CHAPTER VI

IDELE CLASS GROUP AND THE UNIT THEOREM

The ring of adeles. Let $\mathbf{k}$ be an finite extension of the rational number field. An element $\mathbf{a}$ of the direct product $\prod_p \mathbf{k}_p$ of all completions $\mathbf{k}_p$ is an *adele* of $\mathbf{k}$ if every coordinate $\mathbf{a}_p$ is in $\mathbf{o}_p$ except for a finite number of p. Let $\mathbf{A_k}$ denote the set of adeles of $\mathbf{k}$. $\mathbf{A_k}$ is a ring, and the idele group $\mathbf{I_k}$ is the group of units of $\mathbf{A_k}$. As with ideles, $|\mathbf{a}_p|_p$ is denoted simply by $|\mathbf{a}|_p$.

For the topology of $\mathbf{A_k}$, basic neighborhoods are defined as follows. Choose any finite set of primes E of $\mathbf{k}$, and for each prime p in E choose a positive real ϵ_p. Then

$$\big\{\mathbf{b} \in \mathbf{A_k} \;\big|\; |\mathbf{b}-\mathbf{a}|_p < \epsilon_p \text{ for } p \in E \text{ and } |\mathbf{b}-\mathbf{a}|_p \le 1 \text{ for } p \notin E\big\}.$$

is a basic neighborhoods of adele $\mathbf{a}$.

LEMMA 6.1. *Let p be a prime of $\mathbf{k}$, let $\mathbf{K}/\mathbf{k}$ be a finite extension, and let $\wp_1, \dots, \wp_g$ be the primes of $\mathbf{K}$ which divide p. Then there is a natural isomorphism $\sigma : \mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p \to \mathbf{K}_{\wp_1} \oplus \cdots \oplus \mathbf{K}_{\wp_g}$ of algebras over $\mathbf{k}_p$.*

PROOF. Elements of $\mathbf{k}$ are denoted by lower case a, b; elements of finite extension $\mathbf{K}$ by upper case A, X; elements of $\mathbf{k}_p$ by α, β, γ. Let σ_i be the imbedding of $\mathbf{K}$ into completion $\mathbf{K}_{\wp_i}$. Then $\sigma(A, \beta) = \big(\sigma_1(A)\beta, \dots, \sigma_g(A)\beta\big)$ is a $\mathbf{k}$-bilinear mapping of $\mathbf{K} \times \mathbf{k}_p$ to $\mathbf{K}_{\wp_1} \oplus \cdots \oplus \mathbf{K}_{\wp_g}$. There is a $\mathbf{k}$-linear mapping $\sigma : \mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p \to \mathbf{K}_{\wp_1} \oplus \cdots \oplus \mathbf{K}_{\wp_g}$ such that $\sigma(A \otimes \beta) = \big(\sigma_1(A)\beta, \dots, \sigma_g(A)\beta\big)$. Both $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p$ and $\mathbf{K}_{\wp_1} \oplus \cdots \oplus \mathbf{K}_{\wp_g}$ are vector spaces over $\mathbf{k}_p$. We have $\sigma\big((A \otimes \beta)(A' \otimes \beta')\big) = \sigma(A \otimes \beta)\ sigma(A' \otimes \beta')$, and σ is $\mathbf{k}_p$-linear.

Let $X_1, \dots, X_n$ be a basis for $\mathbf{K}$ over $\mathbf{k}$. We want to show that $X_1 \otimes 1, \dots, X_n \otimes 1$ is a basis for $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p$ over $\mathbf{k}_p$. An element of $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p$ is a finite sum $\sum_{k=1}^m A_k \otimes \beta_k$. Let $A_k = \sum_{i=1}^n X_i a_{ik}$. Then

$$\sum_{k=1}^m A_k \otimes \beta_k = \sum_{k=1}^m \left(\left(\sum_{i=1}^n X_i a_{ik}\right) \otimes \beta_k\right) = \sum_{k=1}^m \sum_{i=1}^n X_i \otimes a_{ik}\beta_k = \sum_{i=1}^n X_i \otimes \sum_{k=1}^m a_{ik}\beta_k.$$

Then every element of $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p$ is of the form $\sum_{i=1}^n X_i \otimes \gamma_i$, so $X_1 \otimes 1, \dots, X_n \otimes 1$ span $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p$ over $\mathbf{k}_p$. We will show that $X_1 \otimes 1, \dots, X_n \otimes 1$ are linearly independent over

$\mathbf{k}_p$. Suppose that $\sum_{i=1}^{n} X_i \otimes \gamma_i = 0$. Multiply both sides by $X_j \otimes 1$ for $1 \le j \le n$ to obtain a system of n linear equations.

$$\sum_{i=1}^{n} X_i X_j \otimes \gamma_i = 0 \qquad 1 \le j \le n.$$

The trace $\mathbf{S}_{\mathbf{K}/\mathbf{k}} : \mathbf{K} \to \mathbf{k}$ is $\mathbf{k}$-linear, so we can apply $\mathbf{S}_{\mathbf{K}/\mathbf{k}} \otimes I : \mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p \to \mathbf{k}_p$ to both sides of each equation, obtaining

$$\big(\mathbf{S}_{\mathbf{K}/\mathbf{k}} \otimes I\big) \sum_{i=1}^{n} X_i X_j \otimes \gamma_i = \sum_{i=1}^{n} \mathbf{S}_{\mathbf{K}/\mathbf{k}}(X_i X_j)\gamma_i = 0 \qquad 1 \le j \le n.$$

Matrix $\big(\ \mathbf{S}_{\mathbf{K}/\mathbf{k}}(X_i X_j)\big)$ is non-singular by proposition 4.4, so $\gamma_1 = \cdots = \gamma_n = 0$. This shows that $X_1 \otimes 1, \ldots, X_n \otimes 1$ are linearly independent over $\mathbf{k}_p$.

Since $\sum_{i=1}^{g} [\mathbf{K}_{\wp_i} : \mathbf{k}_p] = [\mathbf{K} : \mathbf{k}] = n$ then algebras $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p$ and $\mathbf{K}_{\wp_1} \oplus \cdots \oplus \mathbf{K}_{\wp_g}$ have the same dimension over $\mathbf{k}_p$. The isomorphism will be established if we can show that $\ker(\sigma) = 0$. If $\sigma\left(X_1 \otimes \gamma_1 + \ldots X_n \otimes \gamma_n\right) = 0$, then multiply both sides of the equation by $\sigma(X_j \otimes 1)$ for $1 \le j \le n$ to obtain the following system of linear equations.

$$\sigma\left(\sum_{i=1}^{n} \left(X_i X_j \otimes \gamma_i\right)\right) = \sum_{i=1}^{n} \sigma\left(X_i X_j \otimes \gamma_i\ \right) = 0 \text{ for } 1 \le j \le n.$$

In $\mathbf{K}_{\wp_1} \oplus \cdots \oplus \mathbf{K}_{\wp_g}$ we have

$$\left(\sum_{i=1}^{n} \sigma_1\left(X_i X_j\right)\gamma_i, \ldots, \sum_{i=1}^{n} \sigma_g\left(X_i X_j\right)\gamma_i\right) = 0. \tag{6.1}$$

The trace function $\mathbf{S}_{\mathbf{K}/\mathbf{k}}$ is the sum of local traces (1.5).

$$\mathbf{S}_{\mathbf{K}/\mathbf{k}}(A) = \sum_{k=1}^{g} \mathbf{S}_{\mathbf{K}_{\wp_k}/\mathbf{k}_p}\big(\sigma_k(A)\big).$$

Each coordinate of (6.1) is zero, so we have

$$\sum_{k=1}^{g} \mathbf{S}_{\mathbf{K}_{\wp_k}/\mathbf{k}_p}\left(\sum_{i=1}^{n} \sigma_k(X_i X_j)\gamma_i\right) = 0 \quad \text{ for } 1 \le j \le n,$$

or

$$\sum_{i=1}^{n}\left(\sum_{k=1}^{g}\mathbf{S}_{\mathbf{K}_{\wp_k}/\mathbf{k}_p}\sigma_k(X_iX_j)\right)\gamma_i=\sum_{i=1}^{n}\mathbf{S}_{\mathbf{K}/\mathbf{k}}(X_iX_j)\gamma_i=0 \quad \text{for } 1\le j\le n.$$

Since $\det\big(\mathbf{S}_{\mathbf{Kk}}(X_iX_j)\big)\neq 0$, we conclude that $\gamma_j=0$ for $1\le j\le n$, and the proof is complete.

Remark on the trace function. If prime p of $\mathbf{k}$ splits into primes $\wp_1,\dots,\wp_g$ in extension $\mathbf{K}$, then for each prime $\wp_i$ we have the embedding $\sigma_i:\mathbf{K}\to\mathbf{K}_{\wp_i}$, and the mapping $\sigma:\mathbf{K}\to\mathbf{K}_{\wp_1}\oplus\cdots\oplus\mathbf{K}_{\wp_g}$, where $\sigma(A)=\big(\sigma_1(A),\dots,\sigma_g(A)\big)$. Consider the function $\mathbf{S}:\mathbf{K}_{\wp_1}\oplus\cdots\oplus\mathbf{K}_{\wp_g}\to\mathbf{k}_p$ defined by

$$\mathbf{S}(A_1,\dots,A_g)=\mathbf{S}_{\mathbf{K}_{\wp_1}/\mathbf{k}_p}(A_1)+\cdots+\mathbf{S}_{\mathbf{K}_{\wp_g}/\mathbf{k}_p}(A_g).$$

Then for A in $\mathbf{K}$ we have $\mathbf{S}_{\mathbf{K}/\mathbf{k}}(A)=\mathbf{S}\big(\sigma(A)\big)$. (Chapter I, *norm and trace functions.*) On $\mathbf{K}\otimes_{\mathbf{k}}\mathbf{k}_p$ we have $\mathbf{k}$-linear transformation $\mathbf{S}_{\mathbf{K}/\mathbf{k}}\otimes I$, which is actually $\mathbf{k}_p$-linear.

$$\text{(6.2)}\qquad \begin{array}{ccc} \mathbf{K}\otimes_{\mathbf{k}}\mathbf{k}_p & \xrightarrow{\sigma\otimes I} & \sum_{i=1}^{g}\mathbf{K}_{\wp_i} \\ \Big\downarrow{\scriptstyle \mathbf{S}_{\mathbf{K}/\mathbf{k}}\otimes I} & & \Big\downarrow{\scriptstyle \mathbf{S}} \\ \mathbf{k}\otimes_{\mathbf{k}}\mathbf{k}_p & \xrightarrow{\iota} & \mathbf{k}_p \end{array}$$

In diagram (6.2), for A in $\mathbf{K}$, on the one hand we have $\iota\left(\mathbf{S}_{\mathbf{K}/\mathbf{k}}\otimes I\right)(A\otimes 1)=\iota\big(\mathbf{S}_{\mathbf{K}/\mathbf{k}}(A)\otimes 1\big)=\mathbf{S}_{\mathbf{K}/\mathbf{k}}(A)$, and on the other we have $\mathbf{S}\big((\sigma\otimes I)(A\otimes 1)\big)=\mathbf{S}\big(\sigma(A)\big)=\mathbf{S}_{\mathbf{K}/\mathbf{k}}(A)$. Therefore $\iota(\mathbf{S}_{\mathbf{K}/\mathbf{k}}\otimes I)$ and $\mathbf{S}(\sigma\otimes I)$ agree on elements $A\otimes 1$ in $\mathbf{K}\otimes_{\mathbf{k}}\mathbf{k}_p$. If $X_1,\dots,X_n$ is a basis for $\mathbf{K}$ over $\mathbf{k}$ then $\iota(\mathbf{S}_{\mathbf{K}/\mathbf{k}}\otimes I)$ and $\mathbf{S}(\sigma\otimes I)$ agree on $X_1\otimes 1,\dots,X_n\otimes 1$, which is a basis for $\mathbf{K}\otimes_{\mathbf{k}}\mathbf{k}_p$ over $\mathbf{k}_p$. Since $\mathbf{K}\otimes_{\mathbf{k}}\mathbf{k}_p$ and $\sum_{i=1}^{g}\mathbf{K}_{\wp_i}$ have the same dimension over $\mathbf{k}_p$, then $\iota(\mathbf{S}_{\mathbf{K}/\mathbf{k}}\otimes I)$ and $\mathbf{S}(\sigma\otimes I)$ agree on all of $\mathbf{K}\otimes_{\mathbf{k}}\mathbf{k}_p$, so we have

$$\text{(6.3)}\quad \sum_{i=1}^{n}\mathbf{S}_{\mathbf{K}/\mathbf{k}}(A_i)\gamma_i=\sum_{j=1}^{g}\mathbf{S}_{\mathbf{K}_{\wp_j}/\mathbf{k}_p}(Y_1)\quad\text{if}\quad(\sigma\otimes I)(\sum_{i=1}^{n}A_i\otimes\gamma_i)=(Y_1,\dots,Y_g).$$

PROPOSITION 6.2. $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{A_k} \simeq \mathbf{A_K}$, *and if* $X_1, \ldots, X_n$ *is a basis for* $\mathbf{K}$ *over* $\mathbf{k}$ *then*

$$X_1\mathbf{A_k} + \cdots + X_n\mathbf{A_k} = \mathbf{A_K}.$$

PROOF. The mapping $\mathbf{A_k}$ to $\mathbf{A_K}$ is defined as follows. Each adele $\mathbf{a}$ in $\mathbf{A_k}$ determines an adele $\tilde{\mathbf{a}}$ in $\mathbf{A_K}$ by $\tilde{\mathbf{a}}_\wp = \mathbf{a}_p$, where p is the prime of $\mathbf{k}$ which $\wp$ divides. An element A of $\mathbf{K}$ is mapped to the diagonal of $\mathbf{A_K}$. Each product $A\tilde{\mathbf{a}}$ is an adele because both $|A|_\wp \leq 1$ and $|\tilde{\mathbf{a}}|_\wp = |\mathbf{a}|_p \leq 1$ except for a finite number of primes $\wp$. The map $\mathbf{K} \times \mathbf{A_k} \to \mathbf{A_K}$ sending $(A, \mathbf{a})$ to $A\tilde{\mathbf{a}}$ induces a homomorphism $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{A_k} \to \mathbf{A_K}$ of algebras over $\mathbf{k}$. We can identify $\mathbf{a}$ with its image $\tilde{\mathbf{a}}$, so the homomorphism may be written simply as $A \otimes \mathbf{a} \to A\mathbf{a}$.

We need to show that $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{A_k}$ is mapped onto $\mathbf{A_K}$. Choose a basis $X_1, \ldots, X_n$ for $\mathbf{K}$ over $\mathbf{k}$. Then $X_1 \otimes 1, \ldots, X_n \otimes 1$ is a basis for $\mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p$. Let $(A_\wp)$ be an element of $\mathbf{A_K}$. For each prime p of $\mathbf{k}$, let $\wp_1, \ldots, \wp_g$ be the primes of $\mathbf{K}$ that divide p. We have the projection $\pi_p : \mathbf{A_K} \to \sum_{i=1}^g \mathbf{K}_{\wp_i}$, and the isomorphism $(\sigma \otimes I) : \mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_p \to \sum_{i=1}^g \mathbf{K}_{\wp_i}$. For each adele $(A_\wp)$ of $\mathbf{A_K}$, there exist unique coefficients $\gamma_i(p)$ in $\mathbf{k}_p$, for $1 \leq i \leq n$, so that

$$(\sigma \otimes I)\left(\sum_{i=1}^n X_i \otimes \gamma_i(p)\right) = \pi_p\big((A_\wp)\big) = \big(A_{\wp_1}, \ldots, A_{\wp_g}\big)\,. \tag{6.4}$$

The $\gamma_i(p)$ determine elements $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in $\prod_p \mathbf{k}_p$ such that the p-coordinate of $\mathbf{a}_i$ is $\gamma_i(p)$. Then $\sum_{i=1}^n X_i \otimes \mathbf{a}_i$ maps to $(A_\wp)$, but we need to check that each $\mathbf{a}_i$ is an adele in $\mathbf{A_k}$, *i.e.*, that $|\gamma_i(p)|_p \leq 1$ except for a finite number of primes p. Multiplying both sides of (6.4) by $(\sigma \otimes I)(X_j \otimes 1) = \big(\sigma_{\wp_1}(X_j), \ldots \sigma_{\wp_g}(X_j)\big)$ for $1 \leq j \leq n$, we obtain a system of n equations for each prime p of $\mathbf{k}$.

$$(\sigma \otimes I)\left(\sum_{i=1}^n X_iX_j \otimes \gamma_i(p)\right) = \big(A_{\wp_1}\sigma_{\wp_1}(X_j), \ldots, A_{\wp_g}\sigma_{\wp_g}(X_j)\big)\,,\ 1 \leq j \leq n. \tag{6.5}$$

Applying identity (6.3), we obtain

$$\sum_{i=1}^n \mathbf{S}_{\mathbf{K}/\mathbf{k}}(X_iX_j)\gamma_i(p) = \sum_{k=1}^g \mathbf{S}_{\mathbf{K}_{\wp_k}/\mathbf{k}_p}\big(A_{\wp_k}\sigma_{\wp_k}(X_j)\big), \qquad 1 \leq j \leq n. \tag{6.6}$$

Let E contain all primes p of $\mathbf{k}$ such that p is infinite, or $\big|\det\big(\mathbf{S}_{\mathbf{K}/\mathbf{k}}(X_iX_j)\big)\big|_p \neq 1$, or p is divisible by a prime $\wp$ of $\mathbf{K}$ for which either $|A|_\wp > 1$ or $|\sigma_\wp(X_j)|_\wp > 1$ for some j, $1 \leq j \leq n$. For all p not in E, the right side of (6.6) satisfies

$$\left|\sum_{k=1}^g \mathbf{S}_{\mathbf{K}_{\wp_k}/\mathbf{k}_p}\big(A_{\wp_k}\sigma_{\wp_k}(X_j)\big)\right|_p \leq \max_{1\leq k\leq g}\left(\Big|\mathbf{S}_{\mathbf{K}_{\wp_k}/\mathbf{k}_p}\big(A_{\wp_k}\sigma_{\wp_k}(X_j)\big)\Big|_p\right) \leq 1 \qquad \text{for } 1 \leq j \leq n.$$

In system (6.6) for p not in E, all the coefficients $\mathbf{S}_{\mathbf{K}/\mathbf{k}}(X_iX_j)$ are in $\mathbf{o}_p$, the determinant $\det\big(\mathbf{S}_{\mathbf{K}/\mathbf{k}}(X_iX_j)\big)$ is a unit of $\mathbf{o}_p$, and the right side terms are all in $\mathbf{o}_p$. Therefore, we have $\gamma_i(p)$ in $\mathbf{o}_p$ for $1 \le i \le n$ and $p \notin E$, showing that $\mathbf{a}_i$ is an adele. Finally, since we identify $\mathbf{a}$ in $\mathbf{A}_\mathbf{k}$ with its image in $\mathbf{A}_\mathbf{K}$, every element of $\mathbf{A}_\mathbf{K}$ is of the form $(\sigma \otimes I)(\sum_{i=1}^n X_i \otimes \mathbf{a}_i) = \sum_{i=1}^n X_i\mathbf{a}_i$. This completes the proof.

Lemma 6.3. *The group $\mathbf{A}_\mathbf{Q}/\mathbf{Q}$ of adele classes is compact and there is a compact subset $\mathbf{C}$ of $\mathbf{A}_\mathbf{Q}$ such that $\mathbf{A}_\mathbf{Q} = \mathbf{Q} + \mathbf{C}$.*

Proof. Since $\mathbf{o}_p$ is compact for each finite rational prime p then the subset $\mathbf{C}$ defined by

$$\mathbf{C} = \prod_{\text{finite } p} \mathbf{o}_p \times \left[-\frac{1}{2}\ ,\ \frac{1}{2}\right] \subset \mathbf{A}_\mathbf{Q}$$

is a compact subset of the adele group $\mathbf{A}_\mathbf{Q}$. If $\mathbf{a}$ is an adele in $\mathbf{A}_\mathbf{Q}$ then there is a finite set E of primes so that $|\mathbf{a}|_p \le 1$ if and only if p is not in E. For a finite prime p in E, we have $\mathbf{a}_p = u_p/p^{n_p}$, where u_p is an element of $\mathbf{o}_p$, and $n_p \ge 0$. Put $u_p = m_p + v_p p^{n_p}$ where m_p is a rational integer, $0 \le m_p < p^{n_p}$, and v_p is an element of $\mathbf{o}_p$. Define α to be the rational number

$$\alpha = \sum_{p \in E} \frac{m_p}{p^{n_p}}.$$

For each finite p not in E we have $|\mathbf{a} - \alpha|_p \le \max\big(|\mathbf{a}|_p, |\alpha|_p\big) = 1$, and for each finite p in E, we have

$$|\mathbf{a} - \alpha|_p \;=\; \left|\frac{m_p + v_p p^{n_p}}{p^{n_p}} - \frac{m_p}{p^{n_p}} - \sum_{q \in E,\ q \ne p} \frac{m_q}{q^{n_q}}\right|_p \;=\; \left|v_p - \sum_{q \in E,\ q \ne p} \frac{m_q}{q^{n_q}}\right|_p \le 1.$$

At the infinite prime $p = \infty$, there exists a rational integer μ such that $|\mathbf{a}-\alpha-\mu|_\infty \le \frac{1}{2}$. At all finite primes p, we have

$$|\mathbf{a} - \alpha - \mu|_p \le \max\left(|\mathbf{a} - \alpha|_p,\ |\mu|_p\right) \le 1.$$

We have shown that there is a rational number $\beta = \alpha + \mu$ so that $\mathbf{a} - \beta \in \mathbf{C}$. Then the continuous homomorphism $\mathbf{A} \to \mathbf{A}/\mathbf{Q}$ maps compact subset $\mathbf{C}$ onto $\mathbf{A}/\mathbf{Q}$, so $\mathbf{A}/\mathbf{Q}$ is compact

Lemma 6.4. *If $\mathbf{k}$ is a finite extension of $\mathbf{Q}$ then the group $\mathbf{A}_\mathbf{k}/\mathbf{k}$ of adele classes is compact, and there is a compact subset $\mathbf{C}$ of $\mathbf{A}_\mathbf{k}$ so that $\mathbf{A}_\mathbf{k} = \mathbf{k} + \mathbf{C}$.*

Proof. Let $x_1, \ldots, x_n$ be a basis for $\mathbf{k}$ over $\mathbf{Q}$. Then $\mathbf{A}_\mathbf{k} = x_1\mathbf{A}_\mathbf{Q} + \cdots + x_n\mathbf{A}_\mathbf{Q}$ by lemma 6.2. If $\mathbf{a}$ is in $\mathbf{A}_\mathbf{k}$, let $\mathbf{a} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ where $\mathbf{a}_i$ is in $\mathbf{A}_\mathbf{Q}$ for

$1 \le i \le n$. By lemma 6.3, there is a compact subset $\mathbf{C}'$ of $\mathbf{A_Q}$ so that $\mathbf{A_Q} = \mathbf{Q} + \mathbf{C}'$, so $\mathbf{a}_i = \beta_i + \mathbf{c}_i$, where β_i is in $\mathbf{Q}$ and $\mathbf{c}_i$ is in $\mathbf{C}'$, for $1 \le i \le n$, and

$$\mathbf{a} = (x_1\beta_1 + \cdots + x_n\beta_n) + (x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n) \ \in \ \mathbf{k} + x_1\mathbf{C}' + \ldots x_n\mathbf{C}'.$$

Subset $\mathbf{C} = x_1\mathbf{C}' + \ldots x_n\mathbf{C}'$ is a compact subset of $\mathbf{A_k}$ and $\mathbf{A_k} = \mathbf{k} + \mathbf{C}$. The continuous homomorphism $\mathbf{A_k} \to \mathbf{A_k}/\mathbf{k}$ maps $\mathbf{C}$ onto $\mathbf{A_k}/\mathbf{k}$, proving that $\mathbf{A_k}/\mathbf{k}$ is compact.

Haar measure. Both $\mathbf{k}_p$ and $\mathbf{A_k}$ are locally compact topological groups so Haar measures may be defined. For infinite primes p, take the ordinary Lebesgue measure on $\mathbf{R}$ or $\mathbf{C}$ for the Haar measure m_p on $\mathbf{k}_p$. For finite primes, the measure m_p is chosen so that $m_p(\mathbf{o}_p) = 1$. The cosets of p^n are open subsets of compact subset $\mathbf{o}_p$, so the measure of each coset should be $\mathrm{N}p^{-n}$. Take the smallest σ-algebra containing all cosets $\alpha + p^n$ for α in $\mathbf{k}_p$. Since every coset of p^n is the disjoint union of cosets of p^{n+k} for $k > 0$, then every union of cosets is equal to a union of cosets of the same power of p.

LEMMA 6.5. *If S is a measurable set of $\mathbf{k}_p$ and α is an non-zero element of $\mathbf{k}_p$ then $m_p(\alpha S) = |\alpha|_p m_p(S)$.*

PROOF. Let $|\alpha|_p = \mathrm{N}p^{-m}$, so $\alpha = u\pi^m$ where u is in $\mathbf{u}_p$, $(\pi) = p$, and m may be positive, zero or negative. S is a union of cosets $\beta + p^n$ and we may take $n \ge \max(-m, 0)$. Then αS is a union of cosets $\alpha\beta + p^{n+m}$ where $n + m \ge 0$, and $m_p(\alpha + p^{n+m}) = \mathrm{N}p^{-n-m} = |\alpha|_p m_p(\beta + p^n)$. This shows that $m_p(\alpha S) = |\alpha|_p m_p(S)$.

Haar measure on the ring of adeles. Take F to be a finite set of primes of $\mathbf{k}$ containing all infinite primes. For each p, let E_p be an open subset of $\mathbf{k}_p$ for which $m_p(E_p)$ is defined and for which $E_p = \mathbf{o}_p$ for all p not in F. Consider subsets $\mathbf{E}$ of $\mathbf{A_k}$ of the form $\mathbf{E} = \prod_p E_p$. Every adele of $\mathbf{A_k}$ is in some $\mathbf{E}$. Define the measure $m(\mathbf{E})$ to be

$$m(\mathbf{E}) = \prod_p m\left(E_p\right).$$

The product is defined since $m_p(E_p) = m_p(\mathbf{o}_p) = 1$ for all but a finite number of p.

LEMMA 6.6. *If $\mathbf{E}$ is a measurable set of $\mathbf{A_K}$ and $\mathbf{i}$ is an element of $\mathbf{I_k}$ then $m_p(\mathbf{iE}) = |\mathbf{i}| m(\mathbf{E})$.*

PROOF. It is enough to check sets of the form $\mathbf{E} = \prod_p E_p$ such that $E_p = \mathbf{o}_p$ for p not in some finite set F_1. Suppose that $|\mathbf{i}|_p = 1$ except for p in finite set F_2. Then

$$\mathbf{iE} = \prod_{p \in F_1 \cup F_2} \mathbf{i}_p E_p \times \prod_{p \notin F_1 \cup F_2} \mathbf{o}_p.$$

We have

$$m(\mathbf{iE}) = \prod_{p\in F_1\cup F_2} m_p(\mathbf{i}_p E_p) = \prod_{p\in F_1\cup F_2} \big(|\mathbf{i}|_p m_p(E_p)\big)$$
$$= \prod_{p\in F_1\cup F_2} |\mathbf{i}|_p \prod_{p\in F_1\cup F_2} m_p(E_p) = |\mathbf{i}| m(\mathbf{E}).$$

Given an $\mathbf{R}$-valued function $f : \mathbf{A_k} \to \mathbf{R}$ such that $\overline{f}(\mathbf{a}) = \sum_{\alpha\in\mathbf{k}} f(\mathbf{a}+\alpha)$ exists, the value $\overline{f}(\mathbf{a})$ depends only the coset $\overline{\mathbf{a}}$ of $\mathbf{a}$ in $\mathbf{A_k}$. Define $\overline{f}(\overline{\mathbf{a}}) = \sum_{\alpha\in\mathbf{k}} f(\mathbf{a}+\alpha)$. If f is an integrable function on $\mathbf{A_k}$ then

$$\int_{\mathbf{A_k}} f(\mathbf{a})\,da = \int_{\mathbf{A_k}/\mathbf{k}} \sum_{\alpha\in\mathbf{k}} f(\mathbf{a}+\alpha)\,d\overline{\mathbf{a}} = \int_{\mathbf{A_k}/\mathbf{k}} \overline{f}(\overline{\mathbf{a}})\,d\overline{\mathbf{a}}$$

$\mathbf{A_k}/\mathbf{k}$ is a compact group, so it must have finite measure.

LEMMA 6.7. *Let* $\mathbf{S}$ *be a measurable subset of* $\mathbf{A_k}$ *such that* $m(\mathbf{S}) > m(\mathbf{A_k}/\mathbf{k})$. *There exist* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *in* $\mathbf{S}$ *so that* $\mathbf{a}_1 \neq \mathbf{a}_2$ *and* $\mathbf{a}_1 - \mathbf{a}_2$ *is an element of* $\mathbf{k}^*$.

PROOF. Let χ be the characteristic function of $\mathbf{S}$. Then $\sum_{\alpha\in\mathbf{k}} \chi(\mathbf{a}+\alpha) > 1$ at some $\mathbf{a}$ because otherwise we would have

$$m(\mathbf{S}) = \int_{\mathbf{A_k}} \chi(\mathbf{a})\,d\mathbf{a} = \int_{\mathbf{A_k}/\mathbf{k}} \left(\sum_{\alpha\in\mathbf{k}} \chi(\mathbf{a}+\alpha)\right) d\overline{\mathbf{a}} \le \int_{\mathbf{A_k}/\mathbf{k}} 1\,d\overline{\mathbf{a}} = m(\mathbf{A_k}/\mathbf{k})$$

If $\sum_{\alpha\in\mathbf{k}} \chi(\mathbf{a}+\alpha) > 1$ then there exist α_1 and α_2 in $\mathbf{k}$ so that $\alpha_1 \neq \alpha_2$, $\mathbf{a}_1 = \mathbf{a}+\alpha_1 \in \mathbf{S}$ and $\mathbf{a}_2 = \mathbf{a}+\alpha_2 \in \mathbf{S}$.

LEMMA 6.8. $\mathbf{k}$ *is a discrete subgroup of* $\mathbf{A_k}$.

PROOF. Let α be an element of $\mathbf{k}$. Choose any prime p_0 of $\mathbf{k}$. Then

$$\mathbf{U} = \left\{\mathbf{a}\in\mathbf{A_k} \;\middle|\; |\mathbf{a}-\alpha|_p \in \mathbf{o}_p \text{ for } p\neq p_0 \text{ and } |\mathbf{a}-\alpha|_{p_0} < \frac{1}{2}\right\}$$

is an open neighborhood of α, and $\mathbf{U}\cap\mathbf{k} = \{\alpha\}$.

PROPOSITION 6.9. *Let* $\mathbf{I}_{\mathbf{k}}^0$ *be the subgroup of* $\mathbf{I}_{\mathbf{k}}$ *consisting of all ideles* $\mathbf{i}$ *such that* $|\mathbf{i}| = 1$*. Then* $\mathbf{I}_{\mathbf{k}}^0$ *contains* $\mathbf{k}^*$*, and the group of idele classes* $\mathbf{I}_{\mathbf{k}}^0/\mathbf{k}^*$ *is compact.*

PROOF. Lemma 6.6 insures that $\mathbf{A}_{\mathbf{k}}$ has arbitrarily large compact subsets, so choose a compact subset $\mathbf{C} \subset \mathbf{A}_{\mathbf{k}}$ so that $m(\mathbf{C}) > m(\mathbf{A}_{\mathbf{k}}/\mathbf{k})$. Subtraction $(\mathbf{a}, \mathbf{a}') \to \mathbf{a} - \mathbf{a}'$ and multiplication $(\mathbf{a}, \mathbf{a}') \to \mathbf{a}\mathbf{a}'$ are continuous functions, so $\mathbf{C}' = \mathbf{C} - \mathbf{C}$ and $\mathbf{C}'' = \mathbf{C}'\mathbf{C}'$ are compact subsets of $\mathbf{A}_{\mathbf{k}}$. By lemma 6.8, $\mathbf{K} \cap \mathbf{C}''$ is a finite set. Let $\mathbf{K} \cap \mathbf{C}'' = \{\xi_1, \dots, \xi_n\}$. Then $\mathbf{V} = \mathbf{C}' \cup \xi_1^{-1}\mathbf{C}' \cup \dots \cup \xi_n^{-1}\mathbf{C}'$ is a compact subset of $\mathbf{A}_{\mathbf{k}}$.

For any finite set E of primes of $\mathbf{k}$, the subset

$$\mathbf{A}_{\mathbf{k}}(E) = \prod_{p \in E} \mathbf{k}_p \times \prod_{p \notin E} \mathbf{o}_p$$

is open in $\mathbf{A}_{\mathbf{k}}$, and $\mathbf{A}_{\mathbf{k}} \subset \cup_E \mathbf{A}_{\mathbf{k}}(E)$. There exists a finite number of sets $E_1, \dots, E_m$ so that compact set $\mathbf{V}$ is contained in $\mathbf{A}_{\mathbf{k}}(E_1) \cup \dots \mathbf{A}_{\mathbf{k}}(E_m)$. If $E_0 = E_1 \cup \dots \cup E_m$ then $\mathbf{A}_{\mathbf{k}}(E_0) = \mathbf{A}_{\mathbf{k}}(E_1) \cup \dots \mathbf{A}_{\mathbf{k}}(E_m)$, so $\mathbf{V}$ is contained in $\mathbf{A}_{\mathbf{k}}(E_0)$. For each p, the function $\mathbf{a} \to |\mathbf{a}|_p$ is continuous, so $|\mathbf{a}|_p$ is bounded on compact set $\mathbf{V}$. Since E_0 is a finite set of primes, there exists a positive bound δ so that $|\mathbf{a}|_p \le \delta$ for $\mathbf{a}$ in $\mathbf{V}$ and p in E_0, and we have

$$\mathbf{V} \subset \prod_{p \in E_0} \big\{\alpha \in \mathbf{k}_p \bigm| |\alpha|_p \le \delta\big\} \times \prod_{p \notin E_0} \mathbf{o}_p. \tag{6.7}$$

Suppose that $\mathbf{c}$ is a unit of $\mathbf{A}_{\mathbf{k}}$ (*i.e.,* an element of $\mathbf{I}_{\mathbf{k}}$) such that $\mathbf{c}$ and $\mathbf{c}^{-1}$ are in $\mathbf{V}$. Then by (6.7) both $\mathbf{c}$ and $\mathbf{c}^{-1}$ are elements of $\mathbf{W}$ defined by

$$\mathbf{W} = \prod_{p \in E_0} \big\{\alpha \in \mathbf{k}^* \bigm| |\alpha|_p \le \delta \text{ and } |\alpha^{-1}|_p \le \delta\big\} \times \prod_{p \notin E_0} \mathbf{o}_p^*, \tag{6.8}$$

which is a compact subset of the idele group $\mathbf{I}_{\mathbf{k}}$. (Group $\mathbf{o}_p^*$ is compact because it is the union of $\mathrm{N}p - 1$ cosets of ideal p, and each coset is compact because ring $\mathbf{o}_p$ is compact.)

Suppose that $\mathbf{i}$ is an idele in $\mathbf{I}_{\mathbf{k}}^0$. If we can show that $\mathbf{i}$ is in $\mathbf{k}^*\mathbf{W}$, then $\mathbf{I}_{\mathbf{k}}^0/\mathbf{k}^*$ will be the image of compact set $\mathbf{W}$, which will prove the proposition. Both $\mathbf{i}\mathbf{C}$ and $\mathbf{i}^{-1}\mathbf{C}$ are compact subsets of $\mathbf{A}_{\mathbf{k}}$. Since $|\mathbf{i}| = 1$, we have $m(\mathbf{i}\mathbf{C}) = m(\mathbf{C})$ and $m(\mathbf{i}^{-1}\mathbf{C}) = m(\mathbf{C})$. By lemma 6.7, there exist elements $\mathbf{i}\mathbf{a}_1$ and $\mathbf{i}\mathbf{a}_2$ in $\mathbf{i}\mathbf{C}$ so that $\mathbf{i}\mathbf{a}_1 - \mathbf{i}\mathbf{a}_2$ is in $\mathbf{k}^*$. Put $\mathbf{c}_1 = \mathbf{a}_1 - \mathbf{a}_2$. Then $\mathbf{c}_1$ is in $\mathbf{C}'$ and $\mathbf{i}\mathbf{c}_1$ is in $\mathbf{k}^*$. Likewise, there exist elements $\mathbf{i}^{-1}\mathbf{b}_1$ and $\mathbf{i}^{-1}\mathbf{b}_2$ in $\mathbf{i}^{-1}\mathbf{C}$ so that $\mathbf{i}^{-1}\mathbf{b}_1 - \mathbf{i}^{-1}\mathbf{b}_2$ is in $\mathbf{k}^*$. Put $\mathbf{c}_2 = \mathbf{b}_1 - \mathbf{b}_2$. Then $\mathbf{c}_2$ is in $\mathbf{C}'$ and $\mathbf{i}^{-1}\mathbf{c}_2$ is in $\mathbf{k}^*$.

The product $\big(\mathbf{i}\mathbf{c}_1\big)\big(\mathbf{i}^{-1}\mathbf{c}_2\big) = \mathbf{c}_1\mathbf{c}_2$ is in $\mathbf{k}^* \cap \mathbf{C}''$, so $\mathbf{c}_1\mathbf{c}_2 = \xi_i$ for some i. We have $\mathbf{c}_1 \in \mathbf{C}' \subset \mathbf{V}$. Also we have $\mathbf{c}_1^{-1} = \xi^{-1}\mathbf{c}_2$ so $\mathbf{c}_1^{-1} \in \xi_i^{-1}\mathbf{C}' \subset \mathbf{V}$. Therefore $\mathbf{c}_1^{-1}$ is in $\mathbf{W}$, and $\mathbf{i} = (\mathbf{i}\mathbf{c}_1)\,\mathbf{c}_1^{-1}$ is in $\mathbf{k}^*\mathbf{W}$, which completes the proof.

LEMMA 6.10. *If E is a finite set of primes of* $\mathbf{k}$*, let* $\mathbf{k}^*(E)$ *be the subgroup of E-units in* $\mathbf{k}$.

$$\mathbf{k}^*(E) = \mathbf{k}^* \cap \mathbf{I_k}(E).$$

Then $\big(\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}}\big)/\mathbf{k}^*(E)$ *is compact.*

PROOF. In the following diagram, the kernel of $\mu\iota$ is $\mathbf{k}^*(E)$, so induced homomorphism ι' is an isomorphism onto a subgroup of $\mathbf{I}^0_{\mathbf{k}}/\mathbf{k}^*$.

$$\begin{array}{ccc} \mathbf{I_k}(E) \cap \mathbf{I}^0_k & \xrightarrow{\ \iota\ } & \mathbf{I}^0_{\mathbf{k}} \\ \Big\downarrow \mu' & & \Big\downarrow \mu \\ \big(\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}}\big)/\mathbf{k}^*(E) & \xrightarrow{\ \iota'\ } & \mathbf{I}^0_{\mathbf{k}}/\mathbf{k}^* \end{array}$$

The map ι' is open because if V is an open subset of $\big(\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}}\big)/\mathbf{k}^*(E)$ then ${\mu'}^{-1}(V)$ is open in $\mathbf{I_k}(E) \cap \mathbf{I}^0_k$, inclusion ι is an open mapping, and the natural homomorphism μ is an open mapping. Therefore the image $\iota'\big(\big(\mathbf{I_k}(E)\cap\mathbf{I}^0_{\mathbf{k}}\big)/\mathbf{k}^*(E)\big)$ is an open subgroup of $\mathbf{I}^0_{\mathbf{k}}/\mathbf{k}^*$. An open subgroup must be closed, so $\iota'\big(\big(\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}}\big)/\mathbf{k}^*(E)\big)$ is a closed subgroup of compact group $\mathbf{I}^0_{\mathbf{k}}/\mathbf{k}^*$. Therefore $\big(\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}}\big)/\mathbf{k}^*(E)$ is isomorphic to a compact subgroup.

LEMMA 6.11. *If E is a finite set of primes containing the infinite primes of* $\mathbf{k}$ *then there exists a positive real number ϵ so that* $\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}} = \mathbf{k}^*(E)C_\epsilon$*, where C_ϵ is the compact set defined by*

$$C_\epsilon = \left\{ \mathbf{i} \in \mathbf{I_K}(E) \cap \mathbf{I}^0_k \;\Big|\; \frac{1}{\epsilon} \leq |\mathbf{i}|_p \leq \epsilon \text{ for } p \in E \right\}. \tag{6.9}$$

PROOF. We need to show $\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}} \subset \mathbf{k}^*(E)C_\epsilon$. We have the natural homomorphism

$$\mathbf{I_k}(E) \cap \mathbf{I}^0_k \xrightarrow{\ \mu'\ } \big(\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}}\big)/\mathbf{k}^*(E)$$

onto a compact group. For any given $\mathbf{i}$ in $\mathbf{I_k}(E) \cap \mathbf{I}^0_k$, the values $|\mathbf{i}|_p$ for p in E are bounded because E is a finite set. For positive real ϵ, the sets C_ϵ form an open covering of $\mathbf{I_k}(E)\cap\mathbf{I}^0_k$, so the images $\mu'(C_\epsilon)$ form an open covering of compact group $\mathbf{I}^0_{\mathbf{k}}/\mathbf{k}^*(E)$. There exist a finite number of the sets $\mu'(C_\epsilon)$ which cover $\mathbf{I}^0_{\mathbf{k}}/\mathbf{k}^*(E)$. If $\epsilon_1 < \epsilon_2$ then $C_{\epsilon_1} \subset C_{\epsilon_2}$. Therefore there exists a single set C_ϵ so that $\mu'(C_\epsilon)$ covers $\mathbf{I}^0_{\mathbf{k}}/\mathbf{k}^*(E)$. For any $\mathbf{i}$ in $\mathbf{I_k}(E)\cap\mathbf{I}^0_k$, there exists an idele $\mathbf{j}$ in C_ϵ so that $\mu'(\mathbf{i}) = \mu'(\mathbf{j})$, so $\mu'(\mathbf{i}\mathbf{j}^{-1}) = 1$. The kernel of μ' is $\mathbf{k}^*(E)$, so there exists an element α in $\mathbf{k}^*(E)$ so that $\mathbf{i} = \alpha\mathbf{j}$. Therefore $\mathbf{I_k}(E) \cap \mathbf{I}^0_{\mathbf{k}} \subset \mathbf{k}^*(E)C_\epsilon$.

LEMMA 6.12. $\mathbf{k}^*$ *is a discrete subgroup of* $\mathbf{I_k}$.

PROOF. The set U defined by

$$U = \left\{ \mathbf{i} \in \mathbf{I_k} \;\middle|\; |\mathbf{i} - 1|_p \le 1 \text{ for } p \text{ finite}, \ |\mathbf{i} - 1|_p < \frac{1}{2} \text{ for } p \text{ infinite} \right\} \tag{6.10}$$

is an open subset of $\mathbf{I}_k$ which contains no element of $\mathbf{k}^*$ other than 1.

PROPOSITION 6.13 (DIRICHLET UNIT THEOREM). *If E is a finite set of pirmes of $\mathbf{k}$ containing all the infinite primes and if the number of elements in E is $s+1$, then $\mathbf{k}^*(E)$ is the product of a finite subgroup (the roots of unity in $\mathbf{k}^*$) and a free abelian group on s generators. That is, there exist in $\mathbf{k}^*(E)$ an m-th root of unity ω and elements $\eta_1, \dots \eta_s$ such that every element η of $\mathbf{k}^*(E)$ may be uniquely expressed as a product*

$$\eta = \omega^{\nu_0} \eta_1^{\nu_1} \dots \eta_s^{\nu_s} \qquad 0 \le \nu_o < m \text{ and } \nu_i \in \mathbf{Z} \ (1 \le i \le s)$$

PROOF. Let E contain infinite primes $p_0, \dots, p_r$. If E contains any finite primes then let them be $p_{r+1}, \dots, p_s$. Let A_s be defined by

$$A_s = \left\{ (a_0, \dots, a_s) \in \left(\mathbf{R}^+\right)^{s+1} \;\middle|\; \prod_{i=0}^{s} a_i = 1 \right\}$$

where $\mathbf{R}^+$ denotes the group of positive real numbers. Let $f : \mathbf{I_k}(E) \cap \mathbf{I_k^0} \to A_s$ be defined by

$$f(\mathbf{i}) = (|\mathbf{i}|_{p_0}, \dots, |\mathbf{i}|_{p_s}) .$$

The kernel of f is the group of $\mathbf{i}$ such that $|\mathbf{i}|_p = 1$ for all primes p, so $\ker(f)$ is compact, and $\ker(f) \cap \mathbf{k}^*(E)$ must be a finite group because $\mathbf{k}^*(E)$ is discrete. Any finite subgroup of $\mathbf{k}^*(E)$ must consist of roots of unity; conversely, any root of unity in $\mathbf{k}^*(E)$ must be in the kernel of f. Let m-th of unity ω generate the group of roots of unity in $\mathbf{k}^*(E)$.

Let B and H be the images in A_s of $\mathbf{I_k}(E) \cap \mathbf{I_k^0}$ and $\mathbf{k}^*(E)$, respectively. H is a discrete subgroup of A_s, because the only elements of $\mathbf{k}^*(E)$ in the open neighborhood

$$\left\{ (a_0, \dots, a_s) \;\middle|\; |a_i - 1| < \frac{1}{2} \quad 0 \le i \le s \right\}$$

of $(1, \dots, 1)$ are in the finite set $\ker(f) \cap \mathbf{k}^*(E)$. For subgroup B we have

$$B = \left\{ (b_0, \dots, b_s) \in A_s \;\middle|\; b_i > 0 \text{ for } 0 \le i \le r; \quad b_i = \mathrm{N}p_i^{u_i}, \ u_i \in \mathbf{Z} \text{ for } r < i \le s \right\}$$

By lemma 6.11, there exists a compact set C_ϵ such $\mathbf{I_k}(E) \cap \mathbf{I_k^0} = \mathbf{k}^*(E) C_\epsilon$. Then

$$B = f(\mathbf{I_k}(E) \cap \mathbf{I_k^0}) = f\big(\mathbf{k}^*(E)\big) f(C_\epsilon) = HC,$$

where $C = f(V_\epsilon)$ is compact.

We next show that $A_s = BV$ where V is compact. Put

$$V = \left\{ (a_0, \ldots, a_s) \in A_s \;\middle|\; a_i = 1 \ (0 \le i < r); \quad \prod_{i=r+1}^{s} (\mathrm{N}p_i)^{-1} \le a_r \le 1; \quad 1 \le a_i \le \mathrm{N}p_i \ (r < i \le s) \right\}.$$

Then V is certainly compact. If $a \in A_s$ then choose $b \in B$ so that

$$\begin{aligned} (ba)_i &= 1 \qquad && 0 \le i < r \\ 1 \le |ba|_i &\le \mathrm{N}p_i && r < i \le s \\ b_r &= \prod_{i=r+1}^{s} b_i^{-1}. \end{aligned}$$

The condition on b_r ensures that $\prod_{i=0}^{s} b_i = 1$. We have $a = b^{-1}(ba)$. To show that ba is in V, it is only necessary to check coordinate $(ba)_r$. We have $a_r = \prod_{i \ne r} a_i^{-1}$ and $b_r = \prod_{i \ne r} b_i^{-1}$, so $(ba)_r = \prod_{i \ne r} (ba)_i^{-1}$. Since $(ba)_i = 1$ for $0 \le i < r$ we have $(ba)_r = \prod_{r < i \le s} (ba)_i^{-1}$. Since $\mathrm{N}p_i^{-1} \le |ba|_i \le 1$ for $r < i \le s$, then

$$\prod_{r < i \le s} \mathrm{N}p_i^{-1} \le (ba)_r \le 1.$$

This shows that ba is in V, and that $A_s = BV$. Combining $A_s = BV$ and $B = HC$ gives

$$A_s = HW,$$

where $W = CV$ is a compact subset of A_s.

Let V_s be the s-dimensional vector space over $\mathbf{R}$ defined by

$$V_s = \left\{ (x_0, \ldots, x_s) \in \mathbf{R}^{s+1} \;\middle|\; \sum_{i=0}^{s} x_i = 0 \right\}$$

We have the isomorphism $\psi : A_s \to V_s$ defined by

$$\psi(a_0, \ldots, a_s) = (\log a_0, \ldots, \log a_s).$$

Since $A_s = HW$, we have $V_s = \psi(A_s) = \psi(HW) = \psi(H) + \psi(W)$. Put $L = \psi(H)$ and $W' = \psi(W)$. Then

$$V_s = L + W'$$

where L is a discrete subgroup and W' is compact. We will show that L is a free abelian group on s generators.

Let $y_1, \ldots, y_t$ be a maximal linearly independent subset of L. For $y \in L$, there are real α_i so that

$$y = \sum_{i=1}^{r} \alpha_i y_i = \sum_{i=1}^{r} [\alpha_i] y_i + \sum_{i=1}^{r} \{\alpha_i\} y_i,$$

where $[\alpha_i] \in \mathbf{Z}$ and $0 \leq \{\alpha_i\} < 1$ for $i = 1, \ldots, t$. The term $\sum_{i=1}^{r} \{\alpha_i\} y_i$ is in the intersection of L and a compact subset of V_s. Therefore, there is a finite set L_0 such that

$$L = \mathbf{Z}y_1 + \cdots + \mathbf{Z}y_t + L_0.$$

If $t < s$, then $y_1, \ldots, y_t$ can be extended to a basis $y_1, \ldots, y_t, y_{t+1}, \ldots, y_s$ of V_s. Since $V_s = L + W'$ with W' compact, there is a constant c so that for any v in V_s, we have

$$v = \sum_{i=1}^{t} m_i y_i + \sum_{i=1}^{s} \alpha_i y_i \quad \text{where } \alpha_i < c.$$

But this is impossible since $\alpha_{t+1} y_{t+1}$ must have unbounded coefficient α_{t+1}. Therefore $t = s$.

Let the elements of finite set L_0 be $z_1, \ldots, z_\nu$. By the pigeon-hole principle, there are two distinct numbers j and j' so that $0 \leq j < j' \leq \nu$ and $jz_1 - j'z_1 = \sum_{i=1}^{s} m_i y_i$ with $m_i \in \mathbf{Z}$. If we replace each y_i by $(j-j')^{-1} y_i$ then z_1 is an element of $\mathbf{Z}y_1 + \ldots \mathbf{Z}y_s$, and we have $L = \mathbf{Z}y_1 + \ldots \mathbf{Z}y_s + L_0'$ where L_0' contains $\nu - 1$ elements. After a finite number of steps, we arrive at a set of free generators $y_1, \ldots, y_s$ for L.

Choose elements $\eta_1, \ldots \eta_s$ in $\mathbf{k}^*(E)$ so that $\psi\big(f(\eta_i)\big) = y_s$. If $\eta \in \mathbf{k}^*(E)$ then there are unique integers $\nu_1, \ldots, \nu_s$ so that $\psi\big(f(\eta)\big) = \sum_{i=1}^{s} \nu_i y_i$, so $\eta \prod_{i=1}^{s} \eta^{-\nu_i}$ is in $\ker(f) =< \omega >$. Therefore

$$\eta = \omega^{\nu_0} \eta_1^{\nu_1} \ldots \eta_s^{\nu_s}.$$

This concludes the proof of the unit theorem.