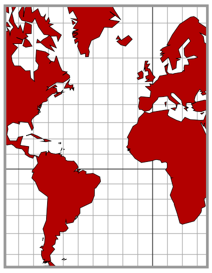
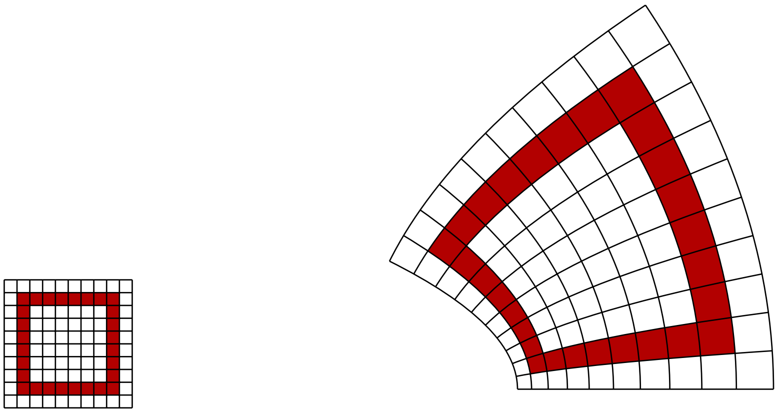
CHAPTER 8



Non-linear 2D transformations: deconstructing paths

Sometimes we want to draw a figure after a non-linear transformation has been applied to it. In image manipulation programs, this is often called **morphing**. For example, here is a morphed 10×10 grid produced by a program in which the basic drawing commands drew a square grid and these were followed by some transforming code before stroking.



In order to apply transformations to paths, we just have to understand (a) transformations and (b) paths!

8.1. Two dimensional transformations

A 2D transformation is a function $f(x, y)$ of two variables which returns a pair of numbers $u(x, y)$ and $v(x, y)$, the coordinates of the transform of the point (x, y). We have already seen affine transformations where

$$f(x, y) = (ax + by + c, dx + ey + f)$$

for suitable constants a, b, etc. But now we want to allow more complicated ones. I should say right at the beginning that these can be very complicated. An affine transformation is not so difficult to visualize because we know what the transformation does everywhere if we know what it does to just a single square. But an arbitrary transformation may have very different effects in different parts of the plane, and this is the source of much difficulty in comprehending it. Indeed, the nature of 2D transformations has been in not-so-distant times the subject of interesting mathematical research. (I am referring to the stability of properties of such transformations under perturbation, part of the so-called 'catastrophe theory'.)

I'll spend some time looking at one which is not too complicated:

$$f(x, y) = (x^2 - y^2, 2xy) \ .$$

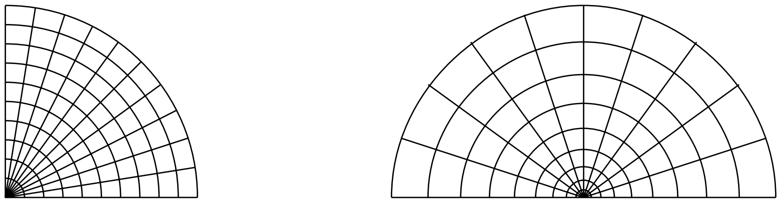
This is not quite a random choice—it is derived from the function of complex numbers that takes z to z^2, since

$$(x + iy)^2 = (x^2 - y^2) + i\,(2xy) \ .$$

The best way to understand what it does is to write (x, y) in polar coordinates as $(r\cos\theta, r\sin\theta)$, since in these terms f takes (x, y) to

$$(r^2\cos^2\theta - r^2\sin\theta, 2r^2\cos\theta\sin\theta) = (r^2\cos 2\theta, r^2\sin 2\theta)\ .$$

In other words, it squares r and doubles θ. We can see how this works in the following figures. On the left is the sector of the unit circle between 0 and $\pi/2$, on the right its image with respect to f:



That seems simple enough. And yet, the image that began this chapter is the transformation with respect to this same f of the square with lower left corner at $(1/2, 0)$, upper right at $(3/2, 1)$, sides aligned with the axes. So even this simple transformation has some interesting effects. We'll see others when we look at what it does to character strings.

A transformation is completely described by the formula for its separate components, i.e. the functions u and v in the expression

$$f(x, y) = (u(x, y), v(x, y))\ .$$

This is all we'll need to know about f in order to apply it. But in order to understand in any deep sense how a transformation in 2D behaves, we also must consider its **Jacobian derivative**. This is a matrix-valued function of x and y whose entries are the partial derivatives of f:

$$\mathrm{Jac}_f(x, y) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{bmatrix}\ .$$

For the example we have already looked at this is

$$\begin{bmatrix} 2x & 2y \\ -2y & 2x \end{bmatrix}\ .$$

The importance of the Jacobian matrix is that it allows us to approximate the map f by an affine map in any small neighbourhood of a point. The exact statement is:

- *If Δx and Δy are small then we have an approximate equality*

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + [\,\Delta x \quad \Delta y\,]\,\mathrm{Jac}_f(x, y)$$

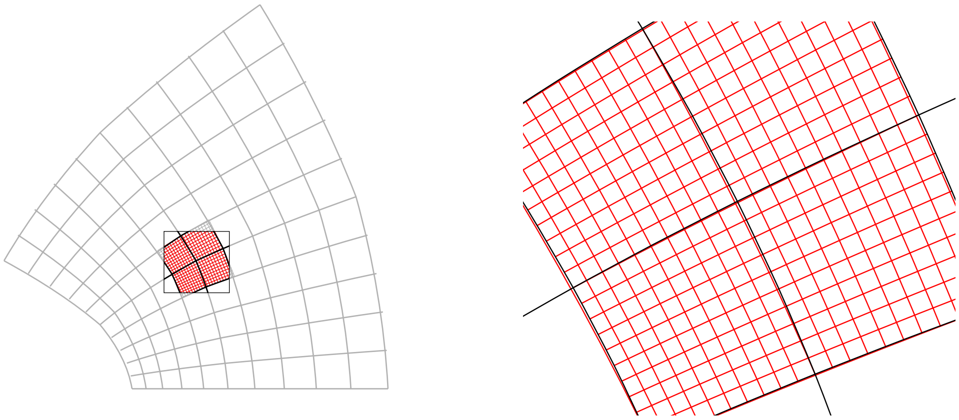
Another way to say it:

- *If (x, y) lies near (x_0, y_0) then we have an approximate equality*

$$f(x, y) \approx f(x_0, y_0) + [\,(x - x_0) \quad (y - y_0)\,]\,\mathrm{Jac}_f(x, y)\ .$$

What this says, roughly, is that locally in small regions the transformation looks as though it were affine.

This phenomenon can be seen geometrically. If we look at the figure that started this chapter, but zoom in, we see:



In other words, if we zoom in to a part of the plane, the transformation looks straighter and straighter as the scale shrinks. I'll not say much more about why this **Jacobian approximation** is valid, but only remark that it is essentially the definition of partial derivatives. It is the generalization to 2D of the linear approximation $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$ encountered in the ordinary calculus of one variable.

In our example, we have an exact equation

$$\begin{aligned} f(x + \Delta x, y + \Delta y) &= ((x + \Delta x)^2 - (y + \Delta y)^2, 2(x + \Delta x)(y + \Delta y)) \\ &= ((x^2 + 2x\Delta x + \Delta x^2) - (y^2 + 2y\Delta y + \Delta y^2), 2(xy + x\Delta y + y\Delta x + \Delta x\Delta y) \end{aligned}$$

which becomes

$$\begin{aligned} ((x^2 + 2x\Delta x) - (y^2 + 2y\Delta y), 2(xy + x\Delta y + y\Delta x) &\approx (x^2 - y^2, 2xy) + [2x\Delta x - 2y\Delta y, 2x\Delta y + y\Delta x] \\ &= f(x, y) + [\,\Delta x \quad \Delta y\,] \begin{bmatrix} 2x & 2y \\ -2y & 2x \end{bmatrix} \end{aligned}$$

if we ignore the terms of order two—i.e. Δx^2, Δy^2, $\Delta x\Delta y$. But this **linear approximation** is the Jacobian approximation. As it always is.

I'll not use the Jacobian derivative in PostScript procedures, although it could conceivably improve quality at the cost of complication. But it is important to realize that the reason our transformation procedures will work so well is precisely because in very small regions the function f looks affine.

In routines to be described later, a 2D transformation f will be given as a procedure with two arguments x and y, and it will return a pair `u v`. In our example it would be

```
% on stack:  x y
/fcn { 2 dict begin
  /y exch def
  /x exch def
  x dup mul y dup mul sub  % x^2-y^2
  2 x mul y mul  % 2xy
end } def
```

It would in principle be possible to get a smoother output if I added a second version that takes advantage of the Jacobian approximation. In this version the function f would return the Jacobian matrix on the stack as well, in the form of an array of 4 variables. In practice this doesn't seem to be important.

8.2. Conformal transforms

As we have seen, the Jacobian matrix of the transform $(x, y) \mapsto (x^2 - y^2, 2xy)$ is

$$\begin{bmatrix} 2x & 2y \\ -2y & 2x \end{bmatrix} .$$

This matrix has the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} ,$$

and such matrices, which I'll call **similarity matrices** for reasons that will become apparent in a moment, have interesting properties. One example of such a matrix is the rotation matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} ,$$

and another is the scalar matrix

$$\begin{bmatrix} r & \\ & r \end{bmatrix} ,$$

So also is the product of these two

$$\begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} .$$

In fact, if S is any similarity matrix

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} ,$$

We can write

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos\theta$$
$$\frac{b}{\sqrt{a^2 + b^2}} = \sin\theta$$

and then

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

where $r = \sqrt{a^2 + b^2}$, so that every similarity matrix is the product of a rotation and a scaling.

- *A matrix is a similarity matrix if and only if (1) it has positive determinant and (2) the linear transformation associated to it is a similarity transformation—that is to say, it preserves the shape of figures.*

In particular, a similarity transformation takes squares to (possibly larger or smaller) squares, which explains what we have seen in the pictures of the map $(x, y) \mapsto (x^2 - y^2, 2xy)$.

In general, a transform from 2D to 2D is called **conformal** if its Jacobian matrix is a similarity matrix at all but a few isolated points. Such a map looks like a similarity transformation in small regions, and hence preserves the angles between paths, even though it may distort large shapes wildly. Thus the map $(x, y) \mapsto (x^2 - y^2, 2xy)$ is conformal, except at the origin (where it doubles angles). As I mentioned before, this map was derived from the map $z \mapsto z^2$ of complex numbers. There is a huge class of similar complex-valued functions of a complex variable which are conformal.

8.3. Transforming paths

The technique for transforming paths involves decomposing them first, and then reassembling them transformed. The following code uses `pathforall` to scan through the current path and build an array setting up the transformed path. Then it annihilates the current path with `newpath` and scans through that array to build the transformed path. The transformation is simply applied to control points to obtain new control points. Stack manipulations are used for efficiency.

```
 % Argument:  the name of the transforming procedure
 % It takes x y -> u v
/ctransform { load  % first we load the procedure onto the stack
1 dict begin
 % and now give it a local name
/f exch def
 % build an array from the current path
[
  {  % x y
    [ 3 1 roll f {moveto}]
  }
  {  % x y
    [ 3 1 roll f {lineto} ]
  }
  {  % x1 y1 x2 y2 x3 y3
    [ 7 1 roll                 % [ P1 P2 P3
      f 6 2 roll               % [ U3 P1 P2
      f 6 2 roll               % [ U2 U3 P1
      f 6 2 roll               % [ U1 U2 U3
      {curveto}
    ]
  }
  {
    [ {closepath} ]
  }
  pathforall
]
 % and then replace the current path
newpath
{
  aload pop exec
} forall
end } def
```

This is generally pretty unsatisfactory unless the components of the path are small. A line segment is just mapped onto another line, and this will usually ignore the non-linearity of f. For this reason it is useful to subdivide the current path one or more times, breaking segments into smaller ones. This is easy to do with a routine `subdivide`, which replaces line segments by Bézier curves and bisects Bézier curves into two smaller curves.

8.4. Maps

The classical examples of 2D transformations, although with an implicit 3D twist, occur in the design of maps of the Earth. The basic problem is that there is no faithful way to render the surface of a sphere on a flat surface. There is no one single kind of map that does for all purposes, and various kinds must be designed to conform to various criteria.

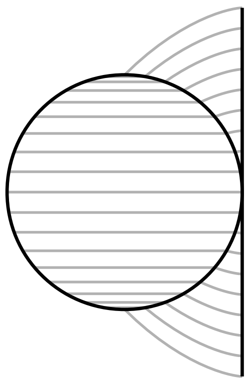
Coordinates on the sphere are East longitude x and latitude y. Assuming the Earth's surface to be a sphere of radius R, the coordinate map takes

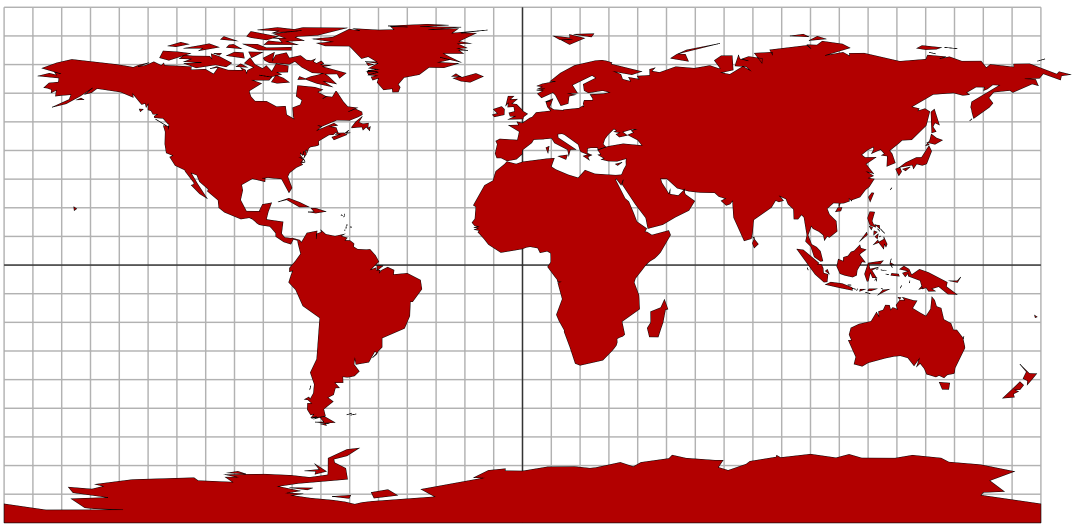
$$(x, y) \longmapsto (R\cos x\ \cos y, R\sin x\ \cos y, R\sin y)\ .$$

The image of a small rectangle $dx \times dy$ is an approximate rectangle in space of dimensions $R\cos y\, dx \times R\, dy$. Of course the coordinates are singular near the poles, since the poles have indeterminate longitude.

I'll assume from now on for convenience that $R = 1$. (This is just a matter of choosing units of lengths correctly.)

The simplest map just transforms longitude and latitude into x and y:



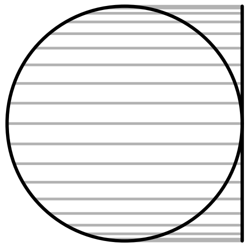


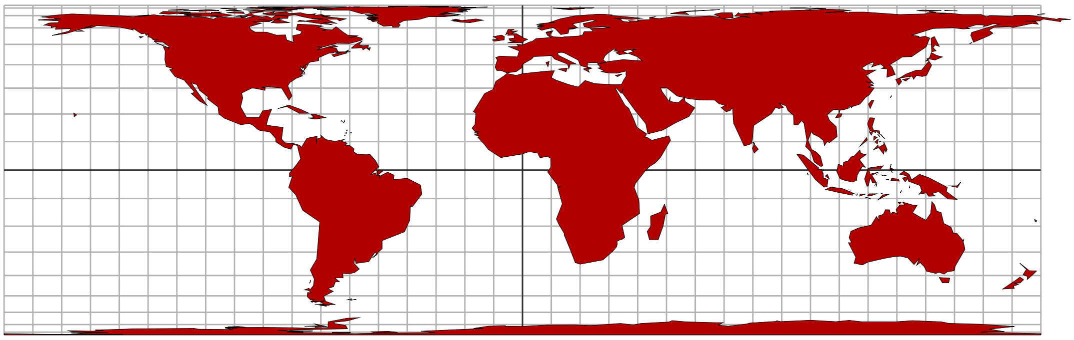
There's not much to be said for it. It preserves distance measured along meridians, but distorts wildly distances along parallels.

The next simplest map is called the **cylindrical projection**, because it projects a point on the Earth's surface straight out to a vertical cylinder wrapped around the Earth, touching at the Equator. Explicitly,

$$(x, y) \longmapsto (x, \sin y) \ .$$

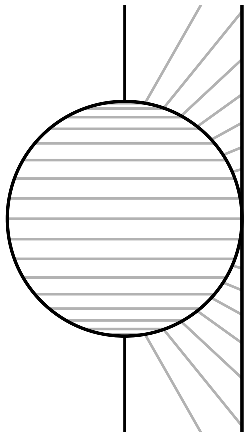
It was proven by Archimedes in his classic 'On sphere and cylinder' that this map preserves areas, although it certainly distorts distances near the poles. The modern proof that area is preserved calculates that the region $\cos y \, dx \times dy$ maps to that of dimensions $dx \times \cos y \, dy$.

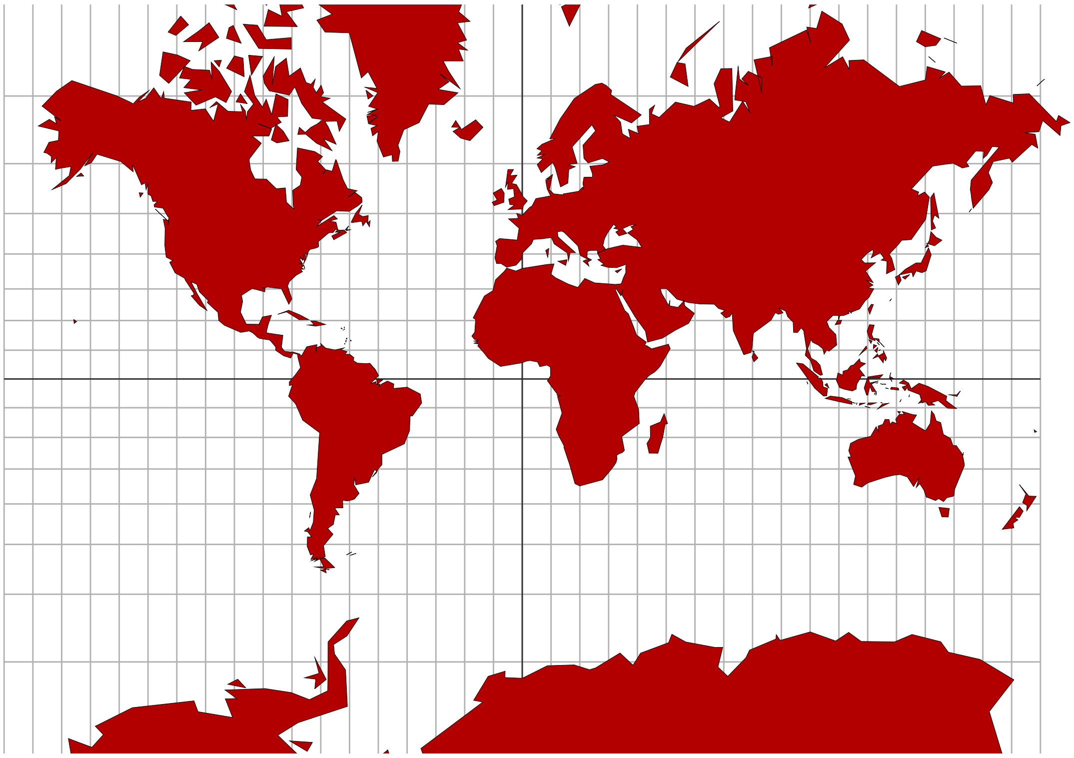




A more interesting one is the Mercator projection

$$(x, y) \mapsto \big(x, \ln \tan(\pi/4 + y/2)\big) \ .$$





Here, distances and areas near the poles are grossly exaggerated, but angles are conserved, so that plotting a route by compass is relatively simple. Indeed this is exactly the purpose for which this projection was designed. The first map of this sort was constructed by the 16th century geographer Gerardus Mercator himself, but it was likely the Englishman Thomas Harriot who understood its mathematical basis thoroughly—well, as thoroughly as could be done without calculus.

Why is the Mercator map conformal? Recall that the image of a small coordinate rectangle dx by dy is very approximately a rectangle on the sphere of dimensions $\cos y\, dx \times dy$. Its image under Mercator's map has size dx by

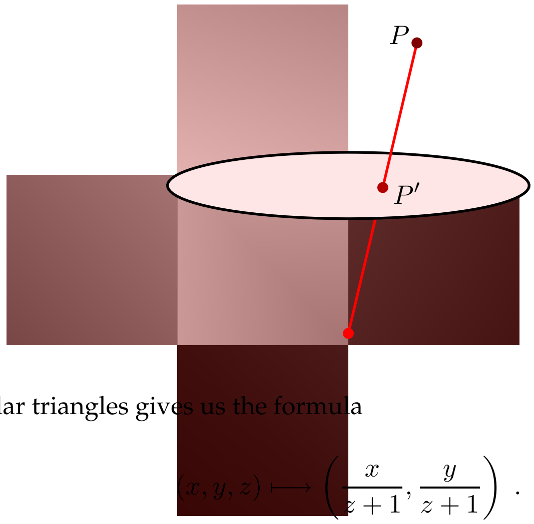
$$\tan'(\pi/4 + y/2)\, dy = \frac{dy}{\cos y}\ .$$

But these two rectangles are similar to each other, with a scale factor of $1/\cos y$. In other words, the function $\tan(\pi/4 + y/2)$ has been chosen precisely because its derivative is $1/\cos y$.

All of the maps exhibited here were obtained from the original map data by applying `ctransform`.

8.5. Stereographic projection

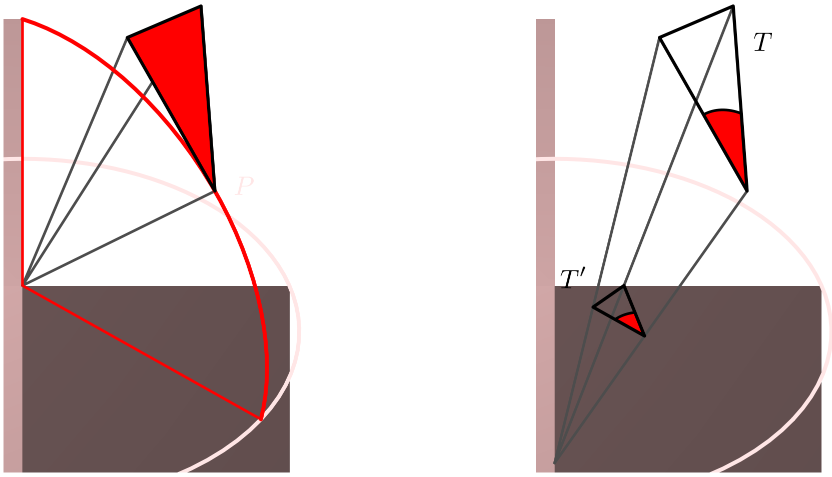
Another map projection, **stereographic projection**, deserves a section all to itself, since it plays an important role in much mathematics. In this projection, a point P on a sphere is mapped to the intersection P' of the line segment from P to the south pole with the equatorial plane.



An argument using similar triangles gives us the formula

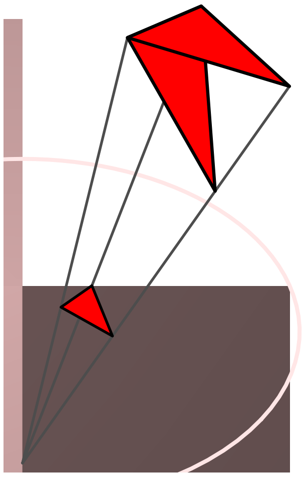
$$(x, y, z) \longmapsto \left(\frac{x}{z+1}, \frac{y}{z+1} \right) .$$

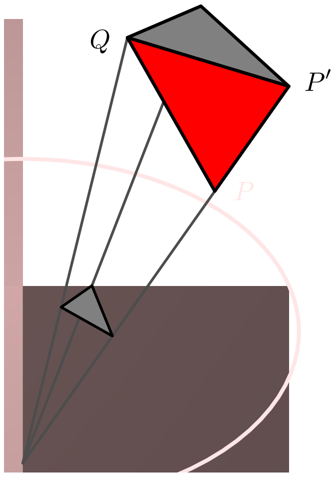
The principal property of stereographic projection is that it is conformal. This can be proven by calculus as was the analogous result for Mercator's projection, but it can also be proven by a geometric argument going back to the English mathematician Thomas Harriot, who might have been the first to discover the result and was almost certainly the first to prove it. The argument I give and the accompanying diagrams follow Harriot's.



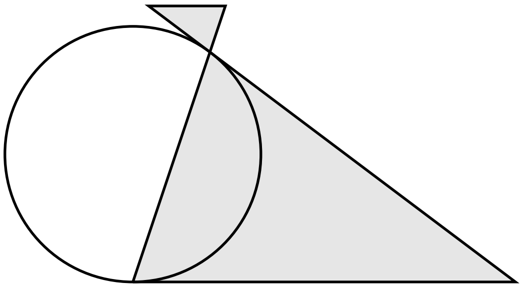
First of all, suppose P to be a given point on the sphere, P' its image under stereographic projection. In order to prove that stereographic projection is conformal, it suffices to show that any angle with respect to a meridian line on the sphere is mapped to an equal angle. Therefore we draw at P a small right-angled triangle, one of whose sides lies tangent to the meridian from P, as in the figure at the left. The proof will show that this triangle T and its image T' under stereographic projection, shown in the figure at the upper right, are similar.

To see why this is so, construct a third triangle T'', sharing a common side with the one whose vertex is P, but parallel to the projected triangle, with a vertex P'' on the line through the south pole, P', and P. It is clearly similar to the projected triangle T', so that in order to show that T and T' are similar, it suffices to show that T'' is congruent to T.





The two triangles T and T'' share a common side, and are both right-angled at Q. Therefore a side-angle-side argument will show that they are congruent, once it is known that the sides PQ and $P'Q$ have the same length. This is because the meridian triangle PQP'' is isosceles, and this can be seen from an elementary two-dimensional diagram:

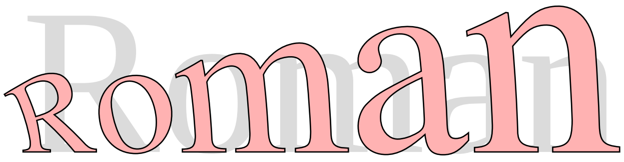


8.6. Fonts want to be free

Text can be transformed, too. After setting up a font and defining procedures `subdivide` and `ctransform`

```
newpath
1 0 moveto
(Roman) true charpath
subdivide
/f ctransform
gsave
1 0.7 0.7 setrgbcolor
fill
grestore
currentlinewidth 2 div setlinewidth
stroke
```

produces (except for the comparison image in light gray)



Subdivision is important here because font paths possess a lot of straight segments which are much better off transformed as curves.

8.7. Code

The procedures `subdivide` and `ctransform` are to be found in the package `transform.inc`. The map data can be found in the PostScript files `coasts.1.inc` etc, which I have made from the `MWDB` files mentioned below. The index indicates level of detail—index 1 is the greatest detail, index 5 the least.

References

1. J. L. Berggren and A. Jones, **Ptolemy's Geography**, Princeton University Press, 1997. Mathematical map-making began with the Greeks. The illustrations here, many taken from medieval manuscripts, are beautiful. The invention of printed books in the West meant not only a wider audience for science but also, ironically, a loss in the quality of graphics and in particular the loss of brilliant colour.

2. Timothy G. Freeman, **Portraits of the Earth—A mathematician looks at maps**, American Mathematical Society, 2002. An appendix includes an account of how what seems to be the same map data that I have used can be used to draw maps using `Maple`.

3. David Hilbert and Stephan Cohn-Vossen, **Geometry and the Imagination**, Chelsea, 1952. Many maps are conformal—preserve angles—and one of the basic theorems in the subject is that stereographic projection is a conformal map. Nowadays this can be proven quickly by calculus, but an elegant geometric proof can be found in §36 of this book. Early Greek astronomers knew that stereographic projection maps circles to circles, applying a theorem of Apollonius on conic sections to deduce it. This fact is closely related to conformality, but conformality itself was apparently first stated and proved by the English mathematician Thomas Harriot, who was among other things Sir Walter Raleigh's navigation expert, in about 1590. This was still in the golden age of map-making stimulated by the discovery and exploration of America, as well as the great Portugese voyages around Africa to the Far East. Harriot's proof was unpublished, but it can be reconstructed from a rather handsome sketch in his manuscripts. A reproduction can be found in the article by J. A. Lohne, 'Thomas Harriot als Mathematiker', *Centaurus* 11 (1965/66), who also discusses Harriot's treatment of the Mercator projection. The first published proof was by Edmund Halley, who is often (and incorrectly) given credit for it. His paper is 'An Easie Demonstration of the Analogy of the Logarithmic Tangents . . . ' in the *Philosophical Transactions* **19** (1695–1697), pp. 202–214 (miraculously available through the electronic library service JSTOR). Although he acknowledges that he learned the statement from De Moivre he claims the proof as his own, but there is suspicion that both came one way one way or another from the remarkable Harriot (as did probably much other mathematics of the 17th century).

4. John McCleary, **Geometry from a differentiable viewpoint**, Cambridge University Press, 1994. There is an interesting account of map projections in Chapter 8 which, like much of this book, would serve nicely as a source of graphics projects.

5. Tristan Needham, **Visual Complex Analysis**, Oxford University Press, 1997. This explains the relationship between complex numbers and conformal maps with lots of illustrations.

6. John P. Snyder, **Flattening the Earth: Two Thousand Years of Map Projections**, University of Chicago Press, 1993. This is a very readable history of map-making, including descriptions of dozens of different ones.

7. For the maps I have used the data from the 'World Database II', which is now included in the file `world.zip` accessible by clicking on the icon at

`http://archive.msmonline.com/1999/12/vis2.htm`

These map data, which are now in the public domain, were compiled by Fred Pospeschil and Antonio Riveria from data originally created by the Central intelligence Agency. The particular files I have used incorporate further modifications by by Paul Anderson of Global Associates, Ltd. The Bodleian Library of Oxford University maintains a convenient web page with links to sources of other map data available without cost.