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Essays on the structure of reductive groups

Root systems

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Root systems arise in the structure of reductive groups. I'll discuss this connection in an introductory section, then go on to a formal exposition.

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1. Introduction

If k is a field of characteristic 0, a **reductive group** defined over k is an algebraic group for which the category of finite-dimensional representations is semi-simple—i.e. extensions always split. For other characteristics, the definition is more technical. In any case, reductive groups are ubiquitous in mathematics, arising in many different contexts, nearly all of great interest. Here are some examples:

• The unitary group $U(n)$ is a connected compact Lie group, and it is also an algebraic group defined over $\mathbb{R}$, since the defining equation

$$ {}^t\overline{Z}\,Z = I $$

translates to

$$ {}^t(X - iY)(X + iY) = \big({}^tX\,X + {}^tY\,Y\big) + i\big({}^tX\,Y - {}^tY\,X\big) = I\,, $$

which dissociates to become a pair of equations for the real matrices X and Y. An arbitrary connected compact Lie group may always be identified with the group of real points on an algebraic group defined over $\mathbb{R}$. It is an example, a very special one, of a reductive group defined over $\mathbb{R}$. It determines in turn the group of its points over $\mathbb{C}$, which is in many ways more tractable. The group $U(n)$ thus corresponds to $\mathrm{GL}_n(\mathbb{C})$, in which it is a maximal compact subgroup. This is related to the factorization $\mathrm{GL}_n(\mathbb{C}) = U(n) \times \mathfrak{H}$, where $\mathfrak{H}$ is the group of positive definite Hermitian matrices. In general, this association reduces the classification, as well as the representation theory, of connected compact groups to the classification of connected reductive groups defined over $\mathbb{C}$.

• All but a finite number of the finite simple groups are either reductive groups defined over finite fields, or at least closely related to such groups. An example would be the finite projective group $\mathrm{PGL}^1_n(\mathbb{F}_q)$, the kernel of the map induced by $\det$ onto $k^\times/(k^\times)^n$. What might be a bit surprising is that the classification of reductive groups over finite fields is not so different from that of groups over $\mathbb{C}$. Even more astonishing, Chevalley's uniform construction of such finite groups passes through the construction of the related complex groups.

• The theory of automorphic forms is concerned with functions on quotients $\Gamma\backslash G(\mathbb{R})$ where G is reductive and Γ an arithmetic subgroup, a discrete subgroup of finite covolume. These quotients are particularly interesting because they are intimately associated to analogues of the Riemann zeta function. Classically one takes $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This is still probably the best known case, but in the past half-century it has become clear that other groups are of equal interest, although unfortunately also of much greater complexity.

Each connected reductive group is associated to a finite subset of a real vector space with certain combinatorial properties, known as its **root system**. The classification of such groups reduces to the classification of root systems, and the structure of any one of these groups can be deduced from properties of its root system. Most of this essay will be concerned with properties of root systems independently of their connections with reductive groups, but in the rest of this introduction I'll give some idea of this connection.

The root system of the general linear group

For the moment, let k be an arbitrary field, and let $G = \mathrm{GL}_n(k)$. This is in many ways the simplest of all reductive groups—perhaps deceptively simple. It is the open subset of $M_n(k)$ where $\det \neq 0$, but may also be identified with the closed subvariety of all (m, d) in $M_n(k) \oplus k$ with $\det(m) \cdot d = 1$, hence is an affine algebraic group defined over k. The matrix algebra $M_n(k)$ may be identified with its tangent space at I. This tangent space is also the Lie algebra of G, but we won't need the extra structure here.

Let A be the subgroup of diagonal matrices $a = (a_i)$, also an algebraic group defined over k. The diagonal entries

$$\varepsilon_i \colon a \mapsto a_i$$

induce an isomorphism of A with with $(k^\times)^n$. These generate the group $X^*(A)$ of algebraic homomorphisms from A to the multiplicative group. Such homomorphisms are all of the form $a \mapsto \prod a_i^{m_i}$, and $X^*(A)$ is isomorphic to the free group $\mathbb{Z}^n$.

Let $X_*(A)$ be the group of algebraic homomorphisms from the multiplicative group to A. Each of these is of the form $x \mapsto (x^{m_i})$, so this is also isomorphic to $\mathbb{Z}^n$. The two are canonically dual to each other—for $\alpha = \prod x^{n_i}$ in $X^*(A)$ and $\beta^\vee = (x^{m_i})$ in $X_*(A)$ we have

$$\alpha\big(\beta(x)\big) = x^{m_1 n_1 + \cdots + m_n n_n} = x^{\langle \alpha, \beta^\vee \rangle} \,.$$

Let $(\widehat{\varepsilon}_i)$ be the basis of $X_*(A)$ dual to (ε_i).

The group A acts on the tangent space at I by the **adjoint action**, which is here conjugation of matrices. Explicitly, conjugation of $(x_{i,j})$ by (a_i) gives $(a_i/a_j \cdot x_{i,j})$. The space $M_n(k)$ is therefore the direct sum of the diagonal matrices $\mathfrak{a}$, on which A acts trivially, and the one-dimensional eigenspaces $\mathfrak{g}_{i,j}$ of matrices with a solitary entry $x_{i,j}$ (for $i \neq j$), on which A acts by the character

$$\lambda_{i,j} \colon a \longmapsto a_i/a_j \,.$$

The characters $\lambda_{i,j}$ form a finite subset of $X^*(A)$ called the **roots** of GL_n. The term comes from the fact that these determine the roots of the characteristic equation of the adjoint action of an element of A. In additive notation (appropriate to a lattice), $\lambda_{i,j} = \varepsilon_i - \varepsilon_j$.

The group GL_n has thus given rise to the lattices $L = X^*(A)$ and $L^\vee = X_*(A)$ as well as the subset Σ of all roots $\lambda_{i,j}$ in $X^*(A)$. There is one more ingredient we need to see, and some further properties of this collection that remain to be explored. To each root $\lambda = \lambda_{i,j}$ with $i \neq j$ we can associate a homomorphism $\lambda^\vee$ from $\mathrm{SL}_2(k)$ into G, taking

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

to the matrix $(x_{k,\ell})$ which is identical to the identity matrix except when $\{k,\ell\} \subseteq \{i,j\}$. For these exceptional entries we have

$$\begin{aligned} x_{i,i} &= a \\ x_{j,j} &= d \\ x_{i,j} &= b \\ x_{j,i} &= c\,. \end{aligned}$$

For example, if $n = 3$ and $(i,j) = (1,3)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}.$$

The homomorphism $\lambda^\vee$, when restricted to the diagonal matrices, gives rise to a homomorphism from the multiplicative group that I'll also express as $\lambda^\vee$. This is the element $\widehat{\varepsilon}_i - \widehat{\varepsilon}_j$ of $X_*(A)$. In the example above this gives us

$$x \longmapsto \begin{bmatrix} x & \\ & 1/x \end{bmatrix} \longmapsto \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/x \end{bmatrix}.$$

The homomorphism $\lambda^\vee$ also gives rise to its differential $d\lambda^\vee$, from the Lie algebra of SL_2 to that of G. In SL_2

$$\begin{bmatrix} a & \\ & 1/a \end{bmatrix} \begin{bmatrix} & x \\ & \end{bmatrix} \begin{bmatrix} 1/a & \\ & a \end{bmatrix} = \begin{bmatrix} & a^2x \\ & \end{bmatrix} = a^2 \begin{bmatrix} & x \\ & \end{bmatrix}.$$

Since the embedding of SL_2 is covariant, this tells us that

$$\lambda\big(\lambda^\vee(x)\big) = x^2$$

or in other terminology

$$\langle \lambda, \lambda^\vee \rangle = 2$$

for an arbitrary root λ.

The set of roots possesses important symmetry. The permutation group $\mathfrak{S}_n$ acts on the diagonal matrices by permuting its entries, hence also on $X^*(A)$ and $X_*(A)$. In particular, the transposition $(i\ j)$ swaps ε_i with ε_j. This is a reflection in the hyperplane $\widehat{\lambda} = \widehat{\lambda}_{i,j} = \widehat{\varepsilon}_i - \widehat{\varepsilon}_j = 0$, with the formula

$$\mu \longmapsto \mu - 2\left(\frac{\mu \bullet \lambda}{\lambda \bullet \lambda}\right)\lambda\,.$$

The dot product here is the standard one on $\mathbb{R}^n$, which is invariant with respect to the permutation group, but it is better to use the more canonical but equivalent formula

$$s_\lambda\colon \mu \longmapsto \mu - \langle \mu, \lambda^\vee \rangle \lambda\,.$$

This is a reflection in the hyperplane $\langle v, \lambda^\vee \rangle = 0$ since $\langle \lambda, \lambda^\vee \rangle = 2$ and hence $s_\lambda \lambda = -\lambda$. So now we have in addition to L, $L^\vee$ and Σ the subset $\Sigma^\vee$. The set Σ is invariant under reflections s_λ, as is $\Sigma^\vee$ under the contragredient reflections

$$s_{\lambda^\vee}\colon \mu^\vee \longmapsto \mu^\vee - \langle \lambda, \mu^\vee \rangle \lambda^\vee\,.$$

The group $\mathfrak{S}_n$ may be identified with $N_G(A)/A$. The short exact sequence

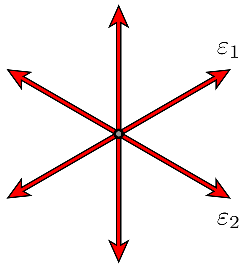
$$1 \longrightarrow A \longrightarrow N_G(A) \longrightarrow N_G(A)/A \cong \mathfrak{S}_n \longrightarrow 1$$

in fact splits by means of the permutation matrices. But this is peculiar to GL_n. What generalizes to other groups is the observation that transpositions generate $\mathfrak{S}_n$, and that the transposition $(i\ j)$ is also the image in $N_G(A)/A$ of the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

under $\lambda^\vee_{i,j}$.

The roots $\varepsilon_i - \varepsilon_j$ all lie in the $n-1$-dimensional space where the sum of coordinates is 0. For $n = 3$ this slice is two-dimensional, and the root system can be pictured:



Root data

So now we can make the definition:

> *A* **root datum** *is a quadruple $(L, \Sigma, L^\vee, \Sigma^\vee)$ where (a) Σ is a finite subset of the lattice L, $\Sigma^\vee$ one of the dual lattice $L^\vee$; (b) there exists a bijection $\lambda \mapsto \lambda^\vee$ from Σ to $\Sigma^\vee$; and (c) both Σ and $\Sigma^\vee$ are invariant under root (and co-root) reflections.*

I am not sure who is responsible for this odd Latin terminology. Many of us have forgotten that 'data' is the plural of 'datum', to be sure (as just about all of us have forgotten that 'agenda' is the plural of 'agendum', meaning *things to be acted upon*). Especially confusing is that a root datum is a collection of several objects. Literally correct, of course, but . . . Mind you, what would we call a set of these arrays if we had called a single array a set of data?

The character lattice L is an important part of the datum. The lattice L for GL_n is that spanned by the ε_i. The group SL_n is the subgroup of matrices with determinant 1, so its character lattice is the quotient of L by the span of $\varepsilon_1 + \cdots + \varepsilon_n$. The projective group PGL_n is the quotient of GL_n by scalars, so the corresponding lattice is the sublattice where the coordinate sum vanishes. The groups $\mathrm{GL}_n(k)$ and $k^\times \times \mathrm{SL}_n(k)$, for example, have essentially the same sets of roots but distinct lattices. It is useful to isolate the roots as opposed to how they sit in a lattice, and for this reason one introduces a new notion: *a* **root system** *is a quadruple $(V, \Sigma, V^\vee, \Sigma^\vee)$ satisfying analogous conditions, but where now V is a real vector space, and we impose the extra condition that $\langle \lambda, \mu^\vee \rangle$ be integral.* Two root systems are called **equivalent** if the systems obtained by replacing the vector space by that spanned by the roots are isomorphic. Sometimes in the literature it is in fact assumed that V is spanned by Σ, but this becomes awkward when we want to refer to root system whose roots are subsets of a given system.

Tori

The simplest reductive groups are those with an empty root set. Connected, compact abelian groups are tori—i.e. products of circles—and over $\mathbb{C}$ become products of the multiplicative group $\mathbb{C}^\times$. For example, the compact torus $x^2 + y^2 = 1$ becomes the complex hyperbola $(x + iy)(x - iy) = 1$. In general, algebraic groups isomorphic to products of copies of the multiplicative group after extension to an algebraic closed field are called **algebraic tori**. Any algebraic representation of a torus decomposes into the sum of one-dimensional eigenspaces. The roots associated to a reductive group defined over an algebraically closed field are the eigencharacters of the adjoint action of a maximal torus. For other fields of definition there are some technical adjustments, as we'll see in a moment.

The symplectic group

The symplectic group Sp_{2n} is that of all $2n \times 2n$ matrices X preserving a non-degenerate anti-symmetric form such as $\sum(x_i y_{i+n} - y_i x_{i+n})$ or, after a coordinate change, satisfying the equation

$$ {}^tX\, J\, X = J $$

where

$$ J = J_n = \begin{bmatrix} 0 & -\omega_n \\ \omega_n & 0 \end{bmatrix} \text{ and } \omega_n = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \dots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}, . $$

Sometimes I is used instead of ω, but there is a good technical reason for the choice here.

The tangent space of this group at I, and indeed that of any algebraic group, may be calculated by a very useful trick. A group defined over k determines also a group defined over any ring extension of k, and in particular the **nil-ring** $k[\varepsilon]$ where $\varepsilon^2 = 0$. The tangent space may be identified with the linear space of all matrices X such that $I + \varepsilon X$ lies in $G(k[\varepsilon])$. Here this gives us the condition

$$ \begin{aligned} {}^t(I + \varepsilon X)J(I + \varepsilon X) &= J \\ J + \varepsilon({}^tXJ + JX) &= J \\ {}^tXJ + JX &= 0\,. \end{aligned} $$

This symplectic group contains a copy of GL_n made up of matrices

$$ \begin{bmatrix} X & \\ & \omega^{-1} \cdot {}^tX^{-1} \cdot \omega \end{bmatrix} $$

for arbitrary X in $\mathrm{GL}_n(k)$, and also unipotent matrices

$$ \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} $$

with ωX symmetric. The diagonal matrices in Sp_{2n} are those like

$$ \begin{bmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & a_3 & & & \\ & & & a_3^{-1} & & \\ & & & & a_3^{-1} & \\ & & & & & a_3^{-1} \end{bmatrix} $$

($n = 3$ here) and make up an algebraic torus of dimension n, again with coordinates ε_i inherited from GL_n. The roots are now the

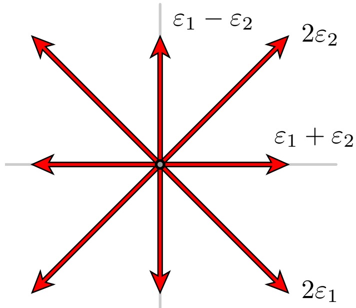
$$\begin{gathered} \pm\varepsilon_i \pm \varepsilon_j \quad (i < j) \\ \pm 2\varepsilon_i \end{gathered}$$

and the co-roots

$$\begin{gathered} \pm\widehat{\varepsilon}_i \pm \widehat{\varepsilon}_j \quad (i < j) \\ \pm\widehat{\varepsilon}_i \,. \end{gathered}$$

It is significant that the two systems differ in the factor 2.

The Weyl group in this case is an extension of $\mathfrak{S}_n$ by $(\pm 1)^n$.



Unitary groups

Suppose ℓ to be a quadratic field extension of k. Let G be the **special unitary group** SU_{ω_n} of $n \times n$ matrices X of determinant 1 with coordinates in ℓ that preserve the Hermitian form corresponding to the matrix ω_n:

$${}^t\overline{X}\,\omega_n X = \omega_n \,.$$

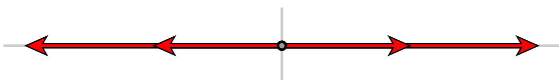
This is an algebraic group defined over k, not over ℓ, whose equations one obtains by translating the matrix equations in ℓ to equations in k by introducing a k-basis in ℓ. The group of diagonal matrices in it is isomorphic to $(\ell^\times)^k$ if $n = 2k+1$ and $(\ell^\times)^k \times N^1_{\ell/k}$ if $n = 2k+2$, where N^1 is the kernel of the norm map from $\ell^\times$ to $k^\times$. For example, if $n = 3$ we have the diagonal matrices

$$\begin{bmatrix} a & & \\ & \overline{a}/a & \\ & & 1/\overline{a} \end{bmatrix}$$

with a in $\ell^\times$. This group is still a torus, since when the field of definition is extended to ℓ it becomes isomorphic to a product of copies of $\ell^\times$. But it is not **split**. The group A in this case is taken to be the subgroup of matrices which are invariant under conjugation—i.e. those with diagonal entries in $k^\times$. The eigenspaces no longer necessarily have dimension one. For odd n the root system is not reduced—for $n = 3$, the roots are

$$a \mapsto a^{\pm 1}, \quad a \mapsto a^{\pm 2} \,.$$

In additive terms it contains four roots $\pm\alpha$, $\pm 2\alpha$:



Further reduction

Not only do reductive groups give rise to root data, but conversely to each root datum is associated a unique reductive group over any field. This is a remarkable result first discovered by Chevalley—remarkable because at the time it was expected that fields of small characteristic would cause serious trouble, perhaps even give rise to exceptional cases. At any rate, natural questions arise: *How to classify root data, or more simply root systems, up to equivalence? How to construct the associated groups and Lie algebras? How to describe their properties, for example their irreducible finite-dimensional representations? How to do explicit computation in these groups and Lie algebras?*

Making the first steps towards answering these requires us to go a bit further in understanding the geometry of root systems.

To each root λ from a root system $(V, \Sigma, V^\vee, \Sigma^\vee)$ is associated a linear transformation of $V^\vee$, the reflection $s_{\lambda^\vee}$ in the root hyperplane $\lambda = 0$. The finite group generated by these reflections is the **Weyl group** of the system. In terms of the group G, it may be identified with the quotient $N_G(A)/A$. One of the most important properties of root systems is that *any one of the connected components of the complement of the root hyperplanes is the interior of a fundamental domain of the Weyl group.*

If we choose one of these, say C, then the **positive roots** associated to that choice are those λ with $\lambda > 0$ on C. A second fundamental fact is that *the set* Σ^+ *of positive roots has as subset a* **basis** Δ *with the property that every positive root is a positive integral combination of elements of* Δ. The **Cartan matrix** of the system is that with rows and columns indexed by Δ and entries $\langle\alpha, \beta^\vee\rangle$. Cartan matrices possess certain characteristic properties I'll explain later. For the moment, it is important to know that a root system can be constructed from any matrix with these properties, which are quite simple to verify. The classification of root systems is thus equivalent to the classification of integral matrices of a certain kind.

Choosing a set of positive roots is related to a choice of **Borel subgroups** in a reductive group. For GL_n, for example, the roots $\lambda_{i,j}$ with $i < j$, hence associated to matrix entries above the diagonal, are related to the choice of upper triangular matrices as Borel subgroup. The corresponding chamber will be the (x_i) with $x_1 \ge x_2 \ge \ \ldots \ \ge x_n$, clearly a fundamental chamber for $\mathfrak{S}_n$. The basis Δ consists of the roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Note that for $i < j$

$$\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1} \, .$$

Something similar occurs for Sp_{2n} and SU_{ω_n}, and indeed the slightly unusual choice of matrix J in the definition of Sp_{2n} was made so that, as with GL_n, the matrices in B are the upper triangular matrices in Sp_{2n}.

A root system is called **irreducible** if it cannot be expressed as a disjoint union of mutually orthogonal subsets. The classification of root systems comes down to the classification of irreducible ones, and this is an interesting combinatorial exercise. The final result is that all simple, connected, reductive algebraic groups can be determined elegantly. Furthermore, the structure of the group associated to a given root datum can be carried out in completely explicit terms from the datum, although this is not a simple process.

I now begin again, more formally, dealing only with root systems and not algebraic groups.

2. Definitions

A **reflection** in a finite-dimensional vector space is a linear transformation that fixes vectors in a hyperplane, and acts on a complementary line as multiplication by -1. Every reflection can be written as

$$v \longmapsto v - \langle f, v\rangle f^\vee$$

for some linear function $f \neq 0$ and vector $f^\vee$ with $\langle f, f^\vee \rangle = 2$. The function f is unique up to non-zero scalar. If V is given a Euclidean norm, a reflection is **orthogonal** if it is of the form

$$v \longmapsto v - 2\left(\frac{v \bullet r}{r \bullet r}\right) r$$

for some non-zero r. This is a reflection since

$$\frac{v \bullet r}{r \bullet r} r$$

is the orthogonal projection onto the line through r.

A **root system** is, by definition:

- a quadruple $(V, \Sigma, V^\vee, \Sigma^\vee)$ where V is a finite-dimensional vector space over $\mathbb{R}$, $V^\vee$ its linear dual, Σ a finite subset of $V - \{0\}$, $\Sigma^\vee$ a finite subset of $V^\vee - \{0\}$;
- a bijection $\lambda \mapsto \lambda^\vee$ of Σ with $\Sigma^\vee$

subject to these conditions:

- for each λ in Σ, $\langle \lambda, \lambda^\vee \rangle = 2$;
- for each λ and μ in Σ, $\langle \lambda, \mu^\vee \rangle$ lies in $\mathbb{Z}$;
- for each λ the reflection
$$s_\lambda \colon\ v \longmapsto v - \langle v, \lambda^\vee \rangle \lambda$$
takes Σ to itself. Similarly the reflection
$$s_{\lambda^\vee} \colon\ v \longmapsto v - \langle \lambda, v \rangle \lambda^\vee$$
in $V^\vee$ preserves $\Sigma^\vee$.

There are many slightly differing definitions of root systems in the literature. Sometimes the condition of finiteness is dropped, and what I call a root system in these notes would be called a **finite root system**. Sometimes the extra condition that Σ span V is imposed, but often in the subject one is interested in subsets of Σ which again give rise to root systems that do not possess this property even if the original does. In case V is spanned by $V(\Sigma)$, the condition that $\Sigma^\vee$ be reflection-invariant is redundant. Sometimes the vector space V is assumed to be Euclidean and the reflections orthogonal. The definition I have given is one that arises most directly from the theory of reductive groups, but there is some justification in that theory for something like a Euclidean structure as well, since a semi-simple Lie algebra is canonically given its **Killing form**. Another virtue of not starting off with a Euclidean structure is that it allows one to keep in view generalizations, relevant to Kac-Moody algebras, where the root system is not finite and no canonical inner product, let alone a Euclidean one, exists.

One immediate consequence of the definition is that if λ is in Σ so is $-\lambda = s_\lambda \lambda$.

The elements of Σ are called the **roots** of the system, those of $\Sigma^\vee$ its **co-roots**. The **rank** of the system is the dimension of V, and the **semi-simple rank** is that of the subspace $V(\Sigma)$ of V spanned by Σ. The system is called **semi-simple** if Σ spans V.

If $(V, \Sigma, V^\vee, \Sigma^\vee)$ is a root system, so is its **dual** $(V^\vee, \Sigma^\vee, V, \Sigma)$.

The **Weyl group** of the system is the group W generated by the reflections s_λ.

The root system is said to be **reducible** if Σ is the union of two subsets Σ_1 and Σ_2 with $\langle \lambda, \mu^\vee \rangle = 0$ whenever λ and μ belong to different components. Otherwise it is **irreducible**.

3. Simple properties

As a group, the Weyl group of the root system $(V, \Sigma, V^\vee, \Sigma^\vee)$ is isomorphic to the Weyl group of its dual system, because:

[reflection-dual] **Proposition 3.1.** *If λ, $\lambda^\vee$ are any vectors in V, $V^\vee$ with $\langle \lambda, \lambda^\vee \rangle = 2$, then the contragredient of s_λ is $s_{\lambda^\vee}$.*

Proof. It has to be shown that

$$\langle s_\lambda u, v \rangle = \langle u, s_{\lambda^\vee} v \rangle .$$

The first is

$$\langle u - \langle u, \lambda^\vee \rangle \lambda, v \rangle = \langle u, v \rangle - \langle u, \lambda^\vee \rangle \langle \lambda, v \rangle$$

and the second is

$$\langle u, v - \langle \lambda, v \rangle \lambda^\vee \rangle = \langle u, v \rangle - \langle \lambda, v \rangle \langle u, \lambda^\vee \rangle . \square$$

Next I introduce a semi-Euclidean structure on V, with respect to which the root reflections will be orthogonal. The existence of such a structure is crucial, epecially to the classiication of root systems (and reductive groups).

Define the linear map

$$\rho: V \longrightarrow V^\vee, \quad v \longmapsto \sum_{\lambda \in \Sigma} \langle v, \lambda^\vee \rangle \lambda^\vee$$

and define a symmetric dot product on V by the formula

$$u \bullet v = \langle u, \rho(v) \rangle = \sum_{\lambda \in \Sigma} \langle u, \lambda^\vee \rangle \langle v, \lambda^\vee \rangle .$$

The semi-norm

$$\|v\|^2 = v \bullet v = \sum_{\lambda \in \Sigma} \langle v, \lambda^\vee \rangle^2$$

is positive semi-definite, vanishing precisely on

$$\mathrm{RAD}(V) = (\Sigma^\vee)^\perp .$$

In particular $\|\lambda\| > 0$ for all roots λ. Since $\Sigma^\vee$ is W-invariant, the semi-norm $\|v\|^2$ is also W-invariant. That $\|v\|^2$ vanishes on $\mathrm{RAD}(V)$ mirrors the fact that the Killing form of a reductive Lie algebra vanishes on the radical of the algebra.

[norms] **Proposition 3.2.** *For every root λ*

$$\|\lambda\|^2 \, \lambda^\vee = 2\rho(\lambda) .$$

Thus although the map $\lambda \mapsto \lambda^\vee$ is not the restriction of a linear map, it is simply related to such a restriction.

Proof. For every μ in Σ

$$\begin{aligned} s_{\lambda^\vee} \mu^\vee &= \mu^\vee - \langle \lambda, \mu^\vee \rangle \lambda^\vee \\ \langle \lambda, \mu^\vee \rangle \lambda^\vee &= \mu^\vee - s_{\lambda^\vee} \mu^\vee \\ \langle \lambda, \mu^\vee \rangle^2 \, \lambda^\vee &= \langle \lambda, \mu^\vee \rangle \mu^\vee - \langle \lambda, \mu^\vee \rangle s_{\lambda^\vee} \mu^\vee \\ &= \langle \lambda, \mu^\vee \rangle \mu^\vee + \langle s_\lambda \lambda, \mu^\vee \rangle s_{\lambda^\vee} \mu^\vee \\ &= \langle \lambda, \mu^\vee \rangle \mu^\vee + \langle \lambda, s_{\lambda^\vee} \mu^\vee \rangle s_{\lambda^\vee} \mu^\vee \end{aligned}$$

But since $s_{\lambda^\vee}$ is a bijection of $\Sigma^\vee$ with itself, we can conclude by summing over μ in Σ. $\square$

[dot-product] **Corollary 3.3.** *For every v in V and root λ*

$$\langle v, \lambda^\vee \rangle = 2 \left(\frac{v \bullet \lambda}{\lambda \bullet \lambda} \right) .$$

Thus the formula for the reflection s_λ is that for an **orthogonal reflection**

$$s_\lambda v = v - 2 \left(\frac{v \bullet \lambda}{\lambda \bullet \lambda} \right) \lambda .$$

[equi-ranks] **Corollary 3.4.** *The semi-simple ranks of a root system and of its dual are equal.*

Proof. The map

$$\lambda \longmapsto \|\lambda\|^2 \, \lambda^\vee$$

is the same as the linear map 2ρ, so ρ is a surjection from $V(\Sigma)$ to $V^\vee(\Sigma^\vee)$. Apply the same reasoning to the dual system to see that $\rho^\vee \circ \rho$ must be an isomorphism, hence ρ an injection as well. ∎

[spanning] **Corollary 3.5.** *The space $V(\Sigma)$ spanned by Σ is complementary to $\mathrm{RAD}(V)$.*

So we have a direct sum decomposition

$$V = \mathrm{RAD}(V) \oplus V(\Sigma) .$$

Proof. Because the kernel of ρ is $\mathrm{RAD}(V)$. ∎

[vsigma-dual] **Corollary 3.6.** *The canonical map from $V(\Sigma)$ to the dual of $V^\vee(\Sigma^\vee)$ is an isomorphism.*

[sigma-lattice] **Corollary 3.7.** *The set Σ is contained in a lattice of $V(\Sigma)$.*

Proof. Because it is contained in the lattice of v such that $\langle v, \lambda^\vee \rangle$ is integral for all $\lambda^\vee$ in some linearly independent subset of $\Sigma^\vee$. ∎

[weyl-finite] **Corollary 3.8.** *The Weyl group is finite.*

Proof. It fixes all v annihilated by $\Sigma^\vee$ and therefore embeds into the group of permutations of Σ. ∎

The formula for s_λ as an orthogonal reflection remains valid for any W-invariant norm on $V(\Sigma)$ and its associated inner product. If we are given such an inner product, then we may set

$$\lambda^\bullet = \left(\frac{2}{\lambda \bullet \lambda} \right) \lambda ,$$

and then necessarily $\lambda^\vee$ is uniquely determined in $V^\vee(\Sigma^\vee)$ by the formula

$$\langle \mu, \lambda^\vee \rangle = \mu \bullet \lambda^\bullet .$$

But there is another way to specify $\lambda^\vee$ in terms of λ, one which works even for most infinite root systems. The following, which I first saw in [Tits:1966], is surprisingly useful.

[tits-uniqueness] **Corollary 3.9.** *The co-root $\lambda^\vee$ is the unique element of $V^\vee$ satisfying these conditions:*

(a) $\langle\lambda,\lambda^\vee\rangle = 2$;
(b) it lies in the subspace of $V^\vee$ *spanned by* $\Sigma^\vee$;
(c) for any μ *in* Σ, *the sum* $\sum_\nu\langle\nu,\lambda^\vee\rangle$ *over the affine line* $(\mu+\mathbb{Z}\,\lambda)\cap\Sigma$ *vanishes.*

Proof. That $\lambda^\vee$ satisfies (a) and (b) is easy; it satisfies (c) since the reflection s_λ preserves $(\mu+\mathbb{Z}\,\lambda)\cap\Sigma$.

To prove that it is unique, suppose ℓ another vector satisfying the same conditions. But then $\lambda^\vee-\ell$ is easily shown to be 0. ▯

[wvee] **Corollary 3.10.** *For all roots* λ *and* μ

$$(s_\lambda\mu)^\vee = s_{\lambda^\vee}\mu^\vee\,.$$

Proof. This is a direct calculation, using the orthogonal reflection formula, but I'll use Tits' criterion instead. Let $\chi = s_\lambda\mu$, $\rho^\vee = s_{\lambda^\vee}\mu^\vee$. According to that criterion, it must be shown that

(a) $\langle\chi,\rho^\vee\rangle = 2$;
(b) $\rho^\vee$ is in the linear span of $\Sigma^\vee$;
(c) for any τ we have

$$\sum_\nu\langle\nu,\rho^\vee\rangle = 0$$

where the sum is over ν in $(\tau+\mathbb{Z}\,s_\lambda\mu)\cap\Sigma$.

(a) We have

$$\begin{aligned}\langle\chi,\rho^\vee\rangle &= \langle s_\lambda\mu, s_{\lambda^\vee}\mu^\vee\rangle\\ &= \langle s_\lambda s_\lambda\mu,\mu^\vee\rangle\\ &= \langle\mu,\mu^\vee\rangle\\ &= 2\,.\end{aligned}$$

(b) Trivial.

(c) We have

$$\begin{aligned}\sum_{(\tau+\mathbb{Z}\,s_\lambda\mu)\cap\Sigma}\langle\nu, s_{\lambda^\vee}\mu^\vee\rangle &= \sum_{(\tau+\mathbb{Z}\,s_\lambda\mu)\cap\Sigma}\langle s_\lambda\nu,\mu^\vee\rangle\\ &= \sum_{(s_\lambda\tau+\mathbb{Z}\,\mu)\cap\Sigma}\langle\nu,\mu^\vee\rangle = 0\,.\end{aligned}$$ ▯

[reflection-reflection] **Corollary 3.11.** *For any roots* λ, μ *we have*

$$s_{s_\lambda\mu} = s_\lambda s_\mu s_\lambda\,.$$

Proof. The algebra becomes simpler if one separates this into two halves: (a) both transformations take $s_\lambda\mu$ to $-s_\lambda\mu$; (b) if $\langle v,(s_\lambda\mu)^\vee\rangle = 0$, then both take v to itself. Verifying these, using the previous formula for $(s_\lambda\mu)^\vee$, is straightforward. ▯

[semi-simple] **Proposition 3.12.** *The quadruple* $(V(\Sigma),\Sigma,V^\vee(\Sigma^\vee),\Sigma^\vee)$ *is a root system.*

It is called the **semi-simple root system** associated to the original.

[intersection] **Proposition 3.13.** *Suppose* U *to be a vector subspace of* V, $\Sigma_U = \Sigma\cap U$, $\Sigma_U^\vee = (\Sigma_U)^\vee$. *Then* $(V,\Sigma_U,V^\vee,\Sigma_U^\vee)$ *is a root system.*

Proof. If λ lies in $\Sigma_U = U\cap\Sigma$ then the reflection s_λ certainly preserves Σ_U. The same is true for $\Sigma_U^\vee$ by
♣ [wvee] Corollary 3.10. ▯

The metric $\|v\|^2$ vanishes on $(\Sigma^\vee)^\perp$, but any extension of it to a Euclidean metric on all of V for which Σ and this space are orthogonal will be W-invariant. Thus we arrive at a Euclidean structure on V such that for every λ in Σ:

(a) $2(\lambda \bullet \mu)/(\lambda \bullet \lambda)$ is integral;
(b) the subset Σ is stable under the orthogonal reflection

$$s_\lambda: v \longmapsto v - 2\left(\frac{v \bullet \lambda}{\lambda \bullet \lambda}\right)\lambda .$$

Conversely, suppose that we are given a vector space V with a Euclidean norm on it, and a finite subset Σ. For each λ in Σ, let

$$\lambda^\bullet = \left(\frac{2}{\lambda \bullet \lambda}\right)\lambda ,$$

and then (as before) define $\lambda^\vee$ in $V^\vee$ by the formula

$$\langle v, \lambda^\vee \rangle = v \bullet \lambda^\bullet .$$

[metric-axioms] **Proposition 3.14.** *Suppose that for each λ in Σ*

(a) $\mu \bullet \lambda^\bullet$ is integral for every μ in Σ;
(b) the subset Σ is stable under the orthogonal reflection

$$s_\lambda: v \longmapsto v - (v \bullet \lambda^\bullet)\lambda .$$

Then $(V, \Sigma, V^\vee, \Sigma^\vee)$ is a root system.

This is straightforward to verify. The point is that by using the metric we can avoid direct consideration of $\Sigma^\vee$.

In practice, root systems can be constructed from a very small amount of data. If S is a finite set of orthogonal reflections and Ξ a finite subset of V, the **saturation** of Ξ with respect to S is the smallest subset of V containing Ξ and stable under S.

[delta-construction] **Proposition 3.15.** *Suppose Λ to be a finite subset of a lattice L in the Euclidean space V such that conditions (a) and (b) hold for every λ in Λ. Then the saturation Σ of Λ with respect to the s_λ for λ in Λ is finite, and (a) and (b) are satisfied for all λ in Σ.*

Without the condition on L the saturation might well be infinite. This condition holds trivially if the elements of Λ are linearly independent.

Proof. Let S be the set of s_λ. We construct Σ by starting with Λ and applying elements of S repeatedly until we don't get anything new. This process has to stop with Σ finite, since the vectors we get are contained in L and bounded in length. At this point we know that (1) $s\Sigma = \Sigma$ for all s in S, and (2) every λ in Σ is obtained by a chain of reflections in S from an element of Ξ.

Let $L^\bullet$ be the lattice of v in V such that $v \bullet \lambda^\bullet$ is integral for every λ in L.

Define the **depth** of λ in Σ to be the smallest n such that for some sequence $s_1, \ldots, s_n$ in S the root $s_n \ldots s_1\lambda$ lies in Λ. The proof of (a) is by induction on depth. If μ lies in $L^\bullet$ then

$$(s_\alpha\mu)^\bullet = s_{\alpha^\bullet}\mu^\bullet = \mu^\bullet - (\mu^\bullet \bullet \alpha)\alpha^\bullet$$

is an integral combination of elements of $L^\bullet$.

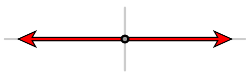
Condition (b) follows from the identity

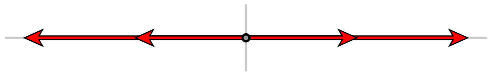
$$s_{s_\lambda\mu} = s_\lambda s_\mu s_\lambda ,$$

and induction on depth. ▯

4. Root systems of rank one

The simplest system is that containing just a vector and its negative. There is one other system of rank one, however, which we have already seen as that of SU_{ω_3}:





Throughout this section and the next I exhibit root systems by Euclidean diagrams, implicitly leaving it
♣ **[metric-axioms]** as an exercise to verify the conditions of Proposition 3.14.

That these are the only rank one systems follows from this:

[non-reduced] **Lemma 4.1.** *If λ and $c\lambda$ are both roots, then $|c| = 1/2, 1,$ or 2.*

Proof. On the one hand $(c\lambda)^\vee = c^{-1}\lambda^\vee$, and on the other $\langle \lambda, (c\lambda)^\vee \rangle$ must be an integer. Therefore $2c^{-1}$ must be an integer, and similarly $2c$ must be an integer. ∎

A root λ is called **indivisible** if $\lambda/2$ is not a root.

5. Root systems of rank two

I start with a situation generalizing that occurring with a root system of rank two:

> *Assume for the moment that we are given a finite set $\mathcal{L}$ of lines through the origin in the Euclidean plane stable under orthogonal reflections in those lines.*

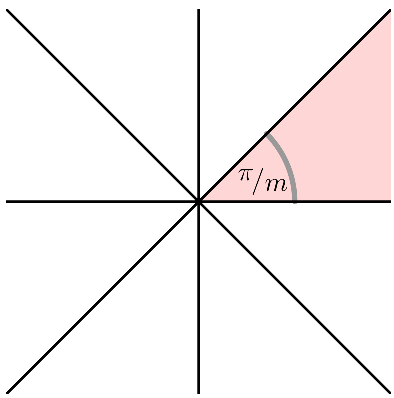
The connected components of the complement of the lines in $\mathcal{L}$ are all acute two-dimensional wedges. Pick one, call it C, and let the rays κ and ℓ be its boundary, say with ℓ in a positive direction from κ, obtained by rotating through an angle of θ, with $0 < \theta < \pi$.

We may choose our coordinate system so that κ is the x-axis. The product $\tau = s_\kappa s_\ell$ is a rotation through angle 2θ. The line $\tau^k \ell$ will lie in $\mathcal{L}$ and lie at angle $2k\theta$. Since $\mathcal{L}$ is finite, $2m\theta$ must be $2\pi p$ for some positive integers m, p and

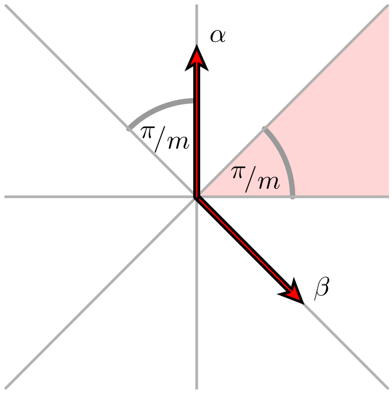
$$\theta = \frac{\pi p}{m}$$

where we may assume p and m relatively prime. Suppose k to be inverse to p modulo m, say $kp = 1+Nm$. The line $\tau^k L_2$ will then lie at angle $\pi/m + N\pi$, or effectively at angle π/m since the angle of a line is only determined up to π. If $p \neq 1$, this gives us a line through the interior of C, a contradiction. Therefore $\theta = \pi/m$ for some integer $m > 1$.

There are m lines in the whole collection. In the following figure, $m = 4$.



Suppose that α and β are vectors perpendicular to κ and ℓ, respectively, and on the sides indicated in the diagram:



Then the angle between α and β is $\pi - \pi/m$, and hence:

[rank-two] **Proposition 5.1.** *Suppose C to be a connected component of the complement of the lines in a finite family of lines stable under reflections. If*

$$C = \{\alpha > 0\} \cap \{\beta > 0\}$$

then

$$\alpha \bullet \beta \le 0\,.$$

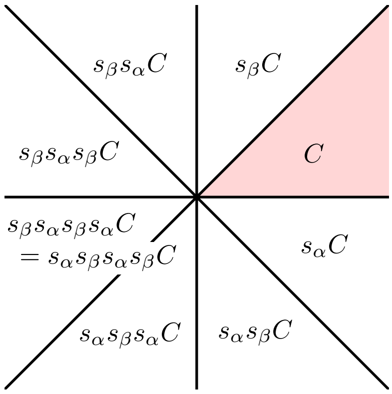
It is 0 if and only if the lines κ and ℓ are perpendicular.

In all cases, the region C is a fundamental domain for W.

As the following figure shows, the generators s_α and s_β satisfy the **braid relation**

$$s_\alpha s_\beta \ldots = s_\beta s_\alpha \ldots \quad (m \text{ terms on each side })\,.$$

This also follows from the identity $(s_\alpha s_\beta)^m = 1$, since the s_* are involutions. Let W^* be the abstract group with generators σ_α, σ_β and relations $\sigma_*^2 = 1$ as well as the braid relation. The map $\sigma_* \mapsto s_*$ is a homomorphism.



[braid2] **Proposition 5.2.** *This map from W^* to W is an isomorphism.*

I leave this as an exercise.

Summarizing for the moment: Let α and β be unit vectors such that C is the region of all x where $\alpha \bullet x > 0$ and $\beta \bullet x > 0$. We have $\alpha \bullet \beta = -\cos(\pi/m)$. Conversely, to each $m > 1$ there exists an essentially unique Euclidean root configuration for which W is generated by two reflections in the walls of a chamber containing an angle of π/m. The group W has order $2m$, and is called the **dihedral** group of order $2m$.

For the rest of this section, suppose $(V, \Sigma, V^\vee, \Sigma^\vee)$ to be a root system in the plane that actually spans the plane. The subspaces $\langle v, \lambda^\vee \rangle = 0$ are lines, and the set of all of them is a finite set of lines stable with respect to the reflections s_λ. The initial working assumption of this section holds, but that the collection of lines arises from a root system imposes severe restrictions on the integer m.

With the wedge C chosen as earlier, again let α and β be roots such that C is where $\alpha \bullet v > 0$ and $\beta \bullet v > 0$. Suppose π/m to be the angle between the two rays bounding C. The matrices of the corresponding reflections with respect to the basis (α, β) are

$$s_\alpha = \begin{bmatrix} -1 & -\langle \beta, \alpha^\vee \rangle \\ 0 & 1 \end{bmatrix}, \quad s_\beta = \begin{bmatrix} 1 & 0 \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix}$$

and that of their product is

$$s_\alpha s_\beta = \begin{bmatrix} -1 & -\langle \beta, \alpha^\vee \rangle \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix} = \begin{bmatrix} -1 + \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle & \langle \beta, \alpha^\vee \rangle \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{bmatrix} .$$

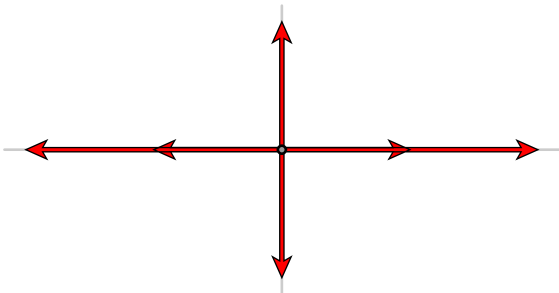
This product must be a non-trivial Euclidean rotation. Because it must have eigenvalues of absolute value 1, its trace $\tau = -2 + \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$ must satisfy the inequality

$$-2 \le \tau < 2 \,,$$

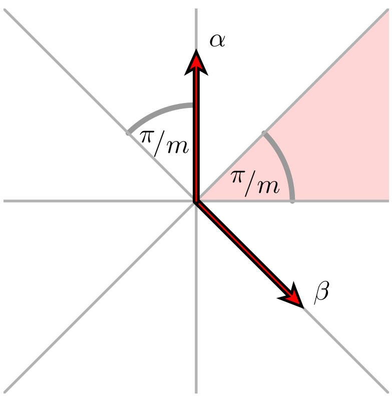
which imposes the condition

$$0 \le n_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle < 4 \,.$$

But $n_{\alpha,\beta}$ must also be an integer. Therefore it can only be 0, 1, 2, or 3. It will be 0 if and only if s_α and s_β commute, which means that Σ is the orthogonal union of two rank one systems:



So now suppose the root system to be irreducible. Recall the picture:



Here, $\alpha \bullet \beta$ will actually be negative. By switching α and β if necessary, we may assume that one of these cases is at hand:

- $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -1$;
- $\langle \alpha, \beta^\vee \rangle = -2$, $\langle \beta, \alpha^\vee \rangle = -1$;
- $\langle \alpha, \beta^\vee \rangle = -3$, $\langle \beta, \alpha^\vee \rangle = -1$.

Since

$$\begin{aligned} \langle \alpha, \beta^\vee \rangle &= 2\left(\frac{\alpha \bullet \beta}{\beta \bullet \beta}\right) \\ \langle \beta, \alpha^\vee \rangle &= 2\left(\frac{\beta \bullet \alpha}{\alpha \bullet \alpha}\right) \\ \|\alpha\|^2 \langle \alpha, \beta^\vee \rangle &= \|\beta\|^2 \langle \beta, \alpha^\vee \rangle \end{aligned}$$

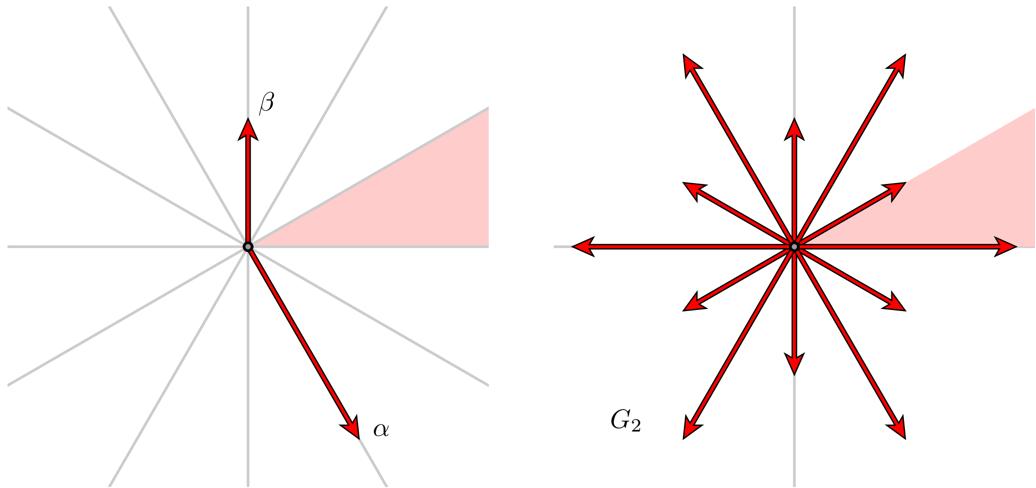
we also have

$$\frac{\langle \alpha, \beta^\vee \rangle}{\|\alpha\|^2} = \frac{\langle \beta, \alpha^\vee \rangle}{\|\beta\|^2}\,.$$

Let's now look at one of the three cases, the last one. We have $\langle \alpha, \beta^\vee \rangle = -3$ and $\langle \beta, \alpha^\vee \rangle = -1$. Therefore $\|\alpha\|^2 / \|\beta\|^2 = 3$. If φ is the angle between α and β,

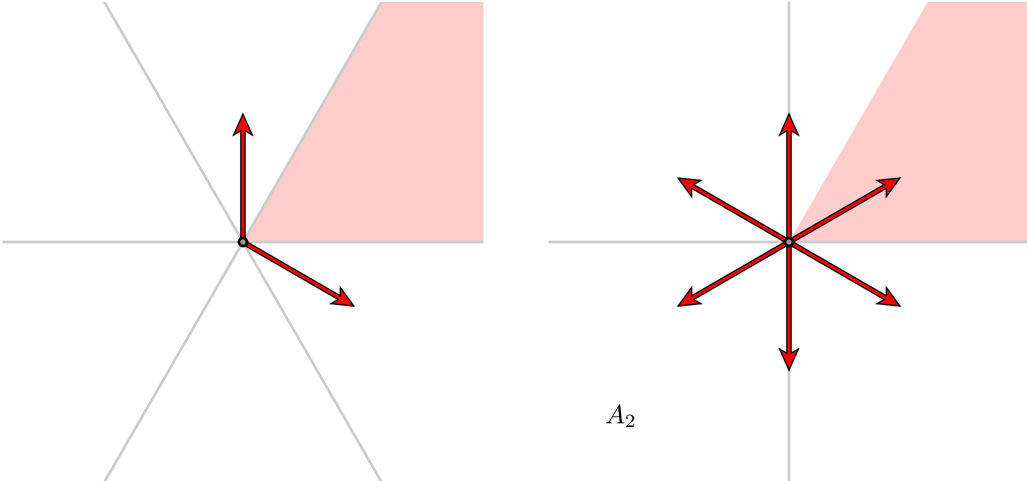
$$\cos \varphi = \frac{\alpha \bullet \beta}{\|\alpha\|\,\|\beta\|} = \frac{1}{2}\,\frac{\langle \beta, \alpha^\vee \rangle \|\alpha\|}{\|\beta\|} = -\sqrt{3}/2\,.$$

Thus $\varphi = \pi - \pi/6$ and $m = 6$. Here is the figure and its saturation (with a different convention as to orientation, in order to make a more compressed figure):

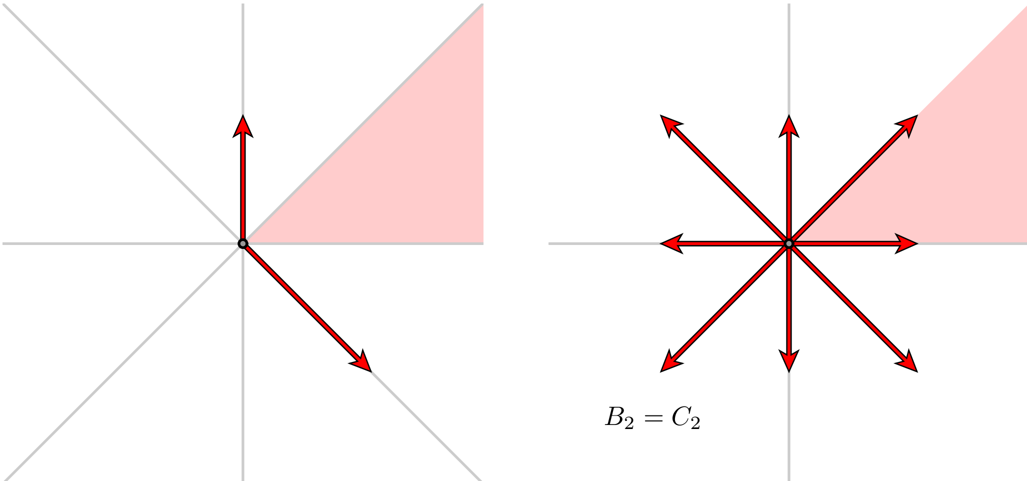


This system is called G_2.

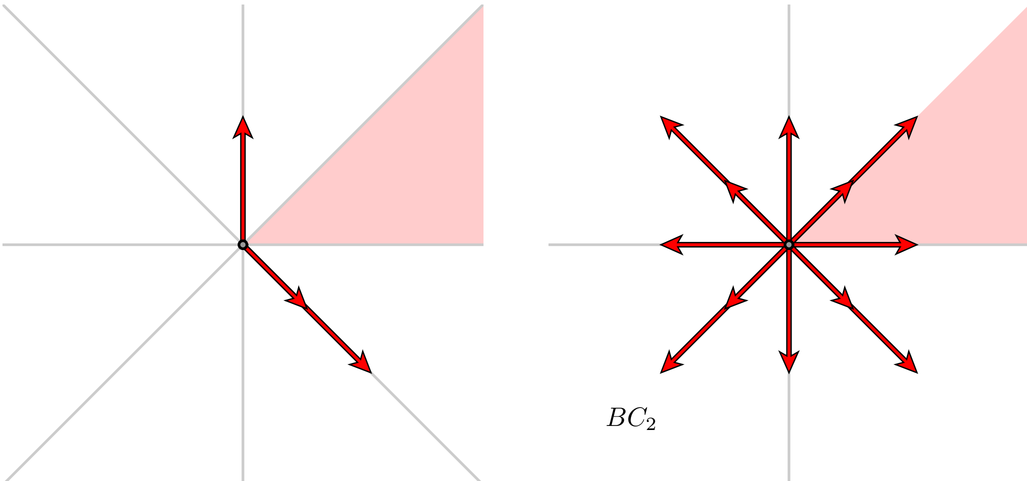
Taking the possibility of non-reduced roots into account, we get all together three more possible irreducible systems:



A_2



$B_2 = C_2$



BC_2

The first three are reduced.

In summary:

[W-roots] **Proposition 5.3.** *Suppose that* $(V, \Sigma, V^\vee, \Sigma^\vee)$ *is a root system of semi-simple rank two. Let* C *be one of the complements of the root reflection lines, equal to the region* $\alpha > 0$ $\beta > 0$ *for roots* α*,* β*. Swapping* α *and* β *if necessary, we have one of these cases:*

$$\begin{aligned} \langle \alpha, \beta^\vee \rangle &= 0, & \langle \beta, \alpha^\vee \rangle &= 0, & m_{\alpha,\beta} &= 2\,; \\ \langle \alpha, \beta^\vee \rangle &= -1, & \langle \beta, \alpha^\vee \rangle &= -1, & m_{\alpha,\beta} &= 3\,; \\ \langle \alpha, \beta^\vee \rangle &= -2, & \langle \beta, \alpha^\vee \rangle &= -1, & m_{\alpha,\beta} &= 4\,; \\ \langle \alpha, \beta^\vee \rangle &= -3, & \langle \beta, \alpha^\vee \rangle &= -1, & m_{\alpha,\beta} &= 6\,. \end{aligned}$$

Another way to see the restriction on the possible values of m is to ask, what roots of unity generate quadratic field extensions of $\mathbb{Q}$?

6. Hyperplane partitions

I now want to begin to generalize the preceding arguments to higher rank.

Suppose $(V, \Sigma, V^\vee, \Sigma^\vee)$ to be a root system. Associated to it are two partitions of $V^\vee$ by hyperplanes. The first is that of hyperplanes $\lambda^\vee = 0$ for $\lambda^\vee$ in $\Sigma^\vee$. This is the one we are primarily interested in. But the structure of reductive groups over p-adic fields is also related to a partition by hyperplanes $\lambda^\vee = k$ in $V^\vee$ where k is an integer. Any of these configurations is stable under Euclidean reflections in these hyperplanes. Our goals in this section and the next few are to show that the connected components of the complement of the hyperplanes in either of these are open fundamental domains for the group generated by these reflections, and to relate geometric properties of this partition to combinatorial properties of this group. In this preliminary section we shall look more generally at the partition of Euclidean space associated to an arbitrary locally finite family of hyperplanes, an exercise concerned with rather general convex sets.

Thus suppose for the moment V to be any Euclidean space, $\mathfrak{h}$ to be a locally finite collection of affine hyperplanes in V.

A connected component C of the complement of $\mathfrak{h}$ in V will be called a **chamber**. If H is in $\mathfrak{h}$ then C will be contained in exactly one of the two open half-spaces determined by H, since C cannot intersect H. Call this half space $D_H(C)$. The following is extremely elementary.

[allh] **Lemma 6.1.** *If* C *is a chamber then*

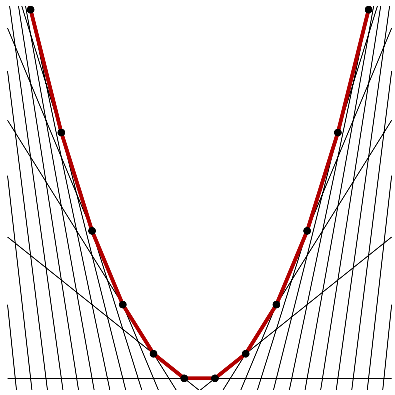
$$C = \bigcap_{H \in \mathfrak{h}} D_H(C)\,.$$

Proof. Of course C is contained in the right hand side. On the other hand, suppose that x lies in C and that y is contained in the right hand side. If H is in $\mathfrak{h}$ then the closed line segment $[x, y]$ cannot intersect H, since then C and y would lie on opposite sides. So y lies in C also. ▯

Many of the hyperplanes in $\mathfrak{h}$ will be far away, and they can be removed from the intersection without harm. Intuitively, only those hyperplanes that hug C closely need be considered, and the next result, only slightly less elementary, makes this precise.

A **panel** of C is a face of C of codimension one, a subset of codimension one in the boundary of $\overline{C}$. A point lies on a panel of C if it lies on exactly one hyperplane in $\mathfrak{h}$, The support of a panel will be called a **wall**. A panel with support H is a connected component of the complement of the union of the $H_* \cap H$ as H_* runs through the other hyperplanes of $\mathfrak{h}$. Chambers and their faces, including panels, are all convex.

A chamber might very well have an infinite number of panels, for example if $\mathfrak{h}$ is the set of tangent lines to the parabola $y = x^2$ at points with integral coordinates.



[chambers] **Proposition 6.2.** *If C is a chamber then*

$$C = \bigcap_{H \in \mathfrak{h}_{\mathfrak{C}}} D_H(C)\,.$$

where now $\mathfrak{h}_C$ is the family of walls of C.

That is to say, there exists a collection of affine functions f such that C is the intersection of the regions $f > 0$, and each hyperplane $f = 0$ for f in this collection is a panel of C.

Proof. Suppose y to be in the intersection of all $D_H(C)$ as H varies over the walls of C. We want to show that y lies in C itself. This will be true if we can pick a point x in C such that the line segment from y to x intersects no hyperplane in $\mathfrak{h}$. Suppose that we can find x in C with the property that the line segment from y to x intersects hyperplanes of $\mathfrak{h}$ only in panels. If it in fact intersects any hyperplanes at all, then the last one (the one nearest x) will be a panel of C whose wall separates C from y, contradicting the assumption on y.

It remains to find such a point x. Pick an arbitrary small ball contained in C, and then a large ball B containing this one as well as y. Since the collection of hyperplanes is locally finite, this large ball intersects only a finite subset $\{H_i\}$ of them. Let $\mathfrak{k}$ be the collection of intersections $H_i \cap H_j$ of pairs of these. The intersection of B with these intersections will be relatively closed in B, of codimension two (possibly empty). Let S be the union of all lines through y and a point in $\mathfrak{k}$, whose intersection with B will be of codimension one. Its complement in B will be open and dense. If x is a point of C in this complement then the intersection points of the line segment from x to y will each be on exactly one hyperplane in $\mathfrak{h}$, hence on a panel. ▯

7. Euclidean root configurations

The motivation for the investigation here is that if Σ is a set of roots in a Euclidean space V, then there are two associated families of hyperplanes: (1) the linear hyperplanes $\alpha = 0$ for α in Σ and (2) the affine hyperplanes $\alpha = k$ for α in Σ and k an integer. Many of the properties of root systems are a direct consequence of the geometry of hyperplane arrangements rather than the algebra of roots, and it is useful to isolate geometrical arguments. Affine configurations play an important role in the structure of $\mathfrak{p}$-adic groups.

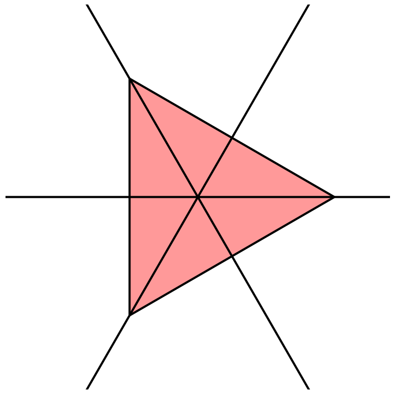
For any hyperplane H in a Euclidean space let s_H be the orthogonal reflection in H. A **Euclidean root configuration** is a locally finite collection $\mathfrak{h}$ of hyperplanes that's stable under each of the orthogonal reflections s_H with respect to H in $\mathfrak{h}$. The group W generated by these reflections is called the **Weyl group** of the configuration. Each hyperplane is defined by an equation $\lambda_H(v) = f_H \bullet v + k = 0$ where f_H may be taken to be a unit vector. The vector $\mathrm{GRAD}(\lambda_H) = f_H$ is uniquely determined up to scalar multiplication by ± 1. We have the explicit formula

$$s_H v = v - 2\lambda_H(v)\, f_H\,.$$

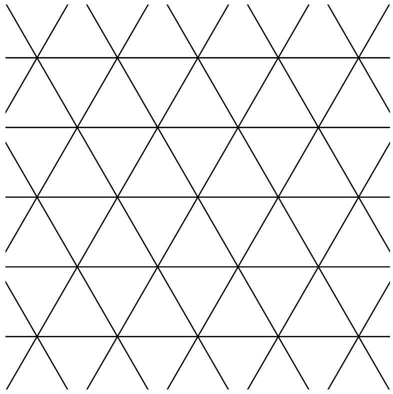
The **essential dimension** of the system is the dimension of the vector space spanned by the gradients f_H.

Here are some examples.

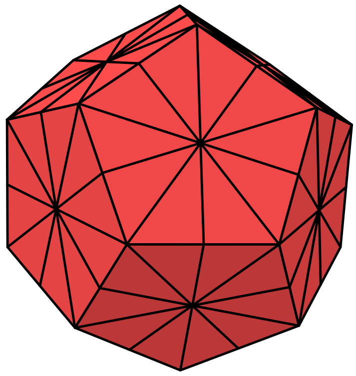
The lines of symmetry of any regular polygon form a planar root configuration. These are exactly the systems explored earlier. The lines at right, for the equilateral triangle, the lines of root reflection for SL_3.



The packing of the plane by equilateral triangles. This is the set of lines $\lambda = k$ for roots λ of SL_3.



The planes of symmetry of any regular figure in Euclidean space form a Euclidean root configuration. Here are shown the planes of symmetry of a dodecahedron and the corresponding triangulation.



A **chamber** is one of the connected components of the complement of the hyperplanes in a Euclidean root configuration. All chambers are convex and open. Fix a chamber C, and let Δ be a set of affine functions α such that $\alpha = 0$ is a wall of C and $\alpha > 0$ on C. For each subset $\Theta \subseteq \Delta$, let W_Θ be the subgroup of W generated by the s_α with α in Θ.

[non-pos] **Proposition 7.1.** *For any distinct α and β in Δ,* $\mathrm{GRAD}(\alpha) \bullet \mathrm{GRAD}(\beta) \le 0$.

Proof. The group $W_{\alpha,\beta}$ generated by s_α and s_β is dihedral. If P and Q are points of the faces of C defined by α and β, respectively, the line segment from P to Q crosses no hyperplane of $\mathfrak{h}$. The region ♣ [rank-two] $\alpha \bullet x > 0$, $\beta \bullet x > 0$ is therefore a fundamental domain for $W_{\alpha,\beta}$. Apply Proposition 5.1. ▯

♣ [chambers] Suppose C to be a chamber of the hyperplane partition. According to Proposition 6.2, C is the intersection of the open half-spaces determined by its walls, the affine supports of the parts of its boundary of codimension one. Reflection in any two of its walls will generate a dihedral group.

[walls-finite] **Corollary 7.2.** *The number of panels of a chamber is finite.*

Proof. If V has dimension n, the unit sphere in V is covered by the $2n$ hemispheres $x_i > 0$, $x_i < 0$. By
♣ [non-pos] Proposition 7.1, each one contains at most one of the $\mathrm{GRAD}(\alpha)$ in Δ. □

[separating-finite] **Lemma 7.3.** *If $\mathfrak{h}$ is a locally finite collection of hyperplanes, the number of H in $\mathfrak{h}$ separating two chambers is finite.*

Proof. A closed line segment connecting them is compact and can meet only a finite number of H in $\mathfrak{h}$. □

[chambers-transitive] **Proposition 7.4.** *The group W_Δ acts transitively on the set of chambers.*

Proof. By induction on the number of root hyperplanes separating two chambers C and C_*, which is
♣ [chambers] finite by the previous Lemma. If it is 0, then $C = C_*$ by Proposition 6.2. Otherwise, one of the walls H of C_* separates them, and the number separating $s_H C_*$ from C will be one less. Apply induction. □

The next several results will tell us that W is generated by the reflections in the walls of C, that the closure of C is a fundamental domain for W, and (a strong version of this last fact) that if F is a face of C then the group of w in W fixing such that $F \cap w(F) \neq \emptyset$ then w lies in the subgroup generated by the reflections in the walls of C containing v, which all in fact fix all points of F. Before I deal with these, let me point out at the beginning that the basic point on which they all depend is the trivial observation that *if $w \neq I$ in W fixes points on a wall H then it must be the orthogonal reflection s_H.*

[generate] **Corollary 7.5.** *The reflections in S generate the group W.*

Proof. It suffices to show that every s_λ lies in W_Δ.

Suppose F to be a panel in hyperplane $\lambda = 0$. According to the Proposition, if this panel bounds C_* we can find w in W_Δ such that $wC_* = C$, hence $w(F)$ lies in $\overline{C}$, and therefore must equal a panel of C. Then $w^{-1}s_\alpha w$ fixes the points of this panel and therefore must be s_λ. □

Given any hyperplane partition, a **gallery** between two chambers C and C_* is a chain of chambers $C = C_0, C_1, \ldots, C_n = C_*$ in which each two successive chambers share a common face of codimension 1. The integer n is called the **length** of the gallery. I'll specify further that any two successive chambers in a gallery are distinct, or in other words that the gallery is not **redundant**. The gallery is called **minimal** if there exist no shorter galleries between C_0 and C_n. The **combinatorial distance** between two chambers is the length of a minimal gallery between them.

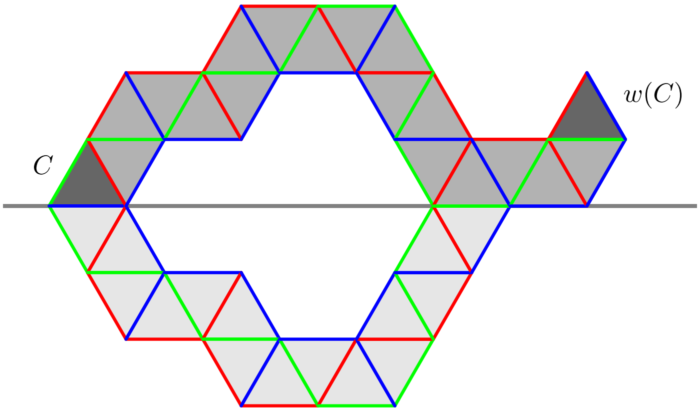
Expressions $w = s_1 s_2 \ldots s_n$ with each s_i in S can be interpreted in terms of galleries. There is in fact a bijective correspondence between such expressions and galleries linking C to $w\,C$. This can be seen inductively. The trivial expression for 1 in terms of the empty string just comes from the gallery of length 0 containing just $C_0 = C$. A single element $w = s_1$ of S corresponds to the gallery $C_0 = C$, $C_1 = s_1 C$. If we have constructed the gallery for $w = s_1 \ldots s_{n-1}$, we can construct the one for $s_1 \ldots s_n$ in this fashion: the chambers C and $s_n C$ share the wall $\alpha = 0$ where $s_n = s_\alpha$, and therefore the chambers wC and $ws_n C$ share the wall $w\alpha = 0$. The pair $C_{n-1} = wC$, $C_n = ws_n C$ continue the gallery from C to $ws_n C$.

This associates to every expression $s_1 \ldots s_n$ a gallery, and the converse construction is straightforward.

[fixC] **Proposition 7.6.** *If $wC = C$ then $w = 1$.*

Proof. If $w = s_1 \ldots s_n$ with $wC = C$, $W \neq 1$ then the corresponding gallery must first cross and recross the reflection hyperplane of s_1. The result now follows by induction from this:

[cross-recross] **Lemma 7.7.** *If the gallery associated to the expression $s_1 \ldots s_n$ with $n > 1$ winds up on the same side of the reflection hyperplane of s_1 as C, then there exists some $1 \leq i \leq n$ with $w = s_2 \ldots s_{i-1} s_{i+1} \ldots s_n$.*



Proof. Let H be the hyperplane in which s_1 reflects. Let $w_i = s_1 \dots s_i$. The path of chambers $w_1 C$, $w_2 C$, $\dots$ crosses H at the very beginning and then crosses back again. Thus for some i we have $w_{i+1}C = w_i s_{i+1} C = s_1 w_i C$. But if $y = s_2 \dots s_i$ we have $s_{i+1} = y^{-1} s_1 y$, and $w_{i+1} = y$, so $w = s_1 y s_{i+1} \dots s_n = y s_{i+2} \dots s_n$. $\square$

8. The Cartan matrix

Now assum $(V, \Sigma, V^\vee, \Sigma^\vee)$ to be a root system. If $\mathfrak{h}$ is the union in $V_{\mathbb{R}}^\vee$ of the **root hyperplanes** $\lambda = 0$, the ♣ [chambers-transitive] connected components of its complement in $V_{\mathbb{R}}^\vee$ are called **Weyl chambers**. According to Proposition 7.4 ♣ [fixC] and Proposition 7.6, these form a principal homogeneous space under the action of W.

Fix a chamber C. Let Δ be the set of indivisible roots α with $\alpha = 0$ defining a panel of C with $\alpha > 0$ on C, $\overline{\Delta}$ the extension to include multiples as well.

[W-transitive] **Proposition 8.1.** *Every root in Σ is in the W-orbit of $\overline{\Delta}$.*

Proof. This is because W acts transitively on the chambers. $\square$

A root λ is called **positive** if $\langle \lambda, C \rangle > 0$, **negative** if $\langle \lambda, C \rangle < 0$. All roots are either positive or negative, since by definition no root hyperplanes meet C. Let S be the set of reflections s_α for α in Δ. The Weyl group W is generated by S.

The matrix $\langle \alpha, \beta^\vee \rangle$ for α, β in Δ is called the **Cartan matrix** of the system. Since $\langle \alpha, \alpha^\vee \rangle = 2$ for all roots α, its diagonal entries are all 2. According to the discussion of rank two systems, its off-diagonal entries

$$\langle \alpha, \beta^\vee \rangle = 2 \left(\frac{\alpha \bullet \beta}{\beta \bullet \beta} \right) = \alpha \bullet \beta^\bullet$$

are all non-positive. Furthermore, one of these off-diagonal entries is 0 if and only if its transpose entry is. This equation means that if D is the diagonal matrix with entries $2/\alpha \bullet \alpha$ and M is the matrix $(\alpha \bullet \beta)$ then

$$A = MD\,.$$

The next few results of this section all depend on understanding the Gauss elimination process applied to M. It suffices just to look at one step, reducing all but one entry in the first row and column to 0. Since $\alpha_1 \bullet \alpha_1 > 0$, it replaces each vector α_i with $i > 1$ by its projection onto the space perpendicular to α_1:

$$\alpha_i \longmapsto \alpha_i^\perp = \alpha_i - \frac{\alpha_i \bullet \alpha_1}{\alpha_1 \bullet \alpha_1} \quad (i > 1)\,.$$

If I set $\alpha_1^\perp = \alpha_1$, the new matrix $M^\perp$ has entries $\alpha_i^\perp \bullet \alpha_j^\perp$. We have the matrix equation

$$L M\, {}^t\!L = M^\perp, \quad M^{-1} = {}^t\!L\, (M^\perp)^{-1}\, L$$

with L a unipotent lower triangular matrix

$$L = \begin{bmatrix} 1 & \ell \\ {}^t\ell & I \end{bmatrix} \qquad \ell_i = -\frac{\alpha_1 \bullet \alpha_i}{\alpha_1 \bullet \alpha_1} \geq 0 \, .$$

This argument and induction proves immediately:

[gauss] **Proposition 8.2.** *If M is a symmetric matrix, then it is positive definite if and only if Gauss elimination yields*

$$M^{-1} = {}^tL\,A\,L$$

for some lower triangular matrix L and diagonal matrix A with positive diagonal entries. If $m_{i,j} \leq 0$ for $i \neq j$ then L and hence also M^{-1} will have only non-negative entries.

[lummox] **Corollary 8.3.** *Suppose $A = (a_{i,j})$ to be a matrix such that $a_{i,i} > 0$, $a_{i,j} \leq 0$ for $i \neq j$. Assume $D^{-1}A$ to be a positive definite matrix for some diagonal matrix D with positive entries. Then A^{-1} has only non-negative entries.*

[titsA] **Lemma 8.4.** *Suppose Δ to be a set of vectors in a Euclidean space V such that $\alpha \bullet \beta \leq 0$ for $\alpha \neq \beta$ in Δ. If there exists v such that $\alpha \bullet v > 0$ for all α in Δ then the vectors in Δ are linearly independent.*

Proof. By induction on the size of Δ. The case $|\Delta| = 1$ is trivial. But the argument just before this handles the induction step, since if $v \bullet \alpha > 0$ then so is $v \bullet \alpha^{\perp}$. ▯

As an immediate consequence:

[tits] **Proposition 8.5.** *The set Δ is a basis of $V(\Sigma)$.*

That is to say, a Weyl chamber is a simplicial cone. Its extremal edges are spanned by the columns ϖ_α in the inverse of the Cartan matrix, which therefore have positive coordinates with respect to Δ. Hence:

[roots-inverse] **Proposition 8.6.** *Suppose Δ to be a set of linearly independent vectors such that $\alpha \bullet \beta \leq 0$ for all $\alpha \neq \beta$ in Δ. If D is the cone spanned by Δ then the cone dual to D is contained in D.*

Proof. Let

$$\varpi = \sum c_\alpha \alpha$$

be in the cone dual to D. Then for each β in Δ

$$\varpi \bullet \beta = \sum c_\alpha \, (\alpha \bullet \beta) \, .$$

If A is the matrix $(\alpha \bullet \beta)$, then it satisfies the hypothesis of the Lemma. If u is the vector (c_α) and v is the vector $(\varpi \bullet \alpha)$, then by assumption the second has non-negative entries and

$$u = A^{-1} v$$

so that u also must have non-negative entries. ▯

[base] **Proposition 8.7.** *Each positive root may be expressed as $\sum_{\alpha \in \Delta} c_\alpha \alpha$ where each c_α is a non-negative integer.*

Proof. The chamber C is the positive span of the basis (ϖ_α) dual to Δ. If $\lambda = \sum c_\alpha \alpha$ then $c_\alpha = \lambda \bullet \varpi_\alpha$, so the positive roots, which are defined to be those positive on C, are also those with non-negative coefficients c_α. If λ is an arbitrary positive root, then there must exist α in Δ with $\langle \lambda, \alpha^\vee \rangle > 0$, hence such that $\lambda - \alpha$ is either 0 or again a root. Suppose it is a root. If it is α itself then $\lambda = 2\alpha$ and the Proposition is proved. Otherwise, some $c_\beta > 0$. The $c_\alpha - 1$ must therefore also be positive. Continuing in this way expresses λ as a positive integral combination of elements of Δ. ▯

One consequence of all this is that the roots in Δ generate a full lattice in $V(\Sigma)$. By duality, the coroots generate a lattice in $V^\vee(\Sigma^\vee)$, which according to the definition of root systems is contained in the lattice of $V^\vee(\Sigma^\vee)$ dual to the root lattice of $V(\Sigma)$.

From this and the discussion of chains, this has as consequence:

[chain-to-Delta] **Proposition 8.8.** *Every positive root can be connected to Δ by a chain of links $\lambda \mapsto \lambda + \alpha$.*

We now know that each root system gives rise to an integral Cartan matrix $A = (a_{\alpha,\beta}) = (\langle \alpha, \beta^\vee \rangle)$ with rows and columns indexed by Δ. It has these properties:

(a) $a_{\alpha,\alpha} = 2$;
(b) $a_{\alpha,\beta} \le 0$ for $\alpha \ne \beta$;
(c) $a_{\alpha,\beta} = 0$ if and only if $A_{\beta,\alpha} = 0$;

but it has another implicit property as well. We know that there exists a W-invariant Euclidean norm with respect to which the reflections are invariant. None of the conditions above imply that this will be possible. We know already know the extra condition

$$0 \le a_{\alpha,\beta} a_{\beta,\alpha} < 4 \,,$$

but this alone will not guarantee that the root system will be finite, as the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -\,2 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

shows. Instead, we construct a candidate metric, and then check whether it is positive definite. We start with the formula we have already encountered:

$$\langle \alpha, \beta^\vee \rangle = 2 \left(\frac{\alpha \bullet \beta}{\beta \bullet \beta} \right) .$$

Suppose A satisfies (a)–(c). Construct a graph from A whose nodes are elements of Δ and with edge between α and β if and only if $\langle \alpha, \beta \rangle \ne 0$. In each connected component of this graph, choose an arbitrary node α and arbitrarily assign a positive rational value to $\alpha \bullet \alpha$. Assign values for all $\beta \bullet \gamma$ according to the rules

$$\beta \bullet \gamma = \frac{1}{2} \langle \beta, \gamma^\vee \rangle \, (\gamma \bullet \gamma)$$
$$\beta \bullet \beta = 2 \, \frac{\beta \bullet \gamma}{\langle \gamma, \beta^\vee \rangle} \, .$$

which allow us to go from node to node in any component. This defines an inner product, and the extra condition on the Cartan matrix is that this inner product must be positive definite, or equivalently

(d) The matrix $(\alpha \bullet \beta)$ must be positive definite.

This can be tested by Gauss elimination in rational arithmetic, as suggested by the discussion at the beginning of this section.

♣ [delta-construction] Given a Cartan matrix satisfying (a)–(d), the saturation process of Proposition 3.15 gives us a root system associated to it. Roughly, it starts with an extended basis $\overline{\Delta}$ and keeps applying reflections to generate all the roots.

In the older literature one frequently comes across another way of deducing the existence of a basis Δ for positive roots. Suppose $V = V(\Sigma)$, say of dimension ℓ, and assume it given a coordinate system. Linearly order V **lexicographically**: $(x_i) < (y_i)$ if $x_i = y_i$ for $i < m$ but $x_m < y_m$. Informally, this is **dictionary order**. For example, $(1, 2, 3) < (1, 3, 2)$. This order is translation-invariant. [Satake:1951] remarks that this is the only way to define a linear, translation-invariant order on a real vector space.

Define Σ^+ to be the subset of roots in Σ that are positive with respect to the given order. Define α_1 to be the least element of Σ^+, and for $1 < k \le \ell$ inductively define α_k to be the least element of Σ^+ that is not in the linear span of the α_i with $i < k$.

The following seems to be first found in [Satake:1951].

[satake] **Proposition 8.9.** *Every root in Σ^+ can be expressed as a positive integral combination of the α_i.*

Proof. It is easy to see that if Δ is a basis for Σ^+ then it has to be defined as it is above. It is also easy to see directly that if $\alpha < \beta$ are distinct elements of Δ then $\langle \alpha, \beta^\vee \rangle \le 0$. Because if not, according to ♣ [chains1] Lemma 11.1 the difference $\beta - \alpha$ would also be a root, with $\beta > \beta - \alpha > 0$. But this contradicts the definition of β as the least element in Σ^+ not in the span of smaller basis elements.

The proof of the Proposition goes by induction on ℓ. For $\ell = 1$ there is nothing to prove. Assume true for $\ell - 1$. Let Σ_* be the intersection of the span of the α_i with $i \le \ell$, itself a root system. We want to show that every λ in the linear span of Σ is a positive integral combination of the α_i. If λ is in Σ_* induction gives this, and it is also true for $\lambda = \alpha_\ell$. Otherwise $\lambda > \alpha_\ell$. Consider all the $\lambda - \alpha_i$ with $i \le \ell$. It suffices to show that one of them is a root, by an induction argument on order. If not, then all $\langle \lambda, \alpha_i \rangle \le 0$. This ♣ [roots-inverse] leads to a contradiction of Proposition 8.6, to be proven later (no circular reasoning, I promise). ▯

The following is found in [Chevalley:1955].

[many-roots] **Corollary 8.10.** *Suppose Δ to be a basis of positive roots in Σ. Let $\Lambda = \{\lambda_i\}$ be a set of linearly independent roots, and Λ_k be the subset containing the λ_i for $i \le k$. There exists w in W and an ordering*

$$\alpha_1 < \ldots < \alpha_n$$

of Δ such that each $w\Lambda_k$ is a positive linear combination of the α_i with $i \le k$.

Proof. Make up an ordered basis of V extending Λ with $\lambda_i < \lambda_{i+1}$. Apply the Proposition to get a basis Δ_* of positive roots. Then λ_1 is the first element of Δ and each λ_k is a positive linear combination of the $\alpha_{*,i}$ with $i \le k$. But then we can find w in W taking Δ_* to Δ. ▯

9. Generators and relations for the Weyl group

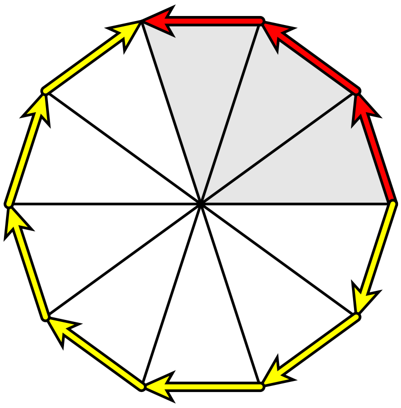
I return to the subject of arbitrary hyperplane partitions. Let C be a chamber, which we know know to be either a simplicial cone or a simplex. Let S be the set of reflections in the walls of C, which we know to generate W. For each $s \ne t$ in S let $m_{s,t}$ be the order of st.

We have a surjection onto W from the abstract group with generators s in S and relations

$$s^2 = 1, \quad (st)^{m_{s,t}} = 1\,.$$

[coxeter] **Proposition 9.1.** *This surjection is an isomorphism.*

Proof sketch. Assume $V = V(\Sigma)$. First of all, we have already seen this for a planar partition, so we may assume the dimension of V to be at least 3. Construct the simplicial complex dual to that determined by the faces of the hyperplane partition. Thus a chamber C give rise to a vertex v_C, panels to edges. This dual complex gives in effect a triangulation of the unit sphere, which is simply connected. If $wC = C$, then an expression $w = s_1 \ldots s_n$ corresponds to a path along edges from v_C back to itelf. Since the unit sphere has trivial fundamental group, standard topological arguments imply that it may be deformed into the trivial path inside the two-skeleton of the complex. But any such deformation corresponds to a sequence of applications of the braid relation.



Combinatorial proofs of this can be found in the literature, showing directly that if $s_1 \ldots s_n = 1$ then it can be reduced by the the given relations to the trivial word.

The group W is a therefore a **Coxeter group**, and many of the properties of W are shared by all Coxeter groups. One of the useful ones is a result due to Jacques Tits:

[Tits] **Proposition 9.2.** *Any two reduced expressions for an element w can be obtained from each other by a succession of braid relations.*

Proof. THe same topological argument applies here, but I'll give instead a direct proof by induction on length, since it is essentially constructive. Suppose

$$w = s_1 \ldots s_m s = t_1 \ldots t_m t$$

are two reduced expressions. Thus $ws < w$, $wt < w$. Hence

$$w = x w_{s,t} s = y w_{t,s} t$$

where $w_{s,t} s$ and $w_{t,s} t$ are the two sides of the braid relation. Then

$$s_1 \ldots s_m = x w_{s,t}, \quad t_1 \ldots t_m = y w_{t,s}$$

and by induction we can find a sequence of braid relations transforming one to the other. But then replacing $w_{s,t} s$ by $w_{t,s} t$ connects $x w_{s,t} s$ to $y w_{t,s} t$. □

[ws] **Proposition 9.3.** *For any w in W and s in S, if $\langle \alpha_s, wC \rangle < 0$ then $\ell(sw) < \ell(w)$ and if $\langle \alpha_s, wC \rangle > 0$ then $\ell(sw) > \ell(w)$.*

♣ [cross-recross] *Proof.* This is essentially a rephrasing of Lemma 7.7, but I'll repeat the argument. Suppose $\langle \alpha_s, wC \rangle < 0$. Then C and wC are on opposite sides of the hyperplane $\alpha_s = 0$. If $C = C_0, \ldots, C_n = wC$ is a minimal gallery from C to wC, then for some i C_i is on the same side of $\alpha_s = 0$ as C but C_{i+1} is on the opposite side. The gallery $C_0, \ldots, C_i, sC_{i+2}, \ldots sC_n = swC$ is a gallery of shorter length from C to swC, so $\ell(sw) < \ell(w)$.

If $\langle \alpha_s, wC \rangle > 0$ then $\langle \alpha_s, swC \rangle < 0$ and hence $\ell(w) = \ell(ssw) < \ell(sw)$. □

[stabilizers] **Proposition 9.4.** *If v and wv both lie in $\overline{C}$, then $wv = v$ and w belongs to the group generated by the reflections in S fixing v.*

Proof. By induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$ and the result is trivial.

If $\ell(w) > 1$ then let $x = sw$ with $\ell(x) = \ell(w) - 1$. Then C and wC are on opposite sides of the hyperplane
♣ [ws] $\alpha_s = 0$, by Proposition 9.3. Since v and wv both belong to $\overline{C}$, the intersection $\overline{C} \cap w\overline{C}$ is contained in the hyperplane $\alpha_s = 0$ and wv must be fixed by s. Therefore $wv = xv$. Apply the induction hypothesis. □

For $\Theta \subseteq \Delta$, let C_Θ be the open face of $\overline{C}$ where $\alpha = 0$ for α in Θ, $\alpha > 0$ for α in Δ but not in Θ. If F is a face of any chamber, the Proposition tells u it will be W-equivalent to a unique $\Theta \subseteq \Delta$. The faces of chambers are therefore canonically labeled by subsets of Δ.

Let

$$R_w = \{\lambda > 0 \mid w\lambda < 0\}$$
$$L_w = \{\lambda > 0 \mid w^{-1}\lambda < 0\}$$

♣ [separating-finite] Of course $L_w = R_{w^{-1}}$. According to Lemma 7.3, the set R_w determines the root hyperplanes separating C from $w^{-1}C$, and $|R_w| = |L_w| = \ell(w)$.

I recall that an expression for w as a product of elements of S is reduced if it is of minimal length. The length of w is the length of a reduced expression for w as products of elements of S. Minimal galleries correspond to reduced expressions. The two following results are easy deductions:

[rw] **Proposition 9.5.** *For x and y in W, $\ell(xy) = \ell(x) + \ell(y)$ if and only if R_{xy} is the disjoint union of $y^{-1}R_x$ and R_y.*

Finally, suppose that the configuration we are considering is one associate to a linear root system, not an affine one. There are only a finite number of hyperplanes in the configuration, and all pass through the origin. Since $-C$ is then also a chamber:

[longest] **Proposition 9.6.** *There exists in W a unique element w_ℓ of maximal length, with $w_\ell C = -C$. For $w = w_\ell$, $R_w = \Sigma^+$.*

10. Subsystems

If Θ is a subset of Δ, let Σ_Θ be the roots which are integral linear combinations of elements of Θ. These, along with V, $V^\vee$ and their image in $\Sigma^\vee$ form a root system. Its Weyl group is the subset W_Θ. Recall that to each subset $\Theta \subseteq \Delta$ corresponds the face C_Θ of $\overline{C}$ where $\lambda = 0$ for $\lambda \in \Theta$, $\lambda > 0$ for $\lambda \in \Delta - \Theta$.
♣ [stabilizers] According to Proposition 9.4, an element of W fixes a point in C_Θ if and only if it lies in W_Θ.

[cosets] **Proposition 10.1.** *In each coset $W_\Theta\backslash W$ there exists a unique representative x of least length. This element is the unique one in its coset such that $x^{-1}\Theta > 0$. For any y in W_Θ we have $\ell(yx) = \ell(y) + \ell(x)$.*

Proof. The region in $V^\vee$ where $\alpha > 0$ for α in Θ is a fundamental domain for W_Θ. For any w in W there exists y in W_Θ such that $xC = y^{-1}wC$ is contained in this region. But then $x^{-1}\alpha > 0$ for all α in Θ. In fact, x will be the unique element in $W_\Theta w$ with this property.

The element x can be found explicitly. Let $[W_\Theta\backslash W]$ be the set of these distinguished representatives, $[W/W_\Theta]$ those for right cosets. These distinguished representatives can be found easily. Start with $x = w$, $t = 1$, and as long as there exists s in $S = S_\Theta$ such that $sx < x$ replace x by sx, t by ts. At every moment we have $w = tx$ with t in W_Θ and $\ell(w) = \ell(t) + \ell(x)$. At the end we have $sx > x$ for all s in S. ▯

Similarly, in every double coset $W_\Theta\backslash W/W_\Phi$ there exists a unique element w of least length such that $w^{-1}\Theta > 0$, $w\Phi > 0$. Let these distinguished representatives be expressed as $[W_\Theta\backslash W/W_\Phi]$.

The next result is motivated by the following considerations. Suppose $\mathfrak{g}$ to be a reductive Lie algebra, $\mathfrak{b}$ a maximal solvable ('Borel') subalgebra, Δ the associated basis of positive roots. If $\mathfrak{p}$ is a Lie subalgebra containing $\mathfrak{b}$, there will exist a subset Θ of Δ such that $\mathfrak{p}$ is the sum of $\mathfrak{b}$ and the direct sum of root spaces $\mathfrak{g}_\lambda$ for λ in Σ_Θ^-. The roots occurring are those in $\Xi = \Sigma^+ \cup \Sigma_\Theta^-$. This set is a **parabolic subset**—(a) it contains all positive roots and (b) it is closed in the sense that if λ and μ are in it so is $\lambda + \mu$. The set Θ is characterized by the subset Ξ, since $-\Theta = -\Delta \cap \Xi$.

For example, if $G = \mathrm{GL}_n$ and $\Theta = \{\varepsilon_m - \varepsilon_{m+1}\}$, then P is the subgroup of matrices where $x_{i,j} = 0$ for $i > m$, $j \le m$. If $n = 4$ and $m = 2$, we get the pattern

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} .$$

Conversely:

[parabolic-set] **Proposition 10.2.** *Suppose Ξ to be a parabolic subset, and let $\Theta = \Delta \cap -\Xi$. Then $\Xi = \Sigma^+ \cup \Sigma^-_\Theta$.*

Proof. We need to show (a) if ξ is in $\Xi \cap \Sigma^-$ then ξ is in Σ^-_Θ and (b) if ξ is in Σ^-_Θ then ξ is in Ξ.

Suppose ξ in Σ^-_Θ. Set $\xi = -\sum_\alpha c_\alpha \alpha$. The proof goes by induction on $h(\xi) = \sum c_\alpha$. Since $-\Theta \subseteq \Xi$, ξ is in Ξ if $h(\xi) = 1$. Otherwise $\xi = \xi_* - \alpha$ with ξ_* also in Σ^-_X. By induction ξ_* is in Ξ, and since Ξ is closed so is ξ in Ξ.

Suppose ξ in $\Xi \cap \Sigma^-$. If $h(\xi) = 1$ then ξ is in $-\Delta \cap \Xi = -\Theta$. Otherwise, $\xi = \xi_* - \alpha$ with α in Δ. Then $\xi_* = \xi + \alpha$ also lies in Ξ since Ξ contains all positive roots and it is closed. Similarly $-\alpha = \xi - \xi_*$ lies in Ξ, hence in Θ. By induction ξ_* lies Σ^-_Θ, but then so does ξ. ▯

In the rest of this section, assume for convenience that $V = V(\Sigma)$ (i.e. that the root system is semi-simple), and also that the root system is reduced. The material to be covered is important in understanding the decomposition of certain representations of reductive groups. I learned it from Jim Arthur, but it appears as Lemma 2.13 of [Langlands:1976], and presumably goes back to earlier work of Harish-Chandra.

At this point we understand well the partition of all of V by chambers. In this discussion we'll look at the induced partitions of lower dimensional linear subspaces in the partition.

Fix the chamber C with associated Δ. For each $\Theta \subseteq \Delta$ let

$$V_\Theta = \bigcap_{\alpha \in \Theta} \ker(\alpha) .$$

The set of roots which vanish identically on V_Θ are those in Σ_Θ. The space V_Θ is partitioned into chambers by the hyperplanes $\lambda = 0$ for λ in $\Sigma^+ - \Sigma^+_\Theta$. One of these is the face C_Θ of the fundamental Weyl chamber $C = C_\emptyset$. If $\Theta = \emptyset$ we know that the connected components of the complement of root hyperplanes are a principal homogeneous set with respect to the full Weyl group. In general, the chambers of V_Θ are the faces of full chambers, and in particular we know that each is the Weyl transform of a unique facette of a fixed positive chamber C. But we want to make this more precise.

The problem to be dealt with here is that the chambers in the partition of V_Θ do not form a principal homogeneous set for any group.

[associates] **Proposition 10.3.** *Suppose Θ and Φ to be subsets of Δ. The following are equivalent:*

(a) there exists w in W taking V_Φ to V_Θ;
(b) there exists w in W taking Φ to Θ.

In these circumstances, Θ and Φ are said to be **associates**. Let $W(\Theta, \Phi)$ be the set of all w taking Φ to Θ.

Proof. That (b) implies (a) is immediate. Thus suppose $wV_\Phi = V_\Theta$. This implies that $w\Sigma_\Phi = \Sigma_\Theta$. Let w_* be of least length in the double coset $W_\Theta w W_\Phi$, so that $w_*\Sigma^+_\Phi = \Sigma^+_\Theta$. Since Φ and Θ are bases of Σ^+_Φ and Σ^+_Θ, this means that $w_*\Phi = \Theta$. ▯

[associate-chambers] **Corollary 10.4.** *For each w in $W(\Theta, \Phi)$ the chamber wC_Φ is a chamber of V_Θ. Conversely, every chamber of V_Θ is of the form wC_Φ for a unique associate Φ of Θ and w in $W(\Theta, \Phi)$.*

Proof. The first assertion is trivial. Any chamber of V_Θ will be of the form wC_Φ for some w in W and some unique $\Phi \subseteq \Delta$. But then $wV_\Phi = V_\Theta$. ▯

We'll see in a moment how to find w and Φ by an explicit geometric construction.

One of the chambers in V_Θ is $-C_\Theta$. How does that fit into the classification? For any subset Θ of Δ, let $w_{\ell,\Theta}$ be the longest element in the Weyl group W_Θ. The element $w_{\ell,\Theta}$ takes Θ to $-\Theta$ and permutes $\Sigma^+\backslash\Sigma^+_\Theta$. The longest element $w_\ell = w_{\ell,\Delta}$ takes $-\Theta$ back to a subset $\overline{\Theta}$ of Σ^+ called its **conjugate** in Δ.

[opposite-cell] **Proposition 10.5.** *If $\Phi = \overline{\Theta}$ and $w = w_\ell w_{\ell,\Theta}$ then $-C_\Theta = w^{-1}C_\Phi$.*

Proof. By definition of the conjugate, $wV_\Theta = V_\Phi$ and hence $w^{-1}V_\Phi = V_\Theta$. The chamber $-C_\Theta$ is the set of vectors v such that $\alpha \bullet v = 0$ for α in Θ and $\alpha \bullet v < 0$ for α in $\Delta\backslash\Theta$. Analogously for C_Φ. ▯

[only] **Lemma 10.6.** *If Θ is a maximal proper subset of Δ then its only associate in Δ is $\overline{\Theta}$.*

Proof. In this case V_Θ is a line, and its chambers are the two half-lines C_Θ and its complement. ▯

If $\Omega = \Theta \cup \{\alpha\}$ for a single α in $\Delta - \Theta$ then the Weyl element $w_{\ell,\Omega}w_{\ell,\Theta}$ is called an **elementary conjugation**.

[elementary-conjugation] **Lemma 10.7.** *Suppose that $\Omega = \Theta \cup \{\alpha\}$ with α in $\Delta - \Theta$. Then the chamber and sharing the panel C_Ω with C_Θ is sC_Φ where Φ is the conjugate of Θ in Ω and $s = w_{\ell,\Omega}w_{\ell,\Phi}$.*

Proof. Let $C_* = wC_\Phi$ be the neighbouring chamber with $s\Phi = \Theta$. Then s fixes the panel shared by C_Θ and C_*, so must lie in W_Ω. But then Φ must be an associate of Θ in Ω. Apply Lemma 10.6. ▯ ♣ [only]

♣ [associate-chambers] A gallery in V_Θ is a sequence of chambers $C_0, C_1, \ldots, C_n$ where C_{i-1} and C_i share a panel. To each C_i we associate according to Corollary 10.4 a subset Θ_i and an element w_i of $W(\Theta, \Theta_i)$. Since C_{i-1} and C_i share a panel, so do $w_{i-1}^{-1}C_{i-1}$ and $w_{i-1}C_i$. But $w_{i-1}C_{i-1}$ is $C_{\Theta_{i-1}}$, so to this we may apply the preceding Lemma to see that Θ_{i1} and Θ_i are conjugates in their union Ω_i, and that $s_i = w_{i-1}^{-1}w_i$ is equal to the corresponding conjugation. The gallery therefore corresponds to an expression$w = s_1 \ldots s_n$ where each s_i is an elementary conjugation. In summary:

[conjugates] **Proposition 10.8.** *Every element of $W(\Theta, \Phi)$ can be expressed as a product of elementary conjugations. Each such expression corresponds to a gallery from C_Θ to wC_Φ.*

For w in $W(\Theta, \Phi)$ its **relative length** is the length of a minimal gallery in V_Θ leading from C_Θ to wC_Φ.

For w in $W(\Theta, \Phi)$, let ψ_w be the set of hyperplanes in V_Θ separating C_Θ from wC_Φ. Then it is easy to see that $\ell_{\mathrm{rel}}(xy) = \ell_{\mathrm{rel}}(x) + \ell_{\mathrm{rel}}(y)$ if and only if $\psi_y \cup y\psi_x \subseteq \psi_{xy}$.

[relative-separates] **Lemma 10.9.** *Suppose w in $W(\Theta, \Phi)$. Then*

(a) If λ is in $\Sigma^+\backslash\Sigma^+_\Theta$ separates C_Θ from wC_Φ, λ separates $wC_\emptyset$ from $C_\emptyset$;
(b) If $\lambda > 0$ separates $wC_\emptyset$ from $C_\emptyset$, either $\lambda \in \Sigma^+_\Theta$ or $\lambda \in \Sigma^+\backslash\Sigma^+_\Theta$.

[longest-relative] **Proposition 10.10.** *If w lies in $W(\Theta, \Phi)$ then*

$$\ell_{\mathrm{rel}}(w_\ell w_\Phi) = \ell_{\mathrm{rel}}(w_\ell w_{\ell,\Phi} w^{-1}) + \ell_{\mathrm{rel}}(w)\,.$$

[relative-lengths] **Proposition 10.11.** *Suppose x in $W(\Theta_3, \Theta_2)$, y in $W(\Theta_2, \Theta_1)$. If the relative length of xy is the sum of the relative lengths of x and y, then $\ell(xy) = \ell(x) + \ell(y)$.*

Proof. By induction on relative length. ▯

If $C_0, C_1, \ldots, C_n$ is a gallery in V_Θ, it is called **primitive** if Θ_{i-1} is never the same as Θ_i.

[primitive] **Proposition 10.12.** *Every $W(\Theta, \Phi)$ has at least one element with a primitive representation.*

Proof. If $w = s_1 \ldots s_{i-1}s_i \ldots s_n$ and $\Theta_{i-1} = \Theta_i$ then $s_1 \ldots \widehat{s}_i \ldots s_n$ is also in $W(\Theta, \Phi)$. ▯

11. Chains

If λ and μ are roots, the λ-**chain** through μ is the set of all roots of the form $\mu + n\lambda$. We already know that both μ and $r_{\lambda^\vee}\mu$ are in this chain. So is everything in between, as we shall see. The basic result is:

[chains1] **Lemma 11.1.** *Suppose λ and μ to be roots.*

(a) *If $\langle \mu, \lambda^\vee \rangle > 0$ then $\mu - \lambda$ is a root unless $\lambda = \mu$.*
(b) *If $\langle \mu, \lambda^\vee \rangle < 0$ then $\mu + \lambda$ is a root unless $\lambda = -\mu$.*

Proof. If λ and μ are proportional then either $\lambda = 2\mu$ or $\mu = 2\lambda$ and the claims are immediate. Suppose they are not. If $\langle \mu, \lambda^\vee \rangle > 0$ then either it is 1 or $\langle \lambda, \mu^\vee \rangle = 1$. In the first case

$$s_\lambda \mu = \mu - \langle \mu, \lambda^\vee \rangle \lambda = \mu - \lambda$$

so that $\mu - \lambda$ is a root, and in the second $s_\mu \lambda = \lambda - \mu$ is a root and consequently $\mu - \lambda$ also. The other claim is dealt with by swapping $-\lambda$ for λ. ▯

Hence:

[chain-ends] **Proposition 11.2.** *If μ and ν are left and right end-points of a segment in a λ-chain, then $\langle \mu, \lambda^\vee \rangle \le 0$ and $\langle \nu, \lambda^\vee \rangle \ge 0$.*

[chains2] **Proposition 11.3.** *Suppose λ and μ to be roots. If $\mu - p\lambda$ and $\mu + q\lambda$ are roots then so is every $\mu + n\lambda$ with $-p \le n \le q$.*

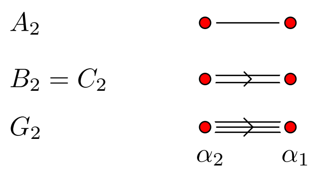
Proof. Since$\langle \mu + n\lambda, \lambda^\vee \rangle$ is an increasing function of n, the existence of a gap between two segments would contradict the Corollary. ▯

12. Dynkin diagrams

The **Dynkin diagram** of a reduced system with base Δ is a labeled graph whose nodes are elements of Δ, and an edge between α and β when

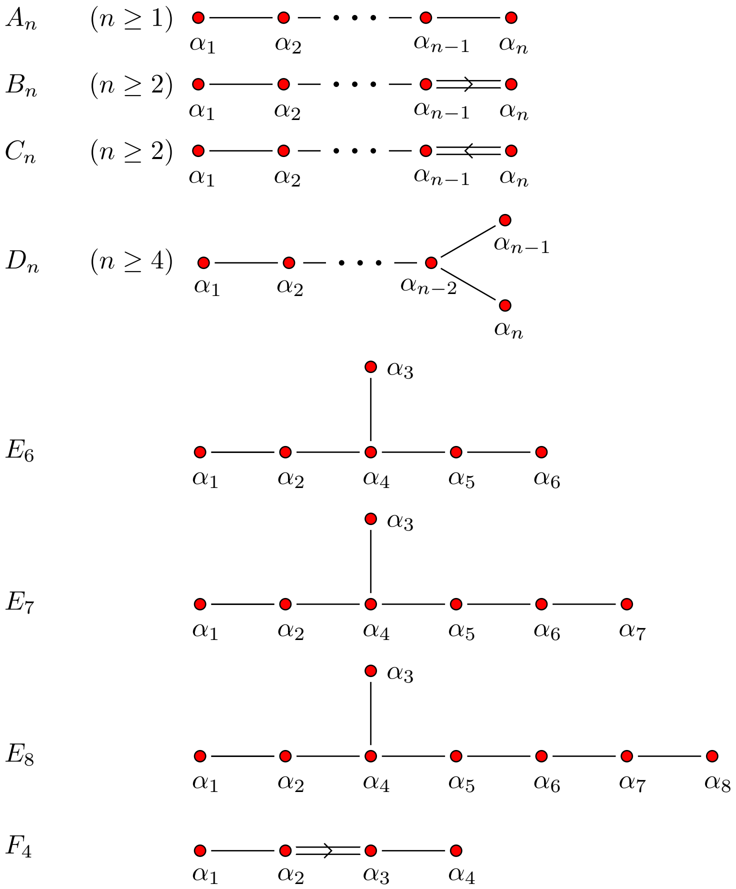
$$\left|\langle \alpha, \beta^\vee \rangle\right| \ge \left|\langle \beta, \alpha^\vee \rangle\right| > 0 \, .$$

This edge is labeled by the value of $\left|\langle \alpha, \beta^\vee \rangle\right|$, and this is usually indicated by a multiple link, oriented from α to β. Here are the Dynkin diagrams of all the reduced rank two systems:



In the diagram for G_2, and in the following diagrams as well, I have noted the conventional labelling of the roots.

The Dynkin diagram determines the Cartan matrix of a reduced system. The complete classification of irreducible, reduced systems is known, and is explained by the following array of diagrams. The numbering is arbitrary, even inconsistent as n varies, but follows the convention of Bourbaki. Note also that although systems B_2 and C_2 are isomorphic, the conventional numbering is different for each of them.



In addition there is a series of non-reduced systems of type BC_n obtained by superimposing the diagrams for B_n and C_n.

13. Dominant roots

A positive root $\widetilde{\alpha}$ is called **dominant** if every other root is of the form

$$\widetilde{\alpha} - \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with all c_α in $\mathbb{N}$.

[dominant] **Proposition 13.1.** *If Σ is an irreducible root system, then there exists a unique dominant root.*

Proof. We can even describe a practical way to calculate it. Start with any positive root λ, for example
♣ [chains1] one in Δ, and as long as some $\lambda + \alpha$ with α in Δ is a root, replace λ by $\lambda + \alpha$. According to Lemma 11.1, at the end we have $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all α in Δ. We then also have $\langle \alpha, \lambda^\vee \rangle \geq 0$ for all α, and at least one of these must actually be > 0.

Suppose $\lambda = \sum n_\alpha \alpha$. Let X be the α with $n_\alpha \neq 0$ and Y its complement in Δ. If Y isn't empty, then because of irreducibility there exists α in X and β in Y with $\langle \alpha, \beta^\vee \rangle < 0$. Hence we get the contradiction

$$0 \leq \langle \lambda, \beta^\vee \rangle = \sum_{\alpha \in X} n_\alpha \langle \alpha, \beta^\vee \rangle < 0$$

So $n_\alpha > 0$ for all α.

If there were a root ν not of the form

$$\lambda - \sum_{\alpha\in\Delta} c_\alpha \alpha$$

then there would exist a second root μ, constructed from ν by the same procedure, with the same property. Then

$$\langle \lambda, \mu^\vee \rangle = n_\alpha \sum \langle \alpha, \mu^\vee \rangle > 0 \,.$$

♣ [chains1] According to Lemma 11.1, $\lambda - \mu$ is a root. But this implies that either $\lambda > \mu$ or $\lambda < \mu$, contradicting maximality.

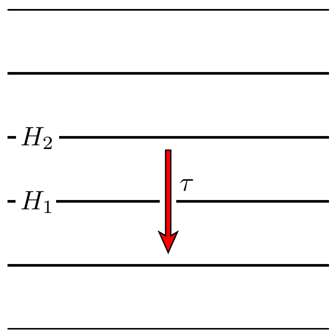
The proof shows that the dominant root is the unique one in the closure of the positive Weyl chamber.

Note that for an irreducible system the dominant root will be the last one in the queue.

14. Affine root systems

Assume a reduced semi-simple root system.

Two configurations will be considered equivalent if they are the same up to an affine transformation. Let's look first at those configurations for which the Weyl group is a **dihedral group**, one generated by orthogonal reflections in two hyperplanes. There are two cases, according to whether the hyperplanes are parallel or not.



The first case is easiest. Let H_1 and H_2 be the two parallel hyperplanes. The product $\tau = s_{H_1} s_{H_2}$ is a translation, and the hyperplanes $\tau^m(H_1)$ and $\tau^n(H_2)$ form a Euclidean root configuration. Conversely, any Euclidean root configuration in which all the hyperplanes are parallel arises in this way. We are dealing here with an essentially one-dimensional configuration. The group W is the infinite dihedral group.

♣ [base] According to Proposition 8.7, the roots are contained in a lattice—in fact, in the free $\mathbb{Z}$-module $\mathcal{R}$ spanned by Δ. The coroots then span a lattice $\mathcal{R}^\vee$ contained in the dual lattice $\mathrm{Hom}(\mathcal{R}, \mathbb{Z})$. In general the inclusion will be proper. The roots will in turn then be contained in the dual $\mathrm{Hom}(\mathcal{R}^\vee, \mathbb{Z})$ of $\mathcal{R}\vee$.

Define W_{aff} to be the group of affine transformations generated by W and translations by elements of $\mathcal{R}^\vee$, $\widehat{W}$ the larger group generated by W and $\mathrm{Hom}(\mathcal{R}, \mathbb{Z})$. Both groups preserve the system of affine hyperplanes $\lambda + k = 0$, where λ is a root, consequently permuting the connected components of the complement, called **alcoves**.

[alcoves] **Proposition 14.1.** *The region C_{aff} where $\alpha > 0$ for all α in Δ, $\widetilde{\alpha} < 1$ is an alcove.*

Proof. It has to be shown that for any root α and integer k, the region C_{aff} lies completely on one side or the other of the hyperplane $\alpha \bullet x - k = 0$. If $k = 0$ this is clear. If $k < 0$ we can change α to $-\alpha$ and k to $-k$, so we may assume that $k > 0$. Since 0 lies in the closure of C_{aff}, it must be shown that $\alpha + k < 0$ on
♣ [dominant] all of C_{aff}. But by Proposition 13.1 we can write $\alpha = \widetilde{\alpha} - \sum_\Delta c_\beta \beta$ with all $c_\beta \geq 0$, so for any x in C_{aff}

$$\alpha \bullet x = \widetilde{\alpha} \bullet x - \sum_\Delta c_\beta (\beta \bullet x) < 1 \leq k \,.$$

Let $\widetilde{\Delta}$ be the union of Δ and $-\widetilde{\alpha}+1$. For any pair α, β in $\widetilde{\Delta}$

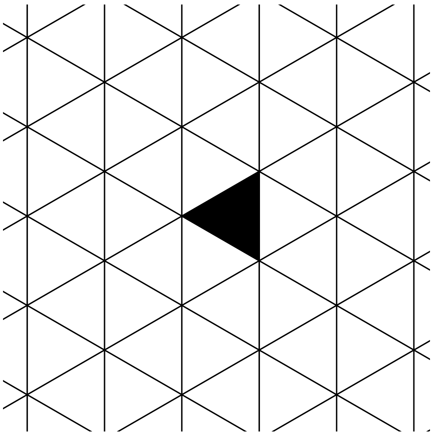
$$-4 < \langle \alpha, \beta^\vee \rangle \le 0$$

The **affine Dynkin diagram** is a graph whose nodes are elements of $\widetilde{\Delta}$ with edges labelled and oriented according to the values of $\langle \alpha, \beta^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle$. Let $\widetilde{s}$ be the affine reflection in the hyperplane $\widetilde{\alpha} = 1$, $\widetilde{S}$ the union of S and $\widetilde{s}$.

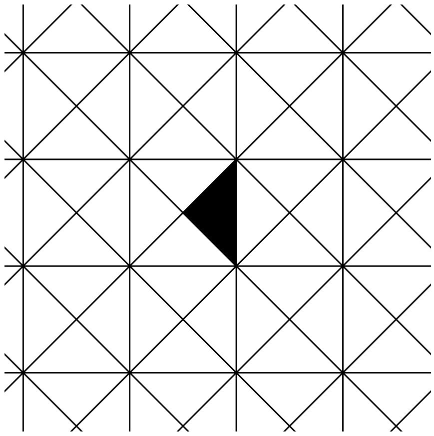
[affines] **Corollary 14.2.** *The group $W_{\rm aff}$ is generated by the involutions in $\widetilde{S}$.*

♣ [stabilizers] According to Proposition 9.4, every face of an alcove is the transform by an element of $W_{\rm aff}$ of a unique face of the alcove $C_{\rm aff}$. Elements of the larger group $\widetilde{W}$ also permute alcoves, but do not necessarily preserve this labelling. If w is an element of $\widetilde{W}$ and α an element of $\widetilde{\Delta}$, then the face wF_α of $wC_{\rm aff}$ will be the transform xF_β for some unique β in $\widetilde{\Delta}$. Let $\iota(w)$ be the map from $\widetilde{\Delta}$ to itself taking α to β.

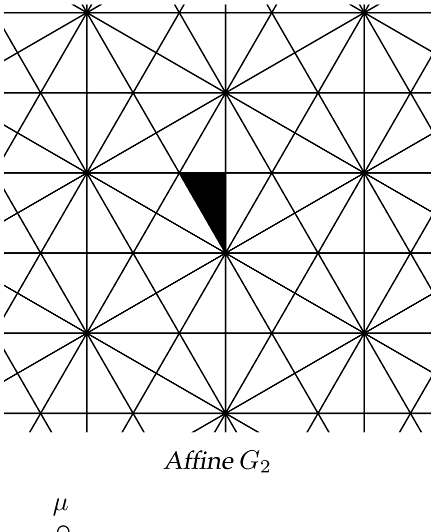
[automorphism] **Proposition 14.3.** *The map from $w \mapsto \iota(w)$ induces an isomorphism of $\widetilde{W}/W_{\rm aff}$ with the group of automorphisms of the affine Dynkin diagram.*



Affine A_2

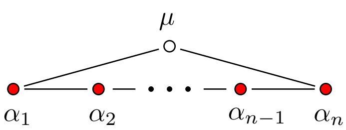


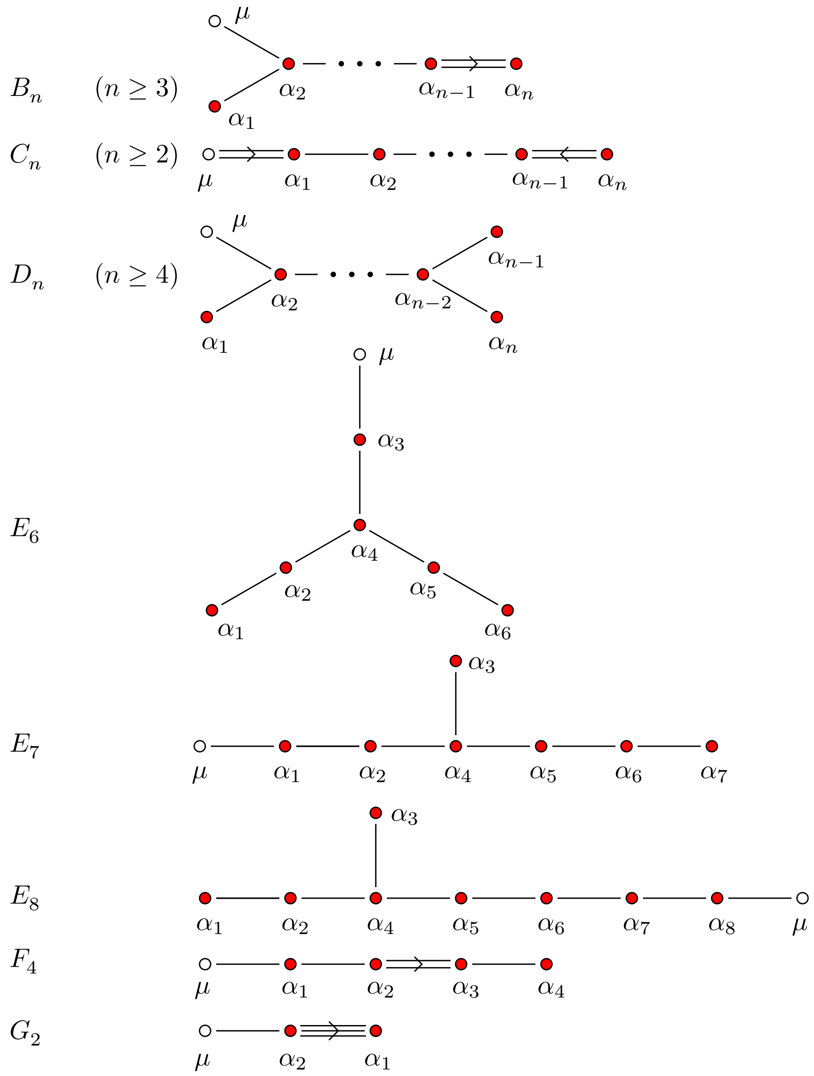
Affine C_2



Affine G_2

A_n $(n \ge 1)$





15. Computation

Working explicitly with a root system of any rank beyond 3 or 4 is tedious, and best done with a computer. I'll mention here only a few techniques.

Here are some computational problems that arise naturally: (1) Given a Cartan matrix, list all the corresponding roots together with data associated to them, such as their reflections; (2) list the entire Weyl group, or at least traverse through it in order to verify something; (3) compute the λ-chains, as well as the numbers $p_{\lambda,\mu}$; They will be useful in dealing with Lie algebras.

The first question that arises is how exactly what data to attach to a root. This might vary with what one wants to calculate, but as a minimum the array of coefficients in an expression

$$\lambda = \sum_{\Delta} \lambda_\alpha \alpha \, .$$

Another is, how to represent an element of W? The obviou schoice is by its matrix, but this will be rather large and in most cases not helpful. A better choice was suggested to me many years ago by David Vogan. For each v in $V^\vee$ define

$$v_\alpha = \langle \alpha, v \rangle$$

for α in Δ. Let $\widehat{\rho}$ be a vector in $V^\vee$ with $\widehat{\rho}_\alpha = \langle \alpha, \widehat{\rho} \rangle = 1$ for all α in Δ. It lies in the chamber C. Represent w by the vector $w\widehat{\rho}$. Reflections can be calculated by this formula:

$$\langle \alpha, s_\beta u \rangle = \langle \alpha, u - \langle \beta, u \rangle \beta^\vee \rangle = \langle \alpha, u \rangle - \langle \beta, u \rangle \langle \alpha, \beta^\vee \rangle = u_\alpha - \langle \alpha, \beta^\vee \rangle u_\beta \ .$$

with $u = \widehat{\rho}$.

We can use this representation directly to solve one of our problems:

ALGORITHM. *Given w, find an expression for it as a product of elementary reflections.*

The vector $u = w\widehat{\rho}$ lies in wC, so $\ell(s_\alpha w) < \ell(w)$ if and only if $\langle \alpha, u \rangle < 0$. So we keep finding such a α and replacing u by $s_\alpha u$, stopping only when $u_\alpha > 0$ for all α. The word expression for w is the product of the s_α applied.

Here's another useful one, amounting to what I call saturation in the construction of the root system:

ALGORITHM. *We start with a Cartan matrix and construct the corresponding root system together with for each root λ a table of all reflections $s_\alpha \lambda$ for α in Δ.*

To each root λ we associate (1) the coefficient array in the expression $\lambda = \sum_\Delta n_\alpha \alpha$ We maintain a queue (first in, first out) of roots whose refection table has not yet been built, as well as a list of all roots so far calculated. We start by putting roots of Δ (basis arrays) on the queue and in the list.

While the queue is not empty, remove a root λ from it Calculate its reflections, adding new roots to the queue and the list of all roots.

And another:

ALGORITHM. *We start with a root λ and produce a product w of elementary reflections with $w^{-1}\lambda \in \Delta$.*

If $\lambda < 0$, record this fact and swap $-\lambda$ for λ. Now

$$\lambda = \sum_\Delta n_\alpha \alpha$$

with all $n_\alpha \geq 0$ and one $n_\alpha > 0$. The proof proceeds by induction on the height $|\lambda| = \sum n_\alpha$ of λ. If $|\lambda| = 1$, then λ lies in Δ, and there is no problem. Since the positive chamber C is contained in the cone spanned by the positive roots, no positive root is contained in the closure of C. Therefore $\lambda, \alpha^\vee >> 0$ for some α in Δ. Then

$$r_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

has smaller height than λ, and we can apply the induction hypothesis.

If the original λ was negative, this gives us w with $w^{-1}\lambda$ in Δ, and just one more elementary reflection has to be applied.

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