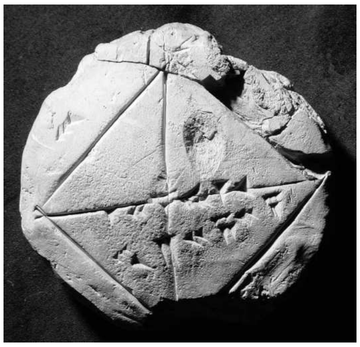
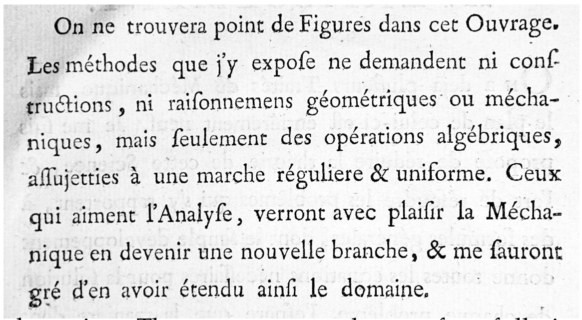
Epilogue



The association between mathematics and graphics is ancient. Indeed, as the scholar of Greek mathematics T. L. Heath has observed, the Greek root of the word 'graphics' seems to mean in places 'to prove'. But the association is far older than even the civilization of classical Greece. One of the very oldest mathematical documents we now possess is YBC 7289, a tablet dating from about 1800 B.C. and now found in the Yale Babylonian Collection. The three numbers written on the tablet (in base 60 notation) express $1/2$, an approximation of $\sqrt{2}$ to about 8 decimal figures, and the corresponding product $\sqrt{2}/2$. The diagram on it is plausibly a part of a simple geometric proof of Pythagoras' theorem for isosceles right triangles, closely related to the well known figure associated with the discussion of this result in Plato's *Meno*. This extraordinary object therefore seems to show that the person who made it knew both that the ratio of the diagonal to the side of a square was a real number whose square was two, and why it was so. I like to think that this diagram tells us that the association of logical reasoning with mathematics originated with deductions from figures, although of course any train of real evidence of how mathematical reasoning came about is to us is beyond recovery.



On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expoſe ne demandent ni conſtructions, ni raiſonnemens géométriques ou méchaniques, mais ſeulement des opérations algébriques, aſſujetties à une marche réguliere & uniforme. Ceux qui aiment l'Analyſe, verront avec plaiſir la Méchanique en devenir une nouvelle branche, & me ſauront gré d'en avoir étendu ainſi le domaine.

The quality of mathematical reasoning made an extraordinary leap among the Greeks, notably with the appearance of the *Elements* of Euclid. Figures were still required, of course, and one small but significant advance was made by linking text and figures with labels. Of course the connection between pictures and reasoning has continued for the entire history of mathematics, although with varying importance. It reached a low point during the 18th century, probably because from the earliest days of modern mathematical analysis it was realized that pictures were inadequate to deal with the complexities of the subject, and could be seriously deceptive. The tone was struck most forcefully in Lagrange's famous boast in the preface to his *Analytical mechanics* that "One will not find any figures in this work. The methods which I explain in it require neither constructions nor geometrical nor mechanical reasoning, but only algebraic operations." Such an attitude towards illustration in mathematics continued to be sounded, and with some justification, through the entire nineteenth century and into the twentieth. During much of that time, it was often amateurs who came up with the best graphics in mathematics, although there were notable exceptions. Technology was poor—even through the nineteenth century technical illustrations were often done with woodcuts. To compensate for this, labour was fairly cheap during much of the century, but eventually costs overtook convenience, and the quality of mathematical illustration went down. Cost presumably explains why we had in the twentieth century otherwise fine books on geometry notoriously lacking illustrations, such as Julian Coolidge's **History of Geometrical Methods**. (In speaking of the origins of perspective drawing (p. 107 in my Dover reprint), he writes "Alberti does not explain himself very clearly, and we may say that . . . the easiest way to understand those early works on perspective is to study the pictures, not to read the text." But Coolidge has no pictures!) Later in that century, computers restored to mathematics the potential of being associated with great graphics, one that has even been occasionally realized. But although computers have made it possible to publish great pictures, technology alone does not guarantee that the worth of a picture reflects how much work went into it. It is important to think carefully about exactly what one wants. Pictures should be drawn with as much care as paragraphs are written.

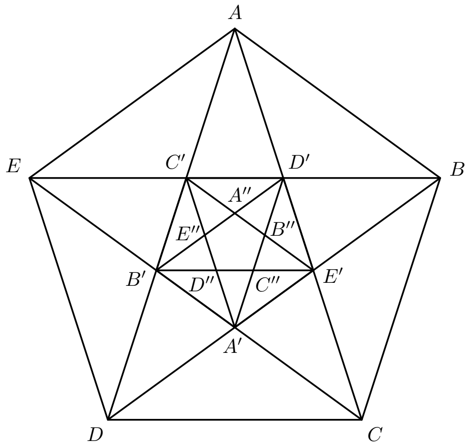
Edward Tufte, now retired from a career as Professor at Yale University, has written several books on what he calls 'information graphics'. They are very beautiful books—produced by his own press, which was founded expressly for the purpose—and indeed these books illustrate that the medium really is often the message. Nonetheless, there are a few more or less practical rules to be extracted from them that could be of value to anybody who wants to explain something by visual means. A lot of Tufte's effort goes into the display of *data*—weather patterns, train schedules, the attrition in Napoleon's army on the long, cold, winter road to Moscow—but many of his ideas apply to the task of mapping out a train of logic. I'll give an example here.

I will do this by comparing what I am embarrassed to call the traditional mode of mathematical exposition with one adapted from Tufte's suggestions.

I am going to look at the classical result, known to the ancient Greeks, that the golden ratio is irrational–cannot be expressed as a ratio of whole numbers. This is of course one of the oldest mathematical discoveries, and perhaps the first truly astonishing one. There is much discussion in the literature—almost entirely speculative, of course—as to how incommensurability was first found. One common and reasonable speculation is that it was arrived at by geometric reasoning—not that geometry provided at first a completely rigourous proof in view of the then primitive state of Greek mathematics, but that it at least provided a convincing chain of reasoning of some kind leading to the result.

The golden ratio is also the ratio between the side and diagonal of a regular pentagon, and what we shall actually demonstrate is that *the side and diagonal of a regular pentagon are incommensurable*. Very briefly, the idea of the argument used here is to see that if the side and diagonal are both multiples of an interval ϵ then so are the side and diagonal of the smaller pentagon at the center of the five-sided star whose vertices are those of the original pentagon. Recursion leads to a contradiction.

I will begin by quoting a very traditional approach to this question, from a 1945 paper by Kurt von Fritz on the discovery of incommensurability. Keep in mind throughout what is to follow that the point, as von Fritz says, is not merely to prove the result, but to make it "almost apparent at first sight". Here is my copy of the figure drawn by von Fritz:



Von Fritz' Figure

Here is what he writes:

> ... the diameters of the pentagon form a new regular pentagon in the centre, ... the diameters of this smaller pentagon will again form a regular pentagon, and so on in an infinite process. It is ... easy to see that in the pentagons produced in this way $AE = AB'$ and $B'D = B'E'$ and therefore $AD - AE = B'E'$, and likewise $AE = ED' = EA'$ and $B'E' = B'D = B'E$ and therefore $AE - B'E' = B'A'$, and so forth ad infinitum, or, in other words, that the difference between the diameter and the side of the greater pentagon is equal to the diameter of the smaller pentagon, and the difference between the side of the greater pentagon and the diameter of the smaller pentagon

> is equal to the side of the smaller pentagon, and again the difference between the diameter of the smaller pentagon and its side is equal to to the diameter of the next smaller pentagon and so forth in infinitum. Since ever new regular pentagons are produced by the diameters it is then evident that the process of mutual subtraction will go on forever, and therefore no greatest common measure of the diameter and the side of the regular pentagon can be found.

There is nothing wrong in the logic of this treatment, although it does stumble around a bit. What we are interested in right now, however, is how the argument relates to the figure. The answer, I think it is fair to say, is "not well". Reading the original article is even more difficult than apparent here because, as often happens, the text and figure are on separate pages. What I claim is that *von Fritz stumbles precisely because he is trying to put in words what could have been far better put in pictures*. His one figure is not really used in a serious way, and essentially does no more than make the argument unambiguous. I also think it is fair to say that von Fritz is far from making the result apparent at first sight. Contrary to what he wants to do, he is preaching to the converted. Of course one might object that a paper over 50 years old cannot be held completely responsible for its graphics, but actually von Fritz does better than many more recent authors.

Let's see what help Tufte might be able to offer. The first step is to decide to take the graphics more seriously—to make the graphics the main part of the narrative, rather than subsidiary to it. The next step is to integrate text and graphics better; this is von Fritz' major failing, because in reading his argument you are constantly forced to go back to the diagram, relocate yourself there, etc. A third is to determine which elements of the illustrations are important, and then to emphasize them. In von Fritz' figure there are only the labelling of the vertices to orient the reader. But in fact the entities involved are not really the vertices at all, but instead various edges and sub-regions of the pentagon.

Perhaps the most succinct application of Tufte's principles is found in Chapter 4 of *Visual Explanations*, 'The Smallest Effective Difference'. It opens up with a diagram of the ear taken from the *Random House Dictionary of the English Language*, which has a remarkable resemblance to von Fritz' diagram! Tufte redraws it to make it clearer, by carrying through the following ideas:

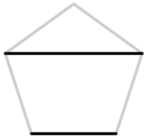
- Tone down the secondary elements of a picture in order to reduce visual clutter, to clarify the primary elements of the figure, and also to eliminate unwanted visual interactions. Tufte calls this **layering** the figure to produce a visual hierarchy.
- Reduce discontinuity in the exposition—replace coding labels in the figure by locally useful information in the figure itself. The general principle is to integrate text and graphics. One point is that unnecessary eye movements are fatal to easy comprehension.
- Produce emphasis by using the smallest possible effective distinctions. In practice this often, but not always, means replacing bold, strongly contrasting colors by quieter shades. This is perhaps the hardest of all sins to avoid, since it is often extremely tempting for the beginner to introduce strong colors whenever he can. In Web graphics, this does often work well.

To this list might be added a few ideas from elsewhere in Tufte's books:

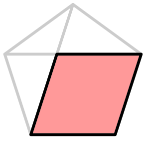
- Eliminate parts of the figure that do not actually add to its content. If they do not add to it then they will subtract from it.
- Use what Tufte calls **small multiples**, numerous repetitions of a single figure with slight variations. Human perception is sharp in making comparisons. Another way to put this, particularly in trying to track a logical argument visually, is to say that the sequence should have **continuity**, in the dramatic sense.
- Make the graphics itself carry the tale, as much as possible. Where text is necessary, elininate one major source of annoyance by placing related text and graphics close to each other.

All of these are nearly self-evident principles, and if the use of graphics in mathematics were more sophisticated than it is now one might consider this an objection to Tufte's books.

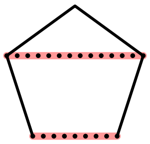
Here now is the argument I have made in up in an attempt to apply these principles:



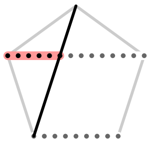
The basic fact is that in a regular pentagon a diagonal and the side opposite to it are parallel. This property in some sense characterizes the regular pentagon.



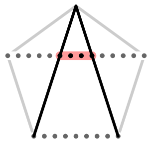
As a consequence, the shaded region shown at left is a parallelogram all of whose sides are equal (a rhombus).



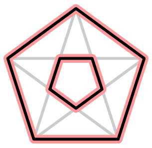
Assume now that the side s and diagonal d in are commensurable, which is to say that they are both multiples of a common interval ϵ.



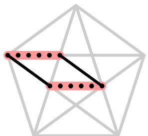
Then $d - s$ is also a multiple of the interval ϵ.



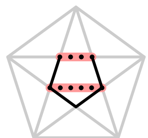
And so is the interval we get in the middle of the diagonal, which has length $d - 2(d - s) = 2s - d$.



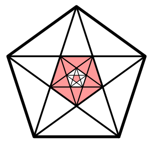
But this interval is the side of the smaller pentagon at the centre of the star we get by drawing all diagonals.



The figure emphasized in the diagram to the left is a parallelogram, since opposite sides are parallel to the same side of the pentagon. Therefore the quantity $d - s$ is the diagonal of the smaller pentagon.



Therefore under the assumption that the diagonal and side of a pentagon are multiples of a common interval ϵ, we deduce that so are the side and diagonal of the smaller pentagon inscribed in it.



We can reason in the same way about the pentagon in its interior in turn, etc. The interval ϵ will divide all the sides and diagonals of the infinite series of pentagons we get. But eventually the sides of those pentagons will be smaller than ϵ, a contradiction.

I imagine that some readers will find my argument distasteful. I am, however, in good company. It is no less than J. E. Littlewood who points out (p. 54 of the *Miscellany*)

> A heavy warning used to be given that pictures are not rigourous; this has never had its bluff called and has permanently frightened its victims into playing for safety.

Littlewood's remark accompanies an elegant 'picture proof' of a lemma of Landau's.

References

1. Bill Casselman, A review of **Visual Explanations** by E. Tufte, *Notices of the AMS* **46**, January 1999, pp. 43–46. Much of this epilogue was taken from this.

2. K. von Fritz, 'The discovery of incommensurability by Hppasus of Metapontium', Annals of Mathematics (1945), pp. 242–264.

3. T. L. Heath, **A History of Greek Mathematics volume I**, Dover, 1981. The discussion of 'drawing' and 'proof' is in the extensive footnote on p. 203. This question has been taken up more recently by Wilbur Knorr in §III.II of **The Evolution of Euclid's Elements**, and again in §2.2 of **The Shaping of Deduction**, an exhaustive investigation of Greek mathematical diagrams by the classical scholar Reviel Netz. Both of these later discussions are very interesting. Especially for those like me who believe on *a priori* grounds that mathematical proofs originated with diagrams, where visualization led inevitably to reasoning.

4. J. E. Littlewood, **Littlewood's Miscellany**, Cambridge University Press, 1988.

5. E. Tufte, **The Visual Display of Quantitative Information**, Graphics Press, Cheshire, Connecticut, 1983.

6. ———, **Envisioning Information**, Graphics Press, 1990.

7. ———, **Visual Explanations**, Graphics Press, 1997.