to root systems and do not possess this property even if the original does.

In case V is spanned by $V(\Sigma)$, the condition that $\Sigma^\vee$ be reflection-invariant is redundant.

One immediate consequence of the definition is that if λ is in Σ so is $-\lambda = s_\lambda \lambda$.

The elements of Σ are called the **roots** of the system, those of $\Sigma^\vee$ its **coroots**. The **rank** of the system is the dimension of V, and the **semi-simple rank** is that of the subspace $V(\Sigma)$ of V spanned by Σ. The system is called **semi-simple** if Σ spans V.

If $(V, \Sigma, V^\vee, \Sigma^\vee)$ is a root system, so is its **dual** $(V^\vee, \Sigma^\vee, V, \Sigma)$.

The **Weyl group** of the system is the group W generated by the reflections s_λ. As a group, it is isomorphic to the Weyl group of the dual system, because:

[wdual] **Proposition 1.1.** *The contragredient of s_λ is $s_{\lambda^\vee}$.*

Proof. It has to be shown that

$$\langle s_\lambda u, v \rangle = \langle u, s_{\lambda^\vee} v \rangle \ .$$

The first is

$$\langle u - \langle u, \lambda^\vee \rangle \lambda, v \rangle = \langle u, v \rangle - \langle u, \lambda^\vee \rangle \langle \lambda, v \rangle$$

and the second is

$$\langle u, v - \langle \lambda, v \rangle \lambda^\vee \rangle = \langle u, v \rangle - \langle \lambda, v \rangle \langle u, \lambda^\vee \rangle \ . \ \square$$

Define the linear map

$$\rho\colon V \longrightarrow V^\vee, \quad v \longmapsto \sum_{\lambda \in \Sigma} \langle v, \lambda^\vee \rangle \lambda^\vee$$

and define a symmetric dot product on V by the formula

$$u \bullet v = \langle u, \rho(v) \rangle = \sum_{\lambda \in \Sigma} \langle u, \lambda^\vee \rangle \langle v, \lambda^\vee \rangle \ .$$

The semi-norm

$$\|v\|^2 = v \bullet v = \sum_{\lambda \in \Sigma} \langle v, \lambda^\vee \rangle^2$$

is positive semi-definite, vanishing precisely on the v with $\langle v, \lambda^\vee \rangle = 0$ for all λ in Σ. In particular $\|\lambda\| > 0$ for all roots λ. Since $\Sigma^\vee$ is W-invariant, the semi-norm $\|v\|^2$ is also W-invariant. Its radical is the space of v annihilated by $\Sigma^\vee$.

[norms] **Proposition 1.2.** *For every root λ*

$$\|\lambda\|^2 \, \lambda^\vee = 2\rho(\lambda) \ .$$

The important consequence of this is that some scalar multiple of $\lambda \mapsto \lambda^\vee$ is the restriction of a linear map to Σ.

Proof. For every μ in Σ

$$\begin{aligned} s_{\lambda^\vee}\mu^\vee &= \mu^\vee - \langle \lambda, \mu^\vee \rangle \lambda^\vee \\ \langle \lambda, \mu^\vee \rangle \lambda^\vee &= \mu^\vee - s_{\lambda^\vee}\mu^\vee \\ \langle \lambda, \mu^\vee \rangle^2 \lambda^\vee &= \langle \lambda, \mu^\vee \rangle \mu^\vee - \langle \lambda, \mu^\vee \rangle s_{\lambda^\vee}\mu^\vee \\ &= \langle \lambda, \mu^\vee \rangle \mu^\vee + \langle s_\lambda \lambda, \mu^\vee \rangle s_{\lambda^\vee}\mu^\vee \\ &= \langle \lambda, \mu^\vee \rangle \mu^\vee + \langle \lambda, s_{\lambda^\vee}\mu^\vee \rangle s_{\lambda^\vee}\mu^\vee \end{aligned}$$

But since $s_{\lambda^\vee}$ is a bijection of $\Sigma^\vee$ with itself, we can conclude by summing over μ in Σ. ▯

[dot-product] **Corollary 1.3.** *For every v in V and root λ*

$$\langle v, \lambda^\vee \rangle = 2 \left(\frac{v \bullet \lambda}{\lambda \bullet \lambda} \right) .$$

Thus the formula for the reflection s_λ is that for an orthogonal reflection

$$s_\lambda v = v - 2 \left(\frac{v \bullet \lambda}{\lambda \bullet \lambda} \right) .$$

[equi-ranks] **Corollary 1.4.** *The semi-simple ranks of a root system and of its dual are equal.*

Proof. The map

$$\lambda \longmapsto \|\lambda\|^2 \lambda^\vee$$

is the same as the linear map 2ρ, so ρ is a surjection from $V(\Sigma)$ to $V^\vee(\Sigma^\vee)$. Apply the same reasoning to the dual system to see that $\rho^\vee \circ \rho$ must be an isomorphism, hence ρ an injection as well. ▯

[spanning] **Corollary 1.5.** *The space $V(\Sigma)$ spanned by Σ is complementary to the space $V_{\Sigma^\vee}$ annihilated by all $\lambda^\vee$ in $\Sigma^\vee$.*

Proof. Because the kernel of ρ is $V_{\Sigma^\vee}$. ▯

[weyl-finite] **Corollary 1.6.** *The Weyl group is finite.*

Proof. It fixes all v annihilated by $\Sigma^\vee$ and therefore embeds into the group of permutations of Σ. ▯

[wvee] **Corollary 1.7.** *For all roots λ and μ*

$$(s_\lambda \mu)^\vee = s_{\lambda^\vee} \mu^\vee .$$

Proof. We have

$$\begin{aligned} 2\rho(s_\lambda\mu) &= \|s_\lambda\mu\|^2 (s_\lambda\mu)^\vee \\ &= \|\mu\|^2 (s_\lambda\mu)^\vee \\ &= \rho(\mu - \langle \mu, \lambda^\vee \rangle \lambda) \\ &= \rho(\mu) - \langle \mu, \lambda^\vee \rangle \rho(\lambda) \\ &= \|\mu\|^2 \mu^\vee - \langle \mu, \lambda^\vee \rangle \|\lambda\|^2 \lambda^\vee \\ \text{so all in all} \quad \|\mu\|^2 (s_\lambda\mu)^\vee &= \|\mu\|^2 \mu^\vee - \langle \mu, \lambda^\vee \rangle \|\lambda\|^2 \lambda^\vee . \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\mu\|^2 s_{\lambda^\vee} \mu^\vee &= \|\mu\|^2 (\mu^\vee - \langle \lambda, \mu^\vee \rangle \lambda^\vee \\ &= \|\mu\|^2 \mu^\vee - \|\mu\|^2 \langle \lambda, \mu^\vee \rangle \lambda^\vee \end{aligned}$$

so it suffices to verify that

$$\|\mu\|^2\langle\lambda,\mu^\vee\rangle = \|\lambda\|^2\langle\mu,\lambda^\vee\rangle$$

which reduces in turn to the known identity

$$\langle\lambda,\rho(\mu)\rangle = \langle\mu,\rho(\lambda)\rangle\,. \ \square$$

[semi-simple] **Proposition 1.8.** *The quadruple* $(V(\Sigma),\Sigma,V^\vee(\Sigma^\vee),\Sigma^\vee)$ *is a root system.*

It is called the **semi-simple root system** associated to the original.

[intersection] **Proposition 1.9.** *Suppose* U *to be a vector subspace of* V*,* $\Sigma_U = \Sigma\cap U$*,* $\Sigma_U^\vee = (\Sigma_U)^\vee$*. Then* $(V,\Sigma_U,V^\vee,\Sigma_U^\vee)$ *is a root system.*

♣ [wvee] *Proof.* The only tricky point is to show that $\Sigma_U^\vee$ is stable under reflections. This follows from Corollary 1.7. $\square$

2. Hyperplane partitions

Suppose $(V,\Sigma,V^\vee,\Sigma^\vee)$ to be a root system. Associated to it are two partitions of $V^\vee$ by hyperplanes. The first is that of hyperplanes $\lambda^\vee = 0$ for $\lambda^\vee$ in $\Sigma^\vee$. The other is by hyperplanes $\lambda = k$ in $V^\vee$ where λ is a root and k an integer. Each of these configurations is stable under Euclidean reflections in these hyperplanes. Our goal in this section and the next few is to show that the connected components of the complement of the hyperplanes in either of these are open fundamental domains for the group generated by these reflections, and to relate geometric properties of this partition to combinatorial properties of this group. In this section we shall look more generally at the partition of Euclidean space associated to an arbitrary locally finite family of hyperplanes, an exercise concerned with rather general convex sets.

Thus suppose for the moment V to be any Euclidean space, $\mathfrak{h}$ to be a locally finite collection of affine hyperplanes in V.

A connected component C of the complement of $\mathfrak{h}$ in V will be called a **chamber**. If H is in $\mathfrak{h}$ then C will be contained in exactly one of the two open half-spaces determined by H, since C cannot intersect H. Call this half space $D_H(C)$.

[allh] **Lemma 2.1.** *If* C *is a chamber then*

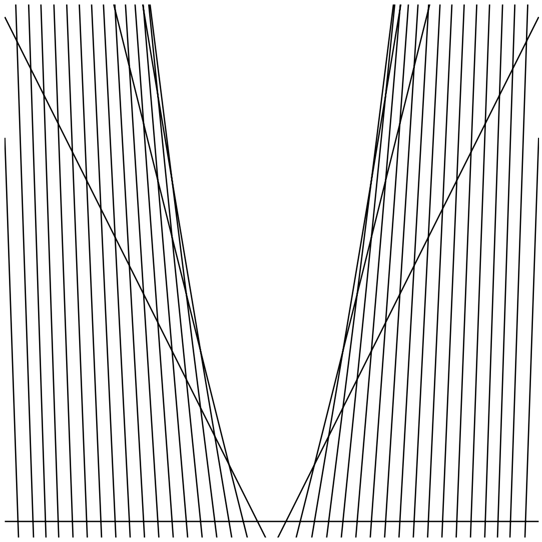
$$C = \bigcap_{H\in\mathfrak{h}} D_H(C)\,.$$

Proof. Of course C is contained in the right hand side. On the other hand, suppose that x lies in C and that y is contained in the right hand side. If H is in $\mathfrak{h}$ then the closed line segment $[x,y]$ cannot intersect H, since then C and y would lie on opposite sides. So y lies in C also. $\square$

Many of the hyperplanes in $\mathfrak{h}$ will be far away, and they can be removed from the intersection without harm. Intuitively, only those hyperplanes that hug C closely need be considered, and the next (elementary) result makes this precise.

A **panel** of C is a face of C of codimension one, a subset of codimension one in the boundary of $\overline{C}$. The support of a panel will be called a **wall**. A panel with support H is a connected component of the complement of the union of the $H_*\cap H$ as H_* runs through the other hyperplanes of $\mathfrak{h}$. Both chambers and their faces are convex.

A chamber might very well have an infinite number of panels, for example if $\mathfrak{h}$ is the set of tangent lines to the parabola $y = x^2$ at points with integral coordinates.



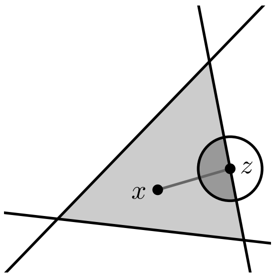
[wallsfirst] **Lemma 2.2.** *Suppose C to be a chamber, H_* in $\mathfrak{h}$, and*

$$C_* = \bigcap_{H \neq H_*} D_H(C)\,.$$

If $C_* \neq C$ then H_* is a bounding hyperplane of C.

The set C_* will be the union of two chambers containing a common panel.

Proof. Pick x in C and y in C_* but not in C. The closed segment $[x, y]$ must meet a hyperplane of $\mathfrak{h}$, but this can only can only be H_*. Let z be the intersection.



Then on the one hand a small enough neighbourhood of z will not meet any other hyperplane of $\mathfrak{h}$, and on the other the interval $[x, z)$ must lie in C. Therefore H_* is a panel of C. ∎

[chambers] **Proposition 2.3.** *A chamber is the intersection of half-spaces determined by its panels.*

That is to say, there exists a collection of affine functions f such that C is the intersection of the regions $f > 0$, and each hyperplane $f = 0$ for f in this collection is a panel of C.

Proof. Suppose H_* to be a hyperplane in $\mathfrak{h}$ that is not a panel of C then by the previous Lemma, $C = \cap_{H \neq H_*} D_H(C)$. An induction argument shows that if S is any finite set of hyperplanes in $\mathfrak{h}$ of which none are walls, then $C = \cap_{H \notin S} D_H(C)$.

Suppose that y lies in the intersection of all the $\cap_H D_H(C)$ as H varies over the walls of C. Choose x in C. Let T be set of hyperplanes that $[x, y]$ crosses. It is finite, since $\mathfrak{h}$ is locally finite. It cannot cross any walls, by choice of y. But then T must be empty and y actually lies in C. ∎

3. Discrete reflection groups

Much of this section can be found most conveniently in §V.3 of [Bourbaki:1968], but originates with Coxeter. The motivation for the investigation here is that if Σ is a set of roots in a Euclidean space V, then there are two associated families of hyperplanes: (1) the linear hyperplanes $\alpha = 0$ for α in Σ and (2) the affine hyperplanes $\alpha + k = 0$ for α in Σ and k an integer. Many of the properties of root systems are a direct consequence of the geometry of hyperplane arrangements rather than the algebra of roots, and it is useful to isolate geometrical arguments. Affine configurations play an important role in the structure of $\mathfrak{p}$-adic groups.

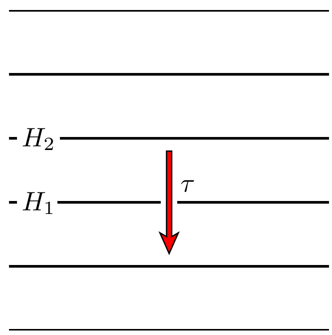
For any hyperplane H in a Euclidean space let s_H be the orthogonal reflection in H. A **Euclidean root configuration** is a locally finite collection $\mathfrak{h}$ of hyperplanes that's stable under each of the orthogonal reflections s_H with respect to H in $\mathfrak{h}$. The group W generated by these reflections is called the **Weyl group** of the configuration. Each hyperplane is defined by an equation $\lambda_H(v) = f_H \bullet v + k = 0$ where f_H may be taken to be a unit vector. The vector $\mathrm{GRAD}(\lambda_H) = f_H$ is uniquely determined up to scalar multiplication by ± 1. We have the explicit formula

$$s_H v = v - 2\lambda_H(v)\, f_H \,.$$

The **essential dimension** of the system is the dimension of the vector space spanned by the gradients f_H.

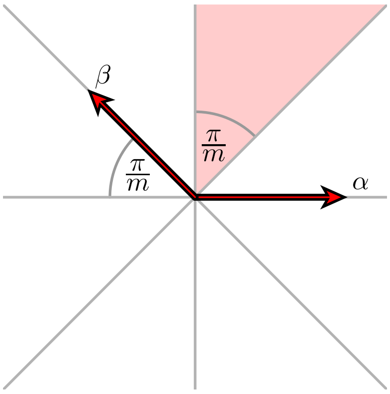
A **chamber** is one of the connected components of the complement of the hyperplanes in the collection. All chambers are convex and open.

Two configurations will be considered equivalent if they are the same up to an affine transformation. Let's look first at those configurations for which the Weyl group is a **dihedral group**, one generated by orthogonal reflections in two hyperplanes. There are two cases, according to whether the hyperplanes are parallel or not.



The first case is easiest. Let H_1 and H_2 be the two parallel hyperplanes. The product $\tau = s_{H_1} s_{H_2}$ is a translation, and the hyperplanes $\tau^m(H_1)$ and $\tau^n(H_2)$ form a Euclidean root configuration. Conversely, any Euclidean root configuration in which all the hyperplanes are parallel arises in this way. We are dealing here with an essentially one-dimensional configuration. The group W is the infinite dihedral group.

Now suppose H_1 and H_2 to be two hyperplanes intersecting in a space L of codimension 2. The entire configuration is determined by that induced on the quotient V/L, so we may as well assume V to be of dimension two. The 'hyperplanes' in this case are just lines. The product $\tau = s_{H_1} s_{H_2}$ is a rotation, say through angle θ. The hyperplanes $\tau^m(H_1)$ and $\tau^n(H_2)$ will form a locally finite collection if and only if τ has finite order. In this case θ will be a multiple of $2\pi/m$ for some $m > 0$, and replacing one of the hyperplanes if necessary we may as well assume $\theta = 2\pi/m$. In this case the angle between H_1 and H_2 will be π/m. There are m hyperplanes in the whole collection. In the following figure, $m = 4$.



Suppose C to be a chamber of the configuration, and let α and β be unit vectors such that C is the region of all x where $\alpha \bullet x > 0$ and $\beta \bullet x > 0$. As the figure illustrates, $\alpha \bullet \beta = -\cos(\pi/m)$. Conversely, to each $m > 1$ there exists an essentially unique Euclidean root configuration for which W is generated by two reflections in the walls of a chamber containing an angle of π/m. The group W has order $2m$, and is called the **dihedral** group of order $2m$.

In summary for the dihedral case:

[rank-two] **Proposition 3.1.** *Suppose α and β to be two affine functions, s_α and s_β the orthogonal reflections in the hyperplanes $\alpha = 0$ and $\beta = 0$. If the region in which $\alpha \geq 0$ and $\beta \geq 0$ is a fundamental domain for the group generated by s_α and s_β, then*

$$\text{GRAD}(\alpha) \bullet \text{GRAD}(\beta) = -\cos(\pi/m)$$

where m is the order (possibly ∞) of the rotation $s_\alpha s_\beta$.

Now assume again that $\mathfrak{h}$ is an arbitrary Euclidean root configuration. Fix a chamber C, and let Δ be a set of affine functions α such that $\alpha = 0$ is a wall of C and $\alpha > 0$ on C. For each subset $\Theta \subseteq \Delta$, let W_Θ be the subgroup of W generated by the s_α with α in Θ.

[non-pos] **Proposition 3.2.** *For any distinct α and β in Δ, $\text{GRAD}(\alpha) \bullet \text{GRAD}(\beta) \leq 0$.*

Proof. The group $W_{\alpha,\beta}$ generated by s_α and s_β is dihedral. If P and Q are points of the faces defined by α and β, respectively, the line segment from P to Q crosses no hyperplane of $\mathfrak{h}$. The region $\alpha \bullet x > 0$,
♣ [rank-two] $\beta \bullet x > 0$ is therefore a fundamental domain for $W_{\alpha,\beta}$. Apply Proposition 3.1. ∎

♣ [chambers] Suppose C to be a chamber of the hyperplane partition. According to Proposition 2.3, C is the intersection of the open half-spaces determined by its walls, the affine supports of the parts of its boundary of codimension one. Reflection in any two of its walls will generate a dihedral group.

[walls-finite] **Corollary 3.3.** *The number of panels of a chamber is finite.*

Proof. If V has dimension n, the unit sphere in V is covered by the $2n$ hemispheres $x_i > 0$, $x_i < 0$. By
♣ [non-pos] Proposition 3.2, each one contains at most one of the $\text{GRAD}(\alpha)$ in Δ. ∎

[separating-finite] **Lemma 3.4.** *If $\mathfrak{h}$ is a locally finite collection of hyperplanes, the number of H in $\mathfrak{h}$ separating two chambers is finite.*

Proof. The closed line segment connecting them can meet only a finite number of H in $\mathfrak{h}$. ∎

[chambers-transitive] **Proposition 3.5.** *The group W_Δ acts transitively on the set of chambers.*

Proof. By induction on the number of root hyperplanes separating two chambers C and C_*, which si
♣ [chambers] finite by the previous Lemma. If it is 0, then by Proposition 2.3 $C = C_*$. Otherwise, one of the walls H of C_* separates them, and the number separating $s_H C_*$ from C will be one less. Apply induction. ∎

The next several results will tell us that W is generated by the reflections in the walls of C, that the closure of C is a fundamental domain for W, and (a strong version of this last fact) that if F is a face of C then the group of w in W fixing such that $F \cap w(F) \neq \emptyset$ then w lies in the subgroup generated by the reflections in the walls of C containing v, which all in fact fix all points of F. Before I deal with these, let me point out at the beginning that the basic point on which they all depend is the trivial observation that *if $w \neq I$ in W fixes points on a wall H then it must be the orthogonal reflection s_H.*

[generate] **Corollary 3.6.** *The reflections in S generate the group W.*

Proof. It suffices to show that every s_λ lies in W_Δ.

Suppose F to be a panel in hyperplane $\lambda = 0$. According to the Proposition, if this panel bounds C_* we can find w in W_Δ such that $wC_* = C$, hence $w(F)$ lies in $\overline{C}$, and therefore must equal a panel of C. Then $w^{-1}s_\alpha w$ fixes the points of this panel and therefore must be s_λ. ▯

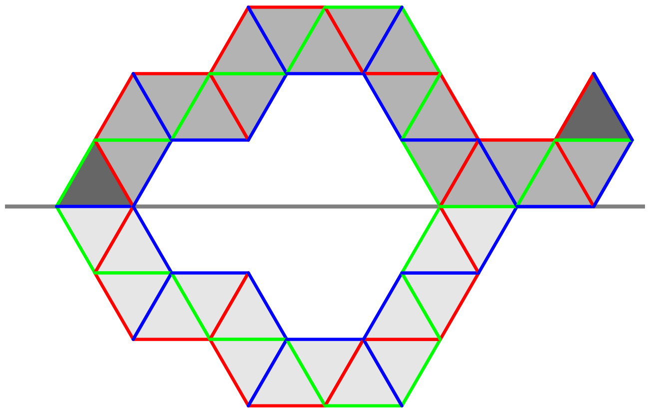
Given any hyperplane partition, a **gallery** between two chambers C and C_* is a chain of chambers $C = C_0, C_1, \ldots, C_n = C_*$ in which each two successive chambers share a common face of codimension 1. The integer n is called the **length** of the gallery. I'll specify further that any two successive chambers in a gallery are distinct, or in other words that the gallery is not **redundant**. The gallery is called **minimal** if there exist no shorter galleries between C_0 and C_n. The **combinatorial distance** between two chambers is the length of a minimal gallery between them.

Expressions $w = s_1 s_2 \ldots s_n$ with each s_i in S can be interpreted in terms of galleries. There is in fact a bijective correspondence between such expressions and galleries linking C to $w\,C$. This can be seen inductively. The trivial expression for 1 in terms of the empty string just comes from the gallery of length 0 containing just $C_0 = C$. A single element $w = s_1$ of S corresponds to the gallery $C_0 = C$, $C_1 = s_1 C$. If we have constructed the gallery for $w = s_1 \ldots s_{n-1}$, we can construct the one for $s_1 \ldots s_n$ in this fashion: the chambers C and $s_n C$ share the wall $\alpha = 0$ where $s_n = s_\alpha$, and therefore the chambers wC and $ws_n C$ share the wall $w\alpha = 0$. The pair $C_{n-1} = wC$, $C_n = ws_n C$ continue the gallery from C to $ws_n C$.

This associates to every expression $s_1 \ldots s_n$ a gallery, and the converse construction is straightforward.

[fixC] **Proposition 3.7.** *If $wC = C$ then $w = 1$.*

Proof. I'll prove that if $w = s_1 \ldots s_n$ with $wC = C$ and $n > 0$ then there exists some $1 \le i \le n$ with $w = s_2 \ldots s_{i-1} s_{i+1} \ldots s_n$. By recursion, this leads to $w = 1$.



Let H be the hyperplane in which s_1 reflects. Let $w_i = s_1 \ldots s_i$. The path of chambers $w_1 C$, $w_2 C$, …crosses H at the very beginning and must cross back again. Thus for some i we have $w_{i+1} C = w_i s_{i+1} C = s_1 w_i C$. But if $y = s_2 \ldots s_i$ we have $s_{i+1} = y^{-1} s_1 y$, and $w_{i+1} = y$, so $w = s_1 y s_{i+1} \ldots s_n = y s_{i+2} \ldots s_n$. ▯

[ws] **Proposition 3.8.** *For any w in W and s in S, if $\langle\alpha_s, wC\rangle < 0$ then $\ell(sw) < \ell(w)$ and if $\langle\alpha_s, wC\rangle > 0$ then $\ell(sw) > \ell(w)$.*

Proof. Suppose $\langle\alpha_s, wC\rangle < 0$. Then C and wC are on opposite sides of the hyperplane $\alpha_s = 0$. If $C = C_0, \ldots, C_n = wC$ is a minimal gallery from C to wC, then for some i C_i is on the same side of $\alpha_s = 0$ as C but C_{i+1} is on the opposite side. The gallery $C_0, \ldots, C_i, sC_{i+2}, \ldots sC_n = swC$ is a gallery of shorter length from C to swC, so $\ell(sw) < \ell(w)$.

If $\langle\alpha_s, wC\rangle > 0$ then $\langle\alpha_s, swC\rangle < 0$ and hence $\ell(w) = \ell(ssw) < \ell(sw)$. ▯

[stabilizers] **Proposition 3.9.** *If v and wv both lie in $\overline{C}$, then $wv = v$ and w belongs to the group generated by the reflections in S fixing v.*

Proof. By induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$ and the result is trivial.

If $\ell(w) > 1$ then let $x = sw$ with $\ell(x) = \ell(w) - 1$. Then C and wC are on opposite sides of the hyperplane ♣ [ws] $\alpha_s = 0$, by Proposition 3.8. Since v and wv both belong to $\overline{C}$, the intersection $\overline{C} \cap w\overline{C}$ is contained in the hyperplane $\alpha_s = 0$ and wv must be fixed by s. Therefore $wv = xv$. Apply the induction hypothesis. ▯

If Θ is a subset of Δ then let C_Θ be the face of $\overline{C}$ where $\alpha = 0$ for α in Θ, $\alpha > 0$ for α in Δ but not in Θ. If F is a face of any chamber, the Proposition tells u it will be W-equivalent to a unique $\Theta \subseteq \Delta$. The faces of chambers are therefore canonically labeled by subsets of Δ.

Let

$$R_w = \{\lambda > 0 \mid w\lambda < 0\}$$
$$L_w = \{\lambda > 0 \mid w^{-1}\lambda < 0\}$$

♣ [separating-finite] Of course $L_w = R_{w^{-1}}$. According to Lemma 3.4, the set R_w determines the root hyperplanes separating C from $w^{-1}C$, and $|R_w| = |L_w| = \ell(w)$.

An expression for w as a product of elements of S is **reduced** if it is of minimal length. The length of w is the length of a reduced expression for w as products of elements of S. Minimal galleries correspond to reduced expressions. The two following results are easy deductions:

[rw] **Proposition 3.10.** *For x and y in W, $\ell(xy) = \ell(x) + \ell(y)$ if and only if R_{xy} is the disjoint union of $y^{-1}R_x$ and R_y.*

Finally, suppose that we are considering a root system, so that there are only a finite number of hyperplanes in the root configuration, and all pass through the origin. Since $-C$ is then also a chamber:

[longest] **Proposition 3.11.** *There exists in W a unique element w_ℓ of maximal length, with $w_\ell C = -C$. For $w = w_\ell$, $R_w = \Sigma^+$.*

This discussion also leads to a simple and useful algorithm to find an expression for w in W as a product of elements in S. For each v in $V^\vee$ define

$$v_\alpha = \langle\alpha, v\rangle$$

for α in Δ. Let $\widehat{\rho}$ be a vector in $V^\vee$ with $\widehat{\rho}_\alpha = \langle\alpha, \widehat{\rho}\rangle = 1$ for all α in Δ. It lies in the chamber C and $u = w\widehat{\rho}$ lies in wC, so $\ell(s_\beta w) < \ell(w)$ if and only if $\langle\beta, u\rangle < 0$. If that occurs, we apply s_β to u, calculating

$$\langle\alpha, s_\beta u\rangle = \langle\alpha, u - \langle\beta, u\rangle\beta^\vee\rangle = \langle\alpha, u\rangle - \langle\beta, u\rangle\langle\alpha, \beta^\vee\rangle = u_\alpha - \langle\alpha, \beta^\vee\rangle u_\beta \,.$$

In effect, we replace the original u by this new one. And then we continue, stopping only when $u_\alpha > 0$ for all α.

I had better make clear some of the consequences of this discussion for a set of roots Σ. First of all, the reflections s_λ for λ in a root system preserve the root hyperplanes associated to a root system. If C is a connected component of the complement of these hyperplanes, then the group W the whole group of root reflections generate is in fact generated by the set S of **elementary reflections** in the walls of C.

[W-transitive] **Proposition 3.12.** *If Δ is the set of roots vanishing on these walls of the chamber C, then every root in Σ is in the W-orbit of Δ.*

Proof. This is because W acts transitively on the chambers.

In the older literature one frequently comes across another way of deducing the existence of a base Δ for positive roots. Suppose $V = V(\Sigma)$, say of dimension ℓ, and assume it given a coordinate system. Linearly order V **lexicographically**: $(x_i) < (y_i)$ if $x_i = y_i$ for $i < m$ but $x_m < y_m$. Informally, this is **dictionary order**. For example, $(1, 2, 2) < (1, 2, 3)$. This order is translation-invariant. [Satake:1951] remarks that this is the only way to define a linear, translation-invariant order on a real vector space.

Define Σ^+ to be the subset of roots in Σ that are positive with respect to the given order. Define α_1 to be the least element of Σ^+, and for $1 < k \le \ell$ inductively define α_k to be the least element of Σ^+ that is not in the linear span of the α_i with $i < k$.

The following seems to be first found in [Satake:1951].

[satake] **Proposition 3.13.** *Every root in Σ^+ can be expressed as a positive integral combination of the α_i.*

Proof. It is easy to see that if Δ is a basis for Σ^+ then it has to be defined as it is above. It is also easy to ♣ [chains1] see directly that if $\alpha < \beta$ are distinct elements of Δ then $\langle \alpha, \beta^\vee \rangle \le 0$. Because if not, according to Lemma 6.1 the difference $\beta - \alpha$ would also be a root, with $\beta > \beta - \alpha > 0$. But this contradicts the definition of β as the least element in Σ^+ not in the span of smaller basis elements.

The proof of the Proposition goes by induction on ℓ. For $\ell = 1$ there is nothing to prove. Assume true for $\ell - 1$. Let Σ_* be the intersection of the span of the α_i with $i \le \ell$, itself a root system. We want to show that every λ in the linear span of Σ is a positive integral combination of the α_i. If λ is in Σ_* induction gives this, and it is also true for $\lambda = \alpha_\ell$. Otherwise $\lambda > \alpha_\ell$. Consider all the $\lambda - \alpha_i$ with $i \le \ell$. It suffices to show that one of them is a root, by an induction argument on order. If not, then all $\langle \lambda, \alpha_i \rangle \le 0$. This ♣ [roots-inverse] leads to a contradiction of Proposition 7.4, to be proven later (no circular reasoning, I promise).

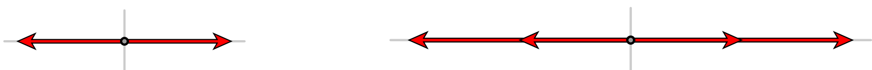
I learned the following from [Chevalley:1955].

[two-roots] **Corollary 3.14.** *Suppose Δ to be a basis of positive roots in Σ. If λ and μ are a linearly independent pair of roots, then there exists w in W such that $w\alpha$ lies in Δ, and $w\mu$ is a linear combination of α and a second element in Δ.*

Proof. Make up an ordered basis of V with λ and μ its first and second elements. Apply the Proposition to get a basis Δ_* of positive roots. Then λ is the first element of Δ. If ν is the second, then μ must be a positive linear combination of λ and $\nu/$. We can find w in W taking Δ_* to Δ.

4. Root systems of rank one

The simplest system is that containing just a vector and its negative. There is one other system of rank one, however:



Throughout this section and the next I exhibit root systems by Euclidean diagrams, implicitly leaving it as an exercise to verify the conditions of the definition.

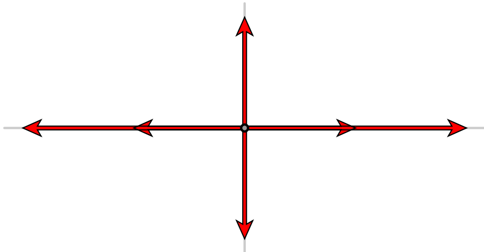
That these are the only ones, up to scale, follows from this:

[non-reduced] **Lemma 4.1.** *If λ and $c\lambda$ are both roots, then $|c| = 1/2, 1,$ or 2.*

Proof. On the one hand $(c\lambda)^\vee = c^{-1}\lambda^\vee$, and on the other $\langle \lambda, (c\lambda)^\vee \rangle$ must be an integer. Therefore $2c^{-1}$ must be an integer, and similarly $2c$ must be an integer.

5. Root systems of rank two

The simplest way to get a system of rank two is to build the orthogonal sum of two systems of rank one. For example:



But more interesting are the **irreducible** systems, those which cannot be expressed as the direct product of smaller systems.

If α and β are two linearly independent roots, the matrices of the corresponding reflections with respect to the basis (α, β) are

$$s_\alpha = \begin{bmatrix} -1 & -\langle \beta, \alpha^\vee \rangle \\ 0 & 1 \end{bmatrix}, \quad s_\beta = \begin{bmatrix} 1 & 0 \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix}$$

and that of their product is

$$s_\alpha s_\beta = \begin{bmatrix} -1 & -\langle \beta, \alpha^\vee \rangle \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix} = \begin{bmatrix} -1 + \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle & \langle \beta, \alpha^\vee \rangle \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix} .$$

This product must be a non-trivial Euclidean rotation, and hence its trace $\tau = -2 + \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$ must satisfy the inequality

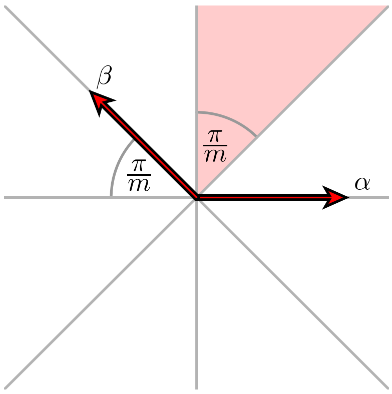
$$-2 \le \tau < 2$$

which imposes the condition

$$0 \le n_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle < 4 .$$

But $n_{\alpha,\beta}$ must also be an integer. Therefore it can only be 0, 1, 2, or 3. It will be 0 if and only if s_α and s_β commute, which means reducibility.

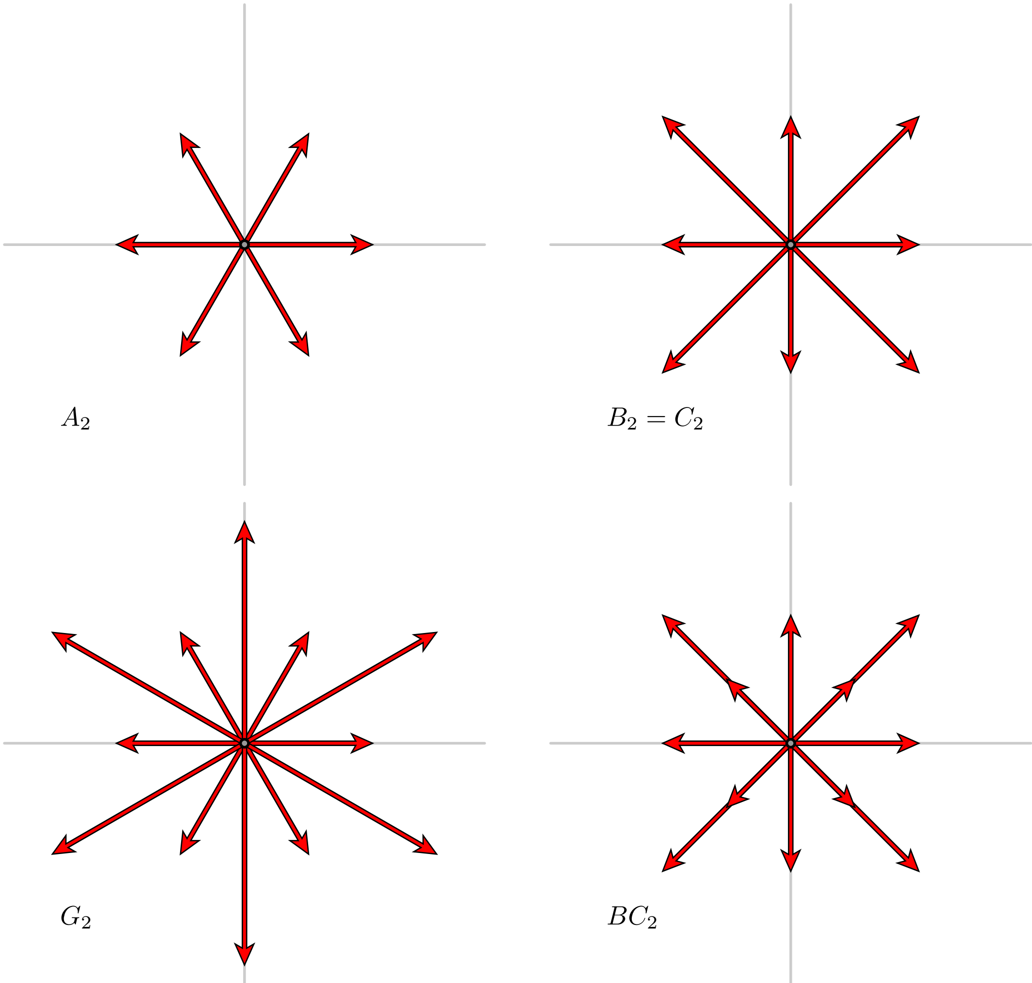
Recall the picture:



Assuming that the root system is irreducible, $\alpha \bullet \beta$ will actually be negative. By switching α and β if necessary, we may assume that one of these cases is at hand:

- $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -1$
- $\langle \alpha, \beta^\vee \rangle = -2$, $\langle \beta, \alpha^\vee \rangle = -1$
- $\langle \alpha, \beta^\vee \rangle = -3$, $\langle \beta, \alpha^\vee \rangle = -1$

Taking the possibility of non-reduced roots into account, we get four possible irreducible systems:



The first three are reduced.

6. Chains

If λ and μ are roots, the μ-chain through λ is the set of all roots of the form $\lambda + n\mu$. We already know that both μ and $r_\lambda \mu$ are in this chain. So is everything in between, as we shall see. The basic result is:

[chains1] **Lemma 6.1.** *Suppose λ and μ to be roots.*

(a) If $\langle \mu, \lambda^\vee \rangle > 0$ then $\mu - \lambda$ is a root unless $\lambda = \mu$.
(b) If $\langle \mu, \lambda^\vee \rangle < 0$ then $\mu + \lambda$ is a root unless $\lambda = -\mu$.

Proof. If λ and μ are proportional, the claims are immediate. Suppose they are not. If $\langle \mu, \lambda^\vee \rangle > 0$ then either it is 1 or $\langle \lambda, \mu^\vee \rangle = 1$. In the first case

$$s_\lambda \mu = \mu - \langle \mu, \lambda^\vee \rangle \lambda = \mu - \lambda$$

so that $\mu - \lambda$ is a root, and in the second $s_\mu \lambda = \lambda - \mu$ is a root and consequently $\mu - \lambda$ also. The other claim is dealt with by swapping $-\lambda$ for λ. ▯

Hence:

[chain-ends] **Proposition 6.2.** *If μ and ν are left and right end-points of a segment in a λ-chain, then $\langle \mu, \lambda^\vee \rangle \le 0$ and $\langle \nu, \lambda^\vee \rangle \ge 0$.*

[chains2] **Proposition 6.3.** *Suppose λ and μ to be roots. If $\mu - p\lambda$ and $\mu + q\lambda$ are roots then so is every $\mu + n\lambda$ with $-p \le n \le q$.*

Proof. Since$\langle \mu + n\lambda, \lambda^\vee \rangle$ is an increasing function of n, the existence of a gap between two segments would contradict the Corollary. ▯

7. The Cartan matrix

If $\mathfrak{h}$ is the union in $V_{\mathbb{R}}^\vee$ of the **root hyperplanes** $\lambda = 0$, the connected components of its complement in ♣ [chambers-transitive] [five] $V_{\mathbb{R}}^\vee$ are called **Weyl chambers**. According to Proposition 3.5 and Proposition 3.7, these form a principal homogeneous space under the action of W.

Fix a chamber C. A root λ is called **positive** if $\langle \lambda, C \rangle > 0$, **negative** if $\langle \lambda, C \rangle < 0$. All roots are either positive or negative, since by definition no root hyperplanes meet C. Let Δ be the set of indivisible roots α with $\alpha = 0$ defining a panel of C with $\alpha > 0$ on C, and let S be the set of reflections s_α for α in Δ. The Weyl group W is generated by S.

The matrix $\langle \alpha, \beta^\vee \rangle$ for α, β in Δ is called the **Cartan matrix** of the system. Since $\langle \alpha, \alpha^{|vee} \rangle = 2$ for all roots α, its diagonal entries are all 2. According to the discussion of rank two systems, its off-diagonal entries

$$\langle \alpha, \beta^\vee \rangle = 2 \left(\frac{\alpha \bullet \beta}{\beta \bullet \beta} \right)$$

are all non-positive. Furthermore, one of these off-diagonal entries is 0 if and only if its transpose entry is.

If D is the diagonal matrix with entries $2/\alpha \bullet \alpha$ then

$$A = DM$$

♣ [rank-two] where M is the matrix $(\alpha \bullet \beta)$. This is a positive semi-definite matrix, and according to Proposition 3.1 its off-diagonal entries are non-positive. The proofs of the results in this section all depend on understanding the Gauss elimination process applied to M. It suffices just to look at one step, reducing all but one entry in the first row and column to 0. Since $\alpha_1 \bullet \alpha_1 > 0$, it replaces each vector α_i with $i > 1$ by its projection onto the space perpendicular to α_1:

$$\alpha_i \longmapsto \alpha_i^\perp = \alpha_i - \frac{\alpha_i \bullet \alpha_1}{\alpha_1 \bullet \alpha_1} \quad (i > 1)\,.$$

If I set $\alpha_1^\perp = \alpha_1$, the new matrix $M^\perp$ has entries $\alpha_i^\perp \bullet \alpha_j^\perp$. We have the matrix equation

$$LM\,{}^tL = M^\perp, \quad M^{-1} = {}^tL\,(M^\perp)^{-1}\,L$$

with L a unipotent lower triangular matrix

$$L = \begin{bmatrix} 1 & \ell \\ {}^t\ell & I \end{bmatrix} \qquad \ell_i = -\frac{\alpha_1 \bullet \alpha_i}{\alpha_1 \bullet \alpha_1} \ge 0\,.$$

This argument and induction proves immediately:

[lummox] **Lemma 7.1.** *Suppose $A = (a_{i,j})$ to be a matrix such that $a_{i,i} > 0$, $a_{i,j} \le 0$ for $i \ne j$. Assume $D^{-1}A$ to be a positive definite matrix for some diagonal matrix D with positive entries. Then A^{-1} has only non-negative entries.*

[titsA] **Lemma 7.2.** *Suppose Δ to be a set of vectors in a Euclidean space V such that $\alpha \bullet \beta \le 0$ for $\alpha \ne \beta$ in Δ. If there exists v such that $\alpha \bullet v > 0$ for all α in Δ then the vectors in Δ are linearly independent.*

Proof. By induction on the size of Δ. The case $|\Delta| = 1$ is trivial. But the argument just before this handles the induction step, since if $v \bullet \alpha > 0$ then so is $v \bullet \alpha^{\perp}$. ▯

[tits] **Proposition 7.3.** *The set Δ is a basis of $V(\Sigma)$.*

That is to say, a Weyl chamber is a simplicial cone. Its extremal edges are spanned by the columns ϖ_i in the inverse of the Cartan matrix, which therefore have positive coordinates with respect to Δ. Hence:

[roots-inverse] **Proposition 7.4.** *Suppose Δ to be a set of linearly independent vectors such that $\alpha \bullet \beta \le 0$ for all $\alpha \ne \beta$ in Δ. If D is the cone spanned by Δ then the cone dual to D is contained in D.*

Proof. Let

$$\varpi = \sum c_\alpha \alpha$$

be in the cone dual to D. Then for each β in Δ

$$\varpi \bullet \beta = \sum c_\alpha \left(\alpha \bullet \beta\right) .$$

If A is the matrix $(\alpha \bullet \beta)$, then it satisfies the hypothesis of the Lemma. If u is the vector (c_α) and v is the vector $(\varpi \bullet \alpha)$, then by assumption the second has non-negative entries and

$$u = A^{-1} v$$

so that u also must have non-negative entries. ▯

[base] **Proposition 7.5.** *Each positive root may be expressed as $\sum_{\alpha \in \Delta} c_\alpha \alpha$ where each c_α is a non-negative integer.*

Proof. Each root is of the form $w\alpha$ for some w in W, α in Δ. This gives such an expression with c_α integral. A root λ is positive if and only if $\lambda \bullet \varpi_i \ge 0$ for all i. But $\lambda \bullet \varpi_i$ is the i-th coordinate of λ. ▯

One consequence of all this is that the roots generate a full lattice in $V(\Sigma)$. By duality, the coroots generate a lattice in $V^\vee(\Sigma^\vee)$, which according to the definition of root systems is contained in the lattice of $V^\vee(\Sigma^\vee)$ dual to the root lattice of $V(\Sigma)$.

Another consequence is a simple algorithm that starts with a given root λ and produces a product w of elementary reflections with $w^{-1}\lambda \in \Delta$.

If $\lambda < 0$, record this fact and swap $-\lambda$ for λ. Now

$$\lambda = \sum_\Delta n_\alpha \alpha$$

with all $n_\alpha \ge 0$ and one $n_\alpha > 0$. The proof proceeds by induction on the height $|\lambda| = \sum n_\alpha$ of λ. If $|\lambda| = 1$, then λ lies in Δ, and there is no problem. Since the positive chamber C is contained in the cone spanned by the positive roots, no positive root is contained in the closure of C. Therefore $\lambda, \alpha^\vee >> 0$ for some α in Δ. Then

$$r_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

has smaller height than λ, and we can apply the induction hypothesis.

If the original λ was negative, this gives us w with $w^{-1}\lambda$ in Δ, and just one more elementary reflection has to be applied.

This has as consequence:

[chain-to-Delta] **Proposition 7.6.** *Every positive root can be connected to Δ by a chain of links $\lambda \mapsto \lambda + \alpha$.*

We now know that each root system gives rise to an integral Cartan matrix $A = (\langle \alpha, \beta^\vee \rangle)$ with rows and columns indexed by Δ. We know that it has these properties:

(a) $A_{\alpha,\alpha} = 2$;
(b) $A_{\alpha,\beta} \le 0$ for $\alpha \ne \beta$;
(c) $A_{\alpha,\beta} = 0$ if and only if $A_{\beta,\alpha} = 0$;

but it has another implicit property as well. We know that there exists a W-invariant Euclidean norm with respect to which the reflections are invariant. This implies the formula we have already encountered:

$$\langle \alpha, \beta^\vee \rangle = 2 \left(\frac{\alpha \bullet \beta}{\beta \bullet \beta} \right) .$$

Construct a graph from A whose nodes are elements of Δ and with edge between α and β if and only if $\langle \alpha, \beta \rangle \ne 0$. For each connected component of this graph, chose an arbitrary node α and arbitrarily assign a positive rational value to $\alpha \bullet \alpha$. Assign values for all $\beta \bullet \gamma$ according to the rules

$$\beta \bullet \gamma = \frac{1}{2} \langle \beta, \gamma^\vee \rangle \gamma \bullet \gamma$$
$$\beta \bullet \beta = 2 \, \frac{\beta \bullet \gamma}{\langle \gamma, \beta^\vee \rangle} .$$

which allow us to go from node to node in any component. This defines an inner product, and the extra condition on the Cartan matrix is that this inner product must be positive definite, or equivalently

(d) The matrix $(\alpha \bullet \beta)$ must be positive definite.

This can be tested by Gauss elimination in rational arithmetic, as suggested by the discussion at the beginning of this section.

If these conditions are all satisfied, then we can construct the root system by the algorithm mentioned earlier.

8. Dynkin diagrams

The Dynkin diagram of a reduced system with base Δ is a labeled graph whose nodes are elements of Δ, and an edge between α and β when

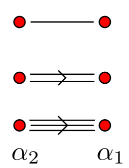
$$\left| \langle \alpha, \beta^\vee \rangle \right| \ge \left| \langle \beta, \alpha^\vee \rangle \right| > 0 .$$

This edge is labeled by the value of $\left| \langle \alpha, \beta^\vee \rangle \right|$, and this is usually indicated by an oriented multiple link. Here are the Dynkin diagrams of all the reduced rank two systems:

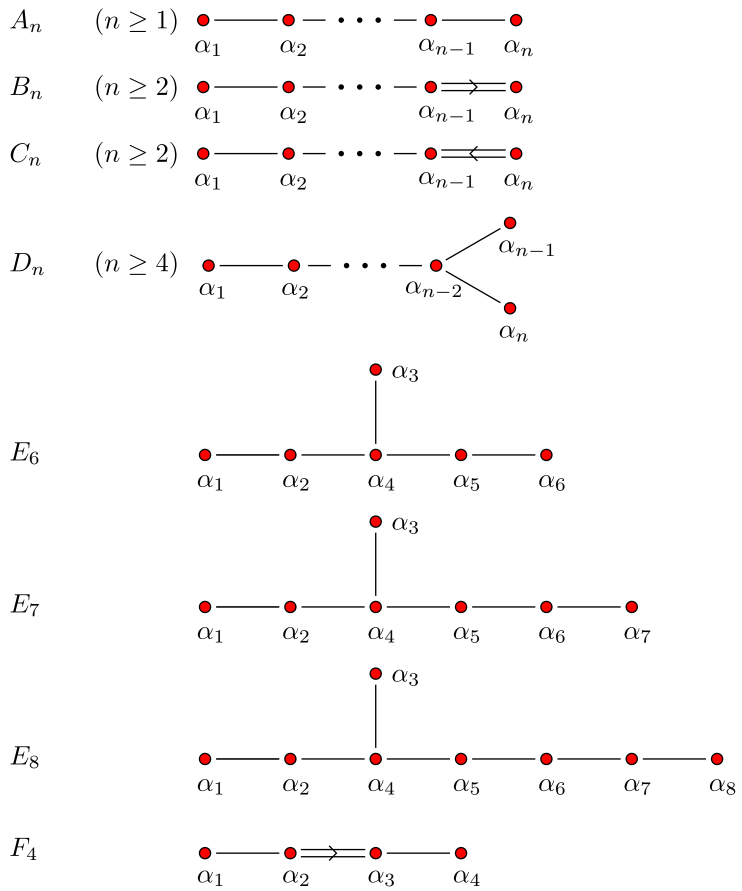
A_2

$B_2 = C_2$

G_2



The Dynkin diagram determines the Cartan matrix of a reduced system. The complete classification of irreducible, reduced systems is known, and is explained by the following array of diagrams. The numbering is arbitrary, even inconsistent as n varies, but follows the convention of Bourbaki. Note also that although systems B_2 and C_2 are isomorphic, the conventional numbering is different for each of them.



In addition there is a series of non-reduced systems of type BC_n obtained by superimposing the diagrams for B_n and C_n.

9. Subsystems

If Θ is a subset of Δ, let Σ_Θ be the roots which are integral linear combinations of elements of Θ. These, along with V, $V^\vee$ and their image in $\Sigma^\vee$ form a root system. Its Weyl group is the subset W_Θ generated by the reflections s_α for α in Θ. Recall that to each subset $\Theta \subseteq \Delta$ corresponds the face C_Θ of $\overline{C}$ where ♣ [stabilizers] $\lambda = 0$ for $\lambda \in \Theta$, $\lambda > 0$ for $\lambda \in \Delta - \Theta$. According to Proposition 3.9, an element of W fixes a point in C_Θ if and only if it lies in W_Θ.

The region in $V^\vee$ where $\alpha > 0$ for α in Θ is a fundamental domain for W_Θ. For any w in W there exists y in W_Θ such that $xC = y^{-1}wC$ is contained in this region. But then $x^{-1}\alpha > 0$ for all α in Θ. In fact, x will be the unique element in $W_\Theta w$ with this property. Hence:

[cosets] **Proposition 9.1.** *In each coset $W_\Theta\backslash W$ there exists a unique representative x of least length. This element is the unique one in its coset such that $x^{-1}\Theta > 0$. For any y in W_Θ we have $\ell(yx) = \ell(y) + \ell(x)$.*

Let $[W_\Theta\backslash W]$ be the set of these distinguished representatives, $[W/W_\Theta]$ those for right cosets. These distinguished representatives can be found easily. Start with $x = w$, $t = 1$, and as long as there exists s in $S = S_\Theta$ such that $sx < x$ replace x by sx, t by ts. At every moment we have $w = tx$ with t in W_Θ and $\ell(w) = \ell(t) + \ell(x)$. At the end we have $sx > x$ for all s in S.

Similarly, in every double coset $W_\Theta\backslash W/W_\Phi$ there exists a unique element w of least length such that $w^{-1}\Theta > 0$, $w\Phi > 0$. Let these distinguished representatives be expressed as $[W_\Theta\backslash W/W_\Phi]$.

Suppose $\mathfrak{g}$ to be a reductive Lie algebra, $\mathfrak{b}$ a maximal solvable ('Borel') subalgebra, Δ the associated basis of positive roots.

If $\mathfrak{p}$ is a Lie subalgebra containing $\mathfrak{b}$, there will exist a subset Θ of Δ such that $\mathfrak{p}$ is the sum of $\mathfrak{b}$ and the direct sum of root spaces $\mathfrak{g}_\lambda$ for λ in Σ^-_Θ. The roots occurring are those in $\Sigma^+ \cup \Sigma^-_\Theta$. This set is a **parabolic subset**—(a) it contains all positive roots and (b) it is closed in the sense that if λ and μ are in it so is $\lambda + \mu$. Conversely:

[parabolic-set] **Proposition 9.2.** *Suppose Ξ to be a parabolic subset, and set $\Theta = \Delta \cap -\Xi$. Then $\Xi = \Sigma^+ + \Sigma^-_\Theta$.*

Proof. We need to show (a) if ξ is in $\Xi \cap \Sigma^-$ then ξ is in Σ^-_Θ and (b) if ξ is in Σ^-_Θ then ξ is in Ξ.

Suppose ξ in Σ^-_Θ, say $\xi = -\sum_\alpha c_\alpha \alpha$. The proof goes by induction on $h(\xi) = \sum c_\alpha$. Since $-\Theta \subseteq \Xi$, ξ is in Ξ if $h(\xi) = 1$. Otherwise $\xi = \xi_* - \alpha$ with ξ_* also in Σ^-_X. By induction ξ_* is in Ξ, and since Ξ is closed so is ξ in Ξ.

Suppose ξ in $\Xi \cap \Sigma^-$. If $h(\xi) = 1$ then ξ is in $-\Delta \cap \Xi = -\Theta$. Otherwise, $\xi = \xi_* - \alpha$ with α in Δ. Then $\xi_* = \xi + \alpha$ also lies in Ξ since Ξ contains all positive roots and it is closed. Similarly $-\alpha = \xi - \xi_*$ lies in Ξ, hence in Θ. By induction ξ_* lies Σ^-_Θ, but then so does ξ. ▯

In the rest of this section, assume for convenience that $V = V(\Sigma)$ (i.e. that the root system is semi-simple), and also that the root system is reduced. The material to be covered is important in understanding the decomposition of certain representations of reductive groups. I learned it from Jim Arthur, but it appears in Lemma 2.13 of [Langlands:1976], and presumably goes back to earlier work of Harish-Chandra.

Fix the chamber C with associated Δ. For each $\Theta \subseteq \Delta$ let

$$V_\Theta = \bigcap_{\alpha \in \Theta} \ker(\alpha) \ .$$

The set of roots which vanish identically on V_Θ are those in Σ_Θ. The space V_Θ is partitioned into chambers by the hyperplanes $\lambda = 0$ for λ in $\Sigma^+ - \Sigma^+_\Theta$. One of these is the face C_Θ of the fundamental Weyl chamber $C = C_\emptyset$. If $\Theta = \emptyset$ we know that the connected components of the complement of root hyperplanes are a principal homogeneous set with respect to the full Weyl group. In general, the chambers of V_Θ are the facettes of full chambers, and in particular we know that each is the Weyl transform of a unique facette of a fixed positive chamber C. But we can make this more precise.

[associates] **Proposition 9.3.** *Suppose Θ and Φ to be subsets of Δ. The following are equivalent:*

(a) there exists w in W taking V_Φ to V_Θ;
(b) there exists w in W taking Φ to Θ.

In these circumstances, Θ and Φ are said to be **associates**. Let $W(\Theta, \Phi)$ be the set of all w taking Φ to Θ.

Proof. That (b) implies (a) is immediate. Thus suppose $wV_\Phi = V_\Theta$. This implies that $w\Sigma_\Phi = \Sigma_\Theta$. Let w_* be of least length in the double coset $W_\Theta w W_\Phi$, so that $w_* \Sigma^+_\Phi = \Sigma^+_\Theta$. Since Φ and Θ are bases of Σ^+_Φ and Σ^+_Θ, this means that $w_*\Phi = \Theta$. ▯

[associate-chambers] **Corollary 9.4.** *For each w in $W(\Theta, \Phi)$ the chamber wC_Φ is a chamber of V_Θ. Conversely, every chamber of V_Θ is of the form wC_Φ for a unique associate Φ of Θ and w in $W(\Theta, \Phi)$.*

Proof. The first assertion is trivial. Any chamber of V_Θ will be of the form wC_Φ for some w in W and some unique $\Phi \subseteq \Delta$. But then $wV_\Phi = V_\Theta$. ▯

We'll see in a moment how to find w and Φ by an explicit geometric construction.

One of the chambers in V_Θ is $-C_\Theta$. How does that fit into the classification? For any subset Θ of Δ, let $W_{\ell,\Theta}$ be the longest element in the Weyl group W_Θ generated by reflections corresponding to roots in Θ. The element $w_{\ell,\Theta}$ takes Θ to $-\Theta$ and permutes $\Sigma^+ \backslash \Sigma^+_\Theta$. The longest element $w_\ell = w_{\ell,\Delta}$ takes $-\Theta$ back to a subset $\overline{\Theta}$ of Σ^+ called its **conjugate** in Δ.

[opposite-cell] **Proposition 9.5.** *If $\Phi = \overline{\Theta}$ and $w = w_\ell w_{\ell,\Theta}$ then $-C_\Theta = w^{-1} C_\Phi$.*

Proof. By definition of the conjugate, $wV_\Theta = V_\Phi$ and hence $w^{-1}V_\Phi = V_\Theta$. The chamber $-C_\Theta$ is the set of vectors v such that $\alpha \bullet v = 0$ for α in Θ and $\alpha \bullet v < 0$ for α in $\Delta\backslash\Theta$. Analogously for C_Φ. ∎

[only] **Lemma 9.6.** *If Θ is a maximal proper subset of Δ then its only associate in Δ is $\overline{\Theta}$.*

Proof. In this case V_Θ is a line, and its chambers are the two half-lines C_Θ and its complement. ∎

If $\Omega = \Theta \cup \{\alpha\}$ for a single α in $\Delta - \Theta$ then the Weyl element $w_{\ell,\Omega} w_{\ell,\Theta}$ is called an **elementary conjugation**.

[elementary-conjugation] **Lemma 9.7.** *Suppose that $\Omega = \Theta \cup \{\alpha\}$ with α in $\Delta - \Theta$. Then the chamber and sharing the panel C_Ω with C_Θ is sC_Φ where Φ is the conjugate of Θ in Ω and $s = w_{\ell,\Omega} w_{\ell,\Phi}$.*

Proof. Let $C_* = wC_\Phi$ be the neighbouring chamber with $s\Phi = \Theta$. Then s fixes the panel shared by C_Θ and C_*, so must lie in W_Ω. But then Φ must be an associate of Θ in Ω. Apply Lemma 9.6. ∎ ♣ [only]

A gallery in V_Θ is a sequence of chambers $C_0, C_1, \ldots, C_n$ where C_{i-1} and C_i share a panel. To each C_i we associate according to Corollary 9.4 a subset Θ_i and an element w_i of $W(\Theta, \Theta_i)$. Since C_{i-1} and C_i share a panel, so do $w_{i-1}^{-1} C_{i-1}$ and $w_{i-1} C_i$. But $w_{i-1} C_{i-1}$ is $C_{\Theta_{i-1}}$, so to this we may apply the preceding Lemma to see that Θ_{i1} and Θ_i are conjugates in their union Ω_i, and that $s_i = w_{i-1}^{-1} w_i$ is equal to the corresponding conjugation. The gallery therefore corresponds to an expression$w = s_1 \ldots s_n$ where each s_i is an elementary conjugation. In summary: ♣ [associate-chambers]

[conjugates] **Proposition 9.8.** *Every element of $W(\Theta, \Phi)$ can be expressed as a product of elementary conjugations. Each such expression corresponds to a gallery from C_Θ to wC_Φ.*

For w in $W(\Theta, \Phi)$ its **relative length** is the length of a minimal gallery in V_Θ leading from C_Θ to wC_Φ.

For w in $W(\Theta, \Phi)$, let ψ_w be the set of hyperplanes in V_Θ separating C_Θ from wC_Φ. Then it is easy to see that $\ell_{\mathrm{rel}}(xy) = \ell_{\mathrm{rel}}(x) + \ell_{\mathrm{rel}}(y)$ if and only if $\psi_y \cup y\psi_x \subseteq \psi_{xy}$.

[relative-separates] **Lemma 9.9.** *Suppose w in $W(\Theta, \Phi)$. Then*

(a) If λ is in $\Sigma^+\backslash\Sigma^+_\Theta$ separates C_Θ from wC_Φ, λ separates $wC_\emptyset$ from $C_\emptyset$;
(b) If $\lambda > 0$ separates $wC_\emptyset$ from $C_\emptyset$, either $\lambda \in \Sigma^+_\Theta$ or $\lambda \in \Sigma^+\backslash\Sigma^+_\Theta$.

[longest-relative] **Proposition 9.10.** *If w lies in $W(\Theta, \Phi)$ then*

$$\ell_{\mathrm{rel}}(w_\ell w_\Phi) = \ell_{\mathrm{rel}}(w_\ell w_{\ell,\Phi} w^{-1}) + \ell_{\mathrm{rel}}(w)\,.$$

[relative-lengths] **Proposition 9.11.** *Suppose x in $W(\Theta_3, \Theta_2)$, y in $W(\Theta_2, \Theta_1)$. If the relative length of xy is the sum of the relative lengths of x and y, then $\ell(xy) = \ell(x) + \ell(y)$.*

Proof. By induction on relative length. ∎

If $C_0, C_1, \ldots, C_n$ is a gallery in V_Θ, it is called **primitive** if Θ_{i-1} is never the same as Θ_i.

[primitive] **Proposition 9.12.** *Every $W(\Theta, \Phi)$ has at least one element with a primitive representation.*

Proof. If $w = s_1 \ldots s_{i-1} s_i \ldots s_n$ and $\Theta_{i-1} = \Theta_i$ then $s_1 \ldots \widehat{s}_i \ldots s_n$ is also in $W(\Theta, \Phi)$. ∎

10. Dominant roots

A positive root $\widetilde{\alpha}$ is called **dominant** if every other root is of the form

$$\widetilde{\alpha} - \sum_{\alpha\in\Delta} c_\alpha \alpha$$

with all c_α in $\mathbb{N}$.

[dominant] **Proposition 10.1.** *If Σ is an irreducible root system, then there exists a unique dominant root.*

Proof. We can even describe an algorithm to calculate it. Start with any positive root λ, for example one ♣ [chains1] in Δ, and as long as some $\lambda + \alpha$ with α in Δ is a root, replace λ by $\lambda + \alpha$. According to Lemma 6.1, at the end we have $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all α in Δ. We then also have $\langle \alpha, \lambda^\vee \rangle \geq 0$ for all α, and at least one of these must actually be > 0.

Suppose $\lambda = \sum n_\alpha \alpha$. Let X be the α with $n_\alpha \neq 0$ and Y its complement in Δ. If Y isn't empty, then because of irreducibility there exists α in X and β in Y with $\langle \alpha, \beta^\vee \rangle < 0$. Hence we get the contradiction

$$0 \leq \langle \lambda, \beta^\vee \rangle = \sum_{\alpha\in X} n_\alpha \langle \alpha, \beta^\vee \rangle < 0$$

So $n_\alpha > 0$ for all α.

If there were a root ν not of the form

$$\lambda - \sum_{\alpha\in\Delta} c_\alpha \alpha$$

then there would exist a second root μ, constructed from ν by the same procedure, with the same property. Then

$$\langle \lambda, \mu^\vee \rangle = n_\alpha \sum \langle \alpha, \mu^\vee \rangle > 0 \,.$$

♣ [chains1] According to Lemma 6.1, $\lambda - \mu$ is a root. But this implies that either $\lambda > \mu$ or $\lambda < \mu$, contradicting maximality. ∎

The proof shows that the dominant root is the unique one in the closure of the positive Weyl chamber. It can be shown moreover that $\langle \alpha, \widetilde{\alpha} \rangle = 0$ or 1, but this will not be needed.

11. Affine root systems

Assume a reduced semi-simple root system.

♣ [base] According to Proposition 7.5, the roots are contained in a lattice—in fact, in the free $\mathbb{Z}$-module $\mathcal{R}$ spanned by Δ. The coroots then span a lattice $\mathcal{R}^\vee$ contained in the dual lattice $\mathrm{Hom}(\mathcal{R}, \mathbb{Z})$. In general the inclusion will be proper. The roots will in turn then be contained in the dual $\mathrm{Hom}(\mathcal{R}^\vee, \mathbb{Z})$ of $\mathcal{R}\vee$.

Define W_{aff} to be the group of affine transformations generated by W and translations by elements of $\mathcal{R}^\vee$, $\widehat{W}$ the larger group generated by W and $\mathrm{Hom}(\mathcal{R}, \mathbb{Z})$. Both groups preserve the system of affine hyperplanes $\lambda + k = 0$, where λ is a root, consequently permuting the connected components of the complement, called **alcoves**.

[alcoves] **Proposition 11.1.** *The region C_{aff} where $\alpha > 0$ for all α in Δ, $\widetilde{\alpha} < 1$ is an alcove.*

Proof. It has to be shown that for any root α and integer k, the region C_{aff} lies completely on one side or the other of the hyperplane $\alpha \bullet x - k = 0$. If $k = 0$ this is clear. If $k < 0$ we can change α to $-\alpha$ and k to

$-k$, so we may assume that $k > 0$. Since 0 lies in the closure of $C_{\rm aff}$, it must be shown that $\alpha + k < 0$ on
♣ [dominant] all of $C_{\rm aff}$. But by Proposition 10.1 we can write $\alpha = \widetilde{\alpha} - \sum_\Delta c_\beta \beta$ with all $c_\beta \ge 0$, so for any x in $C_{\rm aff}$

$$\alpha \bullet x = \widetilde{\alpha} \bullet x - \sum_\Delta c_\beta (\beta \bullet x) < 1 \le k \, .$$

Let $\widetilde{\Delta}$ be the union of Δ and $-\widetilde{\alpha} + 1$. For any pair α, β in $\widetilde{\Delta}$

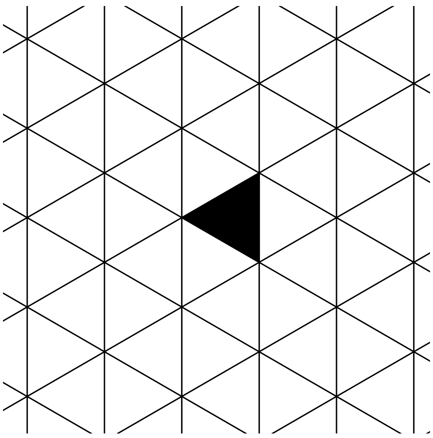
$$-4 < \langle \alpha, \beta^\vee \rangle \le 0$$

The **affine Dynkin diagram** is a graph whose nodes are elements of $\widetilde{\Delta}$ with edges labelled and oriented according to the values of $\langle \alpha, \beta^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle$. Let $\widetilde{s}$ be the affine reflection in the hyperplane $\widetilde{\alpha} = 1$, $\widetilde{S}$ the union of S and $\widetilde{s}$.

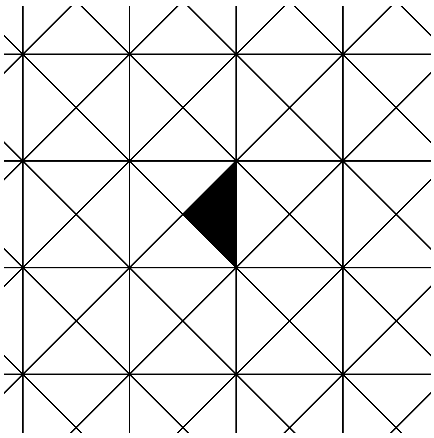
[affines] **Corollary 11.2.** *The group $W_{\rm aff}$ is generated by the involutions in $\widetilde{S}$.*

♣ [stabilizers] According to Proposition 3.9, every face of an alcove is the transform by an element of $W_{\rm aff}$ of a unique face of the alcove $C_{\rm aff}$. Elements of the larger group $\widetilde{W}$ also permute alcoves, but do not necessarily preserve this labelling. If w is an element of $\widetilde{W}$ and α an element of $\widetilde{\Delta}$, then the face wF_α of $wC_{\rm aff}$ will be the transform xF_β for some unique β in $\widetilde{\Delta}$. Let $\iota(w)$ be the map from $\widetilde{\Delta}$ to itself taking α to β.

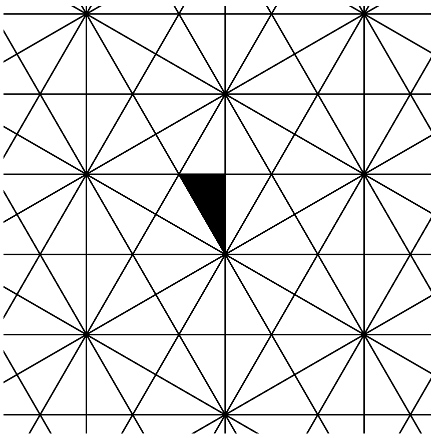
[automorphism] **Proposition 11.3.** *The map from $w \mapsto \iota(w)$ induces an isomorphism of $\widetilde{W}/W_{\rm aff}$ with the group of automorphisms of the affine Dynkin diagram.*



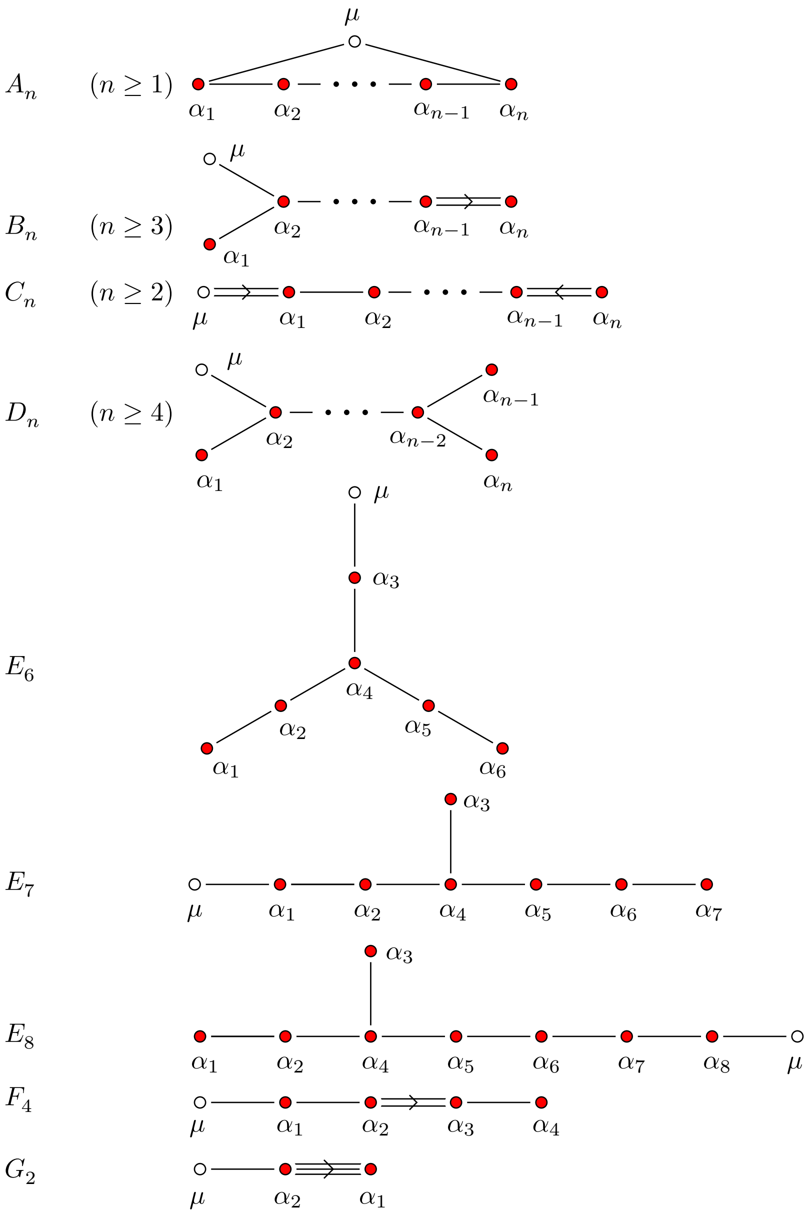
Affine A_2



Affine C_2



Affine G_2



References

N. Bourbaki, Chapitres IV, V, and VI of **Groupes et algèbres de Lie**, Hermann, 1972.

C. Chevalley, 'Sur certains groupes simples', Tohoku Mathematics Journal **48** (1955), 14–66.

James E. Humphreys, **Introduction to Lie algebras and representation theory**, Springer, 1980.

Robert P. Langlands, **On the functional equations satisfied by Eisenstein series**, *Lecture Notes in Mathematics* **544**, Springer, 1976.

Ichiro Satake, 'On a theorem of Cartan', *Journal of the Mathematical Society of Japan* **2** (1951), 284–305.