Classical and adelic automorphic forms

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That interesting new L functions with Euler products arise in the classical theory of modular forms is in some sense an accident, and even a bit deceptive. For algebraic number fields other than $\mathbb{Q}$ the relationship between classical forms and L functions is more complicated. It ought to be no surprise to anyone familiar with John Tate's thesis that the correct groups with which to do automorphic forms are adele groups. In this essay I'll explain roughly how the transition from classical to adelic takes place.

The classical theory is really one about the group $\mathrm{GL}_2(\mathbb{Q})$. This is clear if one looks at Hecke operators. But to understand how things work, I'll have to look at $\mathrm{SL}_2(\mathbb{Q})$ also.

Much of what I am going to say works for very general split reductive groups defined over $\mathbb{Q}$, although I won't amplify much on this remark.

Contents

1. Strong approximation

I begin with recalling some elementary number theory.

1.1. Proposition. (Chinese remainder theorem) *Suppose m and n to be relatively prime positive integers. If a and b are any integers, there exists a single integer c such that*

$$c \equiv a \bmod m, \quad c \equiv b \bmod n\,.$$

Proof. It suffices to prove this when $b = 0$. There exist integers k, ℓ such taht

$$km + \ell n = 1\,.$$

But then

$$akm + a\ell n = a\,,$$

so that $c = a\ell n$ is divisible by n and congruent to a modulo m.

Recall that $\mathbb{Z}_{(p)}$ is the ring of rational numbers a/b with b prime to p. It is a local ring with maximal ideal (p).

1.2. Corollary. (Strong approximation for $\mathbb{Q}$) *Suppose S to be a finite set of primes, and for reach p in S suppose given a rational number a_p. Suppose also given for each p in S a non-negative integer n_p. Then there exists a rational number a such that a lies in $\mathbb{Z}_{(p)}$ for every p not in S and $a - a_p$ lies in $p^{n_p}\mathbb{Z}_{(p)}$ for every p in S.*

Proof. Let m be such that ma_p lies in $\mathbb{Z}_{(p)}$ for all p in S. The Proposition tells us by induction on the size of S that there exists c in $\mathbb{Z}$ such that

$$\begin{aligned} c &\equiv ma_p \bmod mp^{n_p} \quad (p \in S) \\ c &\equiv 0 \bmod m\,. \end{aligned}$$

Let $a = c/m$.

1.3. Lemma. *If F is any field then $\mathrm{SL}_2(F)$ is generated by matrices of the form*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}.$$

Proof. We have all these formulas:

$$\begin{aligned}
\begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} &= \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 1 & x/a \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1/c & a \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} \quad (c \neq 0) \\
\begin{bmatrix} 0 & x \\ -1/x & 0 \end{bmatrix} &= \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/x & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \\
\begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} &= \begin{bmatrix} 0 & x \\ -1/x & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\end{aligned}$$

The first two show that $\mathrm{SL}_2(F)$ is generated by unipotent matrices as in the Lemma, together with diagonal and monomial matrices. The third shows that monomial matrices are products of unipotent matrices as in the Lemma, and the fourth shows that so are the diagonal matrices.

1.4. Corollary. *The group $\mathrm{SL}_2(\mathbb{Z}/p^n)$ is generated by matrices*

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \quad (x \in \mathbb{Z}/p^n)$$

and

$$I + px \quad (x \in M_2(\mathbb{Z}/p^{n-1})).$$

The first approach to strong approximation follows easily from these two results:

1.5. Proposition. *The canonical projection from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/N)$ is surjective.*

Proof. From the the Corollary, by induction on the number of primes dividing N.

The previous result can be formulated as saying that *if one is given for each of a finite number of primes p an integral matrix A_p with $\det(A_p)$ congruent to 1 modulo p^{n_p} then one can find an integral matrix A of determinant 1 simultaneously congruent to each A_p modulo p^{n_p}*. This can be generalized.

1.6. Proposition. (Strong approximation for $\mathrm{SL}_2(\mathbb{Q})$) *Suppose that for each of a finite set S of primes p we are given g_p in $\mathrm{SL}_2(\mathbb{Q}_p)$ and $n_p \geq 0$. There exists γ in $\mathrm{SL}_2(\mathbb{Q})$ such that*

(a) the matrix entries of γ lie in $\mathbb{Z}_p$ for p not in S;
(b) for each p in S, $\gamma^{-1} g_p$ lies in $\mathrm{SL}_n(\mathbb{Z}_p)$ and is congruent to I modulo p^{n_p}.

Proof. By the previous result, it suffices to deal with the case where all $n_p = 0$. What has to be shown now is that there exists γ satisfying (a) with $\gamma^{-1} g_p$ in $\mathrm{SL}_n(\mathbb{Z}_p)$ for all p in S. An induction argument reduces this to the case where S has just one prime.

Let $\mathbb{Q}_{(p)}$ be the ring of fractions whose denominators are powers of p. The image of an element of $\mathbb{Q}_{(p)}$ in $\mathbb{Q}_q$ for $q \neq p$ lies in $\mathbb{Z}_q$. We now want to show that given g in $\mathrm{SL}_2(\mathbb{Q}_p)$ there exists γ in $\mathrm{SL}_n(\mathbb{Q}_{(p)})$ such that $\gamma^{-1} g$ lies in $\mathrm{SL}_2(\mathbb{Z}_p)$.

By the elementary divisors theorem for $M_2(\mathbb{Q}_p)$, we can find matrices δ_1 and δ_2 in $\mathrm{SL}_2(\mathbb{Z}_p)$ such that $g = \delta_1 d \delta_2$, in which

$$d = \begin{bmatrix} p^m & 0 \\ 0 & 1/p^m \end{bmatrix}$$

with $m \geq 0$. According to the Lemma, we can find matrices γ_i in $\mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma_i \equiv \delta_i \bmod p^{2m} \, .$$

If $\gamma = \gamma_1 d \gamma_2$ then $\gamma^{-1} g$ will lie in $\mathrm{SL}_2(\mathbb{Z}_p)$.

Remark. The strong approximation theorem is true for any simply connected algebraic group defined over any global field. The proof in general is very difficult, but that for split groups is only slightly more difficult than that for $\mathrm{SL}_2(\mathbb{Q})$, granting the Bruhat decomposition.

2. Introducing the adèle group

Define the ring of **finite adèles** $\mathbb{A}_f$ over $\mathbb{Q}$ to be the **restricted product** of the p-adic fields $\mathbb{Q}_p$—that is to say, the subring of $\prod \mathbb{Q}_p$ of all (a_p) with a_p in $\mathbb{Z}_p$ for all but a finite number of primes p. The full ring $\mathbb{A}$ of adèles is the direct product $\mathbb{R} \times \mathbb{A}_f$. It is a topological ring in which a basis of neighbourhoods of 0 are subsets $I \times \prod U_p$ where I is a neighbourhood of 0 in $\mathbb{R}$, and each U_p is a subgroup $p^{n_p}\mathbb{Z}_p$ with all but a finite number of $n_p = 0$.

The fields $\mathbb{R}$ and $\mathbb{Q}_p$ are all the reasonable completions of $\mathbb{Q}$, and are usefully considered uniformly. To emphasize this and make notation more efficient I set $\mathbb{Q}_\infty = \mathbb{R}$. In this notation $\mathbb{A}$ is a subring of $\prod_{p \leq \infty} \mathbb{Q}_p$.

The units of $\mathbb{A}$ are the invertible adèles, the (a_p) with all $a_p \neq 0$, and a_p in $\mathbb{Z}_p^\times$ for all but a finite number of p. These are the **idèles** $\mathbb{A}^\times$.

The point is that if V is any variety defined over $\mathbb{Q}$ then one may refer to $V(\mathbb{A})$. This is equal to the set of all (x_p) with x_p in $V(\mathbb{Z}_p)$ for all but a finite number of p.

For example, the group $\mathrm{GL}_2(\mathbb{A})$ consists of the subgroup of elements of $\mathrm{GL}_2(\mathbb{R}) \times \prod \mathrm{GL}_2(\mathbb{Q}_p)$ with all but a finite number of g_p in $\mathrm{GL}_2(\mathbb{Z}_p)$. As we'll see in a moment, the field $\mathbb{Q}$ is a discrete subring of $\mathbb{A}$. Consequently, the group $\mathrm{GL}_2(\mathbb{Q})$ is also a discrete subgroup of $\mathrm{GL}_2(\mathbb{A})$. In the modern theory of automorphic forms, automorphic forms are specified to be functions on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$ rather than on arithmetic quotients $\Gamma\backslash\mathcal{H}$, $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$, or $\Gamma\backslash\mathrm{GL}_2(\mathbb{R})$.

There are many reasons for this change. One good one is that in many problems involving automorphic forms somewhat complicated questions involving number theory are replaced by simpler questions about analysis on the local groups $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{Q}_p)$. This is especially true of Hecke operators, as we'll see. Another good reason is that the discrete group $\mathrm{GL}_2(\mathbb{Q})$ is in most ways much simpler than arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In applying the trace formula, for example, the conjugacy classes of $\mathrm{GL}_2(\mathbb{Q})$ are much simpler to understand than those of $\Gamma(N)$. Another useful thing is that the Bruhat decomposition for $\mathrm{SL}_2(\mathbb{Q})$ is very simple, as opposed to what happens for $\mathrm{SL}_2(\mathbb{Z})$. In any case, what I want to do here is very limited—to provide a simple introduction to the adèles and to explain what is involved in transferring classical automorphic forms to functions on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$. I'll not prove everything in detail.

For any prime p, let $|x|_p$ be the usual p-adic norm, so that

$$|p^n|_p = p^{-n} \, .$$

Then for any rational number x

(2.1)
$$\prod_p |x|_p = 1 \, .$$

2.2. Proposition. *The field $\mathbb{Q}$ is a discrete additive subgroup of $\mathbb{A}$. We have*

$$\mathbb{A} = \mathbb{Q} + \Big([0,1) \times \prod_p \mathbb{Z}_p\Big)\,.$$

In fact the region $[0,1) \times \prod_p \mathbb{Z}_p$ is a fundamental domain for the discrete subgroup $\mathbb{Q}$, and the embedding of $\mathbb{R}$ into $\mathbb{A}$ induces an isomorphism of $\mathbb{Z}\backslash\mathbb{R}$ with $\mathbb{Q}\backslash\mathbb{A}/\prod \mathbb{Z}_p$.

Proof. To see that $\mathbb{Q}$ is discrete in $\mathbb{A}$ we just need to verify that the open subset

$$\big\{|x| < 1\big\} \times \prod \mathbb{Z}_p$$

contains no rational numbers. This follows from (2.1).

As for the second claim, what it means is that if we are given an adèle $a = (a_p)$ there exists a rational number α such that $a - \alpha$ lies in $I \times \mathbb{Z}_p$. Choosing an integer n such that $a_\infty - n$ lies in I, we are reduced to the claim that given a finite set S of primes and for each p in S a p-adic number a_p, there exists a rational number α lying in $\mathbb{Z}_p$ for p not in S with $a_p - \alpha$ lying in $\mathbb{Z}_p$ for p in S. This is Corollary 1.2.

2.3. Proposition. *We have*

$$\mathbb{A}^\times = \mathbb{Q}^\times \cdot \Big(\mathbb{R}^{\mathrm{pos}} \times \prod \mathbb{Z}_p^\times\Big)\,.$$

Proof. Suppose a unit adèle (a_p) to be given, with no $a_p = 0$ and all but a finite number of a_p in $\mathbb{Z}_p^\times$, say for p not in the finite set S. We want to find $\alpha > 0$ in $\mathbb{Q}^\times$ with α in $\mathbb{Z}_p^\times$ for p in S and $a_p\alpha^{-1}$ in $\mathbb{Z}_p^\times$ for p in S. This problem reduces to unique factorization for integers.

The strong approximation theorem for $\mathrm{SL}_2(\mathbb{Q})$ is equivalent to the following result about $\mathrm{SL}_2(\mathbb{A})$:

2.4. Theorem. *Suppose that for each prime p we are given a compact open subgroup $K_p \subseteq \mathrm{SL}_2(\mathbb{Q}_p)$ such that all but a finite number of $K_p = \mathrm{SL}_2(\mathbb{Z}_p)$. Let $K_f = \prod K_p$. Then*

$$\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q}) \cdot \Big(\mathrm{SL}_2(\mathbb{R}) \times K_f\Big)\,.$$

Proof. There is only one small point that should be mentioned—one can replace each K_p by its intersection with $\mathrm{SL}_2(\mathbb{Z}_p)$, which is of finite index, and then assume $K_p \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ for all p.

For each compact open subgroup $K_f = \prod K_p$ of the restricted product of the $\mathrm{SL}_2(\mathbb{Q}_p$, let Γ_{K_f} be the inverse image in $\mathrm{SL}_2(\mathbb{Q})$ of $\mathrm{SL}_2(\mathbb{R}) \times K_f$. For example, if $K_p = \mathrm{SL}_2(\mathbb{Z}_p)$ for all p then $\Gamma_{K_f} = \mathrm{SL}_2(\mathbb{Z})$.

2.5. Corollary. *The canonical map from $\Gamma_{K_f}\backslash\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/K_f$ is a bijection.*

Proof. Just to be careful, I'll do it in stages. Let $H = \mathrm{SL}_2(\mathbb{Q})$, $G = G(\mathbb{R}) \times K_f$. Thus $\Gamma = H \cap G$. The map from $H \cap G\backslash G$ to $H\backslash H \cdot G$ is a bijection. But then we can divide on the right by K_f, using the fact that $\Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \times K_f/K_f$ may be identified with $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$.

2.6. Corollary. *Suppose that for each prime p we are given a compact open subgroup $K_p \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ such that (a) all but a finite number of $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$; (b) $\det(K_p) = \mathbb{Z}_p^\times$ for all p. Then*

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) \cdot \Big(\mathrm{GL}_2^{\mathrm{pos}}(\mathbb{R}) \times \prod K_p\Big)\,.$$

More precisely,

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathrm{SO}_2 \times \prod K_p$$

may be identified with $\Gamma\backslash\mathcal{H}$ if

$$\Gamma = \mathrm{GL}_2(\mathbb{Q}) \cap \left(\mathrm{GL}_2^{\mathrm{pos}}(\mathbb{R}) \times \prod K_p\right).$$

Now I explain how functions on $\Gamma\backslash\mathrm{GL}_2^{\mathrm{pos}}(\mathbb{R})$ become functions on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$. Suppose we are given for each p a compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Z}_p)$, with almost all $K_p = \mathrm{GL}_2(\mathbb{Z}_p$ and with all $\det K_p = \mathbb{Z}_p^\times$. Let $K_f = \prod K_p$. Then

$$U_{K_f} = \mathrm{GL}_2^{\mathrm{pos}}(\mathbb{R}) \times \prod K_p$$

is an open subgroup of $\mathrm{GL}_2(\mathbb{A})$. The group

$$\Gamma_{K_f} = \Gamma \cap U_K$$

will be a congruence subgroup. If $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ for all p, for example, then $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

Every congruence subgroup arises in this fashion. For a given Γ, the intersection $K_p \cap \mathrm{SL}_2(\mathbb{Z}_p)$ is determined by Γ (it is its closure in $\mathrm{SL}_2(\mathbb{Z}_p)$) but there will be several choices possible for K_p itself. If $\Gamma = \Gamma(N)$ the standard choice is the group of all k of the form

$$\begin{bmatrix} * & 0 \\ 0 & 1 \end{bmatrix}$$

modulo N.

2.7. Lemma. *In this situation, the injection*

$$\Gamma_{K_f}\backslash\mathrm{GL}_2^{\mathrm{pos}}(\mathbb{R}) \longrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K_f$$

is a bijection.

This means among other things that automorphic forms of weight m on $\Gamma\backslash\mathcal{H}$ lift uniquely to functions on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$ fixed by K. *It is important to realize that this lifting depends not only on Γ but also on the choice of K.* This phenomenon occurs also in the classical theory, in the definition of Hecke operators for general congruence groups.

3. Hecke operators

The starting point of the classical theory of Hecke operators is the observation that if g lies in $\mathrm{GL}_2^{\mathrm{pos}}(\mathbb{Q})$ and f is an automorphic form for the congruence subgroup Γ then the function $f_g\colon z \mapsto f(gz)$ is an automorphic form for $g^{-1}\Gamma g$, since

$$f_g(g^{-1}\gamma g\, z) = f(gg^{-1}\gamma\, gz) = f_g(z).$$

This should be combined with another simple observation:

3.1. Proposition. *If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and g lies in $\mathrm{GL}_2^{\mathrm{pos}}(\mathbb{Q})$ then $g^{-1}\Gamma g \cap \Gamma$ is also a congruence group.*

Proof. We may assume, by the elementary divisors theorem, that g is diagonal and $\Gamma = \Gamma(M)$. We must show that $g^{-1}\Gamma(M)g \cap \Gamma(M)$ contains some $\Gamma(N)$. This is easy. ▯

Suppose g in $M_2(\mathbb{Z})$ with positive determinant. The double coset $\Gamma g\Gamma$ will be a union of left Γ-cosets Γg_i. Each g_i can be chosen to be of the form $g\gamma_i$ with γ_i in Γ. The map

$$\gamma \longmapsto \Gamma g\gamma$$

is therefore surjective onto $\Gamma g\Gamma$. The isotropy subgroup of g is $\Gamma \cap g^{-1}\Gamma g$, so that the associated map

$$(\Gamma \cap g^{-1}\Gamma g)\backslash\Gamma \longrightarrow \Gamma\backslash\Gamma g\Gamma$$

is a bijection, and as a consequence of the previous result the right hand side is finite. Corresponding to the double coset $\Gamma g\Gamma = \bigcup \Gamma g_i$ we can therefore define the **Hecke operator** $T[g]$

$$f \longmapsto f \,\big|\, [\Gamma g\Gamma]_m = \sum f \,\big|\, [g_i]_m$$

on the space of automorphic forms of weight m for Γ. This definition differs from the classical definition by a power of $\det(g)$. It can be extended consistently to operators $T[g]$ for any rational matrix, since scalar matrices act trivially. If we associate to f the function

$$\Phi(g) = f \,\big|\, [g]_m$$

on $\Gamma\backslash G$, then $f \,\big|\, [\Gamma g\Gamma]_m$ corresponds to $\sum \Phi(g_i g)$ (where $\Gamma g\Gamma = \bigcup \Gamma g_i$).

Cusp forms on $\Gamma\backslash\mathcal{H}$ are square-integrable, and give rise to square-integrable functions on $\Gamma\backslash G$. The representation of G on $\mathrm{L}^2(\Gamma\backslash G)$ is unitary, which means that the adjoint of L_g is $L_{g^{-1}}$. The Hecke operator $T[g]$ is therefore self-adjoint when $T[g] = T[g^{-1}]$.

The classical theory of Hecke operators applies to all congruence groups, but there are complications in the general case that I wish to avoid (related to the problem of choosing K_p for a given Γ), so I'll assume for a while that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Suppose p a prime, and let

$$g = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case $\Gamma g\Gamma$ differs from $\Gamma g^{-1}\Gamma$ by a scalar, so $T[g] = T[g^{-1}]$ and $T[g]$ is self-adjoint on the space of cusp forms. To get an explicit expression, note that

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & p^{-1}b \\ pc & d \end{bmatrix}$$

the intersection $\Gamma \cap g^{-1}\Gamma g$ is the group of all matrices lying in the Borel subgroup

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

modulo p, and the quotient $(g^{-1}\Gamma g \cap \Gamma)\backslash\Gamma$ is in bijection with $\mathbb{P}^1(\mathbb{F}_p)$, a set of $p+1$ elements.

What does the Hecke operator $T[g]$ on $\Gamma\backslash G$ correspond to on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$? The **local Hecke algebra** on the p-adic group $\mathrm{GL}_2(\mathbb{Q}_p)$ is made up of functions of compact support that are left- and right-invariant under $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$.

If φ is in the p-adic Hecke algebra and Φ a function on the adelic quotient fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$, the convolution

$$R_\varphi \Phi(g) = \int_{\mathrm{GL}_2(\mathbb{Q}_p)} \varphi(x)\Phi(gx), dx$$

is defined. In particular, the characteristic function φ_p of the double coset

$$K_p \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} K_p$$

is in this algebra.

3.2. Proposition. *If f is an automorphic form for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and f corresponds to Φ on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$ then $f \,|\, T_p$ corresponds to $R_{\varphi_p}\Phi$.*

How the Hecke operators relate to representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ is the next question to take up, but that won't be done in this essay.

4. Remarks

The transition from classical automorphic forms to functions on the adèle quotient depends, as we have seen, on many features of arithmetic of the rational numbers that do not remain valid for other number fields. For them, the correct way to proceed is to start off immediately with functions on the adèle quotient, making occasional forays into arithmetic quotients for a few questions of real analysis. One loses, perhaps, a certain amount of intuition, but gains enormously in elegance. One sign that this is the correct approach is that notation and basic definitions expanded rapidly in the period 1920–1965 (starting with Ramanujan, passing through Mordell, Hecke, Siegel, and Weil) but that basic notions have remained stable in the longer period since then. The summer symposium in Boulder in 1965 marks the change from what might be called the classical period to the modern one.

The description of $\mathbb{Q}\backslash\mathbb{A}$ generalizes nicely to other number fields, but the factorization of $\mathbb{A}^\times$ depends strongly on the fact that $\mathbb{Z}$ is a principal ideal domain. On the other hand, although the proof of the strong approximation theorem for $\mathrm{SL}_n(\mathbb{Q})$ that I have given depends on unique factorization in $\mathbb{Z}$, the theorem itself remains valid for $\mathrm{SL}_n(F)$ where F is any number field. This result is due to Martin Eichler. The proof is quite different from the one given here, and and a somewhat simplified version can be found in [Kneser:1965]. In fact, strong approximation is valid for any simply connected semi-simple group over a number field. A proof in this generality can be found in [Kneser:1966], on the assumption of the Hasse principle (which was subsequently verified). A complete if rather dense discussion along different lines can be found in [Platonov-Rapinchuk:1994].

5. References

1. J. W. S. Cassels and A. Fröhlich, **Algebraic number theory**, Thompson, 1967.

2. Martin Kneser, 'Starke approximation in algebraische Gruppen', *Journal für die reine and angewandte Mathematik* **218** (1965), 190–203.

3. ——, 'Strong approximation', in **Algebraic groups and discontinuous subgroups**, *Proceedings of Symposia in Pure Mathematics* **IX**, 1966.

4. James Milne, **Modular functions and modular forms**, course notes available at

`http://www.jmilne.org/math/CourseNotes/math678.html`

5. Vladimir Platonov and Andrei Rapinchuk, **Algebraic groups and number theory**, Academic Press, 1994.

6. Michael Pohst and Hans Zassenhaus, **Algorithmic algebraic number theory**, Cambridge University Press, 1989.

7. J-P. Serre, **Cours d'Arithmétique**, Presses Universitaires de France 1970.

8. G. Shimura, **Arithmetic theory of automorphic functions**, Princeton University Press (first edition), 1971.