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Essays in analysis

Elliptic differential equations—analyticity

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The solutions of elliptic differential equations are smooth if the equation is smooth, and analytic if it is analytic. The first is relatively elementary, the second more subtle. In this essay the second result will be proved for homogeneous equations. I follow what seems to be the first proof of the general case, that of [John:1955], but in more modern terms. This makes a more conceptual treatment possible.

Contents

1. Outline

Suppose

$$L = \sum_{|k|\le p} a_k(x)\,\frac{\partial^k}{\partial x^k}$$

to be a linear differential operator on $\mathbb{R}^n$ of order p in dimension n. Here $k = (k_i)$ is a multi-index, and $|k| = \sum k_i$. The p-th order **symbol** of L is the function

$$\sigma_L(x,\xi) = \sum_{|k|=p} a_k(x)\,\xi^k$$

where ξ_j is substituted for $\partial/\partial x_j$. This is to be interpreted canonically as an element of the symmetric algebra $S^p(T_x)$ at every point x, or in other words a homogeneous function of degree p on the cotangent space T_x^*. Thus for $df = \sum(\partial f/\partial x_i)\,dx_i$ the function $\sigma_L(x, df)$ is evaluated by replacing each ξ_i in the symbol by $\partial f/\partial x_i$, since $(\partial/\partial x_i)$ is the basis of T_x dual to (dx_j). One example, historically of much interest and even now not devoid of it, is the Laplacian

$$\Delta = \sum \frac{\partial^2}{\partial x_i^2}$$

whose symbol is $\sum \xi_i^2$.

An **elliptic operator** is one with the property that its symbol vanishes only for real values of ξ at $\xi = 0$. The Laplacian, for example, is elliptic.

The goal of this essay is to prove:

[analyticity] **1.1. Theorem.** *If L is an elliptic operator with analytic coefficients in an open disk U in $\mathbb{R}^n$, then any solution of $LF = 0$ in U is analytic.*

A final section will recall briefly what the role of this result in representation theory is. Roughly speaking, it allows one to relate representations of a reductive group G (global object) to that of its Lie algebra $\mathfrak{g}$ (local object) and a maximal compact K. This is the implicit justification for working with $(\mathfrak{g}, K)$ modules instead of representations of G, which are really the ultimate objects of interest.

This classic theorem ('analytic elliptic regularity') has been known for a long time, but coming in complete generality only after a long development through several special cases. There are, as far as I know, three distinct approaches to proving it. One is to obtain hard estimates on the derivatives of solutions throughout a region, which assures convergence of Taylor series to the solution. This argument is apparently due originally in full generality to [Morrey & Nirenberg:1957], and is the one presented most commonly, for example in [Narasimhan:1968]. The shortest version of this argument can be found in [Bers & Schechter:1968]. I find this proof unilluminating. Another proof is that to be found in [Nelson:1959]. This classic paper is mostly about analytic vectors with respect to operators on Banach space, but derives a version of elliptic analytic regularity as a consequence.

The third proof is due to Fritz John and seems, as I have said, to be the first complete proof. The complete account is in [John:1955]. This proof is rather longer—long-winded, some might think—than those in the other groups, but (I think) much better motivated. John's argument has one great virtue, in that it is the ancestor of the proof of Sato's extension of this theorem to more general differential operators, the one in [Kashiwara et al.:1986]. It fits in well, in other words, with a more general theory, and I'd say that none of the steps in this proof are unimportant or uninteresting. In brief, this second proof uses a variant of the Radon transform to reduce the result to the Cauchy-Kowalevsky theorem concerning solutions of analytic partial differential equations satisfying boundary conditions on non-characteristic hyperplanes. The Cauchy-Kowalevsky theorem is in some sense the most fundamental theorem in all of the theory of partial differential equations, since it characterizes nicely the singular (or, for that matter, non-singular) nature of solutions to a partial differential equation. It is relatively simple to prove, too. The argument is intuitive, proceeding in codimension one in $\mathbb{R}^n$ much as one proceeds with ordinary differential operators in one dimension.

It's a modified version of John's proof that I'll sketch here. What is slightly new over John's treatment is a more thorough use of distributions and the Fourier transform. These tools enable a more succinct account.

The main steps, roughly sketched, are these:

Step 1. A straightforward argument, known for classical elliptic operators for a long time, reduces the problem to the construction of local and locally integrable fundamental solutions $F(x, y)$ that are real analytic off the diagonal $x = y$.

Step 2. Next I'll use a basic idea of John, to apply a version of the Radon transform to express the Dirac delta in $\mathbb{R}^n$ as

$$\delta_0 = \int_{\mathbb{S}^{n-1}} \Phi_\xi \, d\xi \, ,$$

where Φ is a one-dimensional distribution, and Φ_ξ is its translation perpendicularly to the point ξ in $\mathbb{S}^{n-1}$. The Φ_ξ are what John calls **plane wave** distributions. By translation this gives a similar expression for every δ_x.

Step 3. The theorem of Cauchy-Kowalevsky allows one to construct (a) analogues of fundamental solutions of $LF = \delta_{\xi^\perp}$ and then from these (b) solutions $F = F_\xi$ of the equation $LF = \Phi_\xi$.

Step 4. We obtain an analytic fundamental solution of L by combining these steps—we get, for example,

$$\delta_0 = \int_{\mathbb{S}^{n-1}} LF_\xi \, d\xi = L\Big(\int_{\mathbb{S}^{n-1}} F_\xi \, d\xi \Big) ,$$

and similar formulas for all δ_x, with everything varying analytically.

Step 5. Then, finally, Any solution of $L\Phi = \varphi$ with φ analytic can be constructed from analytic boundary data by using a fundamental solution of the adjoint differential operator, which is also elliptic. In the case of the Laplacian in $\mathbb{R}^n$, as I'll recall, this is well known.

2. Adjoint operators

First I must recall what the **adjoint** of a differential operator D is. It is a differential operator D^* such that for f, g in $C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} Df(x) \cdot g(x) \, dx_1 \ldots \, dx_n = \int_{\mathbb{R}^n} f(x) \cdot D^* g(x) \, dx_1 \ldots \, dx_n .$$

Underlying this is an equation involving differential forms. Thus the adjoint of $\partial/\partial x_j$ is $-\partial/\partial x_j$, since

$$\begin{aligned} \big((\partial f/\partial x_j) g + f(\partial g/\partial x_j)\big) \, dx_1 \wedge \ldots \wedge dx_n &= (-1)^{j-1} (\partial fg \, /\partial x_j) \, dx_j \wedge dx_1 \wedge \ldots \widehat{dx_j} \ldots \wedge dx_n \\ &= d\big((-1)^{j-1} fg \, dx_1 \wedge \ldots \widehat{dx_j} \ldots \wedge dx_n\big) . \end{aligned}$$

In other words, for $D = \partial/\partial x_j$ we find that $D^* = -D$ and

$$Df \cdot g \, dx_1 \wedge \ldots \wedge dx_n - f \cdot D^* g \, dx_1 \wedge \ldots \wedge dx_n = d\tau_D(f, g)$$

where

$$\tau_D(f, g) = (-1)^{j-1} fg \, dx_1 \wedge \ldots \widehat{dx_j} \ldots \wedge dx_n .$$

An induction argument gives us a similar formula for operators of all orders:

$$(Df \cdot g - f \cdot D^* g) \, dx_1 \wedge \ldots \wedge dx_n = d\tau_D(f, g)$$

where $\tau_D(f, g)$ is an $(n-1)$-form. This is because of the sequence of formulas

$$\begin{aligned} (D_1 D_2 f \cdot g - D_2 f \cdot D_1^* g) &= d\tau_{D_1}(D_2 f, g) \\ (D_2 f \cdot D_1^* g - f \cdot D_2^* D_1^* g) &= d\tau_{D_2}(f, D_1^* g) \\ (D_1 D_2 f \cdot g - f \cdot D_2^* D_1^* g) &= d\tau_{D_1}(D_2 f, g) + d\tau_{D_2}(f, D_1^* g) \end{aligned}$$

so that

$$\tau_{D_1 D_2}(f, g) = \tau_{D_1}(D_2 f, g) + \tau_{D_2}(f, D_1^* g) .$$

If L is elliptic, so is L^*, since the p-th order symbol of D^* is $(-1)^p \sigma(D)$.

By Stokes' formula, if Ω is a region of $\mathbb{R}^n$ with smooth boundary then

$$\int_\Omega \big(Df \cdot g - f \cdot D^* g\big) \, dx_1 \wedge \ldots \wedge dx_n = \int_{\partial\Omega} \tau_D(f, g) .$$

For example, suppose $D = \Delta$, say in $\mathbb{R}^2$:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

Then

$$
\begin{aligned}
\left(\left(\frac{\partial^2 f}{\partial x^2}\right) g + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right)\right) dx{\wedge}dy &= \quad d\left(\frac{\partial f}{\partial x}\, g\, dy\right) \\
\left(\left(\frac{\partial^2 f}{\partial y^2}\right) g + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)\right) dx{\wedge}dy &= -d\left(\frac{\partial f}{\partial y}\, g\, dx\right) \\
\left(\Delta f \cdot g + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)\right) dx{\wedge}dy &= d\left(\frac{\partial f}{\partial x}\, g\, dy - \frac{\partial f}{\partial y}\, g\, dx\right) \\
\left(f\left(\frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right)\right) dx{\wedge}dy &= \quad d\left(g\frac{\partial f}{\partial x}\, dy\right) \\
\left(f\left(\frac{\partial^2 g}{\partial y^2}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)\right) dx{\wedge}dy &= -d\left(f\,\frac{\partial g}{\partial y}\, dx\right) \\
\left(f \cdot \Delta g + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right)\right) dx{\wedge}dy &= d\left(f\,\frac{\partial g}{\partial x}\, dy - f\,\frac{\partial g}{\partial y}\, dx\right).
\end{aligned}
$$

This leads to

$$
(\Delta f \cdot g - f \cdot \Delta g)\ dx{\wedge}dy = d\left(\left(\frac{\partial f}{\partial x}\, dy - \frac{\partial f}{\partial y}\, dx\right) g - f\left(\frac{\partial g}{\partial x}\, dy - \frac{\partial g}{\partial y}\, dx\right)\right).
$$

which translates into the familiar Green's theorem:

$$
\int_\Omega (\Delta f \cdot g - f \cdot \Delta g)\, dxdy = \int_{\partial\Omega} \left(\frac{\partial f}{\partial \nu}\, g - f\,\frac{\partial g}{\partial \nu}\right) ds\,.
$$

3. Fundamental solutions

Let's now look at the last step of the outline, in which one deduces analyticity of solutions to elliptic equations from analyticity of fundamental solutions. If Ω is an open subset of $\mathbb{R}^n$ and $K(x,y)$ is a distribution on $\Omega \times \Omega$ then for each f in $C_c^\infty(\Omega)$

$$
\int_\Omega K(x,y) f(y)\, dy
$$

is a distribution on Ω defined by the formula

$$
\Big\langle \int_\Omega K(x,y) f(y)\, dy, g(x) \Big\rangle_\Omega = \big\langle K(x,y), g(x) f(y) \big\rangle_{\Omega\times\Omega}\,.
$$

A fundamental solution for an operator L on Ω is a distribution $K(x,y)$ on $\Omega \times \Omega$ such that $L_x K(x,y) = \delta_{x-y}$. Here δ_{x-y} is the distribution defined by integration over the diagonal:

$$
\langle \delta(x-y), f \rangle = \int_\Omega f(x,x)\, dx\,.
$$

I'll show some examples later on. Its important property is that

$$
\int K(x,y) f(y)\, dy = F(x)
$$

satisfies $LF = f$. This is because

$$
\langle L_x F, g \rangle = \Big\langle L_x \int \Phi(x,y) f(y)\, dy, g(x) \rangle = \Big\langle L_x \Phi(x,y), g(x) f(y) \Big\rangle = \int_\Omega f(x) g(x)\, dx\,,
$$

which means that $L_x F = f$ as a distribution.

The crux of John's argument is this:

undamental-elliptic] **3.1. Proposition.** *There exists a locally integrable fundamental solution $K(x,y)$ of L_x which is analytic in the region $x \neq y$.*

Let $K(x,y)$ be a fundamental solution for L^*, which is also an elliptic operator.. For any region Ω with analytic boundary $\partial\Omega$ we have

$$\int_\Omega \big(L_x f(x) \cdot K(x,y) - f(x) \cdot L_x^* K(x,y)\big)\, dx = \int_{\partial\Omega} \tau_L\big(f(x), K(x,y)\big)\, dx\,.$$

If $L_x f = 0$ then the first term inside the volume integral vanishes, and by assumption the remaining integral is $-f(y)$. So

$$f(y) = -\int_{\partial D} \tau_L\big(f(x), K(x,y)\big)\, dx\,.$$

But $K(x,y)$ is analytic in y on ∂D, so the integral is, too.

undamental-elliptic] The point is therefore to prove Proposition 3.1. This will take many steps and much discussion, but with each step—I hope—almost transparent.

In the next two sections I'll look at some examples.

4. Ordinary differential equations

Let L be the linear ordinary differential operator

$$L = a_p(x)\frac{d^p}{dx^p} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)\,.$$

I'll assume the coefficients to be smooth functions on all of $\mathbb{R}$, with $a_p(x)$ never vanishing.

[fund-sol-ode] **4.1. Proposition.** *A function $\Phi(x)$ satisfies $L\Phi = \delta_y$ if and only if:*

(a) Φ is smooth in the region $x \neq y$ and satisfies $L\Phi = 0$ there;
(b) $\Phi^{(k)}$ is continuous at $x = y$ for $k < p - 1$;
(c) $\Phi^{(p-1)}$ increments by $1/a_p(y)$ at y.

Proof. Only the conditions at y need to be checked. By definition of the effect of a differential operator on a distribution

$$\begin{aligned}\langle L\Phi, f\rangle &= \langle \Phi, L^* f\rangle \\ &= \int_{-\infty}^{\infty} \Phi(x)(L^* f)(x)\, dx \\ &= \int_{-\infty}^{y} \Phi(x)(L^* f)(x)\, dx + \int_{y}^{\infty} \Phi(x)(L^* f)(x)\, dx\,,\end{aligned}$$

where L^* is the adjoint of L. How to calculate these last integrals? Repeated integration by parts gives us

$$\begin{aligned}&\int_s^t a(x)\Phi^{(k)} f(x)\, dx \\ &\qquad = \Big[\Phi^{(k-1)}(x)(af)(x) - \Phi^{(k-2)}(x)(af)'(x) + \cdots + (-1)^{k-1}\Phi(x)(af)^{(k-1)}(x)\Big]_s^t \\ &\qquad\qquad + (-1)^k \int_s^t \Phi(x)(af)^{(k)}\, dx\,.\end{aligned}$$

This leads first to the formula

$$\Big(a(x)\frac{d^k}{dx^k}\Big)^* f = (-1)^k \frac{d^k(af)}{dx^k}$$

and then to the equations

$$\begin{aligned}\int_{-\infty}^{y} (L\Phi)(x)f(x)\,dx &= \Big[a_p(x)\Phi^{(p-1)}(x)f(x)\Big]_{-\infty}^{y} \\ &\quad + \Big[\sum_{0\le k<p-1} A_k(x)\Phi^{(k)}(x)\Big]_{-\infty}^{y} \\ &\quad + \int_{-\infty}^{y} \Phi(x)(L^*f)(x)\,dx \\ \int_{y}^{\infty} (L\Phi)(x)f(x)\,dx &= \Big[a_p(x)\Phi^{(p-1)}(x)f(x)\Big]_{y}^{\infty} \\ &\quad + \Big[\sum_{0\le k<p-1} A_k(x)\Phi^{(k)}(x)\Big]_{y}^{\infty} \\ &\quad + \int_{y}^{\infty} \Phi(x)(L^*f)(x)\,dx\end{aligned}$$

where the $A_k(x)$ are smooth on all of $\mathbb{R}$ and vanish at infinity (since f has compact support). This in turn, since $L\Phi = 0$ as a function and the $\Phi^{(k)}$ for $k < p-1$ are continuous, leads to

$$\begin{aligned}\langle L\Phi, f\rangle &= \int_{-\infty}^{y} \Phi(x)\big(L^*f\big)(x)\,dx + \int_{y}^{\infty} \Phi(x)\big(L^*f\big)(x)\,dx \\ &= \big(\Phi^{(p-1)}(y+) - \Phi^{(p-1)}(y-)\big)a_p(y)f(y) \\ &= f(y)\,.\end{aligned}$$

In courses on ordinary differential equations, this result is often presented in a more concrete fashion. Suppose L of order p has constant coefficients, and φ is a function with $\varphi^{(k)}(0) = 0$ for $k < p-1$, $\varphi^{(p-1)}(0) = a_p$. Then $\varphi(x)$ satisfies $L\varphi = \delta_0$, $\varphi(x-s)$ satisfies $L\varphi(x-s) = \delta_s$ and

$$F(x) = \int_a^x \varphi(x-s)f(s)\,ds$$

is the solution of $LF = f$ with $F^{(k)}(a) = 0$ for $0 \le k < p$. This is the formula derived by the technique called 'variation of parameters'. This formula, once guessed, has an easy verification, using this elementary result from calculus:

[int-params] **4.2. Lemma.** *If*

$$F(x) = \int_a^x f(x,s)\,ds$$

then

$$F'(x) = f(x,x) + \int_a^x \frac{\partial f(x,s)}{\partial x}\,ds\,.$$

Proof. I can't resist giving the calculation here. Fix $b > x$ and write formally (but justifiably):

$$\begin{aligned} F(x) &= \int_a^b \chi_{[a,x]}(s) f(x,s)\, ds \\ F'(x) &= (d/dx) \int_a^n \chi_{[a,x]}(s) f(x,s) \\ &= \int_a^n \big((d/dx)\chi_{[a,x]}(s) f(x,s) + \chi_{[a,x]}(s)(d/dx) f(x,s)\big)\, ds \\ &= \int_a^n \big(\delta_x(s) f(x,s)\, ds + \int_a^x \frac{\partial f(x,s)}{\partial x}\, ds \\ & f(x,x) + \int_a^x \frac{\partial f(x,s)}{\partial x}\, ds\,. \end{aligned}$$

5. The Laplacian

Consider the Laplacian Δ on $\mathbb{R}^n$. If $f(r)$ is a function of r alone on $\mathbb{R}^n$ then

$$\begin{aligned} \frac{\partial f}{\partial x_k} &= \frac{\partial r}{\partial x_k}\, f'(r) \\ &= \frac{x}{r}\, f'(r) \\ \frac{\partial^2 f}{\partial x_k^2} &= \left(\frac{r^2 - x_k^2}{r^3}\right) f'(r) + \left(\frac{x_k^2}{r^2}\right) f'(r) \\ \Delta f &= f''(r) + \left(\frac{n-1}{r}\right) f'(r)\,. \end{aligned}$$

Thus the ordinary differential equation $\Delta f(r) = 0$ has a regular singularity at the origin. Solutions are of the form $f(r) = r^s$ with $s\big(s - (n-2)\big) = 0$, except when this equation has equal roots in which 1 and $\log r$ form a basis of solutions.

[fund-sol-laplacian] **5.1. Proposition.** *For every $n > 0$ set*

$$F_n(r) = \begin{cases} r & n = 1 \\ \log r & n = 2 \\ -1/(n-2)r^{n-2} & n > 2 \end{cases}$$

Then $\Delta F_n = \Gamma_{n-1}\, \delta_0$.

Here Γ_{n-1} is the $n-1$-dimensional volume of the unit sphere in $\mathbb{R}^n$. The consequence is that

$$f\big(\|x - y\|\big)/\Gamma_{n-1}$$

is a fundamental solution of Δ.

Proof. Starting with the definition of ΔF_n

$$\begin{aligned}\langle \Delta F_n, f\rangle &= \langle F_n, \Delta f\rangle \\ &= \int F_n(x)\Delta f(x)\,dx \\ &= \lim_{\varepsilon\to 0}\int_{\|x\|\ge\varepsilon} F_n(x)\Delta f(x)\,dx \\ \int_{\|x\|\ge\varepsilon} F_n(x)\Delta f(x)\,dx &= \int_{\|x\|\ge\varepsilon}\Delta F_n(x) f(x)\,dx + \int_{\|x\|=\varepsilon} F_n(x)\,\frac{\partial f}{\partial r} - \frac{\partial F_n}{\partial r}\,f(x)\,dx \\ &= \int_{\|x\|\ge\varepsilon}\Delta F_n(x) f(x)\,dx + \int_{\|x\|=\varepsilon} F_n(x)\,\frac{\partial f}{\partial r} - \int_{\|x\|=\varepsilon}\frac{\partial F_n}{\partial r}\,f(x)\,dx \\ &= \int_{\|x\|\ge\varepsilon}\Delta F_n(x) f(x)\,dx - \int_{\|x\|=\varepsilon} F_n(x)\,\frac{\partial f}{\partial r}\,dx + \int_{\|x\|=\varepsilon}\frac{1}{r^{n-1}}\,f(x)\,dx \\ &\quad\to \Gamma_{n-1} f(0)\,. \ \blacksquare\end{aligned}$$

The two examples I have explained are in fact related. Let H be the hyperplane $x_n = 0$, and let δ_H be the distribution that amounts to integrating over H. If $f(x_n)$ is a function of one variable extended through projection into n dimensions, a solution of the equation $\Delta F = f$ is given by

$$F(x_1, \ldots, x_n) = \int_a^{x_n} (x_n - s) f(s)\,ds\,.$$

It is invariant under translation by vectors in the hyperplane $x_n = 0$. This works because a solution of $\varphi'' = \delta_0$ is

$$\varphi(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{otherwise} \end{cases}\,.$$

In n dimensions $\varphi(x_n - s)$ is a solution of $\Delta\varphi = \delta_{x_n - s}$.

Green's formula in the plane is

$$\int_\Omega (\Delta f\cdot g - f\cdot\Delta g)\,dxdy = \int_{\partial\Omega}\left(\frac{\partial f}{\partial\nu}\,g - f\,\frac{\partial g}{\partial\nu}\right)ds\,.$$

Formally, if we let

$$r_{x,y}(\xi,\eta) = \sqrt{(\xi - x)^2 + (\eta - y)^2}$$

and set

$$g = \log r_{x,y}$$

the $\Delta r_{x,y} = \delta_{x,y}$ and we get

$$\frac{f(x,y)}{2\pi} = -\int_\Omega \Delta f\cdot\log r_{x,y}\,d\xi d\eta - \int_{\partial\Omega}\left(f\,\frac{\partial\log r_{x,y}}{\partial\nu} - \log r_{x,y}\,\frac{\partial f}{\partial\nu}\right)ds$$

for (x, y) inside Ω. This implies that if Δf is analytic, and in particular vanishes, then f itself is analytic. Similar arguments about radially symmetric eigenfunctions of Δ with a logarithmic singularity at 0 (which can be expanded in series easily, and produce Bessel functions), will imply the same about eigenfunctions of Δ. Even the eigenfunctions of the non-Euclidean Laplacian on the upper half-palne can be dealt with similarly. This is a special case of John's argument.

6. Radon's integral

If f is any function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, to each affine hyperplane H in $\mathbb{R}^n$ we can associate the integral

$$\widehat{f}(H) = \int_H f(h)\, dh\,.$$

The function $H \mapsto \widehat{f}(H)$ is called the **Radon transform** of f. I ask, *can f be recovered from its Radon transform? And if so, how?* In recent years these have become practical questions since the Radon transform is essentially what computerized tomography measures, but for us it will remain theoretical.

I define a **plane wave** in $\mathbb{R}^n$ to be a tempered distribution invariant under the translations of a hyperplane through the origin. Each point $\xi \neq 0$ in $\mathbb{R}^n$ and in particular a point of the unit sphere $\mathbb{S}^{n-1}$ corresponds to the hyperplane perpendicular to it. The goal of this section is to explain a generalization of a result originating with Radon about lines in the plane that represents the Dirac δ_0 as an average over $\mathbb{S}^{n-1}$ of distributions K_ξ each of which is a plane waves invariant under $\xi^\perp$.

If ξ is a unit vector in $\mathbb{S}^{n-1}$, associated to it is the map

$$\iota_\xi\colon\, \mathbb{R} \longrightarrow \mathbb{R}^n, \quad t \longmapsto t\xi$$

as well as the projection

$$\pi^\xi\colon\, \mathbb{R}^n \longrightarrow \mathbb{R}, \quad x \longmapsto x \bullet \xi\,.$$

Associated to these are linear maps

$$\iota_\xi^*\colon\, \mathcal{S}(\mathbb{R}^n) \longmapsto \mathcal{S}(\mathbb{R}), \quad [\iota_\xi^* f](t) = f(t\xi)$$

and

$$\pi_*^\xi\colon\, \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}), \quad [\pi_*^\xi f](t) = \int_{h \bullet \xi = 0} f(t\xi + h)\, dh\,.$$

These two constructions are dual to one another in terms of the Fourier transform.

[xi-ft] **6.1. Proposition.** *For f in $\mathcal{S}(\mathbb{R}^n)$*

(a) the Fourier transform of $\pi_*^\xi f$ is $\iota_\xi^* \widehat{f}$;

(b) the Fourier transform of $\iota_\xi^* f$ is $\pi_*^\xi \widehat{f}$.

Proof. The first one is direct. At first let $F = \pi_*^\xi(f)$. Then

$$\begin{aligned}\widehat{F}(x) &= \int_{\mathbb{R}} e^{-2\pi i (xy)} F(y)\, dy \\ &= \int_{\mathbb{R}} e^{-2\pi i (xy)}\, dy \int_{h \bullet \xi = 0} f(y\xi + h)\, dh \\ &= \int_{\mathbb{R}^n} e^{-2\pi i (x\xi \bullet z)} f(z)\, dz \\ &= [\iota_\xi^* \widehat{f}\,](x)\,.\end{aligned}$$

The second follows from this and Fourier duality. Set $f = \widehat{\varphi}$, $\varphi = \mu_{-1}\widehat{f}$. Then Then

$$\begin{aligned}F &= \iota_\xi^* f \\ &= \iota_\xi^* \widehat{\varphi} \\ &= \text{Fourier transform of } \pi_*^\xi \varphi \quad \text{(from the first part)} \\ \widehat{F} &= \mu_{-1} \pi_*^\xi \varphi \\ &= \pi_*^\xi f\,.\end{aligned}$$

Dual to the maps ι_ξ^* and π_*^ξ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R})$ are maps from tempered distributions on $\mathbb{R}$ to those on $\mathbb{R}^n$:

$$\langle \iota_*^\xi \Phi, f\rangle = \langle \Phi, \iota_\xi^* f\rangle$$
$$\langle \pi_\xi^* \Phi, f\rangle = \langle \Phi, \pi_*^\xi f\rangle$$

The distribution $\iota_*^\xi\Phi$ has support on L. The distribution $\pi_\xi^*\Phi$ is invariant under translations in $x \bullet \xi = 0$. Formally, we have

$$\langle \iota_*^\xi \Phi, f\rangle = \int_{\mathbb{R}} \Phi(t) f(t\xi)\, dx$$
$$\langle \pi_\xi^* \Phi, f\rangle = \int_{\mathbb{R}^n} \Phi\big(y \bullet \xi\big) f(y)\, dy$$

where π_L is orthogonal projection onto L. The notation in the following result makes it look like the previous one, from which it follows, although with entirely different meaning:

[xi-ft-dual] **6.2. Proposition.** *For Φ a tempered distribution on $\mathbb{R}$*

(a) the Fourier transform of $\iota_*^\xi\Phi$ is $\pi_\xi^*\widehat{\Phi}$;
(b) the Fourier transform of $\pi_\xi^*\Phi$ is $\iota_*^\xi\widehat{\Phi}$.

I want to express δ_0 as an integral of plane waves. The Fourier transform of δ_0 is the constant function **1**, and I am going to start by representing **1** as an integral of distributions over the sphere. This is straightforward. Expressing **1** in polar coordinates:

$$\langle \mathbf{1}, f\rangle = \int_{\mathbb{R}^n} f(x)\, dx = \int_{\mathbb{S}^{n-1}} d\xi \int_0^\infty r^{n-1} f(r\xi)\, dr\,.$$

The inner integral is the linear distribution corresponding to the function

$$R_n(x) = \begin{cases} 0 & x < 0 \\ x^{n-1} & x \ge 0 \end{cases}$$

Since the Fourier transform of **1** is δ_0 and the Fourier transform commutes with spherical integrals, we have

$$\delta_0 = \int_{\mathbb{S}^n} \widehat{R_\xi}\, d\xi$$

where R_ξ is the distribution on $\mathbb{R}^n$:

$$\langle R_\xi, f\rangle = \int_0^\infty r^{n-1} f(r\xi)\, dr\,.$$

♣ [xi-ft-dual] We must now find the Fourier transform of R_ξ. By Proposition 6.2, this is the pull-back to $\mathbb{R}^n$ of the one-dimensional Fourier transform of $R_n(x)$ on $\mathbb{R}\cdot\xi$.

First of all,

$$R_n(x) = \frac{R_{n,+} + R_{n,-}}{2}$$

where $R_{n,+}$ is even and $R_{n,-}$ is odd. Evaluating these distributions on spherically symmetric functions $f(r^2)$ shows that the spherical integral of the odd ones vanishes. So we have only to deal with $R_{n,+}$. We have

$$R_{n,+} = \begin{cases} x^{n-1} & \text{if } n \text{ is odd} \\ x^{n-1}\mathrm{sgn}(x) & \text{if } n \text{ is even.} \end{cases}$$

[ft-pv] **6.3. Lemma.** *The Fourier transform of* x^n *is*

$$\frac{\delta_0^{(n)}}{(-2\pi i)^n}$$

and the Fourier transform of $x^n \mathrm{sgn}(x)$ *is*

$$\frac{2n!}{(2\pi i)^{n+1}}\,\mathrm{Pf}(1/x^{n+1})\,.$$

Proof. The Fourier transform of δ_0 is **1**, hence the transform of $\delta_0^{(n)}$ is $(2\pi i y)^n$, so the Fourier transform of $(2\pi i y)^n$ is $\mu_{-1}\delta_0^{(n)} = (-1)^n\delta_0^{(n)}$. Therefore the transform of y^n is $(-1)^n\delta_0^{(n)}/(2\pi i)^n$.

Before doing the second half of the proof, I'll first recall some facts about *parties finies* distributions. Suppose f in $\mathcal{S}(\mathbb{R})$, and let

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

be its Taylor series at 0, so $f_m = f^{(m)}(0)/m!$. Then

$$\varphi_n(x) = f - \frac{f_0 + x f_1 + \cdots f_n x^n}{x^{n+1}}$$

is still smooth throughout $\mathbb{R}$, although no longer in in $\mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}\int_\varepsilon^\infty \frac{f(x)}{x^{n+1}}\,dx &= \int_\varepsilon^1 \frac{f(x)}{x^{n+1}}\,dx + \int_1^\infty \frac{f(x)}{x^{n+1}}\,dx \\ &= \int_\varepsilon^1 \frac{f_0 + x f_1 + \cdots + f_n x^n}{x^{n+1}}\,dx + \int_\varepsilon^1 \varphi_n(x)\,dx + \int_1^\infty \frac{f(x)}{x^{n+1}}\,dx\,.\end{aligned}$$

The last integral is independent of ε. As $\varepsilon \to 0$, the second integral has a finite limit. The first integral is

$$\begin{aligned}&\Big[-\frac{f_0}{n x^n} - \frac{f_1}{(n-1)x^{n-1}} - \cdots - f_n \log x\Big]_\varepsilon^1 \\ &\qquad = -\frac{f_0}{n} - \frac{f_1}{(n-1)} - \cdots - f_{n-1} + \frac{f_0}{n\varepsilon^n} + + \frac{f_1}{(n-1)\varepsilon^{n-1}} + \cdots + f_n \log\varepsilon\end{aligned}$$

Therefore the limit

$$\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{f(x)}{x^{n+1}}\,dx - \Big(\frac{f_0}{n\varepsilon^n} + \frac{f_1}{(n-1)\varepsilon^{n-1}} + \cdots + f_n\log\varepsilon\Big)$$

exists, and defines a distribution called the **finite part** $\mathrm{Pf}(1/x^{n+1})$.

The distribution $\mathrm{Pf}(1/x^{n+1})$ behaves covariantly with respect to scalar multiplication. So does $x^n\mathrm{sgn}(x)$ and its Fourier transform must be a multiple of $\mathrm{Pf}(1/x^{n+1})$, since such covariant distributiosn are unique up to scalar. The scalar is easily calculated by considering the effect on various Hermite functions $P(x)e^{-\pi x^2}$. ▯

Let

$$\Phi_n = \begin{cases} \dfrac{\delta_0^{(n)}}{2(-2\pi i)^n} & n \text{ odd} \\ \dfrac{n!}{(2\pi i)^{n+1}}\,\mathrm{Pf}(1/x^{n+1}) & n \text{ even.}\end{cases}$$

[radon-inversion] **6.4. Proposition.** (Radon inversion formula) *We have*

$$\delta_0 = \int_{\mathbb{S}^{n-1}} \Phi_{n-1}(x\bullet\xi)\,d\xi\,.$$

7. The characteristic variety

The **characteristic variety** at x is the projective variety of ξ in $\mathbb{P}(T_x^*)$ determined by the homogeneous equation $\sigma_L(x,\xi)=0$. An **elliptic operator** is one whose real characteristic variety is everywhere empty. The Laplacian, for example, is elliptic.

What is the practical significance of the characteristic variety? Very roughly, it says something about the possible singularities of the solution of an equation $LF=0$. According to this idea, solutions to elliptic equations should therefore have essentially no singularities, and this turns out to be the case. There are two aspects to this—on the one hand, solutions to elliptic equations with smooth coefficients are themselves smooth. On the other, solutions of equations with analytic coefficients are analytic. The first of these results is probably the more important for most purposes, and it is by far the easier to prove. The second, however, is one of the cornerstones of representation theory. It allows an an extrapolation from local to global that makes the basic link between representations of a Lie group and its algebra on infinite-dimensional spaces. It is this second result about analyticity that this note is concerned with.

The identification of symbols with homogeneous functions on T_x^* is canonical. This is most simply seen from a calculation. Suppose we change coordinates from $x=(x_i)$ to $y=(y_i)$. The chain rule gives us

$$\begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \dots \\ \partial/\partial x_n \end{bmatrix} = \begin{bmatrix} \partial y_1/\partial x_1 & \dots & \partial y_n/\partial x_1 \\ \partial y_1/\partial x_2 & \dots & \partial y_n/\partial x_2 \\ \dots & & \\ \partial y_1/\partial x_n & \dots & \partial y_n/\partial x_n \end{bmatrix} \begin{bmatrix} \partial/\partial y_1 \\ \partial/\partial y_2 \\ \dots \\ \partial/\partial y_n \end{bmatrix} .$$

This gives us expressions for higher derivatives as well. Thus

$$\begin{aligned} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) &= \frac{\partial}{\partial x_i}\left(\sum_\ell \frac{\partial y_\ell}{\partial x_j}\frac{\partial f}{\partial y_\ell}\right) \\ &= \sum_\ell \left(\frac{\partial^2 y_\ell}{\partial x_i \partial x_j}\frac{\partial f}{\partial y_\ell} + \frac{\partial y_\ell}{\partial x_j}\frac{\partial}{\partial x_i}\frac{\partial f}{\partial y_\ell}\right) \\ &= \sum_\ell \frac{\partial^2 y_\ell}{\partial x_i \partial x_j}\frac{\partial f}{\partial y_\ell} + \sum_{k,\ell} \frac{\partial y_k}{\partial x_i}\frac{\partial y_\ell}{\partial x_j}\frac{\partial^2 f}{\partial y_k \partial y_\ell} \end{aligned}$$

so that

$$\frac{\partial^2}{\partial x_i \partial x_j} = \sum_{k,\ell} \frac{\partial y_k}{\partial x_i}\frac{\partial y_\ell}{\partial x_j}\frac{\partial^2}{\partial y_k \partial y_\ell} + \text{ differential operators of degree one},$$

in which the coefficient of $\partial^2/\partial y_k^2$ is

$$\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} + \frac{\partial y_k}{\partial x_j}\frac{\partial y_k}{\partial x_i} .$$

If $i \neq j$ this is the same as the symbol of $\partial^2/\partial x_i \partial x_j$ evaluated at dy_k.

What conclusion? The sheaf of differential operators is filtered by order, and an induction argument based on similar calculations shows that coordinate changes preserve this filtration. A closer look shows that coordinate changes act on the graded ring determined by this filtration as they do on the ring of symmetric tensors $S^\bullet T_x$, and that the identification of the symbol of a differential operator with a homogeneous function on T_x^* is canonical.

We shall need to know in a moment, in particular, that the coefficient of $\partial^p/\partial y_1^p$ in the new expression for a differential L of order p at x is $\sigma_L(x, dy_1)$.

If ξ lies in T_x^* then ξ is called **characteristic** for the operator L if $\sigma_L(x,\xi)=0$, non-characteristic otherwise.

[cauchy-k-symbol] **7.1. Proposition.** *The cotangent vector ξ at x is non-characteristic if and only if one can make a coordinate change $y = y(x)$ locally around x with $\xi = dy_1$ so that the operator L takes the form*

$$L = b_*(x)\partial^p/\partial y_1^p + \sum_{|k|\le p}^{*} b_k(y)\,\partial^k/\partial y^k\,,$$

where $b_*(x) \ne 0$ and the sum does not involve $\partial^p/\partial y_1^p$.

Proof. This follows from the remark made a moment ago about the coefficient of $\partial^p/\partial y_1^p$ after a coordinate change. ∎

8. The Cauchy-Kowalevsky theorem

Suppose L to be expressed in non-degenerate form

$$L = \partial^p/\partial x_n^p + \sum_{|k|\le p} b_k(x)\,\partial^k/\partial x^k$$

where the second sum does not involve $\partial^p/\partial x_n^p$.

[formal-CK] **8.1. Lemma.** *Given p formal series $F_i(x_1,\dots,x_{n-1})$ for $0 \le i < p$ there exists a unique formal series $F(x_1,\dots,y_n)$ satisfying $LF = 0$ with*

$$\partial^k F/\partial x_1^k(x_1,\dots,x_{n-1},0) = F_k(x_1,\dots,x_{n-1})\,.$$

Let $\widehat{\mathcal{D}}_n$ be the ring of differential operators whose coefficients are formal power series in the n variables x_j with $1 \le j \le n$. Let (L) be the ideal generated by L. The previous result follows from this, which is easy to verify:

[formal-version] **8.2. Lemma.** *The ring $\mathcal{D}_n/(L)$ is free over $\widehat{\mathcal{D}}_{n-1}$ of rank p.*

This means that given p formal power series in the first $n-1$ variables there exists a unique formal power series Φ in n variables solution of $L\Phi = 0$ restricting suitably to $x_n = 0$. It makes plausible:

[cauchy-kowalevsky] **8.3. Proposition.** (Cauchy-Kowalevsky) *Suppose $x_n = c$ to be a hyperplane H with $\sigma(x, dx_n) \ne 0$, and suppose that the coefficients of L are real analytic in the neighbourhood of x. Given any p analytic functions F_m defined on H near x, there exists a unique real analytic solution of $L\Phi = 0$ in the neighbourhood of x such that*

$$\partial^k\Phi(x_1,\dots,x_{n-1},0)/\partial x_n^k = F_k(x_1,\dots,x_{n-1})$$

for $0 \le k < p$.

A classic theorem of Holmgren extends the uniqueness to smooth solutions as well.

The first step in the proof is to simplify the calculations by changing the given system equation into a system of N first order equations of the form

$$\frac{\partial u_j}{\partial x_n} = \sum_{1\le k\le N,\ell<n} a_{k,\ell}(x)\,\frac{\partial u_k}{\partial x_\ell} + \sum_{1\le k\le N} a_k(x)u_k(x) +$$

This is done by introducing new variables u_i for all partial derivatives $\partial^k\Phi/\partial x^k$ with $k| < p$, and then being careful about the initial conditions to impose. For example, for the Laplacian

$$\frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial x_2^2} = 0$$

we introduce

$$\begin{aligned} u_1 &= \Phi \\ u_2 &= \partial\Phi/\partial x_1 \\ u_3 &= \partial\Phi/\partial x_2 \,. \end{aligned}$$

Then

$$\frac{\partial u_2}{\partial x_2} = \frac{\partial^2\Phi}{\partial x_2 \partial x_1} = \frac{\partial^2\Phi}{\partial x_1 \partial x_2}, \quad \frac{\partial u_3}{\partial x_2} = \frac{\partial^2\Phi}{\partial x_2^2} = -\frac{\partial^2\Phi}{\partial x_1^2}$$

leading to the system

$$\begin{aligned} \frac{\partial u_1}{\partial x_2} &= \quad u_3 \\ \frac{\partial u_2}{\partial x_2} &= \quad \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_3}{\partial x_2} &= -\frac{\partial u_2}{\partial x_1} \,. \end{aligned}$$

For this linear system we are specifying the initial values of the functions u_i on the hyperplane $x_n = 0$, which means three independent functions of $x_1, \ldots, x_{n-1}$. This is in contrast to the initial conditions of the original problem, which required only two. The difference is accounted for by the fact that the original problem specifies the initial value of $u_3 = \partial\Phi/\partial x_1$ on $x_2 = 0$. The original equation is equivalent to the derived first order system of equations, but with restricted initial conditions.

To see how things go, I'll just look at the one of the simplest examples

$$\frac{\partial u}{\partial y} = a(x,y)\frac{\partial u}{\partial x} + b(x,y)u \,.$$

where we are given convergent series

$$\begin{aligned} u(x,0) &= u_0(x) \\ &= \sum u_j x^j \\ a(x,y) &= \sum a_{j,k} x^j y^k \\ b(x,y) &= \sum b_{j,k} x^j y^k \,. \end{aligned}$$

We are looking for a series solution $u(x,y)$. We have

$$\begin{aligned} u &= \sum_{j,k} u_{j,k} x^j y^k \\ \frac{\partial u}{\partial x} &= \sum_{j,k} (j+1) u_{j+1,k} x^j y^k \\ \frac{\partial u}{\partial y} &= \sum_{j,k} (k+1) u_{j,k+1} x^j y^k \end{aligned}$$

and we get an equation

$$\begin{aligned} \sum_{j,k} (j+1) u_{j+1,k} x^j y^k &= \Big(\sum a_{j,k} x^j y^k\Big)\Big(\sum (k+1) u_{j,k+1} x^j y^k\Big) + \Big(\sum b_{j,k} x^j y^k\Big)\Big(\sum u_{j,k} x^j y^k\Big) \\ &= \sum P_{j,k}(a,b,u)\, x^j y^k \end{aligned}$$

where the coefficients in the polynomials $P_{j,k}$ are all non-negative. The differential equation gives us conditions on all the $u_{j,k}$ with $j \geq 1$ but the $u_{0,k}$ are determined by the boundary conditions $u(x,0) = u_0(x)$.

The method to be applied is that of **majorants**, due originally to Cauchy himself. I'll exhibit an equation

$$\frac{\partial U}{\partial y} = A(x,y)\frac{\partial U}{\partial x} + B(x,y)u, \quad U(x,0) = U_0(x)$$

which can be solved explicitly by a relatively simple convergent series and which **majorizes** the original one in that we have inequalities

$$\begin{aligned} U_j &\ge |u_j| \\ A_{j,k} &\ge |a_{j,k}| \\ B_{j,k} &\ge |b_{j,k}| \end{aligned}$$

[majorizing] **8.4. Lemma.** *If the second system majorizes the first, then the solution of the second system majorizes that of the first.*

This means that $U_{j,k} \ge |u_{j,k}|$ and since the series for U converges so does that for u.

Proof. Because the coefficients of P are non-negative. ∎

So now it remains only to find a majorizing equation that can be solved exactly.

[majorizing] **8.5. Lemma.** *Suppose* $A(x) = \sum a_k x^k$ *converges for* $|x_j| \le r$ *where*

$$x^k = x_1^{k_1} \ldots x_n^{k_n}.$$

There exists C *such that* $A(x)$ *is majorized by the series*

$$\frac{C}{1 - \left(\dfrac{x_1 + \cdots + \rho x_n}{r}\right)}.$$

for any $1 \le \rho$.

Proof. This is a straightforward consequence of the convergence hypthesis. ∎

Now we are to find an equation that majorizes the given one, and is explicitly solvable. The secret to making it solvable is to make all the functions of (x,y) functions a single variable z which is to be a linear function of x and y, say $z = x + \rho y$. The partial differential equation becomes

$$\rho U'(z) = A(z)U'(z) + B(z)U(z)$$

which becomes

$$\begin{aligned} U'(\rho - A(z)\beta) &= B(z)U \\ \frac{U'}{U} &= \frac{B(z)}{\rho - A(z)\beta} \\ U &= U(z)e^{C(z)} \\ C(z) &= \int_0^z \frac{B(s)\,ds}{\rho - A(s)} \\ &= \frac{1}{\rho}\int_0^z \frac{B(s)\,ds}{1 - \big(A(s)/\rho\big)} \\ &= \frac{1}{\rho}\int_0^z B(s)\Big(1 + \big(A(s)/\rho\big) + \big(A(s)/\rho\big)^2 + \cdots\Big)\,ds \end{aligned}$$

which makes sense as long as we choose ρ large enough so that $A(s)\beta/\alpha| < 1$ in the interval $[0,z]$ we are concerned with. It now remains only to choose $A(z)$ and $B(z)$ majorizing the coefficients of the equation. This is where the lemma comes in.

9. Its consequences

1. Find solution of $L\Phi = \delta_{\xi \bullet x - s}$.

2. Then of inhomogeneous equations, as for dim. one.

3. Solve inhomogeneous arising in Radon's formula.

4. Find fund. soln. of L, varying analytically with x.

5. Show locally integrable.

10. An application to representation theory

Suppose G to the group of real points on a Zariski-connected reductive group defined over $\mathbb{R}$ (i.e. one whose group of complex points is connected), K a maximal compact subgroup. The group G has a finite number of topological components, and K meets them all. A smooth representation of G is called **admissible** if irreducible representations of K occur with finite multipicity and some ideal of $Z(\mathfrak{g})$ of finite codimension annihilates it. The following can be found in §§3.17–3.23 of [Borel:1974].

[borel-admissible] **10.1. Proposition.** *Suppose (π, V) to be an admissible smooth representation of G. If W is stable under $\mathfrak{g}$ and K, then its closure is stable under G.*

Proof. Let $V_{(K)}$ be the subspace of K-finite vectors in V. It is stable under $\mathfrak{g}$, and of finite length as a module over $(\mathfrak{g}, K)$. The subspace W is hence finitely generated over $U(\mathfrak{g})$, say by the K-stable finite-dimensional subspace U.

Let W_* be the smallest closed, G-stable subspace of V containing U. The vectors $\pi(g)u$ for g in G, u in U are thus dense in W_*. The space W_* is stable under $U(\mathfrak{g})$, hence contains W.

By the Hahn-Banach theorem, it suffices to show that if λ is in the continuous dual of W_* and vanishes on W then it vanishes on all of W_*. From what I have remarked in the previous paragraph, it suffices to show that the function $F_{\lambda,u}$ with

$$F_{\lambda,u}(g) = \langle \lambda, \pi(g)u \rangle$$

vanishs identically. The Casimir element C of $U(\mathfrak{g})$, acting on vectors of a fixed K-type, is elliptic. Therefore $P(C)$ is elliptic for every polynomial $P(x)$. Since $P(C)U = 0$ for some polynomial P. Because of the analyticity of solutions of elliptic differential equations, the function F_u is analytic on G. Its derivative at 1 is determined by the $\langle \lambda, \pi(X)u \rangle$ for X in $U(\mathfrak{g})$. By assumption these vanish, so $F_u(g)$ vanishes on the connected component of G. Since U is K-stable, it vanishes on all of G. But if $\langle \lambda, \pi(g)u \rangle = 0$ for all g in G and u in U then $\lambda = 0$ on W_*. ▯

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