INTRODUCTION

These notes arise from a Seminar held at Harvard and M.I.T. in the Fall of 1962. The main aim was to establish a general formula for the index of elliptic operators on compact manifolds. The reader who is primarily interested in this index theorem will find here all the raw material of the proofs. It should be stressed however that this is far from a final polished version. The sections are presented here in the chronological order in which they were covered in the seminar, and we have made no attempt to reorganize them.

The first half approximately of these notes is concerned with Clifford algebras and K-theory, and this will appear in TOPOLOGY as a joint paper of Atiyah, Bott and Shapiro. By no means all of this part is necessary for the proof of the index theorem. In fact the index theorem uses only the cruder aspects of K-theory: the index being an integer, we may ignore torsion.

Brief outlines of the index theorem can be found in (Bull. A.M.S. (1963), p. 422-433) and in a Bourbaki Seminar (1962/63, No. 253). These can be used as a guide to the present more voluminous notes. We should also refer to a forthcoming paper by R. Seeley (Integro-differential operators on vector bundles: to appear in Trans. A.M.S.) which covers all the analytical background.

Since three authors are piece-wise responsible for these notes, consistency in notation and point of view has been difficult to achieve. In particular we do not guarantee all the correct signs!

The critical theorem is: Let $M_k(\Lambda)$ = the grothendieck ring of ungraded modules over $C_k(\Lambda)$. Then for $l < k$

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$$M_k(\mathbb{R}) \to M_l(\mathbb{R}) \to \tilde{KO}(\mathbb{RP}_k/\mathbb{RP}_l) \to 0$$

is exact

Lectures (1-3). The Spinor Groups. (Raoul Bott).

1. Notation. Let k be a commutative field and let Ω be a quadratic form on the k-module E. Let $T(E) = \Sigma_{i=0}^{\infty} T^i E = k \oplus E + E \oplus E + \cdots$ be the tensor algebra over E, and let $I(\Omega)$ be the two sided ideal generated by the elements $x \oplus x - \Omega(x) \cdot 1$ in $T(E)$. The quotient algebra $T(E)/I(\Omega)$ is called the Clifford algebra of Ω and is denoted by $C(\Omega)$. We also define $i_\Omega : E \longrightarrow C(\Omega)$ to be the canonical map given by the composition $E \longrightarrow T(E) \longrightarrow C(\Omega)$. Then the following proposition relative to $C(\Omega)$ are not difficult to verify.

1.1. $i_\Omega : E \longrightarrow C(\Omega)$ is an injection. (since $\ker \cap T^2(E) = \varphi$)

1.2. Let $\varphi : E \longrightarrow A$ be a linear map of E into a k-algebra with unit A, such that for all $x \in E$, the identity $\varphi(x)^2 = \Omega(x)1$ is valid. Then there exists a unique homomorphism $\tilde{\varphi} : C(\Omega) \longrightarrow B$, such that $\tilde{\varphi} \circ i_\Omega = \varphi$. (We refer to $\tilde{\varphi}$ as the "extension" of φ.)

1.3. $C(\Omega)$ is the universal algebra with respect to maps φ of the type described in (1.2).

1.4. Let $F^q T(E) = \Sigma_{i \le q}^{i} T^i E$ be the filtered structure in $T(E)$. This filtering induces a filtering in $C(E)$, whose associated graded algebra is isomorphic to the exterior algebra $\wedge E$, on E. Thus $\dim_k C(\Omega) = 2^{\dim E}$, and if $\{e_i\}$ $(i = 1, \cdots, n)$ is a base for $i_\Omega(E)$, then 1 together with the products $e_{i_1} \cdot e_{i_2} \cdots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$, form a base $C(\Omega)$.

and actually = $C(\Omega)$ for $\Omega = 0$

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since $x \otimes x - \Omega(x) \equiv x \otimes x$ mod $F^1(E)$

and so it agrees with $C(\Omega)$ for $\Omega = 0$

1.5. Let $C^0(Q)$ be the image of $\Sigma_{i=0}^{i=\infty} T^{2i}(E)$ in $C(Q)$ and set $C^1(Q)$ equal to the image of $\Sigma_0^{\infty} T^{2i+1}(E)$ in $C(Q)$. Then this decomposition defines $C(Q)$ as a $\mathbb{Z}_2$-graded algebra. That is:

a) $$C(Q) = \sum_{i=0,1} C^i(Q)$$

b) If $x_i \in C^i(Q)$, $y_j \in C^j(Q)$ then

$$x_i y_j \in C^k(Q), \qquad k \equiv i + j \mod 2.$$

That the graded structure of $C(Q)$ should not be disregarded is maybe best brought out by the following:

PROPOSITION 1.1. <u>Suppose that $E = E_1 \oplus E_2$ in an orthogonal decomposition of E relative to Q, and let Q_i denote the restriction of Q to E_i. Then there is an isomorphism</u>

$$\psi : C(Q) \simeq C(Q_1) \hat{\otimes}_k C(Q_2)$$

<u>of the graded tensor-product of $C(Q_1)$ and $C(Q_2)$ with $C(Q)$.</u>

Recall first, that the graded tensor product of two graded algebras $A = \Sigma_{\alpha=0,1} A^{\alpha}$, $B = \Sigma_{\alpha=0,1} B^{\alpha}$, is by definition the algebra whose underlying vector space is $\Sigma_{\alpha,\beta=0,1} A^{\alpha} \otimes B^{\beta}$, with multiplication defined by:

$$(u \otimes x_i) \cdot (y_j \otimes v) = (-1)^{ij} uy_j \otimes x_i v, \quad x_i \in C^i(Q), \ y_j \in C^j(Q).$$

This graded tensor product is denoted by $A \hat{\otimes} B$; and is again a graded algebra : $(A \hat{\otimes} B)^k = \Sigma A^i \otimes B^j \quad (i + j \equiv k(2))$.

<u>Proof of the proposition.</u> Define $\psi : E \longrightarrow C(Q_1) \hat{\otimes}_k C(Q_2)$ by the formula, $\psi(e) = e_1 \otimes 1 + 1 \otimes e_2$, where e_1 and e_2 are the orthogonal projections of E on E_1 and E_2. Then

$$\psi(e)^2 = (e_1 \otimes 1 + 1 \otimes e_2)^2 = \{Q_1(e_1) + Q_2(e_2)\}\ (1 \otimes 1) = Q(e)(1 \otimes 1).$$

Hence ψ extends to an algebra homomorphism $\psi : C(Q) \longrightarrow C(Q) \hat{\otimes} C(Q)$, by Proposition 1. Checking the behavior of ψ on basis elements now shows that ψ is a bijection. Note that the graded structure entered through the formula $(e_1 \otimes 1 + 1 \otimes e_2)^2 = e_1^2 \otimes 1 + 1 \otimes e_2^2$ which is valid as $e_i \in C^1(Q_i)$. ∎ $(e_1 \otimes 1)(1 \otimes e_2) + (1 \otimes e_2)(e_1 \otimes 1) = 0$

The algebras $C(Q)$ also inherit a canonical antiautomorphism from the tensor algebra $T(E)$. Namely if $x = x_1 \otimes x_2 \cdots \otimes x_k \in T^k(E)$, then the map $x \longrightarrow x^t$, given by with quaternions: $i \to i$ $j \to j$, $k \to -k$

$x \otimes y \to y \otimes x = -x \otimes y$

$$x_1 \otimes x_2 \otimes \cdots \otimes x_k \longrightarrow x_k \otimes \cdots \otimes x_2 \otimes x_1$$

clearly defines an antiautomorphism of $T(E)$, which preserves $I(Q)$ because $\{x \otimes x - Q(x) \cdot 1\}^t = x \otimes x - Q(x) \cdot 1$. Hence this operation induces a well defined antiautomorphism on $C(Q)$ which we also denote by $x \longrightarrow x^t$ and refer to as <u>the transpose.</u> The transpose is the identity map on $i_Q(E) \subset C(Q)$.

The following two operations on $C(Q)$ will also be useful :

(α is an automorphism)

DEFINITION 1.1. The canonical automorphism of $C(Q)$ is defined as the "extension" of the map $\alpha : E \longrightarrow C(Q)$, given by $\cancel{\alpha}\alpha(x) = -i_Q(x)$. (It is clear that $\{\alpha(x)\}^2 = Q(x)1$ and so α is well-defined by 1.1.) We denote this automorphism by α.

$i \to -i$
$j \to -j$
$k \to k$ ~

DEFINITION 1.2. Let $x \longrightarrow \bar{x}$ be defined by the formula $x \longrightarrow \alpha(x^t)$. This "bar operation" is then an antiautomorphism of $C(Q)$.

$i \to -i,\ j \to -j\ \ k \to -k$

commutativity

Note: 1) The identity $\alpha(x^t) = \{\alpha(x)\}^t$ holds as both are antiautomorphisms which extend the map $E \longrightarrow C(Q)$ given by $x \longrightarrow -i_Q(x)$.

2) The grading on $C(Q)$ may be defined in terms of $\alpha : C^i(Q) = \{\, x \in C(Q) \mid \alpha(x) = (-1)^i x \}$, $i = 1, 2$.

2. The algebra's C_k. We are interested in the algebras $C(Q_k)$, where Q_k is a negative definite form on k-space over the real numbers. Quite specifically, we let R^k denote the space of k-tuples of real numbers, and define $Q_k(x_1, \cdots, x_k) = -\Sigma x_i^2$. Then we define C_k as the algebra $C(Q_k)$ and identify R^k with $i_{Q_k} R^k \subset C_k$ and $\mathbb{R}$ with $\mathbb{R} \cdot 1 \subset C_k$. For $k = 0$, $C_k = \mathbb{R}$.

PROPOSITION 2.1. The algebra C_1 is isomorphic to the complex numbers $\mathbb{C}$ considered as an algebra over $\mathbb{R}$. Further

$$C_k \simeq C_1 \mathbin{\hat{\otimes}} C_1 \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} C_1 \qquad (k \text{ factors}).$$

$\mathbb{H} \cong \mathbb{C} \hat{\otimes} \mathbb{C}$

i, j in $(C_1)^1$, $k = i \otimes j$ in $(C_1)^0$, with $i \otimes j = -j \otimes i$ by graded formula

$e_1 \quad e_2 \quad e_3$

$1 \quad e_1 \quad e_2 \quad e_3 \quad e_1e_2 \quad e_2e_3 \quad e_3e_1 \quad e_1e_2e_3$

Clearly C_1 is generated by 1 and e_1, where 1 denotes the real number 1 in R^1. Hence $e_1^2 = -1$. The formula $C_k \simeq C_1 \hat{\otimes} \cdots \hat{\otimes} C_1$ now follows from repeated application of Proposition 1.

We will denote the k-tuple, $(0, \cdots, 1, \cdots, 0)$ with 1 in the i^{th} position by e_i. The e_i, $i \leq k$ then form a base of $R^k \subset C_k$.

CORОLLARY. The e_i, $i = 1, \cdots, k$, generate C_k multiplicatively and satisfy the relations,

$$(2.1) \qquad e_j^2 = -1, \quad e_ie_j + e_je_i = 0, \qquad i \neq j .$$

C_k may be identified with the universal algebra generated over $\mathbb{R}$ by a unit, 1, and the symbols e_i, $i = 1, \cdots, k$, subject to the relations (2.1).

3. The groups, Γ_k, Pin(k), and Spin(k). Let C_k^* denote the multiplicative group of invertible elements in C_k. = non-zero quaternions if $k=2$, $= C^*$ if $k=1$

DEFINITION 3.1. The Clifford group Γ_k, is the subgroup of those elements $x \in C_k^*$ for which, $y \in R^k$ implies $\alpha(x) y x^{-1} \in R^k$. i.e. either coefficients are 0 (all those C^0)

It is clear enough that Γ_k is a subgroup of C_k, because α is an automorphism. We also write $\alpha(x) R^k x^{-1} \subset R^k$ for the condition defining Γ_k. As α and the transpose map R^k into itself, it is then also evident that

$\alpha(x) R^k x^{-1} \subset R^k$

$\alpha(x^t) R^k (x^t)^{-1} \subset R^k$

[illegible]

PROPOSITION 3.1. The maps, $x \longrightarrow \alpha(x)$, $x \longrightarrow x^t$, preserve Γ_k, and respectively induce an automorphism and an antiautomorphism of Γ_k. Hence $x \longrightarrow \bar{x}$ is also an antiautomorphism of Γ_k.

The group Γ_k comes to us with ready-made homomorphism $\rho : \Gamma_k \longrightarrow \mathrm{Aut}(\mathbb{R}^k)$. By definition $\rho(x)$, $x \in \Gamma_k$ is the linear map $\mathbb{R}^k \longrightarrow \mathbb{R}^k$ given by $\rho(x) \cdot y = \alpha(x) y x^{-1}$. We refer to ρ as the twisted adjoint representation of Γ_k on $\mathbb{R}^k$. This representation ρ turns out to be nearly faithful.

PROPOSITION 3.2. The kernel of $\rho : \Gamma_k \longrightarrow \mathrm{Aut}(\mathbb{R}^k)$ is precisely $\mathbb{R}^*$, the multiplicative group of nonzero multiples of $1 \in C_k$.

Proof: Suppose $x \in \mathrm{Ker}(\rho)$. This implies

$$(3.1) \qquad \alpha(x) y = y x \qquad \text{for all} \quad y \in \mathbb{R}^k .$$

Write $x = x^0 + x^1$, $x^i \in C_k^i$. Then (3.1) goes into

$$(3.2) \qquad x^0 y = y x^0$$

So $x^0 \in$ cntr...

$$(3.3) \qquad x^1 y = -y x^1 .$$

Let $e_1, \cdots, e_k$ be our orthonormal base for $\mathbb{R}^k$, and write $x^0 = a^0 + e_1 b^1$ in terms of this basis. Here $a^0 \in C_k^0$ does not involve e_1 and $b^1 \in C_k^1$ does not involve e_1. By setting $y = e_1$

in (3.2) we get $a^0 + e_1 b^1 = e_1 a^0 e_1^{-1} + e_1^2 b^1 e_1^{-1} = a_0 - e_1 b^1$. Hence $b^1 = 0$. That is, the expansion of x^0 does not involve e_1. Applying the same argument with the other basis elements we see that x^0 does not involve any of them. Hence x^0 is a multiple of 1. Next we write x^1 in the same form: $x^1 = a^1 + e_1 b^0$ and set $y = e_1$. We then obtain $a^1 + e_1 b^0 = -\{e_1 a^1 e_1^{-1} + e_1^2 b^0 e_1^{-1}\} = a^1 - e_1 b^0$. We again conclude that x^1 does not involve the e_i. Hence x^1 is a multiple of 1. On the other hand $x^1 \in C_k^1$ whence $x^1 = 0$. This proves that $x = x_0 \in \mathbb{R}$ and as x is invertible $x \in \mathbb{R}^*$. Q.E.D.

Consider now the function $N : C_k \longrightarrow C_k$ defined by

$N(e_i e_j) = e_i e_j \cdot e_j e_i = 1$

$$(3.4) \qquad N(x) = x \cdot \bar{x} .$$

$N(\lambda x) = \lambda^2 N(x)$

If $x \in R^k$, then $N(x) = x(-x) = -x^2 = -Q_k(x)$. Thus $N(x)$ is the square of the length in R^k relative to the positive definite form $-Q_k$.

PROPOSITION 3.3. If $x \in \Gamma_k$ then $N(x) \in \mathbb{R}^*$.

Proof: We show that $N(x)$ is in the kernel of ρ. Let then $x \in \Gamma_k$, whence for every $y \in R^k$ we have

$$\alpha(x) y x^{-1} = y' \quad , \quad y' = \rho(x) y \in R^k .$$

Applying the transpose we obtain: (as $y^t = y$)

See over

$$(x^t)^{-1} y \ \alpha(x)^t = \alpha(x) y x^{-1}$$

whence $y\,\alpha(x^t)x = x^t \alpha(x)\, y$. This implies that $\alpha(x^t)x$ is in the kernel of ρ, and hence in $\mathbb{R}^*$. It follows that $x^t\alpha(x) \in \mathbb{R}^*$, whence $N(x^t) \in \mathbb{R}^*$. However $x \longrightarrow x^t$ is an antiautomorphism of Γ_k into itself by Proposition 3.1. Hence $N(x) \in \mathbb{R}^*$.

PROPOSITION 3.4. $N : \Gamma_k \longrightarrow \mathbb{R}^*$ *is a homomorphism which commutes with* α.

Proof: $N(xy) = xy\,\overline{y}\,\overline{x} = x\,N(y)\,\overline{x} = N(x) \cdot N(y)$; $N(\alpha(x)) = \alpha(x)x^t$ $= \alpha\, N(x) = N(x)$.

PROPOSITION 3.5. *The image of* $\rho \subset$ *the group of isometries of* $\mathbb{R}^k$.

Proof: $N(\rho(x) \cdot y) = N(\alpha(x) y\, x^{-1}) = N(\alpha(x))\, N(y) N(x^{-1}) = N(y)$.

ie $N = 1$ Q. E. D.

THEOREM 3.1. *Let* $\mathrm{Pin}(k)$ *be the kernel of* $N : \Gamma_k \longrightarrow \mathbb{R}^*$, $k \geq 1$, *and let* $O(k)$ *denote the group of isometries of* $\mathbb{R}^k$. *Then* $\rho|\mathrm{Pin}(k)$ *is a surjection of* $\mathrm{Pin}(k)$ *onto* $O(k)$ *with kernel* $\mathbb{Z}_2$, *generated by* $-1 \in \Gamma_k$: *We thus have the exact sequence*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(k) \xrightarrow{\rho} O(k) \longrightarrow 0 .$$

so that Pin(k) is quite large

funny

How [illegible] a finite covering? [illegible]

$(x \mapsto -x) \to$ some elt $\in O(k)$

Proof: We show first that ρ is onto. For this purpose consider $e_1 \in R^k$. We have $N(e_1) = -e_1 e_1 = +1$, and

$$\alpha(e_1)\, e_i\, e_1^{-1} = \begin{cases} -e_i & \text{if} \quad i = 1 \\ e_i & \text{if} \quad i \neq 1 . \end{cases}$$

Thus, $e_1 \in \mathrm{Pin}(k)$, and $\rho(e_1)$ is the reflection in the hyperplane perpendicular to e_1. Applying the same argument to any orthonormal base $\{e_i\}$ in R^q, we see that the unit sphere $\{x \in R^k \,|\, N(x) = 1\}$ is in $\mathrm{Pin}(k)$ whence all the orthogonal reflection in hyperplanes of R^k are in the $\rho\{\mathrm{Pin}(k)\}$. But these are well known to generate $O(k)$. Thus ρ maps $\mathrm{Pin}(k)$ onto $O(k)$. Consider next the kernel of this map, which clearly consists of intersection $\mathrm{Ker}\,\rho \cap \{N(x) = 1\}$. Thus the kernel of $\rho|\mathrm{Pin}(k)$ consists of the multiples $\lambda \cdot 1$, with $N(\lambda 1) = 1$. Thus $\lambda^2 = +1$ which implies $\lambda = \pm 1$.

DEFINITION 3.2. Let $\mathrm{Spin}(k)$ be the subgroup of $\mathrm{Pin}(k)$ which maps onto $SO(k)$ under ρ; $k \geq 1$.

The groups $\mathrm{Pin}(k)$ and $\mathrm{Spin}(k)$ are double coverings of $O(k)$ and $SO(k)$ respectively. As such they inherit the Lie-structure of the later groups. One may also show that these groups are closed subgroups of C_k^* and get at their Lie structure in this way.

since ρ^k semi-linear $(\mathbb{R}^*)^+ \times \mathrm{Pin}(k)$

so ρ^k splits similarly ...

PROPOSITION. Let $\mathrm{Pin}(k)^i = \mathrm{Pin}(k) \cap C_k^i$. Then $\mathrm{Pin}(k) = \bigcup_{i=0,1} \mathrm{Pin}(k)^i$, and, $\mathrm{Spin}(k) = \mathrm{Pin}(k)^0$.

Proof: Let $x \in \mathrm{Pin}(k)$. Then $\rho(x)$ is equal to the composition of a certain number of reflections in hyperplanes : $\rho(x) = R_1 \circ \cdots \circ R_n$. We may choose elements $x_i \in R^k$, such that $\rho(x_i) = R_i$. Hence $x = \pm x_1, \cdots, x_n$ and is therefore either in C_k^0 or in C_k^1. Finally x is in Spin(k) if and only if the number is in the above decomposition of $\rho(x)$ is even. But then $x \in \mathrm{Pin}(k)^0$. The converse is similar.

PROPOSITION 3.6. When $k \geq 2$, the restriction of ρ to Spin(k) is the nontrivial double covering of SO(k).

Proof: It is sufficient to show that $+1, -1$, the kernel of $\rho|\mathrm{Spin}(k)$, can be connected by an arc in Spin(k). Such an arc is given by:

need to have the extra dimensions to move in.

$$\lambda : t \longrightarrow \cos t + \sin t \cdot e_1 e_2 \qquad 0 \leq t \leq \pi .$$

COROLLARY : When $k \geq 2$, Spin(k) is connected, and when $k \geq 3$ simply connected.

This is clear from the fact that SO(k) is connected for $k \geq 2$, and that $\pi_1\{SO(k)\} = \mathbb{Z}_2$ if $k \geq 3$.

We note finally that $\mathrm{Spin}(1) = \mathbb{Z}_2$, while $\mathrm{Pin}(1) = \mathbb{Z}_4$.

in general, if L a $\mathbb{C}$-module, $\bar{L}$ the conjugate module,

$l \otimes (x+iy)$ (1) $L \otimes_{\mathbb{C}} \mathbb{H} =_{\mathbb{C}} L \oplus \bar{L}$ left Cartan $\quad M \otimes_{\mathbb{H}} \mathbb{H}, \ M \oplus M \oplus M \oplus M$

$\to (lx, ly)$ (2) $L \otimes_{\mathbb{R}} \mathbb{C} =_{\mathbb{C}} L \oplus \bar{L}$ right Cartan $\quad M \otimes_{\mathbb{C}} \mathbb{H}, \ M \oplus M$

works

$l \otimes (x+iy) \to ((x+iy)l, (x-iy)l)$

4. <u>Determination of the algebras C_k</u>. In the following we will write $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ respectively for the real, complex and quarternion number-fields. If F is any one of these fields, $F(n)$ will be the full matrix $n \times n$ algebra over F. The following are well known identies among these:

trivial

$$(4.1)\quad \begin{cases} F(n) = \mathbb{R}(n) \otimes_{\mathbb{R}} F, \quad \mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{R}(m) \simeq \mathbb{R}(nm) \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C} \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}(2) \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq \mathbb{R}(4) \end{cases}$$

not natural, isom. with the isomorphism does not preserve $\mathbb{C}$-module structure (with the second factor, "naturally" $\bar{\mathbb{C}}$)

See notes

see face p 13

$\mathbb{H} \otimes \mathbb{H}$ action $\mathbb{R}^4$ natural way

U.d. Wg.

$h h' \cdot x \to h x \bar{h}'$

$\mathbb{H} \otimes \mathbb{H}^{op} \cong \mathbb{R}(4)$, but $\mathbb{H}^{op} \leftrightarrow \mathbb{H}$, $1 \to 1$, $i \to -i$, $j \to -j$, $k \to -k$

To compute the algebras C_k one now proceeds as follows: Let $\bar{C}'_k$ be the universal $\mathbb{R}$-algebra generated by a unit and the symbols e'_i $(i = 1, \ldots, k)$ subject to the relations $(e'_i)^2 = \cancel{+1}$; $e'_i e'_j + e'_j e'_i = 0$, $i \neq j$. Thus $\bar{C}'_k$ may be identified with $C(-Q_k)$.

Apply Prop 4.1 to get these boundary

PROPOSITION 4.1. <u>There exist isomorphisms:</u>

$C'_2 \cong M_2(\mathbb{R})$

$C_2 \cong \mathbb{H}$

$$(4.2)\quad \boxed{\begin{cases} C_k \otimes_{\mathbb{R}} C'_2 \simeq C'_{k+2} \\ C'_k \otimes_{\mathbb{R}} C_2 \simeq C_{k+2} \end{cases}}$$

ungraded $\otimes$

<u>Proof:</u> We let R^k be the space spanned by the (e_i over $\mathbb{R}$) in C_k, and denote by R'^k the space spanned by the e'_i in C'_k.

Consider the linear map $\psi : \mathbb{R}'^{k+2} \longrightarrow C_k \otimes C'_2$ defined by

$$\psi(e'_i) = \begin{cases} e_i \otimes e'_1 e'_2 & 1 \leq i \leq k \\ 1 \otimes e'_i & 1 \leq i \leq 2 \end{cases} .$$

Then it is easily seen that ψ satisfies the universal property (1.1) for C'_k and hence extends to an algebra homomorphism $\psi : C'_{k+2} \longrightarrow C_k \otimes C'_2$. As the map takes basis elements into basis elements and the space in question have equal dimension, it follows that ψ is a bijection. If we now replace the dashed symbols by the undashed ones and apply the same argument we obtain the second isomorphism.

Now it is clear that

$$C_1 \simeq \mathbb{C} \quad , \quad C'_1 \simeq \mathbb{R} + \mathbb{R}$$

$$C_2 \simeq \mathbb{H} \quad , \quad C'_2 \simeq \mathbb{R}(2) \quad .$$

Hence repeated application of (4.1) and (4.2) yields the following table: (See Table 1 on page 12.)

Note that (4.1) implies $C_4 = C'_4$; $C_k \simeq C_{k+4} \otimes C_4$; $C_{k+8} \simeq C_k \otimes C_8$, further $C_8 = \mathbb{R}_{16}$, whence if $C_k = F(m)$ then, $C_{k+8} \simeq F(16m)$. Thus both columns are in a quite definite sense of period 8. If we move up eight steps, the field is left unaltered, while the dimension is multiplied by 16. Note also the considerably simpler behavior of the complexifications of these algebras, which of course can be interpreted as the Clifford algebra of Q_k – over the complex-numbers. Over the complex field, the periodicity starts with 2 .

TABLE I

$e_i \in Ch \to ie_i \in Ch'$

k	C_k	C'_k	$C_k \underset{\mathbb{R}}{\otimes} \mathbb{C} = C'_k \underset{\mathbb{R}}{\otimes} \mathbb{C}$
1	$\mathbb{C}$	$\mathbb{R} + \mathbb{R}$	$\mathbb{C} + \mathbb{C}$
2	$\mathbb{H}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$
3	$\mathbb{H} + \mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{C}(2) + \mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) + \mathbb{H}(2)$	$\mathbb{C}(4) + \mathbb{C}(4)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
7	$\mathbb{R}(8) + \mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{C}(8) + \mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$

$M_2(\mathbb{C})$

so $A_1 \subset$ of SL_2?

so $SL_2\mathbb{C} \hookrightarrow M_{8^n}(\mathbb{C})$

diagonal?

$\mathbb{C}^n \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong \mathbb{C}^{2n}$

5. Clifford modules. We will now describe the set of $\mathbb{R}$-and $\mathbb{C}$-modules for the algebra's C_k. We write $M(C_k)$ for the free abelian group generated by the irreducible $\mathbb{Z}_2$-graded C_k-modules, and $N(C_k^0)$ for the corresponding group generated by the ungraded C_k^0-modules. The corresponding objects for the complex algebras $C_k \underset{\mathbb{R}}{\otimes} \mathbb{C}$ are denoted by $M^c(C_k)$ and $N^c(C_k^0)$.

PROPOSITION 5.1. Let $R : M \rightsquigarrow M^0$ be the functor which assigns to a graded C_k-module, $M = M^0 + M'$ the ungraded C_k^0-module M^0. Then R induces isomorphisms

(5.1*)

$$M(C_k) \simeq N(C_k^0) \qquad (5.1)$$

$\simeq M(C_{k-1})$ where C_{k-1} is considered ungraded

Proof: If M^0 is an (ungraded!) C_k^0-module, let

$$S(M^0) = C_k \underset{C_k^0}{\otimes} M^0 .$$

The left action of C_k on C_k then defines $S(M^0)$ as a graded C_k-module. We now assert that $S \circ R$ and $R \circ S$ are naturally isomorphic to the identity. In the first case the isomorphism is induced by the "module-map" $C_k \otimes M^0 \longrightarrow M$, while in the second case the map $M^0 \longrightarrow 1 \otimes M^0$ induces the isomorphism.

We of course also have the corresponding formula:

all small

$$M^c(C_k) \simeq N^c(C_k^0) \qquad (5.2)$$

$M(C_k)$ (graded) $\cong M(C_{k-1})$ ungraded

PROPOSITION 5.2. Let $\varphi : R^k \longrightarrow C_{k+1}^0$ be defined by $\varphi(e_i) = e_i e_{k+1}$, $i = 1, \cdots, k$. Then φ extends to yield an isomorphism $C_k \simeq C_{k+1}^0$.

$e_i e_j \to e_i e_j$

$$\begin{array}{ccc} C_k \cong C_{k+1}^0 & & C_{k+1} \\ \downarrow & & \\ C_{k+1} \cong C_{k+2}^0 & \subset & C_{k+2} \end{array}$$

$e_{i+1} \cdot e_i \cdot e_{i+1} \cdot e_i = -1$

<u>Proof:</u> $\varphi(e_i)^2 = \cancel{e_i e_{k+1} \cdot e_i e_{k+1}} = -1$. Hence φ extends. As it maps basis elements onto basis elements the extension is an isomorphism.

In view of these two propositions and Table 1, we may now write down the group $M(C_k)$ etc., explicitly. This is done in Table 2, where we also tabulate the following quantities:

can compute $M(C_k)$ because of (5.1) 5.2, and these that say irreducible modules are minimal left ideals.

Let $i : C_k \longrightarrow C_{k+1}$ be the inclusion which extends the inclusion $R_k \longrightarrow R_{k+1}$, let $i^* : M(C_{k+1}) \longrightarrow M(C_k)$ be the induced homombrphism, and set A_k = cokernel of i^*.

$$M(C_{k+1}) \xrightarrow{i^*} M(C_k) \longrightarrow A_k \to 0$$

to compute A_k, it is important to have the commutativity

we constructed on bottom of p. 14

Similarly define A^c_k as $M^c(C_k)/i^*\{M^c(C_{k+1})\}$ and finally define $a_k[a^c_k]$ as the $\mathbb{R}$ - $[\mathbb{C}]$ dimension of M^0 when M is an irreducible graded module for C_k, $[C_k \underset{\mathbb{R}}{\otimes} \mathbb{C}]$.

= dim of modules/grps of C_{k-1}

Most of the entries in Table 2 follow directly from Table 1, because the algebras $\mathbb{F}(n)$ are simple and hence have only one class or irreducible modules, the one given by the action of $\mathbb{F}(n)$ on the n-tuples of elements in $\mathbb{F}$. The only entries which still need clarification are therefore, A_{4n}, and A^c_{2n}.

uses dimension argument

Before explaining this entry observe that if $M = M^0 + M^1$, then $M^* = M^1 + M^0$ - i.e. the module obtained from M by merely interchanging labels - is again a graded module. This operation therefore induces an involution on $M(C_k)$ and $M^c(C_k)$ which we again denote by $*$.

$\mathbb{R}^2 \longrightarrow \mathbb{C}$

TABLE 2.

k	C_k	$M(C_k)$	A_k	a_k	$M^c(C_k)$ (graded)	A^c_k	
1	$\mathbb{C}(1)$	$\mathbb{Z}$	$\mathbb{Z}_2$	1	$\mathbb{Z}$	0	1
2	$\mathbb{H}(1)$	$\mathbb{Z}$	$\mathbb{Z}_2$	2	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	1
3	$\mathbb{H}(1)+\mathbb{H}(1)$	$\mathbb{Z}$	0	4	$\mathbb{Z}$	0	2
4	$\mathbb{H}(2)$	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	4	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	2
5	$\mathbb{C}(4)$	$\mathbb{Z}$	0	8	$\mathbb{Z}$	0	4
6	$\mathbb{R}(8)$	$\mathbb{Z}$	0	8	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	4
7	$\mathbb{R}(8)+\mathbb{R}(8)$	$\mathbb{Z}$	0	8	$\mathbb{Z}$	0	8
8	$\mathbb{R}(16)$	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	8	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	8

$$M_{k+8} \simeq M_k\,, \quad A_{k+8} = A_k\,, \quad a_{k+8} = 16\,a_k$$

$$M^c_{k+2} \simeq M^c_k\,, \quad A^c_{k+2} = A^c_k\,, \quad a^c_{k+2} = 2a^c_k\,.$$

PROPOSITION 5.3. Let x and y be the classes of the two distinct irreducible graded modules in $M(C_{4n})$. Then

(5.3) $$x^* = y\,, \quad y^* = x\,.$$

COROLLARY. $A_{4n} = \mathbb{Z}$. Indeed if z generates $M(C_{4n})$, then $z^* = z$ as there is only one irreducible graded module for C_{4n+1}. Hence, as $(i^*z)^* = i^*(z^*)$ we see that $i^*z = x + y$, by a dimension count.

Proposition 5.3 follows from the following lemma which is quite straight-forward and will be left to the reader.

LEMMA 5.1. Let $y \in R_k$, $y \neq 0$ and $A(y)$ equal to the inner automorphism of C_k induced by y. Thus $A(y) \cdot w = ywy^{-1}$. We also write $A(y)$ for the induced automorphism on $M(C_k)$. Similarly $A^0(y)$ denotes the restriction of $A(y)$ to C_k^0, as well as the induced automorphism on $N(C_k^0)$. Then we have

$$
(5.4)\qquad
\begin{aligned}
&(a)\quad A(y)\cdot x = x^* \qquad x \in M(C_k)\\
&(b)\quad A^0(y)\cdot R(x) = R(x^*),\\
&(c)\quad A^0(e_k)\,\varphi(w) = \varphi\{\alpha(w)\}.
\end{aligned}
$$

Here $R : M(C_k) \longrightarrow N(C_k^0)$ is the functor introduced earlier, and $\varphi : C_{k-1} \longrightarrow C_k^0$, the map introduced in Proposition 4.2, while α is the canonical automorphism of C_k.

It now follows from these isomorphisms, that $*$ on $M(C_{4n})$ corresponds to the action of α on the ungraded modules of C_{4n-1}. Now the center of C_{4n-1} is spanned by 1 and $w = e_1 \cdots e_{4n-1}$. Further $w^2 = +1$. Hence the projection of C_{4n-1} on the two ideals which make up C_{4n-1} is $(1+w)/2$ and $(1-w)/2$. Hence α interchanges these, and therefore clearly interchanges the two irreducible C_{4n+1} modules.

Finally, the evoluation $A_{2n}^c = Z$ proceeds in an entirely analogous fashion.

$$
\begin{tikzcd}
C_k \arrow[r, "A(e_1)"] & C_k \\
C_k^0 \arrow[u] \arrow[r, "A(e_1)"] & C_k^0 \arrow[u] \\
C_{k-1} \arrow[u, "\varphi"] \arrow[r, "\alpha"] & C_{k-1} \arrow[u, "\varphi"']
\end{tikzcd}
$$

6. The multiplicative properties of the Clifford modules.

If M and N are graded C_k and C_ℓ modules, respectively, then their graded tensor product $M \hat{\otimes} N$ is in a natural way a graded module of $C_k \hat{\otimes} C_\ell$. By definition $(M \hat{\otimes} N)^0 = M^0 \otimes N^0 + M^1 \otimes N^1$ and $(M \hat{\otimes} N)^1 = M^0 \otimes N^1 + M^1 \otimes N^0$, the action of $C_k \hat{\otimes} C_\ell$ on $M \hat{\otimes} N$ being given by :

$$(6.1) \qquad (x \otimes y) \cdot (m \otimes n) = (-1)^{qi}(x \cdot m) \otimes (y \cdot n), \qquad y \in C_\ell^q, \quad m \in M^i \quad (p, i = 0, 1).$$

We also have the isomorphism $\varphi_{k,\ell} : C_{k+\ell} \longrightarrow C_k \hat{\otimes} C_\ell$ defined by the linear extensions of the maps

$$\varphi_{k,\ell}(e_i) = \begin{cases} e_i \otimes 1 & 1 \leq i \leq k \\ 1 \otimes e_{k+i} & k < i \leq n \end{cases} .$$

The operation $(M, N) \longrightarrow M \hat{\otimes} N \longrightarrow \varphi^*_{k,\ell}(M \hat{\otimes} N)$ is easily seen to give rise to a pairing

$$M(C_k) \underset{\mathbb{Z}}{\otimes} M(C_\ell) \longrightarrow M(C_{k+\ell})$$

and thus induces a $\mathbb{Z}$-graded ring structure on the direct sum $M_* = \sum_0^\infty M(C_k)$. We denote this product by $(u, v) \longrightarrow u \cdot v$. It is clearly associative.

PROPOSITION 6.1. The following formulae are valid for $u \in M(C_k)$ $v \in M(C_\ell)$

$$(6.1) \qquad (u \cdot v)^* = u \cdot v^*$$

$$(6.2) \qquad u \cdot v = \begin{cases} v \cdot u & \text{if } k\ell \text{ is even} \\ (v \cdot u)^* & \text{if } k\ell \text{ is odd} \end{cases}$$

so that $k \cdot \ell$ is even, but not divisible by 4

is this true? possibly, ex of the proof is bad

yes see p. 17

(6.3) If $i^* : M(C_k) \longrightarrow M(C_{k-1})$ is the restricted homomorphism, as defined in ( ), then

$$(u \cdot i^* v) = i^*(u \cdot v) \qquad q \geq 1 .$$

The formulae (6.1) and (6.3) follow immediately from the definitions.

Proof of 6.2. We have the diagram :

$$\begin{tikzcd} C_k \hat{\otimes} C_\ell \arrow[d, "T"] & C_{k+\ell} \arrow[l, "\varphi_{k,\ell}"'] \arrow[dl, "\varphi_{\ell,k}"] \\ C_\ell \hat{\otimes} C_k & \end{tikzcd}$$

$1 \otimes e_\ell$ $\quad e_{k+1}$ $\quad e_1$ $\quad e_{\ell+1}$ $\quad e_1 \otimes 1$

where T is the isomorphism $x \otimes y \longrightarrow (-1)^{pq} y \otimes x$, $x \in C_k^p$, $y \in C_\ell^q$.

Now the composition $\varphi_{k,\ell}^{-1} \circ T \circ \varphi_{\ell,k} : C_{k+\ell} \longrightarrow C_{k+\ell}$ is an automorphism σ of $C_{k+\ell}$, which clearly is the linear extension of the map which permutes the first k elements of the basis $\{e_i\}$ with the last ℓ elements

what is true is that a permutation of two elts may be effected by any $x \to \alpha(x)\, y\, x^{-1}$ ... $= -xyx^{-1}$

$$\sigma(e_i) = \begin{cases} e_{\ell+i} & 1 \le i \le \ell \\ e_{i-\ell} & \ell < i \le k+\ell \end{cases}.$$

?

Thus σ is the composition of $k\ell$ inner automorphisms by elements in $R_{k+\ell} = 0$. It follows therefore from (5.4(a)) that the effect of σ on $M(C_k)$ is equal to the effect of the operation (*) applied $k\ell$ times. If we combine this with the fact that $T^*(N \hat{\otimes} M) \simeq M \hat{\otimes} N$, whence

$$\varphi^*_{\ell,k}(N \hat{\otimes} M) = \sigma^* \circ (\varphi_{k,\ell})^* \cdot (M \hat{\otimes} N),$$

we obtain the desired formula.

COROLLARY 1. *Let $\lambda \in M(C_8)$ be the class of an irreducible graded module of C_8. Then multiplication by λ induces an isomorphism:* $M(C_k) \xrightarrow{\simeq} M(C_{k+8})$.

Proof: This follows from our table of the a_k, in all cases except when $k = 4n$. In that case let x, y be the generators corresponding to the two irreducible graded modules of C_k. Then we know that $x^* = y$. Now $\lambda \cdot x \in M(C_{k+8})$ is the class of one of the irreducible graded modules of C_{k+8} by a dimension count. Hence by (6.2) $\lambda \cdot y = \lambda(x^*) = (\lambda x)^*$ corresponds to the other generator.

COROLLARY 2. *The image of* $i^* : M_* \longrightarrow M_*$ *is an ideal, and hence the quotient ring* $A_* = \sum_0^\infty A_k$ *inherits a ring structure from* M_*.

This follows from (6.3). The element λ above projects into a class -- again called λ -- in A_8, and we clearly have:

PROPOSITION 6.2. *Multiplication by* λ *induces an isomorphism* $A_k \simeq A_{k+8}$, $k = 0 \ldots$.

The complete ring-structure of A_* is given by:

THEOREM 6.1. A_* *is the anticommutative graded ring generated by a unit* $1 \in A_0$, *and by elements* $\xi \in A_1$, $\mu \in A_4$, $\lambda \in A_8$ *with relations*: $2\xi = 0$, $\xi^3 = 0$, $\mu^2 = 4\lambda$.

Proof: As $A_1 = \mathbb{Z}_2$, it is clear that $2\xi = 0$. From the fact that $a_1 = 1$, and $a_2 = 2$, we conclude that ξ_1^2 generates A_2. There remains the computation of μ^2. To settle this case we introduce a notion which will be of use later in any case. Let $k = 4n$, and let $\omega = \omega = e_1 \cdots e_{4n}$. Then as we already remarked, the center of $C_k^{\oplus}$ is generated by 1 and ω, whence, as $\omega^2 = +1$, the projection of $C_k^{\oplus}$ on its two ideals is given by $(1 \mp \omega)/2$. It follows that if M is an irreducible graded C_k-module, then ω acts on M^0 as the scalar $\epsilon = \pm 1$. In general we call a graded module for C_k an ϵ-module, $(\epsilon = \pm 1)$ if ω acts as ϵ on M^0.

so if $\omega x = \epsilon x$, then $e_i \epsilon x = -\omega e_i x$

$e_i \omega x = -\omega e_i x$ i.e.

let $x \in M^1$, then $e_i x \in M^0$

Now because $e_i \omega = -\omega e_i$, it follows immediately that if M is an ϵ-module, then M^* is a $(-\epsilon)$-module - i.e., ω acts as $-\epsilon$ on M^1, and finally, that if M is an ϵ-module and M' an ϵ'-module for C_k then $M \hat{\otimes} M'$ is an $\epsilon\epsilon'$-module for C_{2k}.

With this understood, let μ be the class of an irreducible C_4-module M in A_4. Then M is of type ϵ. Hence $M \hat{\otimes} M$ is of type $\epsilon^2 = +1$ in C_8. Now if $\lambda \in A_8$ is chosen as the class of the irreducible $(+1)$-module W of C_8 it follows that $M \hat{\otimes} M \cong 4W$ by a dimension count, and so finally that $\mu^2 = 4\lambda$.

The corresponding propositions for the complex modules are clearly also valid. Thus we may define M_*^c and A_*^c, and now already the generator μ^c corresponding to an irreducible $M^c(C_2)$ module yields periodicity. In fact the following is checked readily.

THEOREM 6.2. _The ring_ A_*^c _is isomorphic to the polynomial ring_ $\mathbb{Z}[\mu^c]$.

[illegible] $\frac{1+\omega}{2}$ $\frac{1-\omega}{2}$ [illegible] $\frac{1+i\omega}{2}$ $\frac{1-i\omega}{2}$

We consider again the element $\omega = e_1 \cdots e_k \in C_k$. For $k = 2\ell$ we have $\omega^2 = (-1)^\ell$. Hence if M is an irreducible complex graded C_k-module then ω acts on M^0 as the complex scalar $\epsilon = \pm i^\ell$. We call a complex graded C_k-module an ϵ-module if ω acts as ϵ on M^0. Let $\mu_\ell^c \in M^c(C_{2\ell})$ denote the generator given by an irreducible i^ℓ-module. Then $\mu_\ell^c = (\mu)^\ell$ when $\mu_1^c = \mu^c$.

the center splits [illegible]

Comparing our conventions in the real and complex cases we see that if M is a real ϵ-module for C_{4n} then $M \otimes_R C$ is a complex $(-1)^n \epsilon$-module for C_{4n}. Now we choose $\mu \in A_4$ to be the class of an irreducible (-1)-module. Then in the homomorphism $A_* \longrightarrow A_*^c$ given by complexification, $\mu \longrightarrow 2(\mu^c)^2$.

[illegible] an $i^2 = -1$ module

From two equations just before, derive $\lambda \to (\mu^c)^4$ under complexification

7. Relation with Grassmann algebra. (Michael Atiyah).

[illegible]

Let E be a Euclidean space of dimension k. We write C(E) for the Clifford algebra of $-Q$ where Q is the quadratic form of the Euclidean metric, and then define Pin(E), Spin(E) and $\rho : \mathrm{Pin}(E) \longrightarrow O(E)$ as in Section 3. As already observed we have an algebra filtration

$$R = C(E)_0 \subset C(E)_1 \subset \cdots \subset C(E)_k = C(E) ,$$

and the associated graded algebra may be identified with the exterior algebra $\Lambda(E)$. Thus

$$C(E)_p / C(E)_{p-1} \cong \Lambda^p(E) .$$

$\rho(\omega)$: $x \in \mathbb{R}^k \to \alpha(\omega) x \omega^{-1}$

$(-1)^k e_1 \cdots e_k \cdot x \cdot (-1)^k e_k \cdots e_1$ where x is some e_i

$= -x$

LEMMA 7.1. <u>The two elements $\omega \in \mathrm{Pin}(E)$ such that</u> $\rho(\omega) = -1$ <u>define generators of</u> $C(E)_k / C(E)_{k-1}$.

<u>Proof:</u> If $e_1, \cdots, e_k$ is an orthonormal basis of E then $\omega = \pm e_1 e_2 \cdots e_k$ and these are generators as required. $\omega^{-1} = (-1)^k e_k \cdot e_{k-1} \cdots e_1$

Hence if E is <u>oriented</u> then there is a canonical element, denoted by $\omega(E)$, such that i.e. orienting amounts to choosing such an element

(i) $\rho(\omega(E)) = -1$

(ii) $\omega(E)$ defines a <u>positive</u> generator of $\Lambda^k(E)$. very mistaken here (wh ...

Putting $C(E)^{k-p} = \omega C(E)_p$ we get another filtration of $C(E)$ by subspaces:

$$\omega\mathbb{R} = C(E)^k \subset C(E)^{k-1} \subset \cdots \subset C(E)^0 = C(E) .$$

In terms of an orthonormal basis $e_1, e_2, \cdots, e_k$ we have

$C(E)_p$ = space spanned by $e_{i_1} e_{i_2} \cdots e_{i_r}$ with $r \leq p$

$C(E)^p$ = space spanned by $e_{i_1} e_{i_2} \cdots e_{i_r}$ with $r \geq p$.

= space spanned by $e_{i_1} \cdots e_{i_r}$ with $r = p$

Putting ${}^pC(E) = C(E)_p \cap C(E)^p$ we get a decomposition $C(E) = \bigoplus_p {}^pC(E)$, isomorphisms ${}^pC(E) \cong \Lambda^p(E)$, and hence an isomorphism $C(E) \cong \Lambda(E)$. This isomorphism is a natural isomorphism of graded vector spaces (not of algebras), $C(E)$ and $\Lambda(E)$ being regarded as functors of the oriented Euclidean space E.

Let $\pi_0 : C(E) \longrightarrow C(E)_0 = R$ be the projection given by this decomposition, and define an inner product in $C(E)$ by

$$\langle x, y\rangle = \pi_0(x\bar{y}) .$$

In terms of an orthonormal oriented basis $e_1, \cdots, e_k$ we find

$$\begin{aligned} \langle e_{i_1} \cdots e_{i_r}, e_{j_1} \cdots e_{j_s}\rangle &= 0 \quad \text{if} \quad (i_1 \cdots i_r) \neq (j_1 \cdots j_s) \\ &= 1 \quad \text{if} \quad (i_1 \cdots i_r) = (j_1 \cdots j_s) . \end{aligned}$$

Hence this is positive definite. Moreover the decomposition $C(E) = \oplus^p C(E)$ is orthogonal with respect to this inner product.

For any $x \in E$ Clifford multiplication by x (on the left) gives a map

$${}^pC(E) \longrightarrow {}^{p+1}C(E) \oplus {}^{p-1}C(E)$$

all terms with x in $\rightarrow$ ${}^{p-1}C(E)$, those without $\rightarrow$ ${}^{p+1}C(E)$

and hence a map

call these d_v, δ_v then

$$\Lambda^p(E) \longrightarrow \Lambda^{p+1}(E) + \Lambda^{p-1}(E) .$$

$x \rightarrow (d_v - \delta_v)x$, since $(d_v - \delta_v)^2 x = -\|v\|^2 x$, a Clifford module,

The first component of this is just the exterior product $y \longrightarrow x \wedge y$. The second component will be denoted by $y \longrightarrow x \vee y$: it is called the interior product. It maps $\Lambda^p(E) \longrightarrow \Lambda^{p-1}(E)$.

with even, odd grading

LEMMA 7.2. For $x \in E$, $y \in \Lambda^p(E)$, $z \in \Lambda^{p-1}(E)$

$$\langle x \vee y, z\rangle = - \langle y, x \wedge z\rangle .$$

i.e. $\vee$ depends on the metric

(and defined on the dual space)

Proof: Take an orthonormal base with $x = e_1$ and let $y = e_1 a + b$ $\quad z = e_1 c + d$. Then $x \vee y = -a$, $x \wedge z = e_1 d$ so that $\langle x \vee y, z \rangle = -\langle a, d \rangle = -\langle y, x \wedge z \rangle$.

since $\langle a, e_1 \rangle = 0$, $\langle -a, e_1 c + d \rangle$ $\quad = -\langle e_1 a + b, e_1 d \rangle$ $(= -\langle a, d \rangle)$
and b orthogonal to e_1 term

If we complexify $C(E)$, $\Lambda(E)$ and take the induced Hermitian inner products then the above identities still hold.

Let $k = 2\ell$ and consider $C(E) \otimes_{R} C$ (or $\Lambda(E) \otimes_{R} C$) as a left $C(E)$-module (the left regular representation). Since $\omega(E)$ is in the center of $C^0(E)$, satisfies $\omega(E)^2 = (-1)^\ell$ and

$$x\,\omega(E) = -\omega(E)x \qquad\qquad x \in E,$$

so if Λ^0 is an ϵ-module, Λ^1 is a $-\epsilon$ module.

it follows that the eigenspaces of $\omega(E)$ will define a Z_2-grading on $\Lambda(E) \otimes_{R} C$. More precisely see p. 45

$$\Lambda(E) \otimes_{R} C = \Lambda^{(0)} + \Lambda^{(1)}$$ with $\dim_{\mathbb{C}} = 2^{2\ell} = 2^k$

where $\Lambda^{(0)}$ corresponds to the eigenvalue i^ℓ, $\Lambda^{(1)}$ to the eigenvalue $-i^\ell$. We shall refer to this Z_2-graded complex module of $C(E)$ as the ω-regular module and denote it by $\Lambda_\omega(E)$. From its definition we see that $\Lambda^{(0)}$ is an i^ℓ-module in the sense of Section 6 and hence, by a dimension count, we deduce

PROPOSITION 7.1. If $\dim E = 2\ell$ then the ω-regular module of $C(E)$ defines the element $2^\ell \mu_\ell^c \in \mathbb{A}^c$.

μ_ℓ^c has dim 2^ℓ

The following is left to the reader:

PROPOSITION 7.2. <u>If</u> $E = \oplus E_i$, $\dim E_i = 2\ell_i$ <u>then we have a natural isomorphism of</u> $\mathbb{Z}_2$<u>-graded vector spaces:</u> $\Lambda_\omega(E) \cong \otimes_i \Lambda_\omega(E_i)$. *

Write $N^p = \Lambda^p(E) \otimes_R C$ for brevity. Then we see that $N^p + N^{k-p}$ is invariant under ω $(p \neq \ell)$ with eigenspaces

$\omega(x \oplus i^{-\ell}\omega x) = \omega^2 x \oplus i^{-\ell}\omega x = i^{\ell}(x \oplus i^{-\ell}\omega x)$

$$\Lambda_p^{(0)} = \Lambda^{(0)} \cap (N^p \oplus N^{k-p}) = \{x \oplus i^{-\ell}\omega x \mid x \in N^p\}$$

$$\Lambda_p^{(1)} = \Lambda^{(1)} \cap (N^p \oplus N^{k-p}) = \{x \oplus -i^{-\ell}\omega x \mid x \in N^p\} \quad .$$

N^ℓ is invariant under ω and so decomposes into

$$\Lambda_\ell^{(0)} = \{x \in N^\ell \mid \omega(x) = i^\ell x\} \quad ,$$

$$\Lambda_\ell^{(1)} = \{x \in N^\ell \mid \omega(x) = -i^\ell x\} \quad .$$

Hence in the Grothendieck ring $RSO(k)$ we have (for $E = R^k$)

$$\Lambda^{(0)} - \Lambda^{(1)} = \Lambda_\ell^{(0)} - \Lambda_\ell^{(1)} \quad .$$

Now take $\ell = 1$, $E = R^2$ with basis e_1, e_2. Then $\Lambda_\ell^{(0)}$ is generated by $e_1 - ie_2$ and $\Lambda_\ell^{(1)}$ by $e_1 + ie_2$. If $SO(2)$ is represented in the usual way by rotations through θ

$$e_1 \longrightarrow e_1 \cos\theta + e_2 \sin\theta$$

$$e_2 \longrightarrow -e_1 \sin\theta + e_2 \cos\theta \quad ,$$

then $e_1 - ie_2 \longrightarrow (\cos\theta + i\sin\theta)(e_1 - ie_2)$. Hence $e_1 - ie_2$ is a weight vector with weight $+x$ in the usual notation. Similarly $e_1 + ie_2$ has weight $-x$, and so (for $\ell = 1$) we have

$$\mathrm{ch}(\Lambda^{(0)} - \Lambda^{(1)}) = e^{x} - e^{-x} .$$

From this and (7.2) we deduce

PROPOSITION 7.3. <u>Let</u> $\Lambda^{(0)} + \Lambda^{(1)}$ <u>denote the ε-regular module of the Clifford algebra</u> C_k $(k = 2\ell)$. <u>Then regarding these as</u> $SO(k)$<u>-modules we have</u>

$$\mathrm{ch}(\Lambda^{(0)} - \Lambda^{(1)}) = \prod_{i=1}^{\ell} (e^{x_i} - e^{-x_i})$$

<u>where we use the Borel-Hirzebruch formalism.</u>

<u>Remark:</u> Since $\prod_{i=1}^{\ell} (e^{x_i} - e^{-x_i}) = 2^{\ell} \prod x_i$ + higher terms we get from (7.3) another proof of (7.1).

8. <u>Sequences of bundles.</u> In this and succeeding sections we shall show how one can give a Grothendieck-type definition for the relative groups $K(X, Y)$. This will apply equally to real or complex vector bundles and we will just refer to vector bundles. For simplicity we shall work in the category of finite CW-complexes (and pairs of complexes).

If $Y \subset X$ we shall consider the set $\mathcal{C}_n(X, Y)$ of sequences nothing (vector) over X' but see later using contractibility /

$$E = (0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \longrightarrow E_1 \xrightarrow{\sigma_1} E_0 \longrightarrow 0)$$

where the E_i are vector bundles on X, the σ_i are homomorphisms defined on Y and the sequence~~s~~ is exact on Y. An isomorphism $E \longrightarrow E'$ in $\mathcal{C}_n$ will mean a diagram

$$\begin{tikzcd}
\arrow[r] & E_i \arrow[r, "\sigma_i"] \arrow[d] & E_{i-1} \arrow[r] \arrow[d] & {} \\
\arrow[r] & E'_i \arrow[r, "\sigma'_i"] & E'_{i-1} \arrow[r] & {}
\end{tikzcd}$$

in which the vertical arrows are isomorphisms on X and the squares commute on Y.

An elementary sequence in $\mathcal{C}_n$ is one in which

$$\begin{array}{lll} E_i = E_{i-1}, & \sigma_i = 1 & \text{for some } i \\ E_j = 0 & & \text{for } j \neq i,\ i-1 . \end{array}$$

The direct sum $E \oplus F$ of two sequences is defined in the obvious way. We consider now the following equivalence relation:

DEFINITION 8.1. $E \sim F \iff$ there exist elementary sequences $P^i, Q^j \in \mathcal{C}_n$ so that

$$E \oplus P^1 \oplus \cdots \oplus P^r \cong F \oplus Q^1 \oplus \cdots \oplus Q^s .$$

In other words this is the equivalence relation generated by isomorphism and addition of elementary sequences. The set of equivalence classes will be denoted by $L_n(X, Y)$. The operation $\oplus$ induces on L_n an abelian semi-group structure. If $Y = \emptyset$ we write $L_n(X) = L_n(X, \emptyset)$.

If $E \in \mathcal{C}_n$ then we can consider the sequence in $\mathcal{C}_{n+1}$ obtained from E by just defining $E_{n+1} = 0$. In this way we get inclusions

$$\mathcal{C}_1 \longrightarrow \mathcal{C}_2 \longrightarrow \cdots \longrightarrow \mathcal{C}_n \longrightarrow$$

and we put $\mathcal{C} = \mathcal{C}_\infty = \varinjlim \mathcal{C}_n$. These induce homomorphisms

$$L_1 \longrightarrow L_2 \longrightarrow \cdots \longrightarrow L_n \longrightarrow$$

and it is clear that

$$L = L_\infty = \varinjlim L_n$$

is obtained from $\mathcal{C}$ by an equivalence relation as above applied now to sequences of finite but unbounded length.

LEMMA 8.1. *Let E, F be vector bundles on X and $f : E \longrightarrow F$ a monomorphism on Y. Then if $\dim F > \dim E + \dim X$, f can be extended to a monomorphism on X and any two such extensions are homotopic* rel. Y.

Proof: Consider the fibre bundle $\mathrm{Mon}(E, F)$ on X whose fibre at $x \in X$ is the space of all monomorphisms $E_x \longrightarrow F_x$. This fibre is homeomorphic to $GL(n)/GL(n - m)$ where $n = \dim F$,

map $GL(n) \to \mathrm{Mon}(E, F)$ by assuming E a subspace of F and taking the map to take A to $E \to A(E)$

$m = \dim E$, and so it is $(n - m - 1)$-connected. Hence cross-sections can be extended and are all homotopic if

$$\dim X \leq n - m - 1 = \dim F - \dim E - 1 .$$

may be extended if $\dim X \leq n - m$ (because obstruction is in π_{n-m-1}(fibre))

But a cross-section of $\mathrm{Mon}(E, F)$ is just a global monomorphism $E \longrightarrow F$.

LEMMA 8.2. $L_n(X, Y) \longrightarrow L_{n+1}(X, Y)$ <u>is an isomorphism</u> for $n \geq 1$.

<u>Proof:</u> Let $\bar{\mathcal{C}}_{n+1}$ denote the subset of $\mathcal{C}_{n+1}$ consisting of sequences E such that

$$(1) \qquad \dim E_n > \dim E_{n+1} + \dim X .$$

If $n \geq 1$ then given any $E \in \mathcal{C}_{n+1}$ we can add an elementary sequence to it so that it will satisfy (1). Hence $\bar{\mathcal{C}}_{n+1} \longrightarrow L_{n+1}$ is surjective. Now let $E \in \bar{\mathcal{C}}_{n+1}$, then by Lemma 8.1, σ_{n+1} can be extended to a monomorphism σ'_{n+1} on the whole of X. Put $E'_n = \mathrm{Coker}\ \sigma'_{n+1}$, let P denote the elementary sequences with $P_{n+1} = P_n = E_{n+1}$, and let $0 \to E_{n+1} \xrightarrow{\text{monom}} E_n \to \mathrm{coker}\ \sigma_{n+1} \to 0$

$$E' = (0 \longrightarrow E'_n \xrightarrow{\rho'_n} E_{n-1} \xrightarrow{\sigma_{n-1}} E_{n-2} \longrightarrow \cdots \xrightarrow{\sigma_1} E_0 \longrightarrow 0) ,$$

where ρ'_n is defined by the commutative diagram on Y :

$$\begin{array}{ccc} E_n & \longrightarrow & E'_n \\ & \searrow^{\sigma_n} & \downarrow{\rho'_n} \\ & & E_{n-1} \end{array} .$$

A splitting of the exact sequence on X Same sequences are defined by adding identical [illegible] on X

$$0 \longrightarrow E_{n+1} \xrightarrow{\sigma'_{n+1}} E_n \longrightarrow E'_n \longrightarrow 0$$

then defines an isomorphism in $\mathcal{C}_{n+1}$

$$P \oplus E' \cong E .$$

If σ''_{n+1} is another extension of σ_{n+1} leading to a sequence E'', then by Lemma 8.1, $E'_n \cong E''_n$ and this isomorphism can be taken to extend the given one on Y, i.e., the diagram

$$\begin{tikzcd} E'_n \arrow[r, "\rho'_n"] \arrow[d] & E_{n-1} \arrow[d, "1"] \\ E''_n \arrow[r, "\rho''_n"] & E_{n-1} \end{tikzcd}$$

commutes on Y. Hence $E' \cong E''$ in $\mathcal{C}_n$ we have a well-defined map $E \longrightarrow E'$ from the isomorphism classes in $\bar{\mathcal{C}}_{n+1}$ to the isomorphism classes in $\mathcal{C}_n$. Moreover, if

$$Q = 0 \longrightarrow Q_{n+1} \longrightarrow Q_n \longrightarrow 0 \ , \quad R = 0 \longrightarrow R_i \longrightarrow R_{i-1} \longrightarrow 0 \qquad (i \leq n)$$

are elementary sequences, then

$$(E \oplus Q)' \cong E' , \quad (E \oplus R)' \cong E' \oplus R .$$

Hence the class of E' in L_n depends only on the class of E in L_{n+1}. Since $\bar{\mathcal{C}}_{n+1} \longrightarrow L_{n+1}$ is surjective it follows that $E \longrightarrow E'$ induces a map $L_{n+1} \longrightarrow L_n$. From its construction it is immediate that note since all vector bundles split topologically, exact sequences give rise to trivial members of $L_n(X,Y)$. In general, members of $L_n(X,Y)$ are trivial over Y.

its composition in either direction with $L_n \longrightarrow L_{n+1}$ is the identity, and this completes the proof.

From Lemma 8.2 we deduce, by induction on n, and then passing to the limit:

PROPOSITION 8.1. The homomorphisms $L_1(X,Y) \longrightarrow L_n(X,Y)$ are ~~mono~~isomorphisms for $1 \le n \le \infty$.

9. Euler characteristics.

DEFINITION 9.1. An "Euler characteristic" for $\mathcal{C}_n$ is a natural homomorphism

$$\chi : L_n(X,Y) \longrightarrow K(X,Y)$$

which for $Y = \emptyset$ is given by

$$\chi(E) = \sum_{i=0}^{n} (-1)^i E_i \quad .$$

Remark: It is clear that, if $Y = \emptyset$, $E \longrightarrow \Sigma(-1)^i E_i$ gives a well-defined map $L_n(X) \longrightarrow K(X)$.

LEMMA 9.1. Let χ be an Euler characteristic for $\mathcal{C}_1$ then

$$\chi : L_1(X) \longrightarrow K(X)$$

is an isomorphism.

<u>Proof:</u> χ is an epimorphism by definition of $K(X)$. Suppose $\chi(E) = 0$, then $E_1 \oplus F \cong E_0 \oplus F$ for some F (in fact
since F for an abelian group
F can be taken trivial). Hence if

$$P = 0 \longrightarrow F \longrightarrow F \longrightarrow 0$$

is the elementary sequence defined by F, $E \oplus P$ is isomorphic to the elementary sequence defined by $E_1 \oplus F$. Hence $E \sim 0$ in $\mathcal{C}_1(X)$ and so $E = 0$ in $L_1(X)$. To conclude we need the following elementary lemma:

LEMMA 9.2. <u>Let A be a semi-group with an identity element 1, B a group, $\varphi : A \longrightarrow B$ an epimorphism with $\varphi^{-1}(1) = 1$. Then φ is an isomorphism.</u>

[illegible] characteristics [illegible] bundles [illegible] ([illegible])

<u>Proof:</u> It is sufficient to prove that A is a group, i.e., has inverses. Let $a \in A$, then from the hypotheses there exists $a' \in A$ so that

$$\varphi(a') = \varphi(a)^{-1} .$$

Hence

$$\varphi(a \cdot a') = \varphi(a) \cdot \varphi(a') = 1 ,$$

and so $a a' = 1$ as required.

LEMMA 9.3. <u>Let χ be an Euler characteristic for $\mathcal{C}_1$, and let Y be a point. Then</u>

$$\chi : L_1(X, Y) \longrightarrow K(X, Y)$$

<u>is an isomorphism.</u>

Proof: Consider the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & L_1(X, Y) & \xrightarrow{\alpha} & L_1(X) & \xrightarrow{\beta} & L_1(Y) & \\
& \big\downarrow \chi & & \big\downarrow \chi & & \big\downarrow \chi & \\
0 \longrightarrow & K(X, Y) & \longrightarrow & K(X) & \longrightarrow & K(Y) & .
\end{array}$$

by ord K-theory, a correspondence of elts of K = homotopy classes of $X \to BU$ or BSO

By (9.1) and (9.2) and the exactness of the bottom line it will be sufficient to show the exactness of the top line. Now $\beta\alpha = 0$ obviously and so we have to show

(i) $\alpha^{-1}(0) = 0$

(ii) if $\beta(E) = 0$ then $E \in \operatorname{Im} \alpha$.

We consider (ii) first. Since Y is a point, and $\chi : L_1(Y) \cong K(Y)$, $\beta(E) = 0$ is equivalent to

$$\dim E_1 | Y = \dim E_0 | Y \; .$$

But then we can certainly find an isomorphism

$$\sigma : E_1 | Y \longrightarrow E_0 | Y \; .$$

Showing that $E \in \operatorname{Im}(\alpha)$. Finally we consider (i). Thus let

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0)$$

be an element of $\mathcal{C}_1(X, Y)$ and suppose $\alpha(E) = 0$ in $L_1(X, Y)$.

in part. they have the same dimensions

Then $\chi\,\alpha(E) = 0$ in $K(X)$, and hence, if we suppose $\dim E_i > \dim X$ (as we may), there is an isomorphism

$$\tau : E_1 \longrightarrow E_0$$ (possibly followed by a reflection)

on the whole of X. Then $\sigma\tau^{-1} \in \mathrm{Aut}(E_0|Y)$. Since Y is a point this since $GL^+(n)$ is connected, and by local contractibility automorphism is homotopic to the identity and hence can be extended to an element $\rho \in \mathrm{Aut}(E_0)$. Then $\rho\tau : E_1 \longrightarrow E_0$ is an isomorphism extending σ. This shows that E represents 0 in $L_1(X, Y)$ as required.

LEMMA 9.4. Let χ be an Euler characteristic for $\mathcal{E}_1$, then χ is an equivalence of functors $L_1 \longrightarrow K$.

Proof: Consider, for any pair (X, Y), the commutative diagram

$$\begin{tikzcd}
L_1(X/Y, Y/Y) \arrow[r, "\chi", "\text{isom}"'] \arrow[d, "\varphi\ \text{epi}"] & K(X/Y, Y/Y) \arrow[d, "\psi\ \text{isom}"] \\
L_1(X, Y) \arrow[r, "\chi"] & K(X, Y)
\end{tikzcd}$$

Since ψ is an isomorphism (by definition) and χ on the top line is an isomorphism by (9.3) it will be sufficient (by (9.2)) to prove that φ is an epimorphism. Now any element ξ of $L_1(X, Y)$ can be represented by a sequence

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0)$$

since Jacobian of E_0/X (so we add this to E_0, E_1) $= E_0$ over Y

where E_0 is a product bundle. But then we can define a "collapsed bundle" $E_1' = E_1/\sigma$ over X/Y and a collapsed sequence $E' \in \mathcal{C}_1(X/Y, Y/Y)$ defining an element $\xi' \in L_1(X/Y, Y/Y)$. Then $\xi = \varphi(\xi')$ and so φ is an epimorphism.

LEMMA 9.5. *Let χ, χ' be two Euler characteristics for $\mathcal{C}_1$. Then $\chi = \chi'$.*

Proof: Let $T = \chi' \chi^{-1}$ (which is well-defined by (9.4)). This is a natural automorphism of $K(X, Y)$ which is the identity when $Y = \emptyset$. Replacing X by X/Y and considering the exact sequence for $(X/Y, Y/Y)$ we deduce that $T = 1$, i.e., that $\chi' = \chi$.

From Lemma 9.5 and Proposition 8.1 we deduce

LEMMA 9.6. *There is a bijective correspondence $(\chi_1 \longrightarrow \chi_n)$ between Euler characteristics for $\mathcal{C}_1$ and $\mathcal{C}_n$ such that the diagram*

$$\begin{tikzcd} L_1 \arrow[r] \arrow[d, "\chi_1"'] & L_n \arrow[dl, "\chi_n"] \\ K & \end{tikzcd}$$

commutes.

These lemmas show that there is at most one Euler characteristic. In the next section we shall prove that it exists by giving a direct construction.

10. <u>The difference bundle.</u> Given a pair (X, Y) define $X_i = X \times \{i\}$ $i = 0, 1$, $A = X_0 \cup_Y X_1$ (obtained by identifying $y \times \{0\}$ and $y \times \{1\}$ for all $y \in Y$). Then we have retractions

$$\pi_i : A \longrightarrow X_i$$

$x \times \{k\} \to x \times \{i\}$
$k = i$ or the other

so that we get split exact sequences:

$$0 \longrightarrow K(A, X_i) \xrightarrow{\rho_i^*} K(A) \underset{j_i^*}{\overset{\pi_i^*}{\rightleftarrows}} K(X_i) \longrightarrow 0 .$$

Also, if we regard the index $i \in \mathbb{Z}_2$, we have maps

$$\sigma_i : (X, Y) \longrightarrow (A, X_{i+1})$$

which induce isomorphisms

$$\sigma_i^* : K(A, X_{i+1}) \longrightarrow K(X, Y) .$$

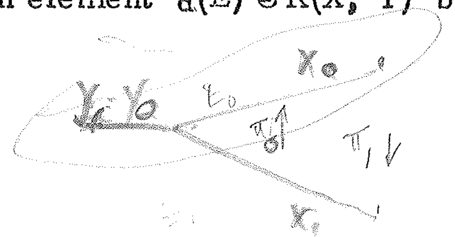
σ_0^* A, X_1

Now let $E \in \mathcal{C}_1(X, Y)$,

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0) ,$$

and construct the vector bundle F on A by putting E_i on X_i and identifying on Y by σ. It is clear that the isomorphism class of F depends only on the isomorphism class of E in $\mathcal{C}_1(X, Y)$. Let $F_i = \pi_i^*(E_i)$. Then $F|X_i \cong F_i$ and so $F - F_i \in \operatorname{Ker} j_i^*$. We define an element $d(E) \in K(X, Y)$ by

$d(E) = \sigma_0^* \rho_1^{*-1}(F - F_1)$



d(E

$$\rho_1^* (\sigma^*)^{-1} d(E) = F - F_1 .$$

It is clear that d is additive:

$$d(E \oplus E') = d(E) + d(E') .$$

Also if E is elementary $F \cong F_1$ so that $d(E) = 0$. Hence d induces a homomorphism

$$d : L_1(X, Y) \longrightarrow K(X, Y)$$

which is clearly natural. Moreover if $Y = \emptyset$ $A = X_0 + X_1$, $F = E_0 + E_1$ (disjoint sum), $F_i = E_i + E_i$ and so

$$d(E) = E_0 - E_1 .$$

Thus d is an Euler characteristic in the sense of Section 9. The existence of this d together with the lemmas of Section 9 lead to the following proposition.

PROPOSITION 10.1. *For any integer* n *with* $1 \leq n \leq \infty$ *there exists a unique natural homomorphism*

$$\chi : L_n(X, Y) \longrightarrow K(X, Y)$$

which, for $Y = \emptyset$, *is given by*

$$\chi(E) = \sum_{i=0}^{i} (-1)^i E_i .$$

Moreover χ *is an isomorphism.*

The unique χ given by (10.1) will be referred to as the Euler characteristic. From (9.6) we see that we may effectively identify the χ for different n.

Two elements $E, F \in \mathcal{C}_n(X, Y)$ are called homotopic if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of an element in $\mathcal{C}_n(X \times I, Y \times I)$.

PROPOSITION 10.2. Homotopic elements in $\mathcal{C}_n(X, Y)$ define the same element in $L_n(X, Y)$.

Proof: This follows at once from (10.1) and the homotopy incariance of $K(X, Y)$.

Proposition 10.1 shows that we could take $L_n(X, Y)$ (for any $n \geq 1$) as a definition of $K(X, Y)$. This would be a Grothendieck-type definition.

We shall now give a method for constructing the inverse of $j : L_1(X, Y) \longrightarrow L_n(X, Y)$. If $E \in \mathcal{C}_n(X, Y)$, then by introducing metrics we can define the adjoint sequence E^* with maps $\sigma_i^* : E_{i-1} \longrightarrow E_i$. Consider the sequence from $\sigma_i : E_i \to E_{i-1}$, get $E_{i-1}^* \to E_i^*$; but $E_{i-1}^* \cong E_{i-1}$ etc.

$$F = (0 \longrightarrow F_1 \xrightarrow{\tau} F_0 \longrightarrow 0)$$

when $F_0 = \bigoplus_i E_{2i}$ $\quad F_1 = \bigoplus_i E_{2i+1}$ and

$0 \to E_2 \to E_1 \to E_0 \to 0$

goes to

$0 \to E_1 \to E_2 \oplus E_0 \to 0$

$v \in E_{i-1,x} \to u \in E_{i,x}$ such that

$\langle u, y \rangle = \langle v, \sigma_i(y) \rangle$

$$\tau(e_1, e_3, e_5, \ldots) = (\sigma_1 e_1, \sigma_2^* e_1 + \sigma_3 e_3, \sigma_4^* e_3 + \sigma_5 e_5, \ldots) .$$

Since, on Y, we have the decomposition

$$E_{2i} = f_{2i+1}(E_{2i+1}) \oplus f_{2i}^*(E_{2i-1})$$

it follows that $F \in \mathcal{C}_2(X, Y)$. If $E \in \mathcal{C}_1$ then $E = F$. Since two choices of metric in E are homotopic it follows by (10. 2) that F will be a representative for $j^{-1}(E)$.

11. Products. In this section we shall consider complexes of vector bundles, i.e., sequences

$$0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \longrightarrow E_0 \longrightarrow 0$$

in which $\sigma_i \sigma_{i-1} = 0$ for all i.

LEMMA 11.1. Let $E_0, \cdots, E_n$ be vector bundles on X,

$$0 \longrightarrow E_n \xrightarrow{\sigma_n} E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow 0$$

a complex on Y. Then the σ_i can be extended so that this becomes a complex on X.

recall: Y a subcomplex of X: take fine enough bary subdivision of X

Proof: Let V be a regular neighborhood of Y in X so that we have Y as a deformation retract of V. Let $\pi : V \longrightarrow Y$ be the retraction and let

or perhaps directly from defn of CW complex by induction

$$\tau_i : E_i | V \longrightarrow E_{i-1} | V$$

be defined over V so that the diagram

$$\begin{tikzcd} E_i | V \arrow[r, "\tau_i"] \arrow[d, "\alpha_i"'] & E_{i-1} | V \arrow[d, "\alpha_{i-1}"] \\ \pi^*(E_i | Y) \arrow[r, "\sigma_i"] & \pi^*(E_{i-1} | Y) \end{tikzcd}$$

commutes, where α_i is an isomorphism (= 1 on Y) given by the homotopy $\pi \simeq 1$. Let ρ be a continuous scalar function with $\rho = 1$ on Y and $\rho = 0$ on $X - V$.

Put

$$\begin{aligned} \lambda_i &= \rho \tau_i \quad \text{on} \quad V \\ &= 0 \quad \text{on} \quad X - V . \end{aligned}$$

Then the sequence

$$0 \longrightarrow E_n \xrightarrow{\lambda_n} E_{n-1} \longrightarrow \cdots \xrightarrow{\lambda_1} E_0 \longrightarrow 0$$

is a complex on X which extends the given complex.

We now introduce the set $\mathcal{D}_n(X, Y)$ of complexes of length n on X acyclic on Y. Two such complexes are homotopic if they are isomorphic to the restriction to $X \times \{0\}$ and $X \times \{1\}$ of an element in $\mathcal{D}_n(X \times I, Y \times I)$. By restricting the homomorphisms to Y we get a natural map

$$\Phi : \mathcal{D}_n(X, Y) \longrightarrow \mathcal{C}_n(X, Y) .$$

LEMMA 11.2. $\Phi : \mathcal{D}_n \to \mathcal{C}_n$ <u>induces a bijective map of homotopy classes.</u>

<u>Proof:</u> Applying (11.1) we see that Φ itself is surjective. Next, applying (11.1) to the pair

$$(X \times I,\ X \times \{0\} \cup X \times \{1\} \cup Y \times I)$$

we see that

$$\Phi(E) \text{ homotopic to } \Phi(F) \Longrightarrow E \text{ homotopic to } F$$

which completes the proof.

If $E \in \mathcal{D}_n(X, Y)$, $F \in \mathcal{D}_m(X', Y')$ then $E \otimes F$ is a complex on X acyclic on $X' \times Y \cup X \times Y'$ so that

$$E \otimes F \in \mathcal{D}_{n+m}(X \times X', X \times Y' \cup X' \times Y) .$$

This product is additive and compatible with homotopies. Hence it induces a bilinear product on the homotopy classes. From (11.2) and (10.2) it follows that it induces a natural product

$$L_n(X, Y) \otimes L_m(X', Y') \longrightarrow L_{n+m}(X \times X', X \times Y' \cup X' \times Y) .$$

PROPOSITION 11.1. <u>The tensor product of complexes induces a natural product</u>

$$L_n(X, Y) \otimes L_m(X', Y') \longrightarrow L_{n+m}(X \times X', X \times Y' \cup X' \times Y)$$

and

$$\chi(a\, b) = \chi(a)\, \chi(b) \tag{1}$$

where χ is the Euler characteristic.

Proof: The formula (1) is certainly true when $Y = Y' = \emptyset$. On the other hand there is a unique natural extension of the product $K(X) \otimes K(X') \longrightarrow K(X \times X')$ to the relative case. Hence, by (10.1), formula (1) is also true in the general case.

Remark. This result is essentially due to Douady.

PROPOSITION 11.2. Let

$$E = (0 \longrightarrow E_1 \xrightarrow{\sigma} E_0 \longrightarrow 0) \in \mathcal{L}_1(X, Y)$$

$$E' = (0 \longrightarrow E_1' \xrightarrow{\sigma'} E_0' \longrightarrow 0) \in \mathcal{L}_1(X', Y')$$

and choose metrics in all the bundles. Let

$$F = (0 \longrightarrow F_1 \xrightarrow{\tau} F_0 \longrightarrow 0) \in \mathcal{L}_1(X \times X', X \times Y' \cup X' \times Y)$$

be defined by

$$F_1 = E_0 \otimes E_1' \oplus E_1 \otimes E_0'$$

$$F_0 = E_0 \otimes E_0' \oplus E_1 \otimes E_1'$$

$$\tau = \begin{pmatrix} 1 \otimes \sigma' , & \sigma \otimes 1 \\ \sigma^* \otimes 1 , & -1 \otimes \sigma'^* \end{pmatrix}$$

$0 \to E^0 \otimes E'^1 \to E^0 \otimes E'^1 + E^1 \otimes E'^0 \to E^1 \otimes E'^0 \to 0$

$\sigma \otimes 1 - 1 \otimes \sigma'$ $\to 1 \otimes \sigma' + \sigma \otimes 1$

where σ^*, σ'^* denote the adjoints of σ, σ'. Then

$$\chi(F) = \chi(E) \cdot \chi(E') .$$

Proof: By Proposition 11.1 $\chi(E) \cdot \chi(E') = \chi(E \otimes E')$. Now the construction of Section 9 for the inverse of $j_2 : L_1 \to L_2$ turns $E \otimes E'$ into F and so $\chi(E \otimes E') = \chi(F)$.

12. Clifford bundles. Let V be a (real) Euclidean vector bundle of dimension k over X. Then we can form the Clifford bundle of V. This is a bundle $C(V)$ of algebras such that, $x \in X$,

$$C(V)_x = C(V_x) .$$

Contained in $C(V)$ are bundles of groups, $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$. All these bundles are associated to the principal $O(k)$-bundle of V by the natural action of $O(k)$ on C_k, $\mathrm{Pin}(k)$, $\mathrm{Spin}(k)$.

By a graded Clifford module of V we shall mean a Z_2-graded vector bundle E (real or complex) over X which is a graded $C(V)$-module. In other words $E = E^0 \oplus E^1$ and we have vector bundle homomorphisms

$$V \otimes_R E^0 \to E^1 , \quad V \otimes_R E^1 \to E^0$$

(denoted simply by $v \otimes e \to v(e)$) such that

Pin(k), recall = elements of Γ^k with Norm = 1
Γ^k = elements of C_k^* (invertibles) with $\alpha(y)\, x\, y^{-1} \in \mathbb{R}^k$ for all $x \in \mathbb{R}^k$

$$v(v(e)) = -\|v\|^2 e \quad .$$

Let $\epsilon(V_x)$ denote the element of $\mathrm{Pin}(V_x)$ which is -1 in the algebra $C(V_x)$. Then $\epsilon(V)$ is a section of $\mathrm{Pin}(V)$. The following facts are then easily verified (as in the case X = point).

(12.1) The inclusion $\mathrm{Pin}(V) \longrightarrow C(V)$ induces a bijection of the classes of graded $C(V)$-modules onto the classes of those graded $\mathrm{Pin}(V)$-modules for which $\epsilon(V)$ acts as -1.

by the above calculation, since $V \subseteq \mathrm{Pin}(V)$

$\mathrm{Pin}\, V = \mathrm{Pin}^0 \cup \mathrm{Pin}^1$

(12.2) The inclusion $\mathrm{Spin}(V) \longrightarrow \mathrm{Pin}(V)$ induces a bijection of the classes of graded $\mathrm{Pin}(V)$-modules onto the classes of $\mathrm{Spin}(V)$-modules.

$\boxed{\mathrm{Pin}^0 = \mathrm{Spin}}$

(12.3) By integration over the fibers of $\mathrm{Spin}(V)$ any $\mathrm{Spin}(V)$-module E can be given a metric invariant under the action of $\mathrm{Spin}(V)$.

From these it follows that if $E = E^0 \oplus E^1$ is a graded $C(V)$-module then it can be given a metric so that E^0 and E^1 are orthogonal complements and for $v \in V_x$, $e \in E_x$

$$\|ve\| = \|v\| \cdot \|e\| \quad . \qquad (a)$$

This implies that the adjoint of

$$v : E_x^0 \longrightarrow E_x^1 \quad \text{is} \quad -v : E_x^1 \longrightarrow E_x^0 \quad .$$

or, $\langle -vx, y \rangle = \langle x, vy \rangle$
let $x = vz$, this becomes
$\|v\|^2 \langle z, y \rangle = \langle vz, vy \rangle$ which confirms (a) directly.

Let $B(V)$, $S(V)$ denote the unit ball and unit sphere bundles of V and let $\pi : B(V) \longrightarrow X$ denote the projection. Let

$$\sigma(E) : \pi^* E^1 \longrightarrow \pi^* E^0$$

be given by multiplication by $-v$, i.e.,

$$\sigma(E)_v(e) = -ve \quad .$$

$= 0$ at origin

Then

$$0 \longrightarrow \pi^* E^1 \xrightarrow{\sigma(E)} \pi^* E^0 \longrightarrow 0$$

yes, since $v \neq 0$ on $S(V)$; only reason ~~[illegible]~~ $X \to B(V)$ where it collapses completely

is an element of $\mathcal{L}_1(B(V), S(V))$ and hence defines an element of $K(B(V), S(V))$ which we will denote by $\chi(E)$. If $A(V)$ denotes the Grothendieck group of graded $C(V)$-modules then we obtain in this way a homomorphism

$A(V) = M_n(V)/M_{n+1}(V)$

$$\chi_V : A(V) \longrightarrow K(B(V), S(V)) \quad .$$

$\cancel{A}$ $S(V)$

This homomorphism plays a basic role in all the theory. Its multiplicative properties are given in the following proposition, where V, V' are bundles over X, X'.

PROPOSITION 12.1. <u>The following diagram commutes</u>

$$\begin{array}{ccc} A(V) \otimes A(V') & \xrightarrow{\mu} & A(V \oplus V') \\ \Big\downarrow \chi_V \otimes \chi_{V'} & & \Big\downarrow \chi_{V \oplus V'} \\ K(B(V), S(V)) \otimes K(B(V'), S(V')) & \xrightarrow{\lambda} & K(B(V \oplus V'), S(V \oplus V')) \end{array}$$

$B(V \oplus V'), S(V \oplus V') \cong B(V) \times B(V'), B(V) \times S(V') \cup S(V) \times B(V)$
(use cubes to see this immediately)

where μ is induced by the graded tensor product of graded modules and λ is induced by the K-product and the two homotopy eigenvalues

equivalently

$$\begin{tikzcd}[column sep=small]
(B(V) \times B(V'), B(V) \times S(V') \cup S(V) \times B(V')) \arrow[dr] & \\
& (B(V \oplus V'), B_0(V \oplus V')) \\
(B(V \oplus V'), S(V \oplus V')) \arrow[ur] &
\end{tikzcd}$$

where B_0 denotes the complement of the zero-section.

Proof: Let E, E' be graded $C(V)$ and $C(V')$ modules and let them both be given invariant metrics as above. Applying Proposition 11.2 it follows that

$$\chi_V(E) \cdot \chi_V(E') \in K(B(V) \times B(V'), B(V) \times S(V') \cup S(V) \times B(V'))$$

is equal to $\chi(F)$ where

$$F \in \mathcal{D}_1(B(V) \times B(V'), B(V) \times S(V') \cup S(V) \times B(V'))$$

is defined by

$$F_1 = E^0 \otimes E^{1'} \oplus E^1 \otimes E^{0'}$$

$$F_0 = E^0 \otimes E^{0'} \oplus E^1 \otimes E^{1'}$$

and $\tau : F_1 \longrightarrow F_0$ is given by

$$\tau = \begin{pmatrix} 1 \otimes \sigma(E') & , & \sigma(E) \otimes 1 \\ -\sigma(E) \otimes 1 & , & 1 \otimes \sigma(E') \end{pmatrix} \quad \text{on } F_1 \to E^0 \otimes E^{0'}$$

$$\sigma^* \otimes 1 \qquad -1 \otimes \sigma'^*$$

(since $\sigma(E)^* = -\sigma(E)$, $\sigma(E')^* = -\sigma(E')$). Thus, at a point $v \oplus v' \in V \oplus V'$, τ is given by the matrix

$$\tau_{v \oplus v'} = \begin{pmatrix} 1 \otimes -v', & -v \otimes 1 \\ v \otimes 1, & 1 \otimes -v' \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \otimes v', & v \otimes 1 \\ v \otimes 1, & -1 \otimes v' \end{pmatrix}$$

where v, v' denote module multiplication by v, v'. Hence

when $\sigma(E \otimes E')$ $E^0 \otimes E'^0 \oplus E^1 \otimes E'^1 \longleftrightarrow E^0 \otimes E'^1 \oplus E^1 \otimes E'^0$

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sigma(E \hat{\otimes} E') \quad .$$

Since τ and $\sigma(E \hat{\otimes} E')$ are both isomorphisms on $B_0(V \oplus V')$ (while $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an isomorphism on all of $V \oplus V'$) it follows that $\chi(F) = \chi_{V \oplus V'}(E \hat{\otimes} E')$ and hence

$$\chi_V(E) \cdot \chi_{V'}(E') = \chi_{V \oplus V'}(E \hat{\otimes} E')$$

where we have identified $K(B(V \oplus V'), S(V \oplus V'))$ and $K(B(V) \times B(V'), S(V) \times B(V') \cup B(V) \times S(V'))$.

Suppose now that P is a principal $\mathrm{Spin}(k)$ bundle over X, $V = P \times_{\mathrm{Spin}(k)} R^k$ the associated vector bundle. If M is a graded C_k-module then $E = P \times_{\mathrm{Spin}(k)} M$ will be a graded $C(V)$-module. $M = M^0 \oplus M^1$, say

In this way we obtain a homomorphism of Grothendieck groups

$$\rho_P : A_k \longrightarrow A(V) \quad .$$

for $(\gamma, v) \in P \times_{\mathrm{Spin}(k)} \mathbb{R}^k$

$(\gamma, m) \in P \times_{\mathrm{Spin}(k)} M$, let $(\gamma, v)(\gamma, m) = \gamma, vm$

then $(\gamma\beta^{-1}, \beta v)\cdot(\gamma\beta^{-1}, \beta m) \to (\gamma\beta^{-1}, (\beta v)\cdot\beta m))$, but $\beta v = \beta v \beta^{-1}$, [illegible]

PROPOSITION 12.2. Let P, P' be Spin k, Spin ℓ bundles over X, X' and let $V = P \times_{\mathrm{Spin}\,k} R^k$, $V' = P' \times_{\mathrm{Spin}\,\ell} R^\ell$. Let P'' be the Spin$(k+\ell)$-bundle over $X \times X'$ induced from $P \times P'$ by the standard inclusion

$$\begin{array}{ccc} C_k \hat{\otimes} C_\ell & \longrightarrow & C_{k+\ell} \\ \uparrow & \text{commutes} & \uparrow \\ \mathrm{Spin}\,k \times \mathrm{Spin}\,\ell & \longrightarrow & \mathrm{Spin}\,(k+\ell)\,. \end{array}$$

pat. easy to see with use of coord transformations to define bundles

also by our real see $P \times P' \times_{\mathrm{Spin}\,k \times \mathrm{Spin}\,\ell} \mathrm{Spin}(k+\ell)$

Then if $a \in A_k$, $b \in A_\ell$, we have

$$\beta_{P''}(ab) = \beta_P(a)\,\beta_{P'}(b)\,.$$

The verification of this result is left to the reader.
over a pt will do

Let $\alpha_P : A_k \longrightarrow K(B(V), S(V))$ be defined by $\alpha_P = \chi_V\,\rho_P$.
Then from Propositions 12.1 and 12.2 we deduce

PROPOSITION 12.3. With the notation of 12.2 we have

$$\alpha_{P''}(ab) = \alpha_P(a)\,\alpha_{P'}(b)\,.$$

If we apply all the preceding discussion to the case when X is a point (and P denotes the trivial Spin(k)-bundle) we get maps
by means of $(B_k / S_{k-1}) \simeq S_k$

$$\alpha : A_k \longrightarrow \widetilde{KO}(S^k) \qquad \text{in the real case}$$

$$\alpha^c : A_k^c \longrightarrow \widetilde{K}(S^k) \qquad \text{in the complex case.}$$

Proposition 12.3 then yields the following corollary, as a special case:

COROLLARY 1. The maps

$$\alpha : A_* \longrightarrow \sum_{k \geq 0} KO^{-k}\,(\text{point})$$

$$\alpha^c : A_*^c \longrightarrow \sum_{k \geq 0} K^{-k}(\,(\text{point})$$

are ring homomorphisms.

Now the rings A_* and A_*^c were explicitly determined in Section 6 (Theorems 6.1 and 6.2). Also the rings $B_* = \Sigma\, KO^{-k}(\text{point})$ and $B_*^c = \Sigma K^{-k}(\text{point})$ are known and are in fact abstractly isomorphic to A_* and A_*^c respectively. Moreover this abstract isomorphism is compatible with the complexifications

Key to deep K-theory (check it all)

$$A_* \longrightarrow A_*^c \;,\; B_* \longrightarrow B_*^c \quad .$$

Prove this; then the [illegible] (i.e. prove relations between B_* and B_*^c given for A_* and A_*^c on p 23)

In view of this and of the special structure (periodicity) of A_* and A_*^c the maps α and α^c will be isomorphisms provided

i.e. totally follows (i.e. it shows that the only necessary thing to check is [illegible] and above [illegible])

(i) $\alpha : A_1 \longrightarrow B_1$ is an isomorphism

and

(ii) $\alpha : A_2^c \longrightarrow B_2^c$ is an isomorphism.

These are trivially verified since they amount to showing that the in dim 1 Hopf bundles on $P_1(R)$ and $P_1(C)$ are the generating bundles. Hence we have proved:

$P_1(R) = S^1$, $P_1(C) = S^2$

Since all else agrees

See remarks on p. 33

THEOREM 12.1. <u>The maps</u>

$$\alpha : A_* \longrightarrow \overline{\sum_{k \geq 0}} KO^{-k} \text{ (point)}$$

<u>and</u>

$$\alpha^c : A_*^c \longrightarrow \overline{\sum_{k > 0}} K^{-k} \text{ (point)}$$

[illegible]

<u>are ring isomorphisms.</u>

13. <u>The Thom isomorphism.</u> We begin with some general remarks on the Thom isomorphism for general cohomology theories.

Let F be a generalized cohomology theory with products. Thus $F^{\#}(X) = \Sigma F^q(X)$ is a graded anti-commutative ring with identity and $F^{\#}(X, Y)$ is a graded $F^{\#}(X)$-module. Moreover the product
$F^{\#}(Y)$ as well, by restriction
must be compatible with the coboundary in the sense that

$a, b \in F^{\#}(Y)$?

$$\delta(ab) = \delta(a) \cdot b + (-1)^{\alpha} a \, \delta b$$

where $\alpha = \deg a$ and $a \cdot b$ belong to suitable F-groups. ?

say $K^n(S^n)$ ($\cong K^{[illegible]+n}(S^n)$)

In $\tilde{F}^n(S^n)$ we have a canonical element σ^n which corresponds to the identity element $1 = \sigma^0 \in F^0$ (point) $= \tilde{F}^0(S^0)$ under suspension. $\tilde{F}^{\#}(S^n)$ is then a free module over $F^{\#}$ (point) generated by σ^n.

Suppose now that V is a real vector bundle of dimension n over X. We choose a metric in V and introduce the pair $(B(V), S(V))$ (or the Thom complex $B(V)/S(V)$). For each point $x \in X$ we consider the inclusion

$$i_x : (B(V_x), S(V_x)) \longrightarrow (B(V), S(V))$$

and the induced homomorphism

generator, $K^{-n} = K^{8-n}$ etc satisfy

$$i_x^* : F^n(B(V), S(V)) \longrightarrow F^n(B(V_x), S(V_x)) .$$

$F^n(S^n) \cong K(S^0)$

Suppose now that V is oriented, then for each $x \in X$ we have a well-defined suspension isomorphism

F^n

$$S_x : F^0(\{x\}) \longrightarrow F^n(B(V_x), S(V_x)) .$$

We let $\sigma_x^n = S_x(1)$. We shall say that V is F-orientable if there exists an element $\mu_V \in F^n(B(V), S(V))$ such that, for all $x \in X$,

$$i_x^*(\mu_V) = \sigma_x^n .$$

A definite choice of such a μ_V will be called an F-orientation of V. Then we have the following general Thom isomorphism theorem:

THEOREM 13.1. *Let V be an F-oriented bundle over X with orientation class μ_V. Then $F^{\#}(B(V), S(V))$ is a free $F^{\#}(X)$-module with generator μ_V.*

Proof: Multiplication by μ_V defines a homomorphism of the F-spectral sequence of X into the F-spectral sequence of $(B(V), S(V))$ which is an isomorphism on E_2 (the Thom isomorphism

E_2^{pq} $H^p(X, K^q(F)) \Rightarrow K^*(B)$ filtered for cohomology) and hence on E_∞. Hence

$$a \longrightarrow \mu_V a$$

gives an isomorphism $F^{\#}(X) \longrightarrow F^{\#}(B(V), S(V))$ as stated. [For further details see various unpublished notes of Atiyah, Dold, G. W. Whitehead].

Applying 13.1 to the special theories K, KO we obtain.

THEOREM 13.2. *Let* V *be an oriented real vector bundle of dimension* n *over* X. *Then*

(i) *if* $n \equiv 0 \mod 2$ *and there is an element* $\mu_V \in K(B(V), S(V))$ *whose restriction to each* $K(B(V_x), S(V_x))$ *is the generator, then* $K^*(B(V), S(V))$ *is a free* $K^*(X)$*-module generated by* μ_V,

(ii) *if* $n \equiv 0 \mod 8$ *and there is an element* $\mu_V \in KO(B(V), S(V))$ *whose restriction to each* $KO(B(V_x), S(V_x))$ *is the generator, then* $KO^*(B(V), S(V))$ *is a free* $KO^*(X)$*-module generated by* μ_V.

Remark: Since $K^0(\text{point}) \cong KO^0(\text{point}) \cong \mathbb{Z}$ these groups are generated by the identity element of the ring. This element and its suspensions are what we mean by *the generator*.

Suppose now that V has a Spin-structure, i.e., that we are given a principal Spin(n)-bundle P and an isomorphism

$$V \cong P \times_{Spin(n)} R^n .$$

Then from Section 12 we have homomorphisms

$$\alpha_P : A_n \longrightarrow KO(B(V), S(V))$$

$$\alpha_P^c : A_n^c \longrightarrow K(B(V), S(V)) .$$

In the real case assume $n = 8k$ and in the complex case $n = 2k$, and put

$$\mu_V = \alpha_P(\lambda^k)$$

$$\mu_V^c = \alpha_P^c((\mu^c)^k) .$$

Then by the naturality of α_P, α_P^c and Theorem 12.1 we see that μ_V, μ_V^c define KO and K orientations of V and hence 13.2 gives:

THEOREM 13.3. *Let* P *be a* Spin(n) - *bundle over* X, $V = P \times_{Spin(n)} R^n$. *Then*

(i) *if* $n = 8k$ $KO^*(B(V), S(V))$ *is a free* $KO^*(X)$-*module generated by* μ_V

of a $Spin^c(n)$ bundle (i.e. any complex bundle)

(ii) *if* $n = 2k$, $K^*(B(V), S(V))$ *is a free* $K^*(X)$-*module generated by* μ_V^c.

Remark: It is easy to see that $\omega_2(V) = 0$, i.e., the existence of a Spin structure for V, is necessary for KO-orientability. 13.3, (i) shows that it is also sufficient. In the complex case $\omega_2(V) = 0$

is stronger than K-orientability and (13.3)(ii) has a generalization which we do not enter on here.

(13.3) together with (12.3) shows that, for Spin bundles, we have a Thom isomorphism for KO and K with all the good formal properties. It is then easy to show now that for Spin-manifolds one can define a functorial homomorphism

see Lawson p 104

$$f_! : KO^*(Y) \longrightarrow KO^*(X) \qquad \text{for maps } f : Y \longrightarrow X .$$

If one is only interested in $K(X) \otimes Q$ then one gets a Thom isomorphism without any need of Spin-structures. In fact since

$$ch : K^*(X) \otimes Q \longrightarrow H^*(X; Q)$$

see Atiyah-Hirzebruch

is an isomorphism which is functorial the ordinary Thom isomorphism for cohomology will at once give a Thom isomorphism for $K^*(X) \otimes Q$. However this procedure does not give us a nice generator from the point of view of K-theory. On the other hand for any oriented Euclidean vector bundle V of dimension 2ℓ we have the ω-regular $C(V)$-module $\Lambda_\omega(V)$ constructed in Section 7 and hence an element $\nu_V = \chi_V(\Lambda_\omega(V)) \in K(B(V), S(V))$. Proposition 7.1 shows that the restriction of ν_V to $K(B(V_x)), S(V_x))$ is 2^ℓ times the generator. Hence we deduce:

THEOREM 13.4. *Let V be an oriented Euclidean vector bundle of dimension 2ℓ over X, then $K^*(B(V), S(V)) \otimes Q$ is a free $K^*(X) \otimes Q$-module generated by ν_V.*

The multiplicative properties of ν are not quite as simple as those of μ and they will be dealt with by characters in the next section.

14. Character computations. Let G be a compact connected Lie group, W a real oriented Euclidean G-module, M^0, M^1 two complex G-modules and let $\mathrm{Iso}(M^1, M^0)$ denote the space of all vector space isomorphisms of M^1 onto M^0. Let $\theta : S(W) \longrightarrow \mathrm{Iso}(M^1, M^0)$ be an equivariant map with respect to the operations of G, i.e.,

$$\theta(g(\omega)) \cdot gm) = g(\theta(\omega) \cdot m) \qquad g \in G, \ \omega \in S(W), \ m \in M^1 .$$

Next let P be any principal G-bundle over a space X, then the above data defines an element of $\mathcal{C}_1(B(V), S(V))$ and hence an element $\Phi(P) \in K(B(V), S(V))$, where $V = P \times_G W$. Then Φ is a functor which depends on M^0, M^1, θ (W is supposed fixed throughout).

THEOREM 14.1. *With the notation above suppose further that $\dim W = 2\ell$ and that the image of G in $\mathrm{Aut}(W)$ has rank ℓ. Then Φ depends only on M^0, M^1 and not on θ. Moreover, if φ_* denotes the Thom isomorphism in rational cohomology, the functor $P \rightsquigarrow \varphi_*^{-1} \operatorname{ch} \Phi(P) \in H^*(X, Q)$ is the characteristic class*

$$\frac{\operatorname{ch} M^0 - \operatorname{ch} M^1}{\prod_{i=1}^{\ell} \alpha_i} ,$$

ch M^0, ch M^1 denote the characters of these G-modules $\alpha_1, \cdots, \alpha_\ell$ are the positive weights of the real oriented G-module W, and we use the Borel-Hirzebruch description of the cohomology of B_G.

Proof: It is sufficient to work in the universal case, i.e., to suppose X is (a finite approximation to) the classifying space B_G. Now the Euler class of the bundle $V = P \times_G W$ is just $\prod_{i \geq 1}^{\ell} \alpha_i$, and this is non-zero by the assumption on the rank of G, and hence not a zero-divisor (since $H^*(B_G ; Q)$ is a polynomial ring). Thus we get a short exact sequence

$$0 \longrightarrow H^*(B(V), S(V) ; Q) \xrightarrow{j^*} H^*(B_G ; Q) \longrightarrow H^*(B_H ; Q) \longrightarrow 0$$

where $H \subset G$ is the isotropy subgroup of a point in W, and the image of j^* is the principal ideal generated by $\prod_{i=1}^{\ell} \alpha_i$. Now

$$\mathrm{ch}\, \Phi(P) \in H^*(B(V), S(V); Q)$$

and $j^* \,\mathrm{ch}\, \Phi(P) = \mathrm{ch}\, M^0 - \mathrm{ch}\, M^1$. Since

$$j^* \varphi_*(x) = \left(\prod_{i=1}^{\ell} \alpha_i\right) x$$

we deduce that

$$\varphi_*^{-1}\, \mathrm{ch}\, \Phi(P) = \frac{\mathrm{ch}\, M^0 - \mathrm{ch}\, M^1}{\prod_{i=1}^{\ell} \alpha_i}$$

the right hand side being a well-defined element in $H^{**}(B_G ; Q)$. This shows that $\operatorname{ch} \Phi(P)$ does not depend on θ. Since G is connected we know that

$$\operatorname{ch} : K(B_G) \longrightarrow H^*(B_G ; Q)$$

is injective. Hence $\Phi(P)$ does not depend on θ.

Applying (14.1) to the case $G = SO(2\ell)$ and $M^0 \oplus M^1 = \Lambda_\omega(R^{2\ell})$ and using (7.3) we deduce

THEOREM 14.2. *Let* V *be an oriented Euclidean vector bundle of dimension* 2ℓ *over* X. *Then, if* ν_V *denotes the element of* $K(B(V), S(V))$ *of Section 13, we have*

$$\operatorname{ch} \nu_V = \varphi_* \prod_{i=1}^{\ell} \left(\frac{e^{x_i} - e^{-x_i}}{x_i} \right)$$

where φ_* *is the Thom isomorphism and the Pontrijagin classes of* V *are the elementary symmetric functions in the* x_i^2.

15. The sphere. The purpose of these next sections is to identify the generators of $KO(B(V), S(V))$ (for a V with Spinor structure and $\dim \equiv 0 \mod 8$) given in Section 13 with those given in Bott's lecture notes. Essentially it all comes down to the two basic ways of describing the sphere: as the compactification of R^n or as a homogeneous space.

$e_i \to e_i e_{k+1}$
or
$e_i \to e_{i+1} e_1$

We recall first the existence of an isomorphism $\varphi : C_k \longrightarrow C^0_{k+1}$ (Proposition 5.2). We introduce the following notation:

Pin+

$$K = \mathrm{Spin}\,(k+1)\,, \qquad H = \varphi(\mathrm{Pin}(k)) = H^0 + H^1\,, \qquad H^0 = \varphi(\mathrm{Spin}(k))$$

(where $+$ here denotes disjoint sums of the two components).

S^k = unit sphere in R^{k+1}

$S_+ = S^k \cap \{x_{k+1} \geq 0\}\,, \quad S_- = S^k \cap \{x_{k+1} \leq 0\}$

$S^{k-1} = S^+ \cap S^-$.

We consider S^k as the orbit space of e_{k+1} for the group K operating on R^{k+1} by the representation ρ. Thus $K/H^0 = S^k$ and we have the principal H^0-bundle

$$K \xrightarrow{\ \pi\ } K/H^0 \quad .$$

Let $K_+ = \pi^{-1}(S_+)$, $K_- = \pi^{-1}(S_-)$. We shall give explicit trivializations of K_+ and K_-, and the identification will then give the "characteristic map" of the sphere.

We parametrize S_+ by use of "polar coordinates":

$$(x, t) = \mathrm{Cos}\ t\ e_{k+1} + \mathrm{Sin}\ t\ x \qquad x \in S_{k-1}, \quad 0 \leq t \leq \pi/2 \ .$$

Now define a map $\beta_+ : S_+ \times H^0 \longrightarrow K_+$ by

$$\beta_+(x, t, h^0) = (-\ \mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\, e_{k+1})\, h^0 \ .$$

Since $\rho(y)\cdot x = \alpha'(y)\cdot x \cdot y^{-1}$

$$\rho((-\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})h^0)e_{k+1}$$

$$= (-\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})\ e_{k+1}(-\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})^{-1}$$

$$= (-\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})^2\ e_{k+1}$$

$$= \mathrm{Cos}\ t\ e_{k+1} + \mathrm{Sin}\ t\ x \ = \ (x,\ t)\ ;$$

it follows that β_+ is an H^0-bundle isomorphism.

Similarly we parametrize S_- by

$$(x,\ t) \ = \ -\mathrm{Cos}\ t\ (-e_{k+1}) + \mathrm{Sin}\ t\ \ x \qquad\qquad 0 \leq t \leq \pi/2,\ \ x \in S_{k-1}\ .$$

Note that for points of S_{k-1} the two parametrizations agree (putting $t = \pi/2$).

Now define a map $\beta_- : S_- \times H^1 \longrightarrow K_-$ by

$$\beta_-(x,\ t,\ h^1) \ = \ (\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})h^1\ .$$

Since

$$\rho((\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})h^1)\ e_{k+1}$$

$$= (\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})\ -\ e_{k+1}(\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})^{-1}$$

$$= -(\mathrm{Cos}\ t/2 + \mathrm{Sin}\ t/2\ x\ e_{k+1})^2\ e_{k+1} \ = \ -\ \mathrm{Cos}\ t\ e_{k+1} + \mathrm{Sin}\ t\ \ x\ ,$$

it follows that β_- is an H^0-bundle isomorphism.

Putting $t = \pi/2$ above we get

$$\beta_+(x, \pi/2, h^0) = (-\mathrm{Cos}\,\pi/4 + \mathrm{Sin}\,\pi/4\, x\, e_{k+1})\, h^0$$

$$\beta_-(x, \pi/2, h^1) = (\mathrm{Cos}\,\pi/4 + \mathrm{Sin}\,\pi/4\, x\, e_{k+1})\, h^1 \quad .$$

These are the same point of $K_+ \cap K_-$ if

$$\begin{aligned} h^1 &= -(\mathrm{Cos}\,\pi/4 - \mathrm{Sin}\,\pi/4\, x\, e_{k+1})^2\, h^0 \\ &= +x\, e_{k+1}\, h^0 \quad . \end{aligned}$$

Thus we have a commutative diagram

$$\begin{tikzcd} S_{k-1} \times H^0 \arrow[r, "\beta_+"] \arrow[d, "\delta"] & K_+ \cap K_- \arrow[d, "1"] \\ S_{k-1} \times H^1 \arrow[r, "\beta_-"] & K_+ \cap K_- \end{tikzcd}$$

where

$$\delta(x, h^0) = (x, x\, e_{k+1}\, h^0) \quad . \tag{1}$$

LEMMA 15.1. _If we regard H^0 as (left) operating on both factors of $S_+ \times H^0$ and $S_- \times H^1$, then β_+ and β_- are compatible with left operation._

Proof: (i) $\beta_+ g(x, t, h^0) = \beta_+(g(x), t, g h^0)$

$$= (-\mathrm{Cos}\, t/2 + \mathrm{Sin}\, t/2\, gxg^{-1} e_{k+1})gh^0$$

$$= g\, \beta_+ (x, t, h^0)$$

where $g \in H^0$ and $g(x) = \rho_{k+1}(g) \cdot x = gxg^{-1}$.

(ii) $\beta_- g(x, t, h^1) = \beta_-(\mathrm{Cos}\, t/2 - \mathrm{Sin}\, t/2\, gxg^{-1} e_{k+1})gh^1$

$$= g\, \beta_-(x, t, h^1) \ .$$

Since $\varphi(x) = x\, e_{k+1}$ for $x \in R^k$ formula (1) above can be rewritten

$$\delta(x, g) = (x, xg) \qquad x \in R^k, \quad g \in \mathrm{Spin}(k) \ .$$

Summarizing our results therefore we get:

PROPOSITION 15.1. <u>The principal</u> Spin(k)-<u>bundle</u> $\mathrm{Spin}\,(k+1) \longrightarrow S^k$ <u>is isomorphic to the bundle obtained from the two bundles</u>

$$S_+ \times \mathrm{Pin}^0(k) \longrightarrow S_+$$
$$S_- \times \mathrm{Pin}^1(k) \longrightarrow S_-$$

<u>by the identification</u>

$$(x, g) \longleftrightarrow (x, xg) \qquad \underline{\text{for}}\ x \in S^{k-1}, \quad g \in \mathrm{Pin}^0(k) \ .$$

<u>Moreover this isomorphism is compatible with left multiplication by</u> Spin (k) .

Here $\mathrm{Pin}^0(k) = \mathrm{Spin}(k)$ and $\mathrm{Pin}^1(k)$ are the two components of $\mathrm{Pin}(k)$.

16. Spinor bundles. Let P^0 be a principal $\mathrm{Spin}(k)$-bundle over X and put

$$P^1 = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Pin}^1(k) \;, \quad Q = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Spin}(k+1)$$

$$T^k = P^0 \times_{\mathrm{Spin}(k)} S^k = T_+ \cup T_- \;, \text{ where}$$

$$T_+ = P^0 \times_{\mathrm{Spin}(k)} S_+ \;, \quad T_- = P^0 \times_{\mathrm{Spin}(k)} S_-$$

$$\pi_+ : T_+ \longrightarrow X \;, \quad \pi_- : T_- \longrightarrow X \quad \text{the projections.}$$

Consider now the two commutative diagrams

$$\begin{tikzcd} P^0 \times_{\mathrm{Spin}(k)} (S_+ \times \mathrm{Spin}(k)) \arrow[r] \arrow[d] & P^0 \arrow[d] \\ T_+ \arrow[r, "\pi_+"] & X \end{tikzcd}$$

$$\begin{tikzcd} P^0 \times_{\mathrm{Spin}(k)} (S_- \times \mathrm{Pin}^1(k)) \arrow[r] \arrow[d] & P^1 \arrow[d] \\ T_- \arrow[r, "\pi_-"] & X \end{tikzcd} .$$

These allow us to identify the two Spin(k) bundles occurring in the first column with $\pi_+^*(P^0)$ and $\pi_-^*(P^1)$ respectively. Now because of the left compatibility in (15.1) we immediately get

PROPOSITION 16.1. *The principal* Spin (k)-*bundle* $\Omega \longrightarrow T^k$ *is isomorphic to the bundle obtained from the two bundles*

$$\pi_+^*(P^0) \longrightarrow T_+ , \quad \pi_-^*(P^1) \longrightarrow T_-$$

by the identification

$$(p, s, g) \longleftrightarrow (p, s, sg)$$

for $s \in S^{k-1}$, $g \in \mathrm{Spin}(k)$ *and* $p \in P^0$.

Now suppose that $M = M^0 \oplus M^1$ is a graded C_k-module. Then we have a natural isomorphism

$$M^1 \cong \mathrm{Pin}^1(k) \times_{\mathrm{Spin}(k)} M^0 .$$

Hence

$$P^1 \times_{\mathrm{Spin}(k)} M^0 = P^0 \times_{\mathrm{Spin}(k)} \mathrm{Pin}^1(k) \times_{\mathrm{Spin}(k)} M^0$$
$$\cong P^0 \times_{\mathrm{Spin}(k)} M^1 .$$

From (16.1) and this isomorphism we obtain:

PROPOSITION 16.2. <u>The vector bundle</u> $\Omega \times_{Spin(k)} M^0$ <u>over</u> T^k <u>is isomorphic to the bundle obtained from the two bundles</u>

$$\pi_+^*(P^0 \times_{Spin(k)} M^0) \longrightarrow T_+ , \quad \pi_-^*(P^0 \times_{Spin(k)} M^1) \longrightarrow T_-$$

<u>by the identification</u>

$$(p, s, m) \longleftrightarrow (p, s, sm) \qquad \underline{\text{for}}\ p \in P^0,\ s \in S^{k-1},\ m \in M^0.$$

Let us consider now the construction of Section 12 which assigned to any graded C_k-module M and any Spin(k)-bundle P^0 an element $\alpha_{P^0}(M) \in KO(B(V), S(V))$ where $V = P^0 \times_{Spin(k)} R^k$. This construction depended on the "difference bundle" of Section 10. In our present case the spaces A, X_0 , X_1 of Section 10 can be effectively replaced by T^k , T_+ , T_- and we see from (16.2) (and the fact that $s^2 = -1$ for $s \in S_{k-1}$) that the bundle F of Section 10 is isomorphic to the bundle $\Omega \times_{Spin(k)} M^0$. Now from the split exact sequence of the pair (T^k , T_-) and the isomorphisms

$$KO(T^k, T_-) \cong KO(T_+ , T^{k-1}) \cong KO(B(V), S(V))$$

we obtain a natural projection

$$KO(T^k) \longrightarrow KO(B(V), S(V)) .$$

Then what we have shown may be stated as follows:

THEOREM 16.1. Let P^0 be a principal $Spin(k)$-bundle M a graded C_k-module, $Q = P^0 \times_{Spin(k)} Spin(k+1)$, $V = P^0 \times_{Spin(k)} R^k$, $T^k = Q/Spin(k)$, $E^0 = Q \times_{Spin(k)} M^0$, $p : KO(T^k) \longrightarrow KO(B(V), S(V))$ the natural projection. Then

$$\alpha_{P^0}(M) = p(E^0) .$$

Remarks. This ties up the two definitions of the basic map α_P. For some purposes, such as the behaviour under products, the first definition (i.e., $\alpha_{P^0}(M)$) is most appropriate. For others, such as computing the effect of representations, the second definition (i.e., $p(E^0)$) is better.

We need [35]: Geometric interpretation

Presumably, a connection allows splitting of

$\to \ldots \to (S_k T) \to 0$

As example, at given splitting of

$\to T^*(X) \otimes E \to J_1(X,E) \to E \to 0$

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(continued)

(I. M. Singer)

17. Differential operators on vector bundles.

17.1. Introduction. Locally, and for functions, a differential operator is a linear combination of operators of the type $\Sigma_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k}(x_1,\ldots,x_n)\frac{\partial^k}{\partial x_{i_1}\ldots\partial x_{i_k}}$. For m-tuples of functions, a differential operator is an $\ell \times m$ matrix whose entries are differential operators. Since vector bundles are locally trivial, it is possible to define differential operators on smooth cross-sections of vector bundles as operators which locally can be represented as above. One can also give a more invariant treatment via jets. In our approach we will use connections with covariant derivative in the i^{th} direction playing the role of $\frac{\partial}{\partial x_i}$. Though this treatment depends on a connection, in many geometric situations there is a natural connection to use. It also has the advantage of allowing one to define homogeneous differential operators of a given order.

17.2. Notation. Let X be an n-dimensional manifold and E_i , i=1,2, two complex vector bundles over X .

Unless otherwise specified, all manifolds will be C^∞. Let C_i, $i=1,2$, be two principal bundles associated to the vector bundles E_i, with groups G_i and projection maps π_i. Thus there exist G_i-modules M_i, i.e., representations $\tilde{\rho}_i$ of G_i on vector spaces M_i so that $E_i = C_i \times_{G_i} M_i$, [that is, $E_i = C_i \times M_i$ with the equivalence relation $(c_i, m_i) \sim (c_i g_i^{-1}, \tilde{\rho}_i(g_i)m_i)$]. Let C_0 be a principal bundle over X with group G_0 and a representation $\tilde{\rho}_0$ on $R^n = M_0$ (Euclidean n-space) such that $C_0 \times_{G_0} M_0 = T(X)$, the tangent bundle of X. For example, C_0 could be the bundle of bases over X so that $G_0 = G\ell_R(n)$; any principal subbundle would also do. Unless otherwise stated, C_0 will be chosen to be the bundle of bases.

$$C = \left\{(c_0, c_1, c_2) \in C_0 \times C_1 \times C_2,\ \pi_0(c_0) = \pi_1(c_1) = \pi_2(c_2)\right\}.$$

Let C be as above. It is easy to verify that C is a principal bundle over X with group $G = G_0 \times G_1 \times G_2$. We denote its projection on X by π. The vector spaces M_j, $j=0,1,2$ can be made into G-modules via the representations ρ_j, where $\rho_j(g_0, g_1, g_2) = \tilde{\rho}_j(g_j)$. Then $C \times_G M_j = E_j$, $j=0,1,2$ with $E_0 = T(X)$. Thus, given the vector bundles E_1, E_2, and $E_0 = T(X)$, we have constructed a principal bundle C with group G, and G-modules M_j, such that

$C \times_G M_j = E_j$, $j=1,2,0$.

Suppose M is a G-module and $E = C \times_G M$. Let $\Gamma(E) = [f : C \to M ; f \in C^\infty , f(cg) = \rho(g^{-1})(f(c))]$, i.e., $\Gamma(E)$ is the set of M-valued C^∞ functions on C equivariant under G . $\Gamma(E)$ is naturally isomorphic to the C^∞-cross sections of E . The isomorphism is given by $f \to \tilde{f}$ where $\tilde{f}(x) = (c,f(c))$ and $\pi(c) = x$. $\tilde{f}$ is well defined on $C \times_G M$ since $(cg,f(cg)) = (cg,\rho(g^{-1})f(c)) \sim (c,f(c))$

Finally, note that since the duals and tensor products of G-modules are G-modules, so are M_j^* ,

$M_0^{*k} = \underbrace{M_0^* \otimes \cdots \otimes M_0^*}_{k}$, and $M_2 \otimes M_1^* \otimes M_0^{*k}$.

17.3. Differential operators of order k . Fix a C^∞ connection h on C^*, and suppose $f \in \Gamma(E)$. Then Df , the total differential of f relative to the connection h , is an equivariant horizontal one form on C with values in M , i.e., $Df(c) : H_c \to M$ where H_c denotes the horizontal space of h at c . The equivariance of Df means

[illegible] G = any gp, E_1, E_2, T_X all assoc to a G-bundle over X.

$$\text{(I)} \quad Df(cg) = \rho(g^{-1}) \quad Df(c) \quad dr_{g^{-1}} ,$$

where r_g denotes the operation of G on C . We can interpret this total differential as an element of $\Gamma(E \otimes T^*(X))$ in the following way. The map $d\pi_c$

is an isomorphism of H_c with $X_{\pi(c)}$, the tangent space of X at $\pi(c)$. On the other hand, if p is the identification map of $C \times M_0$ onto $T(X) = C \times_G M_0$, then p_c, the restriction of p to (c, M_0) gives an isomorphism of M_0 with $X_{\pi(c)}$. Let $\tau_c : M_0 \to H_c$ be $(d\pi_c)^{-1} \circ p_c$. Then

$$\text{(II)} \quad \tau_{cg} = dr_g \circ \tau_c \circ \rho_0(g)$$

for $\tau_{cg}(m_0) = d\pi_{cg}^{-1}(p(cg, m_0)) = d\pi_{cg}^{-1}(p(c, \rho_0(g)m_0) = dr_g \circ d\pi_c^{-1} \circ p_c(\rho_0(g)(m_0)) = dr_g \circ \tau_c \circ \rho_0(g)(m_0)$.

Consider the map $(\tilde{D}f)(c) = Df(c) \circ \tau_c : M_0 \to M$.

From I and II we obtain, $\tilde{D}f(cg) = \rho(g^{-1})Df(c)\rho_0(g)$.

Consequently, $\tilde{D}f$ is an equivariant function with values in the G-module $\operatorname{Hom}(M_0, M) = M \otimes M_0^*$, i.e., $\tilde{D}f \in \Gamma(E \otimes T^*(X))$. With repeated applications, we find $\tilde{D}^k : \Gamma(E) \to \Gamma(E \otimes T^*(X)^k)$, $k=1,2\ldots$. Let $\tilde{D}^0 = I : \Gamma(E) \to \Gamma(E)$.

It is necessary here to have ... $E \otimes T^*$, etc.

$T^*(X)^k$ is k-tensor product not just the symmetric one

Let s^k denote the G-module of symmetric tensors in M_0^k and let $S^k(X) = C \times_G s^k$. We will now assign to each $\underline{a} \in \Gamma(E_2 \otimes E_1^* \otimes S^k(X))$, a differential operator $D(\underline{a}) : \Gamma(E_1) \to \Gamma(E_2)$. Before we do so, note that $M_2 \otimes M_1^* \otimes s^k$ is the linear space of symmetric k-linear maps of M_0^* into $\operatorname{Hom}(M_1, M_2)$. Hence, we can view $\underline{a} \in \Gamma(E_2 \otimes E_1^* \otimes S^k(X))$ as a symmetric

Note: the connection $\tilde{D}$ is in the bundle E_1

$\tilde{D}^k : \Gamma(E_1) \to \Gamma(E_1 \otimes T^*(X)^k)$

$\to \Gamma(E_2)$ by contraction

k-linear fibre map of the cotangent bundle $T^*(X)$ into $\mathrm{Hom}(E_1, E_2)$. Furthermore, $M_2 \otimes M_1^* \otimes s^k =$ $\mathrm{Hom}(M_1 \otimes (S^k)^*, M_2) \subset \mathrm{Hom}(M_1 \otimes M_0^{*k}, M_2)$, so that $\underline{a} \in \Gamma(\mathrm{Hom}(E_1 \otimes T^*(X)^k, E_2))$. Consequently $\underline{a} \circ \tilde{D}^k : \Gamma(E_1) \to \Gamma(E_2)$; we denote this map by $D^{(k)}(\underline{a})$ so that the map $\underline{a} \to D^{(k)}(\underline{a})$ is a linear map $D^{(k)} : \Gamma(E_2 \otimes E_1^* \otimes S^k(X)) \to \mathrm{Hom}(\Gamma(E_1), \Gamma(E_2))$. Since the space s of all symmetric tensors equals $\sum_k \oplus s^k$, $\Gamma(E_2 \otimes E_1^* \otimes S(X)) = \sum_k \oplus \Gamma(E_2 \otimes E_1^* \otimes S^k(X))$, and the linear maps $D^{(k)}$ give a map $\hat{D} = \sum_k \oplus D^{(k)} : \Gamma(E_2 \otimes E_1^* \otimes S(X)) \to \mathrm{Hom}(\Gamma(E_1), \Gamma(E_2))$. The range of $\hat{D}$ will be called the space of differential operators and will be denoted by $\mathrm{Diff}(E_1, E_2)$. The range of $\sum_{\ell=0}^{k} D^{(\ell)}$ will be called the space of differential operators of order $\leq k$ and will be denoted by $\mathrm{Diff}^k(E_1, E_2)$. Thus $\mathrm{Diff}(E_1, E_2)$ is a linear subspace of $\mathrm{Hom}(\Gamma(E_1), \Gamma(E_2))$ with a filtration given by $\mathrm{Diff}^k(E_1, E_2)$.

17.4. Change of connection. Local representation.

Suppose another connection h_1 on C is chosen. Then $h_1 - h$ is an equivariant one form on C with values in g , the Lie algebra of G . As above, we can interpret $h_1 - h$ as an equivariant 0-form τ with values in $g \otimes M_0^*$. For any G-module M , τ gives rise to a bundle map

map $W_\tau : \Gamma(E) \to \Gamma(E \otimes T^*(X))$, where $W_\tau(f)(c) = b(\tau(c) \otimes f(c))$ and $b(X \otimes m_0^* \otimes m) = d\rho(X)(m) \otimes m_0^*$, $X \in g$. Similarly, any equivariant μ with values in $g \otimes (M_0^*)^k$ gives a map $W_\mu : (E \otimes T^*(X)^\ell) \to (E \otimes T^*(X)^{\ell+k})$. If D_1 is the total differential relative to the connection h_1, then $\tilde{D}_1 - \tilde{D} = W_\tau$, an elementary computation in connection theory.

THEOREM: $\mathrm{Diff}^k(E_1,E_2)$ and hence $\mathrm{Diff}(E_1,E_2)$, are independent of the choice of connection.

Proof: Since $\tilde{D}_1 = \tilde{D} + W_\tau$, $\tilde{D}_1^\ell = (\tilde{D}+W_\tau)\cdots(\tilde{D}+W_\tau)$. However, $\tilde{D}$ is a derivation and hence $\tilde{D}(W_\tau) = W_\tau \tilde{D} + W_{\tilde{D}\tau}$. Hence $\tilde{D}_1^\ell = \tilde{D}^\ell + W_{\tau_{\ell-1}}\tilde{D}^{\ell-1} + W_{\tau_{\ell-2}}\tilde{D}^{\ell-2} + \cdots + W_{\tau_0}$ where τ_j are equivariant with values in $g \times (M_0^*)^j$. Now let $C_{\ell,j}$ be the linear map of $(M_2 \otimes M_1^* \otimes s^\ell) \times (g \otimes M_0^{*j}) \to M_2 \otimes M_1^* \otimes s^{\ell-j} = \mathrm{Hom}(M_1 \otimes (s^{\ell-j})^*, M_2)$ given by

$C_{\ell,j}(m_2 \otimes m_1^* \otimes s \otimes X \otimes \{\phi_1 \otimes \cdots \otimes \phi_j\}) = m_2 \otimes \rho_1^*(X)(m_1^*) \otimes s(\phi_1,\ldots,\phi_j,\cdot,\cdot,\cdot)$ where $X \in g$, $\phi_i \in M_0^*$.

Note that if $\underline{a}(c) \in M_2 \otimes M_1^* \otimes s^\ell$ and $\tau_{\ell-j}(c) \in g \otimes M_0^{*\,\ell-j}$, then $C_{\ell,j}(a(c) \otimes \tau_{\ell-j}(c)) = a(c) \circ W_{\tau_{\ell-j}}(c)$ as elements of $\mathrm{Hom}(M_1 \otimes (s^j)^*, M_2)$. Hence

$\underline{a} \circ \tilde{D}_1^\ell = \underline{a} \circ \tilde{D}^\ell + \underline{a}_{\ell-1} \circ \tilde{D}^{\ell-1} + \cdots + \underline{a}_1 \circ \tilde{D}^1 + \underline{a}_0$ where

$\underline{a}_j \in \Gamma(E_2 \otimes E_1^* \otimes S^j(X))$, $\underline{a}_j(c) = C_{\ell,j}(\underline{a}(c) \otimes \tau_{\ell-j}(c))$ and $\underline{a}_\ell = \underline{a}$. In particular $D_1^{(\ell)}(\underline{a}) = \sum_{j=0}^{\ell} D^{(j)}(\underline{a}_j)$ where $D_1^{(\ell)}$ is the map $D^{(\ell)}$ relative to the connection h_1 . q.e.d.

COROLLARY: $D_1^{(\ell)}(\underline{a}) - D^{(\ell)}(\underline{a}) \in \mathrm{Diff}^{\ell-1}(E_1,E_2)$.

THEOREM: The map $\hat{D}$ is injective.

Proof: Suppose $\sum_{j=0}^{k} D^{(j)}(\underline{a}_j) = 0$, $\underline{a}_j \in \Gamma(E_2 \otimes E_1^* \otimes S^j(X))$ Then $D^{(k)}(\underline{a}_k)$ is a differential operator of order $k-1$. By the corollary, relative to any other connection h_1 , $D_1^{(k)}(\underline{a}_k)$ is a differential operator of order $k-1$. Therefore it suffices to show that, for some connection h_1 , $D_1^k(\underline{a}_k)$ of order $k-1$ implies $\underline{a}_k = 0$. Fix a point $x^0 \in X$ and choose a coordinate neighborhood U of x^0 , with coordinate functions $x_1,\ldots,x_n$. Choose a local cross section $\psi : U \to C$ such that for each $x \in U$, $p_{\psi(x)}$ maps the natural basis of M_0 into the basis $\left\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right\}$ at X_x . (If C_0 is the bundle of basis, this cross section is obtainable via the coordinate cross section in C_0 . In general, one must, in fact, enlarge the bundle C as follows. Let $\tilde{p} : C \to B$, (the bundle of basis over X) be the bundle map which p_c induces. Let $C' = [(c,b) \in C \times B , \pi(c) = \pi(b)]$. C' is a principal bundle over X with group $G \times G\ell_R(n)$ and the graph of $\tilde{p}$ imbeds C as a sub-bundle of C'. C' can be used in place of C for the consideration of this chapter. In particular, a cross section of the desired type exists in C'.) The cross section ψ gives a connection h_1 on $\psi^{-1}(U)$ with horizontal space at $\psi(x)$ equal to $d\psi(X_x)$. For any G-module M, the restriction map $f \to f \circ \psi$ gives an isomorphism of $\Gamma(E)_{\pi^{-1}U}$ with $[g;\ g : U \to M_0]$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the natural dual base in M_0^*. Then the special nature of the cross section implies that $D_1 f \circ \psi = \sum_{i=1}^{n} \frac{\partial(f \circ \psi)}{\partial x_i} dx_i$, $\tilde{D}_1 f \circ \psi = \sum_{i=1}^{n} \frac{\partial(f \circ \psi)}{\partial x_i} \otimes \alpha_i$ and $\tilde{D}_1^k f \circ \psi = \sum_{i_1, \ldots, i_k} \frac{\partial^k (f \circ \psi)}{\partial x_{i_1} \cdots \partial x_{i_k}} \otimes \alpha_{i_1} \otimes \cdots \otimes \alpha_{i_k}$.

Let $(\underline{a}_k)_{i_1, \ldots, i_k}(x) = \underline{a}_k(\psi(x))(\alpha_{i_1}, \ldots, \alpha_{i_k})$, an element of $\mathrm{Hom}(M_1, M_2)$. Hence, if $f \in \Gamma(E_1)|_{\pi^{-1}(U)}$, we have

$$\text{(III)} \quad ((D_1^k(\underline{a}_k)f) \circ \psi)(x) = \Sigma(\underline{a}_k)_{i_1, \ldots, i_k}(x)\left(\frac{\partial^k (f \circ \psi)}{\partial x_{i_1}, \ldots, \partial x_{i_k}}\right)$$

Now choose f so that the support of f vanishes outside $\pi^{-1}(U)$ and in a small neighborhood of x, $(f \circ \psi)(x) = x_{i_1} \cdots x_{i_k} m_1$, m_1 a fixed non-zero element of M_1. Then if L is any differential operator of order lower than k, $(Lf)(\psi(x)) = 0$. If $D_1^k(a_k)$ is of order $k-1$,

$$0 = (D_1^k(a_k)f)(\psi(x)) = \Sigma(\underline{a}_k)_{i_1\ldots i_k}(x)\cdot\frac{\partial^k(x_{i_1}\ldots x_{i_k})m_1}{\partial x_{i_1}\ldots\partial x_{i_k}}$$

$$= (\underline{a}_k)_{i_1,\ldots,i_k}(x))(m_1) \ .$$

Hence $\underline{a}_k(\psi(x)) = 0$. q.e.d.

Remarks: (a) Formula (III) gives the local representation of a differential operator in terms of partial derivatives.

Vh. with splitting

(b) Since s is graded, and D is injective, any connection makes $\mathrm{Diff}(E_1,E_2)$ into a graded linear space, i.e., $\mathrm{Diff}(E_1,E_2) = \sum_{\ell} \oplus$ range of $D^{(\ell)}$ and $\mathrm{Diff}^k(E_1,E_2) = \sum_{\ell\leq k} \oplus$ range of $D^{(\ell)}$. We shall call the range of $D^{(k)}$, differential operators homogeneous of order k . This depends upon the connection, of course.

!!!

17.5. The symbol of a differential operator.

Suppose now that $d \in \mathrm{Diff}^k(E_1,E_2)$ but $d \notin \mathrm{Diff}^{k-1}(E_1,E_2)$. Then $d = \sum_{\ell=0}^{k} D^{(\ell)}(\underline{a}_\ell)$, with $\underline{a}_k \not\equiv 0$.

In fact, $\underline{a}_k$ is independent of the connection for if

$$d = \sum_{\ell=0}^{k} D^{(\ell)}(\underline{a}_\ell) = \sum_{\ell=0}^{k} D_1^{(\ell)}(\underline{b}_\ell) \text{ then } D^{(k)}(\underline{a}_k-\underline{b}_k)$$

$$= D_1^{(k)}(\underline{b}_k) - D^{(k)}(\underline{b}_k) + \text{lower order} \in \mathrm{Diff}^{k-1}(E_1,E_2) \ .$$

Hence, $\underline{a}_k = \underline{b}_k$.

Now $\underline{a}_k$ can be interpreted as a bundle map of $S^k(X)^* \to \mathrm{Hom}(E_1, E_2)$. Let $S^*(X)$ denote the unit sphere bundle in $T^*(X)$ (relative to some Riemannian metric in X) and let $\tilde{E}_1$, $\tilde{E}_2$ be the bundles E_1, E_2 pulled back to $S^*(X)$ relative to the projection of $S^*(X)$ onto X. $T^*(X)$ and hence $S^*(X)$ is imbedded in $S^k(X)^*$ as the diagonal, so that $i^{-k}\underline{a}_k|_{S^*(X) \subset S^k(X)^*} \in \Gamma(\mathrm{Hom}(\tilde{E}_1, \tilde{E}_2))$. We denote this element of $\Gamma(\mathrm{Hom}(\tilde{E}_1, \tilde{E}_2))$ by $\sigma(d)$ and call it the <u>symbol</u> of d. The differential operator d is said to be <u>elliptic</u> if $\sigma(d) \in \mathrm{Iso}(\tilde{E}_1, \tilde{E}_2)$. Note that $\dim(E_1)$ must equal $\dim(E_2)$ in order for d to be elliptic. Also, $\sigma(d)$ is independent of the connection chosen because $\underline{a}_k$ is independent of the connection.

We leave to the reader the verification that (i) the composition of differential operators is a differential operator, (ii) $\sigma(d_1 \circ d_2) = \sigma(d_1) \circ \sigma(d_2)$, (iii) if d_1, d_2, $d_1 + d_2$ are in $\mathrm{Diff}^k(M_1, M_2)$ but not in $\mathrm{Diff}^{k-1}(M_1, M_2)$, then $\sigma(d_1 + d_2) = \sigma(d_1) + \sigma(d_2)$.

17.6. <u>Some examples</u>. A vector field V on X gives rise to a differential operator of the first order, namely covariant differentiation in the direction of V, for any vector bundle E. In this case, the operator is $D'(\underline{a})$ where $\underline{a} : T(X)^* \to \mathrm{Hom}(E, E)$ with

Different from the Lie derivative

$\Gamma(T) \otimes \Gamma(E) \to \Gamma(E)$

$\Gamma(E) \to \Gamma(T^* \otimes E)$

$= \Gamma(\mathrm{Hom}\, T, E))$

$\underset{\sim}{a}(\phi(x)) = \phi(x)(V(x))I$, $\phi(x)$ a dual tangent vector at x . If we denote by D_t the covariant derivative in the direction t , t a tangent vector, then the differential operator associated to V is $D_{V(x)}$, $x \in X$. More generally, if $\underset{\sim}{a} \in \Gamma(E_2 \otimes E_1^* \otimes S^1)(X)) = \Gamma(E_2 \otimes E_1^* \otimes T(X))$

$$= \Gamma(\mathrm{Hom}(T^*(X), \mathrm{Hom}(E_1, E_2)) \text{ , and}$$

$\{t_1, \ldots, t_n\}$, $\{\phi_1, \ldots, \phi_n\}$ are a basis and a dual basis in X_x, X_x^* , then (in coordinates: $\sum a_j D_{t_j}$ if $V(x) = \sum a_j t_j$)

$$D(\underset{\sim}{a})_x = \sum_{j=1}^{n} \underset{\sim}{a}(\phi_j) \quad D_{t_j} .$$

This formula is obtained by interpreting the definition of a differential operator of first order in terms of a basis.

Some special and geometrically interesting first order differential operators arise by imposing a geometric structure on X . Let G be a Lie group and let M_0 be a fixed real G-module of dimension equal to $\dim X$. We shall call a <u>G-structure on</u> X a fixed principal bundle P on X with structure group G such that $P \otimes_G M_0 = T(X)$. Suppose M_1 and M_2 are two G-modules and suppose

$M_0 \cong \mathbb{R}^n$, $n = \dim X$

$a : S^k(M_0)^* \to \mathrm{Hom}(M_1, M_2)$ is a G-map. Then a induces the <u>constant section</u> $\underset{\sim}{a} \in \Gamma(E_2 \times E_1^* \times S^k(X))$.

i.e., from a on modules, a on sections of associated bundles

The corresponding differential operator $D^k(\underline{a}) : \Gamma(E_1) \to \Gamma(E_2)$ is said to be associated to the G-structure. Some examples of such first order operators are:

(i) <u>The total differential</u>. M_1 = any G_1-module; $G = G_1 \times G\ell(n,R)$; $M_0 = R^n$, a G-module with G_1 acting trivially on R^n and $G\ell(n,R)$ acting in the usual way on R^n ; $M_2 = M_1 \times M_0^*$. Choose $a : M_0^* \to \mathrm{Hom}(M_1, M_1 \otimes M_0^*)$ to be $a(\phi)(m_1) = m_1 \otimes \phi$. Clearly a is a G-map. The associated differential operator $D'(\underline{a})$ is just the

?= total differential D of the connection for $\underline{a} \in \Gamma(E_1 \otimes T^*(X) \otimes E_1^* \otimes T(X)) = \Gamma(\mathrm{Hom}(T^*(X) \otimes E_1 , T^*(X) \otimes E_1))$ is the identity transformation.

$E \to T^*(X) \otimes E$ $\quad \Omega \otimes T \otimes E \to E$

$E \to E$

(ii) <u>The ordinary differential on forms</u>. Let X be an oriented manifold, B the bundle of bases so that $G = G\ell(n)$, $M_0 = R^n$. Let $M_1 = \sum_k \Lambda^{2k}(M_0^*) \otimes C$. $M_2 = \sum_{k=0} \Lambda^{2k+1}(M_0^*) \otimes C$ where $\Lambda^k(M_0^*) \otimes C$ denotes the G-module consisting of the homogeneous elements of degree k in the complex Grassman algebra over M_0^* .

$T \otimes E \to E$

$\Omega \otimes E$ $\downarrow Id$ $\Omega \otimes E$

$E \to \Omega \otimes E$

Let $a : M_0^* \to \mathrm{Hom}(M_1, M_2)$ given by $a(\phi) = \ell_\phi$ where ℓ_ϕ denotes multiplication by ϕ in the Grassman algebra. It is easy to verify that a is a G-map. It turns out that $D'(\underline{a})$ is the ordinary differential mapping forms

$\nabla_X Y - \nabla_Y X = [X, Y]$

$dw = \sum \phi_i \wedge D_{t_j} w$

but doesn't depend on the connection —!

of even degree into forms of odd degree, if the connection h has zero torsion. This follows from the classical fact that for a connection with zero torsion

$dw = \sum_{j=1}^{n} \phi_j \wedge D_{t_j} w$ for any form w, i.e., ϕ_j, t_j dual bases

$d = \sum_j \ell_{\phi_j} \circ D_{t_j} = D(\underline{a})$. $dw = \sum \left(\frac{\partial w}{\partial x_i}\right) \cdot dx_i \wedge w$

(iii) The Riemannian case. If X is an oriented Riemannian manifold we could have taken h to be the Riemannian connection, P the bundle of frames so that $G = SO(n)$. Choose M_0, M_1, and M_2 as in example (ii). Now, however, M_1 and M_2 inherit an inner product from M_0 and ℓ_ϕ has an adjoint ℓ^*_ϕ . Now let $\underline{a}(\phi) = \ell_\phi - \ell^*_\phi$; then $D'(\underline{a}) = d + \delta$ mapping even forms into odd forms. Here δ is the adjoint of the differential d which maps odd forms to even. This fact will become clearer after a later discussion of adjoints of differential operators. Note that (α) $d + \delta$ is elliptic because $\underline{a}(\phi)$ is an isomorphism, $\phi \neq 0$, (β) $(d + \delta)^2$ = Laplacian on even forms. (γ) The kernel (cokernel) of $d + \delta$ is the space of harmonic even (odd) forms so that $\sigma(\Box)$:
dim kernel $D'(\underline{a})$ - dim kernel $D'(a)^* = \chi(X)$, the Euler characteristic.

Presumably, $\Box$ comes from $[(\phi, \psi) \to -(\phi, \psi)] \otimes I$. $= \Box$

(iv) The Riemannian case, a different decomposition. Besides the decomposition of forms into even and odd,

$\sigma(\Box)$

there is a second decomposition leading to another elliptic differential operator. Suppose $n = 2\ell$ so that $M_0 = R^{2\ell}$. Let $*$ denote the extension to the complex Grassman algebra of the usual star operator.
Suppose $\tau \in \mathrm{Hom}(\Lambda(M_0^*) \otimes C, \Lambda(M_0^*) \otimes C)$ with $\tau = \Sigma \oplus \tau_p$, $\tau_p \in \mathrm{Hom}(\Lambda^p(M_0^*) \otimes C, \Lambda^{2\ell-p}(M_0^*) \otimes C)$, and $\tau_p = i^{p(p+1)-\ell} *$. Then $\tau^2 = I$. Take M^+ and M^- to be the $+1$ and -1 eigenspaces of τ. As above, choose a so that $a(\phi) = \ell_\phi - \ell_\phi^*$. A bit of algebra shows that (i) M_1 and M_2 are invariant under G, (ii) $a(\phi)(M^+) \subset M^-$; (iii) a is a G-map. In fact, in terms of the vector space isomorphism of $\Lambda(M_0) \otimes C$ with the Clifford algebra $C(M_0) \otimes C$ exposed in section 7 (p.26), the operator τ = Clifford multiplication by $i^\ell w(M_0)$ and $a(\phi)$ = Clifford multiplication by ϕ.

We obtain a differential operator $D'(\underline{a}) = d + \delta$ but which now maps $\Gamma(E^+) \to \Gamma(E^-)$. This operator is still elliptic.

It is not hard to compute dim kernel $D'(\underline{a})$ − dim kernel $D'(\underline{a})^*$. These kernels will consist of the harmonic forms in $\Gamma(E^\pm)$ because $(d + \delta)^2$ = Laplacian. If ℓ is even, $\tau_\ell = *$ and the kernel (cokernel) of $d + \delta$ contains the space of harmonic ℓ forms $h_\ell^\pm$ invariant (anti-invariant) under $*$. If w is a harmonic p-form $p \neq \ell$, the map $\alpha_p : w \to w \pm \tau(w)$ is injective. Consequently, the complement to h_ℓ^+ in dim kernel $D'(\underline{a})$ and the complement to h_ℓ^- in dim kernel $D'(a)^*$ have the same dimension so that dim kernel $D'(\underline{a})$ - dim kernel $D'(\underline{a})^*$ $=$ dim h_ℓ^+ - dim h_ℓ^- = Hirzebruch index of X. If ℓ is odd, it is easy to see this integer is zero.

This example can be generalized slightly by using a complex vector bundle $W = P \times_G C^m$ as coefficients. Let $G = SO(2\ell) \times G\ell(m,c)$, $M_0 = R^n$, with $G\ell(m,c)$ acting trivially on M_0. Let $\overline{M}^{\pm} = M^{\pm} \otimes C^m$ with $M^{\pm}$ as in the previous paragraph. The $\overline{M}^{\pm}$ are G-modules via the tensor representation of $SO(2\ell)$ on $M^{\pm}$ and $G\ell(m,c)$ on C^m. Finally choose $a : M_0^* \to \mathrm{Hom}(\overline{M}^+, \overline{M})$ by $a(\phi) = (\ell_\phi - \ell_\phi^*) \otimes I$. a is again a G-map and the corresponding operator $D(\underline{a})$ is still elliptic mapping $\Gamma(\overline{E}^+) = \Gamma(E^+ \otimes W) \to \Gamma(E^- \otimes W) = \Gamma(\overline{E}^-)$.

(v) <u>Hermitian structure</u>. Let X be a complex Kaehler manifold of dimension ℓ and W a holomorphic vector bundle of dimension m, with a Hermitian metric. Then the principal bundle of the complex tangent bundle and the principal bundle of W gives a principal bundle P with group $G = U(\ell) \times U(m)$. Take $M_0 = C^\ell$ with $V(m)$ acting trivially on M_0. Let $M_1 = \sum_k \Lambda^{2k}(C^\ell) \otimes C^m$

and $M_2 = \sum_k \wedge^{2k+1}(C^{\ell}) \otimes C^m$, both G-modules via the tensor action. Now $\Gamma(E_i)$, $i=1,2$, can be interpreted as the space of forms of type (0, even) $\{(0, \text{odd})\}$ with coefficients in $W = P \times_G C^m$. Choose $a(\phi) = (\ell_\phi - \ell_\phi^*) \otimes I$, $\phi \in M_0^*$. Again a is a G-map and the corresponding differential operator $D'(\underline{a}) = \bar{\partial} + \bar{\partial}^* : \Gamma(E_1) \to \Gamma(E_2)$ where $\bar{\partial}$ is the (0,1) component of exterior differentiation, i.e.,

$$\bar{\partial} = \sum_j d\bar{z}_j \wedge \frac{\partial}{\partial \bar{z}_j} .$$

This stems from the fact that in a Kähler manifold the Riemannian connection lives in the bundle of complex bases. Since $(D'(\underline{a}))^2$ = Laplacian, dim kernel $D'(\underline{a})$ - dim kernel $D'(\underline{a})^*$ = dim harmonic forms of type (0, even)- dim harmonic forms of type (0, odd), and this integer equals the Euler characteristic of the cohomology of X with coefficients in the sheaf of germs of holomorphic sections of W.

(vi) Spinor structure. Suppose X is an oriented Riemannian manifold of dimension 2ℓ whose second Stiefel Whitney class vanishes. Let P be a principal bundle with G = Spin (2ℓ) covering the bundle of frames of X, giving X a Spinor structure. Choose $M_0 = R^{2\ell}$

with G acting on M_0 via its image $SO(2\ell)$. Let M_1 and M_2 be the two half spin irreducible representation spaces of Spin (2ℓ). As exposed in section 5, $M_1 \oplus M_2$ is a z_2-graded irreducible module of the Clifford algebra $C(R^{2\ell})$ so that the odd elements $C'(R^{2\ell})$ in $C(R^{2\ell})$ map M_1 into M_2. In fact $M_1 \oplus M_2$ can be taken to be a minimal left ideal in the simple algebra $C(R^{2\ell})$ so that $C(R^{2\ell})$ acts on $M_1 \oplus M_2$ via left multiplication. In particular, if $t \in R^{2\ell} \subset C^1(R^{2\ell})$, $tm_1 \in M_2$. Since M_0 has an inner product, we can identify M_0 with the dual G-module M_0^*. Hence we seek a G-map $a : M_0 \to \mathrm{Hom}(M_1, M_2)$. Using the Clifford multiplication, we can choose a by $a(t)(m_1) = tm_1, t \in M_0, m_1 \in M_1$. It is easy to verify that a is a G-map. In terms of an orthonormal base $\{e_1, \ldots, e_{2\ell}\}$, the corresponding first order differential operator $D'(\underline{a})$ is the <u>Dirac operator</u>:

$$D(f) = \Sigma\, e_i \cdot D_{e_i} f$$

where e_i denotes Clifford multiplication by e_i and $f \in \Gamma(E_1)$.

Again, this construction can be generalized to include the case of spinors with coefficients in a vector bundle W.

(vii) Odd dimensional Clifford structure. Let X be an oriented odd dimensional Riemannian manifold, P the bundle of frames so that again $G = SO(2\ell+1)$ and $M_0 = R^{2\ell+1} = M_0^*$. Let $M_1 = M_2 = M$ be the G-module $C^0(R_{2\ell+1}) \otimes C$, the subalgebra of even elements of the complex Clifford algebra. Let $\{e_1, \ldots, e_{2\ell+1}\}$ be an oriented orthonormal basis and let $w \in C(R_{2\ell+1})$ be $e_1 \cdots e_{2\ell+1}$. w is independent of the choice of basis. Define $a : M_0 \to \mathrm{Hom}(M,M)$ by $a(t)(m) = i^{\ell} twm$. Again a is a G-map, and we obtain the differential operator $D(\underline{a})$. As in section 7, M has a natural inner product inherited from M_0 in which $a(t)$ is self-adjoint and consequently the corresponding differential operator $D(\underline{a}(\alpha))$ is skew adjoint. This example can also be generalized to allow a complex vector bundle W as coefficients.

17.7. The formal adjoint of a differential operator. Stokes' Theorem.

In this section we wish to show that the formal adjoint L^* of a differential operator L is a differential operator and that the symbol of L^* is the adjoint of the symbol of L. In addition, we show that the adjoints of differential operators associated to a G-structure take a special form.

So, again, let C be the principal bundle with group G, M_1 and M_2 G-modules, L a differential operator from $\Gamma(E_1) \to \Gamma(E_2)$. Let $\Gamma_0(E)$ denote the cross sections in $\Gamma(E)$ with compact support and let ω denote a volume form of X, an oriented manifold.

THEOREM. <u>There exists a unique differential operator</u> $L^* : \Gamma(E_2^*) \to \Gamma(E_1^*)$ <u>such that for every</u> $f \in \Gamma(E_1)$, $g \in \Gamma(E_2^*)$, n-form

(A) $\langle Lf , g \rangle \omega - \langle f , L^*g \rangle \omega = d\tau$,

τ <u>an</u> n-1 <u>form</u>. <u>In particular if</u> $f \in \Gamma_0(E_1)$, <u>then</u>

$$\int_X \langle Lf , g \rangle \omega = \int_X \langle f , L^*g \rangle \omega .$$

$E^* = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$

Or, one can state this for the defn of E^* as $\mathrm{Hom}(E, \Omega^{*n})$

One would like to have a way of describing τ so as to vary "smoothly" with f and g in (as in the diff. only locally $d\tau$!)

<u>Proof</u>: One can show that the formal adjoint of a differential operator is a differential operator by using the representation in local coordinates. We proceed differently in order to obtain, as well, a Stokes' formula for differential operators associated to a G-structure. The basic idea we adapt to our situation is this: Let V be a vector field on a manifold Y with volume form α; let f, g be functions on Y, let $\theta(V)$ denote the Lie derivative with respect to V, and $i(V)$ the derivation on forms which is interior product. Then $\theta(V)$ is a derivation and $\theta(V) = di(V) + i(V)d$.

* Occurs in Uhlenbeck.
It can be extended to $\mathbb{R}^n$ and $\mathbb{R}^{n,k}$ as well.

$$V\cdot(g\alpha) = Vg\cdot\alpha + g\cdot V\alpha$$
$$= Vg\cdot\alpha + g\cdot r\cdot\alpha = \boxed{(Vg+gr)}\alpha$$

Thus $\quad \Theta(f)\cdot(g\alpha) + f\cdot(\Theta(V)(g\alpha)) = \Theta(V)(fg\alpha)$

$$(*) \quad Vf\cdot g\alpha + f\cdot(Vg)\alpha + fg\,\underbrace{\Theta(V)\alpha}_{r\alpha} = (di(V)+i(V)d)fg\alpha$$
$$= d(i(V)fg\alpha)\ .$$

Now $\Theta(V)\alpha = r\alpha$, r a function on Y so that if we let $V^* = -V$ - mult. by r, we get
(V on $g\alpha$)

* $\boxed{Vf\cdot g\alpha - f.(V^*g)\alpha = d(i(V)fg\alpha)}$.* special case for the differential operator V on the trivial bundle.

To apply this to the present setup, we first deal with the special case $L = \tilde{D} : \Gamma(E_1) \to \Gamma(E_1 \otimes T^*(X))$, D the total differential relative to a fixed connection h on C. The map τ_c^{-1} of section 17.3 gives vector fields $\{E_1,\ldots,E_n\}$ on C ~~such~~ horizontal vectors such that (i) $\{E_1(c),\ldots,E_n(c)\}$ is a basis of H_c and $\tau_c^{-1}(E_j(c))$ is the standard fixed basis $\{e_j\}$ in M_0, (ii) If $f \in \Gamma(E_1)$, then $\tilde{D}f(c) = \sum_j (E_jf)(c) \otimes \phi_j$ where ϕ_j is the dual basis in M_0^*. If $g \in \Gamma((E_1 \otimes T^*(X))^*) = \Gamma(E_1^* \otimes T(X))$, then $g = \sum_k g_k \otimes e_k$ and $<\tilde{D}f(c), g(c)> = \sum_j < E_jf(c), g_j(c)>$. The equivariance of f and g shows that this function is constant on the fibres of ~~C~~. M ought to be.. Let μ denote the vertical invariant form of degree = dim (G), whose restriction to a fibre is the volume form of G and set $\alpha = \mu \wedge \tilde{\omega}$, $\tilde{\omega}$ the lift of ω from X. Then applying $*$, we find

$$D: \Gamma(E) \to \Gamma(E\otimes T^*)$$
$$D^*: \Gamma(E\otimes T^*) \to \Gamma(E)$$

$$\mathcal{L}_X(f\cdot\mu) = (Xf)\mu + f\mathcal{L}_X\mu$$

$$\hat{X}\mu = -\mathcal{L}_X\mu$$

$$-Xf\cdot\mu - f\cdot\hat{X}\mu = d(i_X\mu) = \mathcal{L}_X\mu$$
$$= d(i_X(f\mu))$$

The bundles become trivial on C, so this is O.K. — these are "real" inner products.

$$< \tilde{D}f, g > \alpha = \sum_j < E_j f, g_j > \alpha = - < f, \sum_j E_j g_j > \alpha$$

$\mu \wedge \tilde{\omega}$

$$- \Sigma < f, r_j g_j > \alpha + d(\sum_j < f, g_j > i(E_j)\alpha)$$

r_j is a function. Since $d\mu = 0$?

where $r_j \alpha = \theta(E_j)\alpha$. Since $d\mu \wedge i(E_j)\tilde{\omega} = 0$ and $i(E_j)\alpha = \mu \wedge i(E_j)\tilde{\omega}$ the last term can be written as $\mu \wedge d(\sum_j < f, g_j > i(E_j)\tilde{\omega})$, so that

$$(**) \qquad < \tilde{D}f, g > \tilde{\omega} = - < f, \sum_j E_j g_j > \tilde{\omega} - < f, \sum_j r_j g_j > \tilde{\omega}$$
$$+ d(\sum_j < f, g_j > i(E_j)\tilde{\omega}) .$$

Now let $\tilde{D}^* : \Gamma(E_1^* \otimes T(X)) \to \Gamma(E_1^*)$ be the transformation $\tilde{D}^*(g) = \tilde{D}^*(\sum_j g_j \otimes e_j) = - \sum_j E_j g_j - \Sigma r_j g_j$ so that $< \tilde{D}f, g > \omega - < f, \tilde{D}^* g > = d(\sum_j < f, g_j > i(E_j)\tilde{\omega}) = d\tau$.

does τ exist globally?

But $\tilde{D}^*$ is a differential operator. In fact, choose $\underline{a}_1 \in \Gamma(E_1^* \otimes E_1 \otimes T^*(X) \otimes T(X))$ to be $-I \otimes I$ and choose $\underline{a}_0 \in \Gamma(E_1^* \otimes E_1 \otimes T^*(X))$ to be $-I \otimes \Sigma r_j \phi_j$. Then $\tilde{D}^* = D^1(\underline{a}_1) + D^0(\underline{a}_0)$. Thus (A) is verified for the basic first order operator $\tilde{D}$.

(r_j as above)

If L is a differential operator of 0^{th} order, i.e., if $L \in Hom(E_1, E_2)$, then L^* is the usual adjoint in $Hom(E_2^*, E_1^*)$ and $< Lf, g > - < f, L^* g > = 0$. It is easy to check that if (A) holds for L_1 and L_2 , then it holds for $L_1 \circ L_2$ with $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$.

and τ = ?

$D: \Gamma(E) \to \Gamma(E \otimes T^*)$

$D^*: \Gamma(E^* \otimes T) \to \Gamma(E^*)$

$D: \Gamma(T^* \otimes E^* \otimes T) \to \Gamma(E^*)$

$T^* \otimes T \to$ ~~[illegible]~~ contr

D: function E → F

= $\Gamma(E \otimes T^*) \to F$

$(D\varphi, \psi) - (\varphi, D^*\psi)$

$= d(\sim)$

Since the composition of differential operators is a differential operator, $(L_1 \circ L_2)^*$ is a differential operator. Since any differential operator is a linear combination of compositions of $\tilde{D}^k$ and 0^{th} order operators, (A) holds in general. ▨

Note that for $L = \tilde{D}$ or L of the 0^{th} order, support $(\tau) \subset$ support (f). Similarly, under composition so that for any L, support $(\tau) \subset$ support (f). Hence an elementary use of the ordinary Stokes theorem implies $\int_X \langle Lf, g \rangle \omega = \int_X \langle f, L^*g \rangle \omega$ for $f \in \Gamma_0(E_1)$. Uniqueness of L^* follows from this last formula.

Let $\sigma^*(d) \in \mathrm{Hom}(\tilde{E}_2^*, \tilde{E}_1^*)$ be the element defined by $\sigma^*(d)(\phi) = \sigma(d)(\phi)^*$, $\phi \in S^*(X)$. Since $\sigma(d_1 \circ d_2) = \sigma(d_1)\,\sigma(d_2)$, the verification that $\sigma(L^*) = \sigma^*(L)$ reduces to the case $L = \tilde{D}$.

In the case of first order differential operators associated to a G-structure, the formula (A) can be made more explicit. The examples of section 17.6, were of the type $D^1(\underline{a})$ where $\underline{a}$ was a constant cross section of $\Gamma(E_2 \otimes E_1^* \otimes T(X)) = (\mathrm{Hom}(T^*(X), \mathrm{Hom}(E_1, E_2))$ coming from a G-map $a : M_0^* \to \mathrm{Hom}(M_1, M_2)$. Furthermore, in almost all the examples, X has a Riemannian structure, and one can put metrics on M_1, M_2 invariant under G. Since $D^1(\underline{a}) = \underline{a} \circ \tilde{D}$, for $f \in \Gamma(E_1)$ and $g \in \Gamma(E_2^*)$, we have

$$< \underline{a} \circ \tilde{D}f, g > \omega = < \tilde{D}f, (\underline{a})^*(g) > \omega = < f, \tilde{D}^*(\underline{a}^*(g)) > \omega$$

$$+ d(\sum_j < f, \underline{a}^* g)_j > i(E_j)\tilde{\omega}) .$$

But $\tilde{D}^* \circ \underline{a}^* : \Gamma(E_2^*) \xrightarrow{a^*} (E_1^* \otimes T(X)) \xrightarrow{\tilde{D}^*} \Gamma(E_1^*)$ and $\tilde{D}^* \circ (\underline{a})^* = (\underline{a}_1 \circ \tilde{D} + \underline{a}_0) \circ (\underline{a})^*$ with

$\underline{a}_1 = -I \otimes I \in (E_1^* \otimes E_1 \otimes T^*(X) \otimes T(X))$.

Since $(\underline{a})^*$ is a constant cross section, $\tilde{D} \circ (\underline{a})^* = (\underline{a})^* \otimes I \circ \tilde{D}$

so that $\tilde{D}^* \circ (\underline{a})^* = \underline{a}_1 \circ (\underline{a})^* \otimes I \circ \tilde{D} + \underline{a}_0 \circ (\underline{a})^*$.

$a_0 = I \otimes \sum_j r_j \varphi_j$

But the map $\underline{a}_1 \circ (\underline{a})^* \otimes I : \Gamma(E_2^* \otimes T^*(X))$

$$\xrightarrow{a^* \otimes I} \Gamma(E_1^* \otimes T(X) \otimes T^*(X)) \xrightarrow{a_1} \Gamma(E_1^*))$$

is just $-\underline{a}^*$ where $a^* : M_0^* \to \mathrm{Hom}(M_2^*, M_1^*)$ with $a^*(\phi) = (a(\phi))^*$, $\phi \in M_0^*$. Thus

$(D^1(\underline{a}))^* = \tilde{D}^* \circ (\underline{a})^* = -\underline{a}^* \circ \tilde{D} + \underline{a}_0 \circ (\underline{a})^* = -D^1(\underline{a}^*) + a_0 \circ (\underline{a})^*$.

We now show that for an appropriate choice of connection h and volume element ω, $\underline{a}_0$ equals 0. Let C_0 be the orthonormal frame bundle of the oriented Riemannian space X, let h_0 be the Riemannian connection, and ω the Riemannian volume form. Let $\{\omega_{ij}\}$ ($\omega = \sum_{i<j} \omega_{ij} E_{ij}$ since in anti-symmetric) be the Lie algebra valued one forms on C_0 of the

connection, $\{\omega_i\}$ the usual tautological 1-forms on C_0, and $\mu_0 = \bigwedge_{i<j} \omega_{ij}$, whose restriction to the fiber is the invariant volume form. If $\{E^0_{ij}, E^0_j\}$ is the dual basis to $\{\omega_{ij}, \omega_j\}$, then

$\Theta(E^0_j)(\mu_0 \wedge \tilde{\omega}) = \Theta(E^0_j)(\mu \wedge \omega_1 \wedge \cdots \wedge \omega_n) = 0$ for

$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j$, i.e., the Riemannian connection has zero torsion.

Let us now return to the bundle C. If the representation ρ_0 of G on M_0 is consistent with the Riemannian structure, i.e., if $\rho_0(G) \subset SO(n)$, then ρ_0 induces a projection $\pi_0 : C \to C_0$ and one can find a connection h on $C \ni \pi_0 \circ h = h_0$. Then relative to (no torsion) this connection, $\alpha\pi_0(E_i) = E^0_i$, and $\Theta(E^0_j)(\mu_0 \wedge \tilde{\omega}) = 0$. Hence $r_j = 0$ and $\underline{a}_0 = 0$. [In the examples (iii)-(vii) of the previous section, C either equals C_0, is a double covering of C_0, or is a subbundle of C_0].

We have proved the

THEOREM. 2. *Let X be an oriented Riemannian manifold. Let $D^1(\underline{a})$ be a first order differential operator: $\Gamma(E_1) \to \Gamma(E_2)$ associated to a G-structure P on X with a a G-map : $M_0^* \to \mathrm{Hom}(M_1, M_2)$. Suppose there exists a connection h on P with 0-torsion. Then $D(\underline{a})^* = -D(\underline{a}^*)$ where $\underline{a}^* : M_0^* \to \mathrm{Hom}(M_2^*, M_1^*)$ is*

This proves that $D(l_\varphi^*) = \delta = d^*$

the map $a^*(\phi) = (a(\phi))^*$, $\phi \in M_0^*$. *Moreover, if* ω *is the Riemannian volume element,* $f \in \Gamma(E_1)$, $g \in \Gamma(E_2)$, *then* $< D^1(\underline{a})f,g > \omega + < f,D^1(\underline{a}^*)g > \omega = d(\sum_j < f, \underline{a}^*(\phi_j)g > i(E_j)\tilde{\omega})$ *where* ϕ_j *is the dual basis to* $\pi(E_j)$. $(e,f) \to \tau$ is linear

Remark: In the examples (ii)-(vi) a connection h with 0 torsion does exist. Each case can be checked directly.

Suppose now X is a compact Riemannian manifold with smooth boundary ∂X . We can apply Stokes theorem to the above and obtain

$$\int_X < D^1(\underline{a})f,g > \omega + \int_X < f,D(\underline{a}^*)g > \omega = \int_{\partial X} \sum_j < f, \underline{a}^*(\phi_j)g > i(E_j)\tilde{\omega}.$$

Observe, however, that the integrand on the right is independent of the oriented orthonormal base chosen, and that because of the metric we can identify the tangent space with its dual. Choose the first vector $\pi(E_1) = n$ the inward normal. Then $\int_{\partial X} i(E_j)\tilde{\omega} = 0$, for $j \neq 1$, so that $\int_{\partial X} \sum_j < f,\underline{a}^*(\phi_j)g > i(E_j)\tilde{\omega} = \int_{\partial X} < f,\underline{a}^*(n)g > \nu$ where $\nu = i(E_1)\omega$, the volume element on ∂X . Consequently, we have the

COROLLARY. *Let* X *be a compact oriented Riemannian manifold with smooth boundary* ∂X , ω , *the Riemannian volume element on* X , n *the inward normal field at*

∂X , ν the induced Riemannian volume element on ∂X .

Then $\int_X < D^1(\underline{a})f,g > \omega + \int_X < f,D^1(\underline{a}^*)g > \omega = \int_{\partial X} < f,\underline{a}^*(n)g > \nu$.

Remark: If, in addition, we assume that

$M_1 = M_2 = M_2^* = M_1^*$ (with metrics on M_1 , M_2) and

$a^* = a$, then

$$\int_X < D^1(\underline{a})f,g > \omega + \int_X < f,D^1(\underline{a})g > \omega = i \int_{\partial X} < \sigma(D^1(\underline{a}))(n)f,g > \nu$$

for $\frac{1}{i}\,\underline{a}(n) = \sigma(D^1(\underline{a}))(n)$.

This situation holds in example (vii) of the previous section.

a^*

$T^k(\mathbb{R}^n) \leftrightarrow T^k(\mathbb{R})^r$

$a^*(m^k)$ in general?

T^k $\quad a \leftarrow \delta^k \otimes$ — part OK —
but 1st parts? —

18. Singular integral operators and the index.

18.1. Some definitions. In the previous chapter we defined the symbol $\sigma(d)$ of a differential operator d as an element of $\mathrm{Hom}(\widetilde{E}_1, \widetilde{E}_2)$. We shall soon wish to consider homotopies of symbols. Unfortunately, the set of symbols of differential operators is not a wide enough class in which to perform homotopies, for these symbols restricted to a spherical fiber S_x^* come from polynomial maps on the cotangent space. We in fact want a class of operators whose symbols will be all of $\mathrm{Hom}(\widetilde{E}_1, \widetilde{E}_2)$. The class in question is the class of singular integral operators and their symbols involve the functional calculus of Calderon-Zygmund. We need the extension of these ideas to vector bundles. This has been done recently by R. Seeley in a paper to appear in the TAMS. We give a resume of what we need of that theory and refer the reader to this paper for proofs.

We suppose X is an oriented Riemannian compact manifold with volume element ω, M a complex G-module with G-invariant Hermitian inner product, $E = B \times_G M$ the vector bundle associated to the principal G-bundle B. The C^∞ cross sections $\Gamma(E)$ is a complex pre-Hilbert space with Hermitian inner product $\int_X \langle f, g \rangle \omega$ so that

$\|f\|_0^2 = \int_X \langle f, f\rangle\,\omega$. Let $H^0(E)$ denote their Hilbert space completion. The total differential $\tilde{D}$ maps $\Gamma(E) \to \Gamma(E \otimes T^*(X))$ and $\tilde{D}^k$ maps $\Gamma(E) \to \Gamma(E \otimes \underbrace{T^*(X) \otimes \cdots \otimes T^*(X)}_{k \text{ times}})$. We define a series of pre-Hilbert space norms on $\Gamma(E)$ by

$$\|f\|_r^2 = \sum_{k=0}^{r} \|\tilde{D}^k f\|_0^2 \ , \quad f \in \Gamma(E) ,$$

and let $H^k(E)$ denote the Hilbert space completion of $\Gamma(E)$ in this norm. We collect the relevant facts into a theorem.

THEOREM. ($\Gamma(E)$ = C^∞ sections)

(1) $\Gamma(E) \subset H^r(E)$ $r=0,1,2,\ldots$

(2) $H^{r+1}(E) \subset H^r(E)$. The identity map $\Gamma(E) \to H^r(E)$ is norm decreasing and is therefore extendable to a bounded operator: $H^{r+1}(E) \to H^r(E)$. (even compact)

(3) $\tilde{D} : H^1(E) \to H^0(E \otimes T^*(X))$ is a bounded operator. i.e. $\tilde{D}$ is of order one, this must be clear...

(4) There exists a compact (completely continuous) self-adjoint operator J on $H^0(E)$ such that

(a) $J^2 = (\tilde{D}^*\tilde{D} + I)^{-1}$ on $\Gamma(E)$ $\to \Gamma(E^*) = \Gamma(E)$. Question: what significance does D^*D have?

(b) $J : H^r(E) \to H^{r+1}(E)$ $r=0,1,2,$

(c) $J^{-r} : H^r(E) \to H^0(E)$ is $1 - 1$ onto. ([illegible])

(d) If V is a smooth vector field on X ,

then $D_V J$ is a bounded operator on $H^0(M)$;
if d is a differential operator of order r, dJ^r is a bounded operator on $H^0(E)$.

Definition. Let M_1 and M_2 be two complex G-modules. A smooth singular integral operator S is a bounded operator from $H^0(E_1)$ into $H^0(E_2)$ mapping $\Gamma(E_1)$ into $\Gamma(E_2)$ such that

(1) If $\phi, \psi \in C^\infty(X)$ with disjoint compact support and m_ϕ, m_ψ denote the operators multiplication by ϕ and ψ respectively, then $m_\phi S m_\psi$ is a compact operator: $H^0(E_1) \to H^0(E_2)$ which in addition maps $H^r(E_1) \to H^{r+1}(E_2)$.

(2) If $\phi \in C^\infty(X)$ with support in a small coordinate neighborhood U, then $m_\phi S m_\phi = R + \tilde{S}$ where R is a compact operator of $H^0(E_1) \to H^0(E_2)$ mapping $H^r(E_1) \to H^{r+1}(E_2)$, and $\tilde{S}$ is a singular integral operator as usually defined on Euclidean space. That is, in restricting our attention to U, the vector bundles are trivial bundles so that if the support of $f \subset U$, then f and $\tilde{S}f|_U$ are $\dim(M_1)$ and $\dim(M_2)$ tuples of functions. And

$$(\tilde{S}f)(x) = a(x)(f(x)) + \lim_{\varepsilon \to 0} \int_{d(x,y) > \varepsilon} h(x, x-y) f(y)\,dy, \quad x \in U .$$

Here a is a smooth mapping of U into $\dim(M_2) \times \dim(M_1)$

matrices, and $h(x,z)$ is a map from $T^*(U) - U$ into $\dim(M_2) \times \dim(M_1)$ matrices which is homogeneous of degree $-n$ in z and for which $\frac{\partial^k h}{\partial z^k}$, $k=0,1,\ldots$ are smooth mappings for $\|z\| \geq 1$.

Definition. A singular integral operator is an operator in the norm closure of the linear space of smooth singular integral operators.

Thus the set of singular integral operators $\mathcal{S}$ is a linear space of bounded operators: $H^0(M_1) \to H^0(M_2)$ closed in the norm topology. It is easy to see that it contains the compact operators and also $\Gamma(\mathrm{Hom}(E_1,E_2))$. If S is a smooth singular integral operator, one can define its symbol $\sigma(S) \in \Gamma(\mathrm{Hom}(\tilde{E}_1,\tilde{E}_2))$ where $\sigma(S)(x,\phi) = a(x) +$ the Fourier transform in the z-variable of $h|_{S^*_x}$. This turns out to be independent of the local representation of S. Furthermore, the symbol can be extended to all of $\mathcal{S}$. a little mysterious?

THEOREM. Let $\Gamma_0(\mathrm{Hom}(\tilde{E}_1,\tilde{E}_2))$ denote the continuous cross sections of the vector bundle $\mathrm{Hom}(\tilde{E}_1,\tilde{E}_2)$ in the uniform norm. Then (i) the symbol σ : is a continuous linear map of $\mathcal{S}$ onto $\Gamma_0(\mathrm{Hom}(\tilde{E}_1,\tilde{E}_2))$ whose kernel is the set of compact operators. (ii) If d is a differential operator of order r, then $d = SJ^{-r}$, S

<u>a smooth singular integral operator</u> [<u>as maps from</u> $H^r(E_1) \to H^0(E_2)$] <u>and</u> $\sigma(S) = \sigma(d)$. (iii) <u>If</u> $S \in \mathcal{S}(E_1,E_2)$, <u>then</u> $S^* \in \mathcal{S}(E_2^*,E_1^*)$ <u>and</u> $\sigma(S^*) = \sigma(S)^*$ <u>where</u> $\sigma(S)^*(x,\phi) = (\sigma(S)(x,\phi))^*$. (iv) <u>If</u> $S \in \mathcal{S}(E_1,E_2)$ <u>and</u> $T \in \mathcal{S}(E_2,E_3)$, <u>then</u> $TS \in \mathcal{S}(E_1,E_3)$ <u>and</u> $\sigma(TS) = \sigma(T)\sigma(S)$.

<u>Definition.</u> A singular integral operator S is <u>elliptic</u> if $\sigma(S)(x,\phi)$ is 1 - 1 onto for all $(x,\phi) \in S^*(X)$, i.e., $\sigma(S) \in \Gamma((\mathrm{ISO}\ \tilde{E}_1,\tilde{E}_2))$.

18.2. <u>The index.</u> If $d : \Gamma(E_1) \to \Gamma(E_2)$ is an elliptic differential operator of order r , then as a mapping from $H^r(E_1) \to H^0(E_2)$ it has finite dimensional kernel, closed range, and finite dimensional cokernel. The regularity theorems show that, as a mapping from $\Gamma(E_1) \to \Gamma(E_2)$, d has finite dimensional kernel and cokernel and dim(ker(d)), dim(cok(d)) are independent of which domain and range spaces are chosen. We define the index of d, $i(d) = \dim \ker(d) - \dim \operatorname{cok}(d)$. It is the aim of these notes to find an explicit formula for $i(d)$ in topological terms involving $\sigma(d)$.

If $S \in \mathcal{S}(E_1,E_2)$ is elliptic, then $S : H^0(E_1) \to H^0(E_2)$ also has finite dimensional kernel and cokernel, and a closed range. Again, we define the index of S , $i(S) = \dim \ker(S) - \dim \operatorname{cok}(S) = \dim(\ker S) - \dim(\ker S^*)$.

We collect the properties of the index in the

THEOREM. (i) *If* d *is an elliptic differential operator of order* r , *and* $d = SJ^{-r}$, *as in the previous theorem, then* $i(d) = i(S)$.

(ii) $i(S^*) = -i(S)$, S *elliptic.*

(iii) *If* $\sigma(S_1) = \sigma(S_2)$, *with* S_1 *and* S_2 *elliptic singular integral operators, then* $i(S_1) = i(S_2)$, *i.e., the index depends only on the symbol.*

(iv) *The map* $\sigma(S) \to i(S)$ *is a continuous integer valued function on* $\Gamma_0^*(\mathrm{Hom}(\tilde{E}_1, \tilde{E}_2))$, *the non-singular cross-sections of* $\Gamma_0(\mathrm{Hom}(\tilde{E}_1, \tilde{E}_2))$.

(v) *If* $S \in \mathcal{S}(E_1, E_2)$ *and* $T \in \mathcal{S}(E_2, E_3)$ *are elliptic, then* TS *is elliptic, and* $i(TS) = i(T) + i(S)$.

(vi) *If* $S \in \mathcal{S}(E_1, E_2)$ *and* $T \in \mathcal{S}(E_3, E_4)$ *are elliptic, then* $S \oplus T \in \mathcal{S}(E_1 \oplus E_3, E_2 \oplus E_4)$ *is elliptic, and* $i(S \oplus T) = i(S) + i(T)$.

We are now ready to tie up the analytic facts concerning the symbol and index with the topology of the symbol. Let $B(X)$ denote the unit ball in the cotangent bundle and p the projection: $B(X) \to X$. Using the notation of section 8, every $\sigma \in \Gamma_0^* \mathrm{Hom}(\tilde{E}_1, \tilde{E}_2)$ gives an element $\phi(\sigma) : 0 \to p^*(E_1) \overset{\sigma}{\to} p^*(E_2) \to 0$ of $C_1(B(X), S^*(X))$.

COROLLARY. <u>If</u> $\phi(\sigma)$ <u>is isomorphic to</u> $\phi(\sigma')$, <u>then</u> $i(\sigma) = i(\sigma')$.

Proof. $\phi(\sigma)$ isomorphic to $\phi(\sigma')$ means

$$\begin{array}{ccccccc} 0 & \to & p^*(E_1) & \overset{\sigma}{\to} & p^*(E_2) & \to & 0 \\ & & \downarrow \alpha & & \downarrow \alpha' & & \\ 0 & \to & p^*(E_1') & \overset{\sigma'}{\to} & p^*(E_2') & \to & 0 \end{array}$$

where α and α' are isomorphisms on $B(X)$. Let c denote the 0 cross-section: $X \to B(X)$. Then the symbols $\alpha|_{S^*(X)}$ and $\alpha'|_{S^*(X)}$ are homotopic to the symbols $\alpha \circ c \circ p$ and $\alpha' \circ c \circ p$ respectively. But $\alpha \circ c \circ p$ $(\alpha' \circ c \circ p)$ is a nonsingular element of $\Gamma(\mathrm{Hom}(E_1, E_2))$ and therefore an invertible 0^{th} order differential operator. Its index is zero. But $\alpha_{|S^*(X)}$ is homotopic to $\alpha \circ c \circ p$ so by (iv) of the previous theorem, $i(\alpha_{|S^*(X)}) = 0$. Hence

$$i(\sigma) = i(\alpha'_{|S^*(X)} \circ \sigma' \circ \alpha_{|S^*(X)}) = i(\alpha'_{|S^*(X)}) + i(\sigma') + i(\alpha_{|S^*(X)}) = i(\sigma').$$

Using this corollary, we can extend the index to be an integer valued function on $C_1(B(X), S^*(X))$. For, since $B(X)$ is homotopic to X, every element $E \in C_1(B(X),S^*(X))$ is isomorphic to $d(\sigma)$ for some $\sigma \in \Gamma_0^*(\mathrm{Hom}(\tilde{E}_1,\tilde{E}_2))$. Define $i(E) = i(\sigma)$. The corollary shows $i(E)$ is well defined.

COROLLARY. *The index* i *is a map of* $C_1(B(X),S^*(X))$ *into the integers such that* $i(E \oplus F) = i(E) + i(F)$ *and* $i(E) = i(F)$ *if* $E \sim F$. *Hence the index induces a homomorphism* $\tilde{i}$ *of the semigroup* $L_1(B(X),S^*(X))$ *into the integers.*

Proof. $i(E \oplus F) = i(E) + i(F)$ by (v) of the previous theorem. If E is isomorphic to F, then $i(E) = i(F)$ by the previous corollary. If $P : 0 \to P_1 \xrightarrow{I} P_2 \to 0$ is an elementary sequence in $C_1(B(X),S^*(X))$, then $i(P) = 0$, for $P = \phi(\sigma)$, $\sigma = I$. Thus if $E \sim F$, then $i(E) = i(F)$ because this equivalence is generated by isomorphisms and the addition of elementary sequences.

Now, by Proposition 10.1, there exists a unique natural isomorphism $\chi : L_1(B(X),S^*(X)) \to K(B(X),S^*(X))$. Hence, we can view $\gamma = \tilde{i} \circ \chi^{-1}$ as a homomorphism of the abelian group $K(B(X),S^*(X))$ into the integers. We now use Theorem 13.4 with $V = T^*(X)$. Then $K^*(B(X),S^*(X)) \otimes Q$ is a free $K^*(X) \otimes Q$ module generated by an element ν, $\dim X = 2\ell$. Since

$\nu \in K(B(X), S^*(X))$, this theorem implies that $K(B(X), S^*(X)) \otimes Q$ is a free $K(X) \otimes Q$ module generated by ν. But the definition ν [see section 7] shows that $\nu = -\chi \circ d(\sigma(d))$ where $\sigma(d)$ is the symbol of the operator given in example (iv) of 17.6. Extend γ to a homomorphism: $K(B(X), S^*(X)) \otimes Q \to Q$ and let $\sigma_0 W$ denote the symbol $\sigma(d)$ of 17.6 (iv) with the vector bundle W as coefficients. We have the

COROLLARY. Let X be a compact oriented manifold of even dimension. If μ is a homomorphism: $K(B(X), S^*(X)) \otimes Q \to Q$ such that $\mu \circ \chi \circ \phi(\sigma_0 W) = \gamma \circ \chi \circ \phi(\sigma_0 W)$, for all complex vector bundles W, then $\mu = \gamma$.

This corollary shows that to find a formula for the index, it suffices to find one which agrees with the index on the basic first order operators whose symbols are $\sigma_0 W$.

18.3. Cobordism. So far we have kept the base manifold X fixed. We now vary X and consider the set Σ of pairs (X,W), X a compact oriented even dimensional manifold and W a complex vector bundle over X. To emphasize the dependence on X, we will denote $\sigma_0 W$ by $\sigma_0(X,W)$ and we let $X \cdot W$ denote the element $\chi \circ \phi(\sigma_0(X,W))$ of $K(B(X), S^*(X))$.

THEOREM: $\gamma(X_1 \times X_2 \cdot W_1 \otimes W_2) = \gamma(X_1 \cdot W_1)\gamma(X_2 \cdot W_2)$.

Proof: Let $\dim X_i = 2k_i$, $i=1,2$. Now the symbol $\sigma_0(X,W)$ arises from the G-map $a_{2k} : R^{2k} \to \mathrm{Hom}(M_{2k}^+ \otimes W, M_{2k}^- \otimes W)$ of section 17.6 (iv). Here $a_{2k}(t)$ = Clifford multiplication by $t \otimes I$. Because $R^{2(k_1+k_2)} = R^{2k_1} \oplus R^{2k_2}$ and because of the multiplicative properties of Clifford modules exposed in sections 6 and 12, one has

$$M_{2(k_1+k_2)}^+ \otimes W_1 \otimes W_2 = (M_{2k_1}^+ \otimes W_1) \otimes (M_{2k_2}^+ \otimes W_2) \oplus$$

$$(M_{2k_1}^- \otimes W_1) \otimes (M_{2k_2}^- \otimes W_2) \text{ and } M_{2(k_1+k_2)}^- \otimes W_1 \otimes W_2$$

$$= (M_{2k_1}^- \otimes W_1) \otimes (M_{2k_2}^+ \otimes W_2) \oplus (M_{2k_1}^+ \otimes W_1) \otimes (M_{2k_2}^- \otimes W_2)$$

while $$a_{2(k_1+k_2)}(t_1,t_2) = \begin{pmatrix} a_{2k_1}(t_1) \otimes I & -I \otimes a_{2k_2}(t_2)^* \\ I \otimes a_{2k_2}(t_2) & a_{2k_1}(t_1)^* \otimes I \end{pmatrix}$$

where $a_{2k_1}(t_1) \otimes I \in \mathrm{Hom}(M_{2k_1}^+ \otimes W_1 \otimes M_{2k_2}^+ \otimes W_2,$ $M_{2k_1}^- \otimes W_1 \otimes M_{2k_2}^+ \otimes W_2)$, etc.. Passing to the symbols, one gets

$$\sigma_0(X_1 \times X_2, W_1 \otimes W_2) = \begin{pmatrix} \sigma_0(X_1,W_1) \otimes I & -I \otimes \sigma_0^*(X_2,W_2) \\ I \otimes \sigma_2(X_2,W_2) & \sigma_0^*(X_1,W_1) \otimes I \end{pmatrix}.$$

Now $\sigma_0(X,W)$ is the symbol of an elliptic first order differential operator $d(X,W)$ so that $d(X_1 \times X_2, W_1 \otimes W_2)$ has the same symbol as $d' = \begin{pmatrix} d(X_1,W_1) \otimes I & -I \otimes d^*(X_2,W_2) \\ I \otimes d(X_2,W_2) & d^*(X_1,W_1) \otimes I \end{pmatrix}$

(In fact, if the product Riemannian metric were chosen on $X_1 \times X_2$ and the Riemannian connection used throughout, these two first order differential operators are equal.) Hence $\gamma(X_1 \times X_2 \cdot W_1 \otimes W_2) =$ index $d(X_1 \times X_2, W_1 \otimes W_2) =$ index d'. We now apply section 13 of Seeley (developed to handle this situation). In his notation, $d(X_i,W_i)$ $i=1,2$ are elliptic A_∞ operators of order 1 and $d' = d(X_1,W_1) \quad d(X_2,W_2)$ so that by theorem 13.2,
index $d' =$ index $d(X_1,W_1) \cdot$ index $d(X_2,W_2)$. Hence $\gamma(X_1 \times X_2, W_1 \otimes W_2) =$ index $d(X_1,W_1) \cdot$ index $d(X_2,W_2) =$ $\gamma(X_1,W_1)\gamma(X_2,W_2)$.

<u>Definition</u>. $(X,W) \sim 0$ if there exists a compact manifold Y and vector bundle $\widetilde{W}$ over Y such that $\partial Y = X$ and $\widetilde{W}|_X = W$.

THEOREM. <u>If</u> $(X,W) \sim 0$, <u>then</u> $\gamma(X,W) = 0$.

<u>Proof</u>. Choose $(Y,\widetilde{W})$ Y oriented so that $\partial Y = X$ and $\widetilde{W}|_X = W$. Consider the differential operator L

of example 17.6 (vii). Let $C_{2\ell+1}(Y)$ denote the complex Clifford vector bundle associated to the $SO(2\ell+1)$ module $C(R_{2\ell+1}) \otimes C$. Similarly, let $C^0_{2\ell+1}(Y)$ be the subbundle of even elements so that $L : (C^0_{2\ell+1}(Y) \otimes \tilde{W}) \to (C^0_{2\ell+1}(Y) \otimes \tilde{W})$.

We apply the Stokes formula of section 17.7 and find

$$\begin{aligned} < Lf,g >_Y + < f,Lg >_Y &= i^{\ell} <(\eta w \otimes I)f,g >_X \\ &= -< i^{\ell}(w' \otimes I)f,g >_X \end{aligned}$$

where η is the inward unit normal and $w' = \eta w = e_1 \cdots e_{2\ell} \in C^0_{2\ell}(X) \subset C^0_{2\ell+1}(Y)|_X$. But $(i^{\ell} w' \otimes I)^2 = I \otimes I$ so that over each point of X, $C^0_{2\ell+1}(Y) \otimes \tilde{W}|_X$ splits into the orthogonal vector bundles $C^+_{2\ell+1}(Y) \otimes \tilde{W}$ and $C^-_{2\ell+1}(Y) \otimes \tilde{W}$, the ± 1 eigenspaces of $i^{\ell} w' \otimes I$. Let $B^{\pm}$ denote the projection of $\Gamma(C^0_{2\ell+1}(Y) \otimes \tilde{W})|_X$ onto $\Gamma(C^{\pm}_{2\ell+1}(Y) \otimes \tilde{W})_X$. Consider now the two boundary value problems (L,B^+) and (L,B^-). These are coercive boundary value problems in the sense of Agmen-Douglas-Nirenberg II (to appear). See also Agranovic-Dynin [Soviet Math. vol 3 #5 (1962) pp. 1320-1323] and Hörmander [Linear Partial Diff. Operators, Chap. X]. We shall not go into the general definition here, but in our case for an operator of order 1 the general definition reduces to this: At any point x in ∂Y, let t be a unit vector of $T(\partial Y)$, n be the inward normal in $T(\partial Y)$, and let $p_t(\lambda)$ denote the polynomial of degree $2k = \dim(C^0_{2\ell+1}(Y) \otimes \tilde{W})$ which is determinant of $(\sigma(L)(t) + \lambda\sigma(L)(n))$. Since L is elliptic $p_t(\lambda)$ has no real roots and $C^0_{2\ell+1}(Y) \otimes \tilde{W}|_x$ splits into two subspaces M^+_t and M^-_t of dim k spanned by the generalized eigenvectors of $\sigma(L)(t) + \lambda\sigma(L)(n)$ corresponding to eigenvalues λ in the upper and lower halfplanes respectively. A boundary value problem (L,B) is coercive (elliptic) if $B \varepsilon \Gamma(\mathrm{Hom}(C^0_{2\ell+1}(Y) \otimes \tilde{W}|_{\partial Y}, U))$, U a vector bundle of dim k, and at each $x \varepsilon \partial Y$, $M^+_t \cap (\text{null space of } B) = (0)$ for all $t \varepsilon T(\partial Y)$.

In our special situation $\sigma(L)(t) + \lambda\sigma(L)(n) = i^{\ell+1}(tw - \lambda w') \otimes I$. It is easy to verify that the eigenvalues are $\pm i$ and that $M^{\pm}_t = \{(w' \pm iwt)C^0_{2\ell+1}(Y) \otimes \tilde{W}\}$. Since $w'(tw + iw') = -(tw - iw')w'$, $w'(M^+_t) = M^-_t$ and hence $M^+_t \cap (C^{\pm}_{2\ell+1}(Y) \otimes \tilde{W}) = (0)$. Thus the boundary value problems $(L,B^{\pm})$ are coercive.

We now follow Agranovicz-Dynin. The pairs $(L,B^{\pm})$ define linear operators from

$$H^1(C^0_{2\ell+1}(Y) \otimes \tilde{W}) \to H^0(C^0_{2\ell+1}(Y) \otimes \tilde{W}) + H^{\frac{1}{2}}(C^{\pm}_{2\ell+1}(Y) \otimes \tilde{W}|_{\partial Y})$$

by $(L,B^{\pm})f = (Lf, B^{\pm}f)$. Here the $\frac{1}{2}$-space $H^{\frac{1}{2}}(E)$, E a vector bundle on X is defined in terms of J^{-1}, the positive square root of $\widetilde{D}^*\widetilde{D} + I$ (see 18.1). If $g \in \Gamma(E)$, define $\|g\|^2_{\frac{1}{2}} = \langle J^{-1}g, g \rangle$. $H^{\frac{1}{2}}(E)$ is the completion of $\Gamma(E)$ in this norm and $H^{\frac{1}{2}}(E) \subset H^0(E)$. Since the boundary value problems, $B^{\pm}$, are coercive, the operators $(L,B^{\pm})$ have finite dimensional kernel and cokernel and therefore index $(L,B^{\pm})$ exists.

Now there exists a well-defined singular integral operator $S : H^0(C^+_{2\ell+1}(Y) \otimes \widetilde{W}|_{\partial Y}) \to H^0(C^-_{2\ell+1}(Y) \otimes \widetilde{W}|_{\partial Y})$ which maps $H^{\frac{1}{2}}(C^+_{2\ell+1}(Y) \otimes \widetilde{W}|_{\partial Y}) \to H^{\frac{1}{2}}(C^-_{2\ell+1}(Y) \otimes \widetilde{W}|_{\partial Y})$ such that index (L,B^-) = index S + index (L,B^+). Furthermore, the symbol of S can be computed as follows. Since both the null spaces of $B^{\pm}$ are complementary to M_t^+, the projection of M_t^+ on the ranges $C^{\pm}_{2\ell+1}(Y) \otimes \widetilde{W}|_{\partial Y}$ are isomorphisms $\alpha_t^{\pm}$. Then it turns out that $\sigma(S)(t) = \alpha_t^-(\alpha_t^+)^{-1}$. In our case, because $\alpha_t^{\pm} = \dfrac{I \pm i^{\ell} w' \otimes I}{2}$, it is easy to show that

$(-it\,\pi X \otimes I)\quad \alpha_t^+ = \alpha_t^-$ so that $\sigma(S)(t) = -it \otimes I$.

We now wish to relate $\sigma(S)$ with the symbol we are interested in, namely $\sigma_0(X,W)$. By proposition 5.2, the map $i : R^{2\ell} \to R^{2\ell+1}$ yields an isomorphism

$\phi : C_{2\ell} \to C^0_{2\ell+1}$ where $\phi(u^0+u^1) = u^0 + e_{2\ell+1}u^1$ with $u^i \in C^i_{2\ell}$, $i=0,1$ and $e_{2\ell+1} \perp R^{2\ell}$. Using the unit normal π along X, we have the isomorphism $\phi \otimes I$ of vector bundles $C_{2\ell}(X) \otimes W$ with $C^0_{2\ell+1}(Y) \otimes \widetilde{W}|_X$. This induces an isomorphism

$\widetilde{\phi} : \Gamma(C_{2\ell}(X) \otimes \widetilde{W}) \to \Gamma(C^0_{2\ell+1}(Y) \otimes \widetilde{W}|_X)$ by

$\widetilde{\phi}f = (\phi \otimes I) \circ f$. The operator $i^\ell w' \otimes I$ on

$C^0_{2\ell+1}(Y) \otimes \widetilde{W}|_X$ transforms into

$(\phi^{-1} \otimes I)(i^\ell w' \otimes I)(\phi \otimes I) = i^\ell w' \otimes I$ on $C_{2\ell}(X) \otimes W$

because $w' \in C_{2\ell}(X)$. Let $C^{\pm}_{2\ell}(X) \otimes W$ denote the $\pm$ eigenspaces of $i^\ell w' \otimes I$ on $C_{2\ell}(X) \otimes W$ so that

$\phi^{\pm} : C^{\pm}_{2\ell}(X) \otimes W \to C^{\pm}_{2\ell+1}(Y) \otimes \widetilde{W}|_{\partial Y}$ are isomorphisms,

where $\phi^{\pm} = \phi \otimes I|_{C^{\pm}_{2\ell}(X) \otimes W}$. The singular integral operator S induces on $\widetilde{S} = (\phi^-)^{-1} \circ S \circ \phi^+$ and

$\sigma(\widetilde{S})(t) = (\phi^-)^{-1}\sigma(S)(t)\phi^+ = it \otimes I$. Note, however, that the bundles $C^{\pm}_{2\ell}(X) \otimes W$ are exactly those that occur in example (iv) of 17.6 and the differential operator of that example has symbol $\sigma_0(X,W)(t) = \frac{1}{i}\, t \otimes I$, so that $\sigma(\widetilde{S}) = -\sigma_0(X,W)$. Since $i(\widetilde{S}) = i(S)$, to prove that $\gamma(X,W) = 0$, we must show that $i(S) = 0$.

Since $i(S) = \text{index}\ (L,B^-) - \text{index}\ (L,B^+)$, it suffices to show that index $(L,B^{\pm}) = 0$.

We first note that the kernel of $(L,B^{\pm}) = [f \in H^1(C^0_{2\ell+1}(Y) \otimes \widetilde{W})\ ;\ Lf = 0 \text{ and } f|_X = 0\ ]$. For $f \in$ kernel of $(L,B^{\pm})$ means that $Lf = 0$ and $B^{\pm}f|_X = 0$. By Stokes, $B^{\mp}f = 0$ so that $f|_X = 0$. Let us now examine the orthogonal complement of the range of (L,B^+) , namely $R^+ = [g + a;\ < Lf,g >_Y + < B^+f|_X, a >_{\frac{1}{2}} = 0,$ $g = H^0(C^0_{2\ell+1}(Y) \otimes \widetilde{W})$, $a \in H^{\frac{1}{2}}(C^+_{2\ell+1}(Y) \otimes \widetilde{W}|_X)$. However, because of the regularity of the coercive problem, $g + a \in R^+$ implies $g \in \Gamma(C^0_{2\ell+1}(Y) \otimes \widetilde{W})$ and $a \in \Gamma(C^+_{2\ell+1}(Y) \otimes \widetilde{W}|_X)$. See Hörmander, Linear Partial Differential Operators, p. 273. Hence by Stokes, for all f with compact support, $g + a \in R^+$ implies $< f,Lg >_Y = 0$, i.e., $Lg = 0$. Consequently, for any $f \in H^1(C^0_{2\ell+1}(Y) \otimes \widetilde{W})$, $< Lf,g >_Y + < f,Lg >_Y = < Lf,g >$

$= i^{\ell}(< B^-f|_X, g|_X >_X - < B^+f|_X, g|_X >_X) = - < B^+f|_X, a >_{\frac{1}{2}}$,

or $< B^-f|_X, g_X >_X = < B^+f|_X, g|_X - i^{\ell}J^{-\frac{1}{2}}a >_X$. Since

one can extend any element of $\Gamma(C^{\pm}_{2\ell+1}(Y) \otimes \widetilde{W}|_X)$ to an

element of $\Gamma(C^0_{2\ell+1}(Y) \otimes \widetilde{W})$, we conclude, $B^-g|_X = 0$

and $a = \frac{1}{i^{\ell}} J^{\frac{1}{2}}(B^+g)|_X$. But $Lg = 0$ and $B^-g|_X = 0$

implies $B^+g|_X = 0$ so that $a = 0$. Hence $R^+ = [g \neq 0 ; Lg = 0$ and $g|_X = 0]$ and $\dim R^+ = \dim$ kernel of (L,B^+). Hence index $(L,B^+) = 0$; similarly for (L,B^-). q.e.d.

We remark that our original proof of this theorem required analyticity. It ran as follows. The uniqueness of the Cauchy problem shows that $[Lf = 0 ; f|_X = 0] = (0)$. Hence, by the argument of the previous paragraph, $R^{\pm} = 0$; in particular, for any $a \in \Gamma(C^{\pm}_{2\ell+1}(Y) \otimes \tilde{W}|_X)$, there exists an $f \in \Gamma(C^0_{2\ell+1}(Y) \otimes \tilde{W}) \ni Lf = 0$ and $B^{\pm}f = a$.

Now, for all $f \in \Gamma(C^0_{2\ell+1}(Y) \otimes \tilde{W})$ with $Lf = 0$, consider the operator $T : B^+f \to B^-f$. This operator is well defined by Stokes' theorem which in fact shows that $\| B^+f \| = \| B^-f \|$; i.e., T is an isometry where defined. The previous paragraph shows that T has dense domain and range and hence can be extended to a unitary operator of $H^0(C^+_{2\ell+1}(Y) \otimes \tilde{W}|_X) \to H^0(C^-_{2\ell+1}(Y) \otimes \tilde{W}|_X)$.

By using the estimates in Agmen-Douglas-Nirenberg one can show directly that T is a singular integral operator and $\sigma(T) = \sigma_0(X,W)$. Since T is unitary, index $T = 0$, so that $\gamma(X,W) = 0$.

Actually, for the proof of the index formula, the analytic case suffices for one can show that X is diffeomorphic to an analytic manifold X^1, the boundary of an analytic manifold Y^1 and one find analytic vector bundles W^1 and $\widetilde{W}^1$ equivalent to W and $\widetilde{W}$.

One final observation before we discuss the formula for the index. If we denote by $X_1 \dotplus X_2$ the disjoint union of two compact oriented even dimensional manifolds and $W_1 \dotplus W_2$ the complex vector bundle which is W_i on X_i, then clearly the index can be extended to satisfy $\gamma(X_1 \dotplus X_2, W_1 \dotplus W_2) = \gamma(X_1,W_1) + \gamma(X_2,W_2)$.

19. The index theorem.

19.1. Statement of the theorem. In what follows considerable care has to be taken with sign conventions, orientation etc. We hope that our choices of sign are the right ones!

Let X be a compact oriented manifold. Then its tangent bundle $T(X)$ has an induced orientation. A choice of metric gives an isomorphism

$$T(X) \cong T^*(X)$$

and hence an induced orientation on $T^*(X)$. This orientation does not depend on the choice of metric. We shall always take this orientation[1] of $T^*(X)$. If $B(X)$, $S(X)$ denote the unit ball and unit sphere bundle in $T^*(X)$, then the orientation of $T^*(X)$ defines a fundamental class:

$$U = U_X \in H^n(B(X), S(X); Q) \qquad n = \dim X$$

and the Thom isomorphism

$$\psi_* = \psi_*^X : H^*(X;Q) \to H^*(B(X), S(X); Q)$$

is given by $\psi_*(x) = Ux$, so that $U = \psi_*(1)$.

The Chern character gives an isomorphism

$$\mathrm{ch}: K^*(B(X), S(X)) \otimes Q \to H^*(B(X), S(X); Q) .$$

---

[1] There are good reasons for choosing another "natural" orientation of $T^*(X)$, which would simplify signs later, but we shall stick to the simple orientation given here. (presumably the one coming from the 'symplectic' structure)

Now if S is an elliptic operator with symbol $\sigma(S)$, then we get an element

$$\phi\sigma(S) \in L_1(B(X), S(X))$$

and then an element

$$\chi\phi\sigma(S) \in K(B(X), S(X)) .$$

We define our basic cohomological invariant $\operatorname{ch} S \in H^*(X; Q)$ by the formula

$$\operatorname{ch} S = \varepsilon(n)\ \psi_*^{-1}\ \operatorname{ch}(\chi\sigma(S))$$

where $n = \dim X$ and[1]

$$\begin{aligned} \varepsilon(n) &= +1 \quad \text{if} \quad n = 1 \text{ or } 2 \mod 4 \\ &= -1 \quad \text{if} \quad n = 0 \text{ or } 3 \mod 4 , \end{aligned}$$

or $\varepsilon(n) = (-1)^{\rho(n)}$ with $\rho(n) = \frac{1}{2}n(n+1) + 1$.

We note the multiplicative property of $\operatorname{ch} S$:

PROPOSITION 1. *Let* S, T *be elliptic operators of order* $r > 0$ *on* X, Y *respectively, so that* $S \# T$ *is an elliptic operator on* $X \times Y$. *Then we have*

$$\operatorname{ch}(S \# T) = \operatorname{ch} S \cdot \operatorname{ch} T$$

Proof: By (11.2) we have

$$\chi\phi\sigma(S \# T) = -\chi\phi\sigma(S \# T)^1 = -\chi\phi\sigma(S) \cdot \chi\phi\sigma(T) .$$

[1] This sign factor amounts of course to taking a new orientation of $T^*(X)$. One minus sign is naturally accounted for by the fact that our definition of χ was designed for complexes with decreasing degrees:

$$\to E_n \to E_{n-1} \to \cdots$$

whereas the definition of the index is more appropriate for increasing degrees.

Hence, putting $\dim X = k$, $\dim Y = \ell$ and $Z = X \times Y$,

$$\operatorname{ch} \chi\phi\sigma(S \ast T) = - [U_X(\psi_*^X)^{-1} \chi\phi\sigma(S)] \quad [U_Y(\psi_*^Y)^{-1} \chi\phi\sigma(T)]$$

$$= - \varepsilon(k)\, \varepsilon(\ell)\, [U_X \operatorname{ch} S \cdot U_Y \cdot \operatorname{ch} T]$$

$$= - \varepsilon(k)\, \varepsilon(\ell)(-1)^{k\ell}[U_Z \operatorname{ch} S \cdot \operatorname{ch} T] .$$

Then

$$\operatorname{ch}(S \ast T) = - \varepsilon(k + \ell)\, \varepsilon(k)\, \varepsilon(\ell)(-1)^{k\ell} \operatorname{ch} S \cdot \operatorname{ch} T$$

$$= \operatorname{ch} S \cdot \operatorname{ch} T ,$$

since $\rho(k+\ell) + \rho(k) + \rho(\ell) = k(k+1) + \ell(\ell+1) + k\ell + 3$

$$= k\ell + 1 \mod 2 .$$

Recall next that for any complex vector bundle ξ the Todd class $\tau(\xi)$ is a polynomial in the Chern classes of ξ defined by

$$\tau(\xi) = \prod_i \frac{x_i}{1-e^{-x_i}}$$

where the Chern classes $c_k(\xi)$ are as usual the elementary symmetric functions in the x_i . For a differential manifold X we then define $\tau(X)$ by

$$\tau(X) = \tau(T(X) \otimes_R C) .$$

Thus $\tau(X)$ is a polynomial in the Pontrjagin classes of X , given by

$$\tau(X) = \prod_i \frac{x_i}{1-e^{-x_i}} \cdot \prod_i \frac{-x_i}{1-e^{x_i}}$$

where the Pontrjagin classes $p_k(x)$ are the elementary symmetric functions in the x_i^2. Note that we can also write

$$\tau(X) = \prod_i \left(\frac{x_i}{e^{x_i/2} - e^{-x_i/2}}\right)^2$$

Finally, for any $a \in H^*(X; Q)$, we denote by $a[X]$ the value of the top-dimensional component of a on the fundamental class of X.

Now we are in a position to state the main theorem.

INDEX THEOREM: *Let* S *be an elliptic operator on the compact oriented manifold* X. *Then its index is given by the formula:*

$$\text{index } (S) = \{\text{ch } S \cdot \tau(X)\} [X] .$$

We shall say that the index theorem holds for X if it holds for all elliptic operators on X. Then from Proposition 1 and that fact that

$$\tau(X \times Y) = \tau(X)\tau(Y)$$

we deduce

PROPOSITION 2. *Suppose the index theorem holds for* $X \times Y$ *and for* Y, *and suppose further that there exists an elliptic operator on* Y *with non-zero index. Then the index theorem holds for* X.

19.2. Some special cases. (a) Suppose $\dim X = 2\ell$ and let S be the differential operator

$$d + \delta : \text{even forms} \to \text{odd forms}$$

of Section (17.2) Ex. (iii). Then as observed in (17.2) we have

$$\text{index } S = \sum (-1)^q b_q = \text{Euler number of } X$$

where b_q is the q-th Betti number of X. On the other hand using Theorem (14.1) and the formula for the characters of the exterior powers we get

$$\text{ch } S = \varepsilon(2\ell) \cdot (-1) \prod_{i=1}^{\ell} \frac{(1-e^{x_i})(1-e^{-x_i})}{x_i}$$

where $p_k(X)$ are the elementary symmetric functions of the $x_i{}^2$ and $\prod_{i=1}^{\ell} x_i$ is the Euler class $E(X)$ of X. Hence

$$\text{ch } S = (-1)^{\ell+1} \varepsilon(2\ell) E(X) = E(X) .$$

Since $\tau(X) = 1 +$ higher terms we obtain

$$\{\text{ch } S \cdot \tau(X)\} [X] = E(X) [X] .$$

But $E(X) [X]$ is the Euler number of X, so that the index theorem is verified in this case.

(b) Suppose $\dim X = 4k$ and let S be the basic differential operator

$$d + \delta : \Gamma(E^+) \to \Gamma(E^-)$$

of (17.2) Ex. (iv). Then as pointed out there we have

$$\text{index } S = \text{Hirzebruch index of } X .$$

As this case is important we fill in the details here. By Hodge's theorem we identify $H^{2k}(X; R)$ with $H^{2k}(X)$, the space of harmonic 2k-forms. The Hirzebruch index is defined to be the index (number of + elements minus the number of - elements in the diagonal form) of the symmetric bilinear form on $H^{2k}(X; R)$ given by

$$f(\alpha,\beta) = \alpha\,\beta[X] .$$

In terms of harmonic forms this becomes

$$f(\alpha,\beta) = \int_X \alpha \wedge \beta .$$

On the other hand the positive definite inner product on $H^{2k}(X)$ is given by

$$\langle \alpha,\beta \rangle = \int_X \alpha \wedge *\beta .$$

It follows that

$$\text{index } f = \dim H^{2k}_{+} - \dim H^{2k}_{-}$$

where H^{2k}_{+} and H^{2k}_{-} are the $\pm$ 1-eigenspaces of $*$ acting on H^{2k}.

On the other hand using (14.2) and taking careful account of all the sign convections[1] we get

---

[1] The element ν of (14.2) is equal to $-\chi\phi\sigma(S)$, and $\varepsilon(4k) = -1$.

$$\operatorname{ch} S = \prod_{i=1}^{2k} \frac{(e^{x_i} - e^{-x_i})}{x_i} \ ,$$

where the x_i have the same significance as in (a) .

Thus

$$\operatorname{ch} S \cdot \tau(X) = \prod_{i=1}^{2k} x_i/\tanh x_i/2$$

$$= \prod_{i=1}^{2k} \frac{x_i/2}{\tanh x_i/2}$$

$$= 2^{2k} \prod_{i=1}^{2k} \frac{x_i/2}{\tanh x_i/2}$$

Recall next that the Hirzebruch L-genus is defined by

$$L(X) = \Big(\prod_{i=1}^{2k} \frac{x_i}{\tanh x_i}\Big) [X] \ .$$

Thus we get

$$\operatorname{ch} S \cdot \tau(X) \quad [X] = 2^{2k} \cdot 2^{-2k} L(X) = L(X) \ .$$

Hence our index theorem reduces in this special case to the Hirzebruch index theorem.

(c) X is the circle and S is a singular integral operator. To be quite precise on sign conventions we take X as the circle $|Z| = 1$ in the complex plane with its standard orientation, and we consider the integral operator S acting on functions

$$(S\phi)(z) = a(z)\,\phi(z) - \frac{b(z)}{\pi i} \int_{|\zeta|=1} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta$$

Here $a(z)$ and $b(z)$ are continuous functions on X such that $a(z)^2 - b(z)^2$ is never zero. Then according to a classical formula of F. Noether (cf. Mihlin "Singular Integral Equations", A.M.S. Translations (1) Vol. 10) we have

$$\text{index } S = \frac{1}{2\pi i} \int_{|\zeta|=1} d \log \frac{a(\zeta) + b(\zeta)}{a(\zeta) - b(\zeta)} \, d\zeta$$

To compute $\sigma(S)$ let us write $S = aI + bL$ where I is the identity operator. Now putting $\zeta = \exp(is), z = \exp(it)$ and $\phi(z) = \psi(t)$ we get

$$(L\psi)(t) = \frac{1}{\pi i} \int_{s=0}^{2\pi} \frac{i \exp(is) \, \psi(s) \, ds}{\exp(is) - \exp(it)} ,$$

so that $L = H + K$, where K is a compact operator and H is defined by

$$(H\psi)(t) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{\psi(s) \, ds}{t - s} \qquad \text{(defined as a principal value)}$$

Now by definition of the symbol we have

$$\sigma(H)(\xi) = \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{\varepsilon^{-1} > |z| > \varepsilon} \exp(i\xi z) \frac{dz}{z}$$

$$= \operatorname{sgn}(\xi)$$

and so $\sigma(L)(\xi) = \operatorname{sgn}(\xi)$, where ξ is the coordinate in the cotangent bundle of X given by the isomorphism

$$T^*(X) \cong T(X) \cong X \times R^1$$

(using the natural orientations and metric). Thus $\sigma(S)$ is the function on $X \times R^1$ given by

$$\sigma(S)(z,\xi) = a(z) + b(z)\ \mathrm{sgn}(\xi) .$$

To compute we shall simplify matters and consider only the special case in which

$$a(z) = \tfrac{1}{2}(z + 1) \qquad b(z) = \tfrac{1}{2}(z - 1)$$

so that $\dfrac{a(z) + b(z)}{a(z) - b(z)} = z$, and hence

$$\text{index } S = 1 .$$

Let us now go through the construction of Section 10 for the difference bundle. We find that the bundle F of Section 10 is obtained from the trivial bundle on $X \times [-1, +1]$ by the identification

$$(z, -1, u) \to (z, +1, zu)$$

Now let D denote the unit disc $|\omega| \leq 1$ in C and define maps

$$f_{\pm}: X \times [0, \pm 1] \to D \quad \{\pm 1\}$$

by $f_{\pm}(z,\xi) = (\pm\xi z, \pm 1)$. Let P denote the 2-sphere obtained by identifying $D \times \{-1\}$ with $D \times \{+1\}$ along their boundaries and let Y be obtained from P by further identifying $\{0\} \times \{-1\}$ with $\{0\} \times \{+1\}$, so that we have a map $p : P \to Y$. Then the maps $f_{\pm}$ induce a map

$$f : X \times [-1, +1]/X \times \{\pm 1\} \to Y ,$$

and we have

$$F = f^*G \quad , \qquad L = p^*G$$

where L is the bundle on P obtained from the trivial bundle on $D \times \{\pm 1\}$ by the identification

$$(\omega, -1, u) \to (\omega, +1, \omega u) .$$

If we orient P so that $D \times \{+1\} \subset P$ has the natural complex orientation then L is the positive generating bundle i.e.,

$$c_1(L) = + \text{ generator of } H^2(P_1 Z)$$

[in fact the best definition of the positive generator of $K(D, \partial D)$ is that it is $\chi\phi(E)$ where

$$E = (0 \to 1 \xrightarrow{\omega} 1 \to 0) \in \mathcal{C}_1(D, \partial D)] .$$

If we now orient

$$B(X)/S(X) = X \times [-1, +1]/X \times \{\pm 1\}$$

using the orientation coming from that of X by the Thom isomorphism (i.e., orienting the product as $[-1, +1] \times X$ with the product orientation) then we see that f preserves orientation. Hence finally we obtain

$$ch(S) = \psi_*^{-1}\, ch(\chi\phi\sigma(S)) = g$$

where $g \in H^1(X,Z)$ is the positive generator.

Since $\tau(X) = 1$ trivially we see that

$$\{ch(S) \cdot \tau(X)\} [X] = 1$$

so that the index theorem holds in this case for this particular operator. In fact, for $X = S^1$, we have

$$K(B(X), S(X)) \cong Z ,$$

and since both index S and $ch(S)[X]$ are homomorphisms

$$K(B(X), S(X)) \to Q ,$$

the index theorem follows for all operators on S^1. Alternatively with a little more work we could go through the above verification in the general case.

From the examples (a) and (b) applied to the case $X = S^{2\ell}$ we can deduce

PROPOSITION 1. *The index theorem holds for every elliptic operator on* $S^{2\ell}$.

Proof: As shown in Section 18 the index is essentially a homomorphism

$$K(B(X), S(X)) \to Z .$$

Now when $X = S^{2\ell}$ the group $K(B(X), S(X)) \cong K(X)$ is free on two generators. Thus it is sufficient to know that the index theorem holds for any two operators S_1, S_2 for which $ch\, S_1$ and $ch\, S_2$ generate $H^*(X; Q)$. Taking S_1 and S_2 as operators of examples (a) and (b) we see (using (14.2)) that

$$ch(S_1) = E(X) = 2g$$

$$ch(S_2) = 2^{\ell} + \text{constant} \cdot g$$

where g generates $H^{2\ell}(X)$. But we saw in (a) and (b) that the index theorem held for these two operators. Hence it holds in general.

From Ex. (c) and Prop. 2 of (19.1) (with $Y = S^1$) we deduce

PROPOSITION 2. *If the index theorem holds for all even-dimensional manifolds then it holds for all manifolds.*

19.3. *More on cobordism.* In this section we recall some of the results of the generalized cobordism theory of Connor and Floyd (Ergebnisse 1963).

We consider pairs (X,W) where X is a compact oriented differentiable n-manifold and W is a complex vector bundle of dimension k. The notion of cobordism for such pairs was explained in Section (18.3). The cobordism group we get is denoted by $\Omega_n(k)$. Characteristic Pontrjagin-Chern numbers of (X,W) may be defined as follows. Let

$$p^{\alpha} = \prod p_i^{\alpha_i} \qquad c^{\beta} = \prod c_j^{\beta_j}$$

be monomials in the Pontrjagin classes of X and the Chern classes of W. Then if

$$4\ \Sigma\alpha_i + 2\ \Sigma\beta_j = n$$

we can define the characteristic number

$$p^{\alpha}c^{\beta}[X] .$$

It is clear that these are cobordism invariants. Conversely Conner and Floyd have proved:

PROPOSITION 1. *Suppose that* (X_1, W_1) *and* (X_2, W_2) *have the same Pontrjagin-Chern numbers. Then*

$$(X_1, W_1) - (X_2, W_2)$$

is a torsion element of the cobordism group, i.e., *for some integer* $m \neq 0$

$$m(X_1, W_1) \sim m(X_2, W_2) .$$

We shall now need some lemmas concerning characteristic numbers of particular manifolds and bundles. We shall regard the set of Pontrjagin numbers in a given dimension $4n$ as a vector with components $\phi_{\underline{\alpha}}$, where $\underline{\alpha}$ runs over all partitions of n and call it the Pontrjagin vector. Similarly for Chern numbers.

LEMMA 1. *Consider the manifolds*

$$P_{2\underline{k}} = \prod_{i=1}^{r} P_{2k_i}(C) \qquad \underline{k} = (k_1,\ldots,k_r)$$

over all partitions $\underline{k}$ *of* n. *Then the Pontrjagin* vectors of these manifolds are linearly independent. This is proved in Hirzebruch's book (Ergebnisse 1963 pp. 78-79).

LEMMA 2. Let ξ_k denote the generating vector bundle on S^{2k} (so that $c_k(\xi_k) = g_k \neq 0$). Consider on $S^{2\underline{k}} = \prod_{i=1}^{r} S^{2k_i}$, $\underline{k} = (k_1, \ldots, k_r)$ the bundle $\xi_{\underline{k}} = \bigoplus_{i=1}^{r} \pi_i^* \xi_{k_i}$ where $\pi_i : S^{2\underline{k}} \to S^{2k_i}$ is the projection. Then, as $\underline{k}$ runs over all partitions of n, the chern vectors of the $\xi_{\underline{k}}$ are linearly independent.

Proof: Since $c(\xi_{k_i}) = 1 + g_{k_i}$ we deduce $c(\xi_{\underline{k}}) = \prod_{i=1}^{r} (1 + g_{k_i})$. Let us now write each partittion $\underline{k} = (k_1, \ldots, k_r)$ with $k_1 \leq k_2 \leq \cdots \leq k_r$ and order them lexicographically. Then since $g_{k_i}^2 = 0$ we easily find that

$$C_{\underline{\ell}}(\xi_{\underline{k}}) = 0 \quad \text{if} \quad \underline{\ell} < \underline{k}$$

$$= A \prod_{i=1}^{r} g_{k_i} \quad \text{if} \quad \ell = k$$

where A is a non-zero constant. Thus the matrix whose entries are $C_{\underline{\ell}}(\xi_{\underline{k}})$ is non-singular and so the chern vectors of the ξ_k are linearly independent.

For any integer $2n$ consider all the manifolds $P_{2\underline{k}} \times S^{2\underline{\ell}}$ with

$$4\ \Sigma k_i + 2\ \Sigma\ \ell_i = 2n\ .$$

Choose any integer[1] N so that $N > \dim \xi_{\underline{\ell}}$ for all $\underline{\ell}$, and let $\pi : P_{\underline{2k}} \times S^{2\underline{\ell}} \to S^{2\underline{\ell}}$ denote the projection. Consider the bundle of dimension N over $P_{\underline{2k}} \times S^{2\underline{\ell}}$ defined by

$$\eta_N(\underline{k},\underline{\ell}) = (N - \dim \xi_{\underline{\ell}}) \oplus \pi^*\xi_{\underline{\ell}}\ .$$

Then we have

PROPOSITION 2. *Let* $f : \Omega_{2n}(N) \to Q$ *be a homomorphism. Then* f *is determined by the values* $f(P_{\underline{2k}} \times S^{2\underline{\ell}},\ \eta_N(\underline{k},\underline{\ell}))$.

Proof: By lemmas 1 and 2 the Pontrjagin-Chern vectors of the pairs $(P_{\underline{2k}} \times S^{2\underline{\ell}},\ \eta_N(\underline{k},\underline{\ell}))$ are linearly independent. Hence, by Proposition 1, they form a basis for $\Omega_{2n}(N) \otimes Q$. Thus f is determined by its values on them.

We are now in a position to formulate the results on cobordism in a manner convenient for our purpose.

PROPOSITION 3. *Let* $f_\alpha(X,W)$ $(\alpha = 1, 2)$ *be two functions with values in* Q *, defined for any even-dimensional compact oriented differentiable manifold* X *and any complex vector bundle* W *over* X . *Suppose* f_1, f_2 *have the following properties:*

---

[1] If we chose $\xi_{\underline{\ell}}$ to have dimension ℓ, as we could, then we could simply take $N = n$.

(i) $f_\alpha(X,W) = 0$ if $(X,W) \sim 0$

(ii) $f_\alpha(X_1 + X_2, W_1 + W_2) = f_\alpha(X_1,W_1) + f_\alpha(X_2,W_2)$
(*where* + *means disjoint sum*)

(iii) $f_\alpha(X,W_1 \oplus W_2) = f_\alpha(X,W_1) + f_\alpha(X,W_2)$

(iv) $f_\alpha(X_1 \times X_2, W_1 \otimes W_2) = f_\alpha(X_1,W_1)\, f_\alpha(X_2,W_2)$,

and suppose further that

(a) $f_1(P_{2k}(C), 1) = f_2(P_{2k}(C), 1)$

(b) $f_1(S^{2k},\xi_k) = f_2(S^{2k},\xi_k)$.

Then $f_1 = f_2$.

Proof: For any fixed integers n, k properties (i) and (ii) imply that the f_i induce homomorphisms $\Omega_{2n}(k) \to Q$. Now take $k = N$ as in Proposition 2. Then we see that f_1 and f_2 will coincide on all (X,W) with $\dim X = 2n$, $\dim W = N$ if they coincide on the special pairs of Proposition 2. But we have

$$f_\alpha(P_{\underline{2k}} \times S^{\underline{2\ell}}, \eta_N(\underline{k},\underline{\ell}))$$

$$= (N - \dim \xi_{\underline{\ell}})f_\alpha(P_{\underline{2k}} \times S^{\underline{2\ell}}, 1) = \sum_j f_\alpha(P_{\underline{2k}} \times S^{\underline{2\ell}},\pi^*\pi_j^* \xi_{\ell_j})$$

by (iii)

$$= (N - \dim \xi_{\underline{\ell}}) - \prod_j f_\alpha(P_{2k_i}, 1) \prod_j f_\alpha(S^{2\ell_j}, 1)$$

$$+ \sum_j \prod_j f_\alpha(P_{2k_i}, 1) \cdot \prod_{p\neq j} f_\alpha(S^{2\ell_p}, 1) \cdot f_\alpha(S^{2\ell_j}, \xi_{\ell_j})$$

by (iv) .

Also $f_\alpha(S^{2m}, 1) = 0$ by (i). Hence applying (a) and (b) we see that f_1 and f_2 coincide on the sequence of Proposition 2 and so they coincide on all pairs of these dimensions. It remains to treat the case of pairs (X,W) with $\dim X = 2n$ but where $\dim W$ is not large. We do this as follows (N denotes the trivial bundle of dimension N)

$$\begin{aligned} f_1(X,W) &= f_1(X,W \oplus N) - f_1(X,N) && \text{by (iii)} \\ &= f_2(X,W \oplus N) - f_2(X,N) && \text{by what has been proved} \\ &= f_2(X,W) \quad \text{by (iii)}. \end{aligned}$$

This completes the proof.

19.4. Proof of the index theorem. By Proposition 2 of (19.2) it is sufficient to consider the case with $\dim X$ even. Then as in Section 18 it is sufficient to consider the indices of the special first order operators with symbols $\sigma_0 W$. We introduce there the notation $\gamma(X,W)$ for the index of these operators. Let us write

$$\mu(X,W) = \{ch\, \sigma_0 W \cdot \tau(X)\}[X]$$

so that the index theorem asserts that $\gamma = \mu$. We are now in a position to apply Proposition 3 of 19.3 with $f_1 = \gamma$, $f_2 = \mu$. Property (ii) is trivial, (iii) is clear since the index and ch are both additive. Property (i) is elementary for μ and for γ it was proved as a theorem in 18.3. Property (iv) for γ was proved in 18.3 which for μ it follows from Proposition 1 of (19.1) and the formula $d(X_1 \times X_2, W_1 \otimes W_2) = d(X_1, W_1) \# d(X_2, W_2)$ already verified in 18.3 ($d(X,W)$ is the basic operator whose symbol is $\sigma_0(X,W)$) .

Finally (a) and (b) follow from the fact that the index theorem has been verified in these special cases (Proposition 1 of 19.2 and the Hirzebruch index formula for $P_{2k}(C)$ (Ex. b)) . Thus we can apply Proposition 3 and deduce that the index theorem holds for X . This completes the proof.