Conjugacy classes of parabolic subgroups

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Suppose G to be a Zariski-connected reductive group. Fix a minimal rational parabolic subgroup $P_\emptyset$ and a maximal split torus $A_\emptyset$ contained in it. Let Δ be the set of simple roots associated to these choices, Σ the set of all roots. To each subset $\Theta \subseteq \Delta$ is associated a parabolic subgroup P_Θ whose unipotent radical is generated by root subgroups N_λ with λ in $\Sigma^+ - \Sigma^+_\Theta$. The classification of rational parabolic subgroups is very simple: *Every rational parabolic subgroup is conjugate in G to exactly one P_Θ.*

This essay is mainly interested in the analogous question about pairs of rational parabolic subgroups, and will produce an analogous answer. One of the basic notions involved is that of **associated parabolic subgroups**. Two parabolic subgroups of a reductive group are associates if their Levi factors are conjugate. In analysis on reductive groups over local and global fields, they play an important role. They are needed in particular for understanding the structure of representations induced from parabolic subgroups, for computing the constant terms of Eisenstein series, in Jim Arthur's analysis of truncation operators, and for computing Plancherel measures.

The final discussion requires a few basic facts about Weyl groups of root systems, but I shall begin in the first section by dealing with general Coxeter groups. In the second W will be a finite Coxeter group, and in the last the Weyl group of a finite integral root system.

The standard reference for this subject is [Bourbaki:1968], but many major results there are only exercises, and only for finite systems. I have taken some important material from [Bergeron et al.:1992].

Contents

1. Cosets 1
2. Associates 4
3. Parabolic pairs 6
4. References 8

1. Cosets

Throughout, W will be a Coxeter group of finite rank. and suppose we are given a Cartan realization on the vector space $V^\vee$. Let $\mathcal{C}$ be the Tits cone, C a choice of positive chamber. For example, if W is finite $\mathcal{C}$ is all of $V^\vee$.

In this section, W is arbitrary.

Let Σ be the set of roots, which are to be identified with certain closed half-spaces $\lambda \geq 0$, or equivalently rays in the dual V of $V^\vee$. Let Δ be the simple roots, those tightly enclosing C. For a root λ, let s_λ be the corresponding reflection, and let S_Δ be the set of all of the s_α for α in Δ. A root is positive if it contains C, or equivalently if its ray is contained in the closed cone spanned by Δ.

SINGLE COSETS. Every w on W may be expressed as a product of elementary reflections s_α in the walls of C. Let $\ell(w)$ be the minimum length of such an expression. It is immediate from the definition that $\ell(ws_\alpha) = \ell(w) \pm 1$. One basic fact relating combinatorics and geometry of W is that *if α is in Δ and $w\alpha < 0$ then $\ell(ws_\alpha) = \ell(w) - 1$.*

For w in W, let

$$\begin{aligned} R_w &= \{\lambda > 0 \,|\, w\lambda < 0\} \\ L_w &= R_{w^{-1}} \\ &= \{\lambda > 0 \,|\, w^{-1}\lambda < 0\}\,. \end{aligned}$$

A second basic fact is that $R_{s_\alpha} = \{\alpha\}$.

From these we deduce:

- *If $w(\alpha) > 0$ (α in Δ) then $\ell(ws_\alpha) = \ell(w) + 1$.*
- *If $w(\alpha) > 0$ (α in Δ) then $R_{ws_\alpha} = s_\alpha R_w \sqcup \{\alpha\}$.*

For suppose α in Δ, $w(\alpha) > 0$, $s = s_\alpha$. It is immediate that $R_{ws} \supseteq sR_w \cup \{\alpha\}$. Suppose $\lambda > 0$, $ws\lambda < 0$, $\lambda \neq \alpha$. Then $s\lambda > 0$, hence $s\lambda$ is in R_w.

- *For all w, $\ell(w) = |R_w|$.*

By induction from previous claims.

These lead to:

1.1. Proposition. *The following are equivalent:*

(a) $\ell(xy) = \ell(x) + \ell(y)$;
(b) R_{xy} is the disjoint union of R_y and $y^{-1}R_x$;
(c) $y^{-1}R_x \subseteq \Sigma^+$.

Something similar holds for L_w.

Proof. The only difficult point is to show that if $w^{-1}R_y > 0$ then $\ell(yw) = \ell(y) + \ell(w)$. This can be done by induction on $\ell(y)$. If $y = s_\alpha$ this claim is the basic fact I have already referred to. Otherwise, write $y = xs_\alpha$ with $y > x$. Then

$$w^{-1}R_y = w^{-1}\big(\{\alpha\} \cup R_x\big) > 0\,.$$

By induction $\ell(sw) = 1 + \ell(w)$, and $\ell(xsw) = \ell(x) + \ell(sw)$. ▮

1.2. Corollary. *The set R_w determines w.*

Proof. By induction on $\ell(w)$, because if $w(\alpha) < 0$ and $x = ws_\alpha$ then $\ell(x) < \ell(w)$ and $R_x = s_\alpha(R_w - \{\alpha\}$. ▮

Recall that to each subset $\Theta \subseteq \Delta$ corresponds the face C_Θ of $\overline{C}$ where $\lambda = 0$ for $\lambda \in \Theta$, $\lambda > 0$ for $\lambda \in \Delta - \Theta$. An element of W fixes a point in C_Θ if and only if it lies in W_Θ. The roots in Θ generate a root system in $V/\mathrm{Ker}(\Theta$. Its Coxeter group is the subgroup W_Θ.

1.3. Corollary. *An element w in W lies in W_Θ if and only if $R_w \subset \Sigma^+_\Theta$.*

1.4. Corollary. *In each coset $W_\Theta\backslash W$ there exists a unique representative x of least length. This element is the unique x in its coset such that $x^{-1}\Theta > 0$. For any y in W_Θ we have $\ell(yx) = \ell(y) + \ell(x)$.*

Proof. Existence: start with $x = w$, $t = 1$, and as long as there exists s in $S = S_\Theta$ such that $sx < x$ replace x by sx, t by ts. At every moment we have $w = tx$ with t in W_Θ and $\ell(w) = \ell(t) + \ell(x)$. At the end we have $sx > x$ for all s in S.

Uniqueness: suppose $w = yx$ with y in W_Θ and $x^{-1}\Theta > 0$. Corollary 1.3 and Proposition 1.1 imply that $L_x \subseteq \Sigma^+ - \Sigma^+_\Theta$, and $L_y \subseteq \Sigma^+_\Theta$. Hence $yL_x > 0$ and therefore $L_w = yL_x \sqcup L_y$. The set L_y is therefore $L_w \cap \Sigma_\Theta$, and L_x is $L_w \cap (\Sigma^+ - \Sigma^+_\Theta)$. Corollary 1.2 implies that x and y are also determined. ▮

From now on:

$$[W_\Theta\backslash W] = \{w \in W \mid w^{-1}\Theta > 0\}$$
$$[W/W_\Theta] = \{w \in W \mid w\Theta > 0\}\,.$$

According to Corollary 1.4, the product maps

$$W_\Theta \times [W_\Theta\backslash W] \longrightarrow W$$
$$[W/W_\Theta] \times W_\Theta \longrightarrow W$$

are bijections.

Among the elements of $[W_\Theta\backslash W]$ are those in the set

$$[\![W_\Theta\backslash W]\!] = \{w \mid w^{-1}\Theta \subseteq \Delta\}\,.$$

DOUBLE COSETS. Next I want to do something similar for double cosets. The clearest approach to this apparently first appeared as Lemma 2.2 in [Bergeron et al.:1992], whom I'll follow.

1.5. Lemma. *Suppose $\Theta \subseteq \Delta$. If $w\Theta > 0$ then $\Delta \cap w\Sigma_\Theta = \Delta \cap w\Theta$.*

Proof. Let D_Θ be the region where $\alpha \ge 0$ for all $\alpha \in \Theta$. It is convex, and the cone $\mathcal{D}$ of functions that are non-negative on D_Θ is spanned by Θ. Therefore $w\mathcal{D}$ is spanned by $w\Theta$, and its extremal points are the $w(\alpha)$ with α in Θ. By assumption on w, the cone $w\mathcal{D}$ is contained in the cone $\widehat{C}$ spanned by Δ.

Suppose β to in $\Delta \cap w\Sigma_\Theta$. Since $w\Theta > 0$, this is actually in $w\Sigma_\Theta^+$. It certainly spans an extremal ray of $\widehat{C}$. But since $w\mathcal{D} > 0$, it also spans an extremal ray of $w\mathcal{D} > 0$, which means that it must lie in $w\mathcal{D}$.

Let $[W_\Theta\backslash W/W_\Omega] = [W_\Theta\backslash W] \cap [W/W_\Omega]$.

1.6. Proposition. *Suppose $\Theta, \Omega \subseteq \Delta$.*

(a) *If*

$$u \in [W_\Theta/W_{\Theta\cap v(\Omega)}],\ v \in [W_\Theta\backslash W/W_\Omega]\,, \qquad (*)$$

then $uv \in [W/W_\Omega]$.

(b) *Conversely, suppose that x in $[W/W_\Omega]$ is expressed as*

$$x = uv \quad (u \in W_\Theta,\ v \in [W_\Theta\backslash W])\,.$$

Then u and v satisfy conditions $(*)$.

Proof. Suppose that conditions $(*)$ hold. We want to deduce that $uv\Omega > 0$.

The assumptions on u and v mean that

$$\begin{aligned} u\big(\Sigma^+ - \Sigma_\Theta^+\big) &> 0 \\ u\big(\Sigma_{\Theta\cap v\Omega}^+\big) &> 0 \\ v(\Omega) &> 0 \\ v^{-1}(\Theta) &> 0 \end{aligned}$$

Since $\Sigma^+ = (\Sigma^+ - \Sigma_\Theta^+) \cup \Sigma_\Theta^+$,

$$\Omega \subseteq v^{-1}\big(\Sigma^+ - \Sigma_\Theta^+\big) \cup v^{-1}\big(\Sigma_\Theta^+\big)\,.$$

But by Lemma 1.5

$$\Omega \cap v^{-1}\Sigma_\Theta^+ = \Omega \cap v^{-1}\Theta\,,$$

so

$$v\Omega \subseteq (\Sigma^+ - \Sigma_\Theta^+) \cup \Sigma_{\Theta\cap v\Omega}^+$$

and

$$uv\Omega \subseteq u\big(\Sigma^+ - \Sigma_\Theta^+\big) \cup u\big(\Sigma_{\Theta\cap v\Omega}^+\big) \subseteq \Sigma^+\,.$$

This concludes the proof of (a).

For (b), factor $x = uv$ as in Corollary 1.4, with $u \in W_\Theta$, $v^{-1}(\Theta) > 0$. Since $\ell(uv) = \ell(u) + \ell(v)$, $R_x = R_v \cup v^{-1}R_u$. This implies that (i) $R_v \subseteq R_x \subseteq \Sigma^+ - \Sigma_\Omega^+$, hence $v(\Omega) > 0$ and

$$v \in [W_\Theta\backslash W/W_\Omega]\,;$$

(ii) by assumption on x, $u\big(v(\Omega)\big) > 0$, hence in particular

$$u\big(\Theta \cap v(\Omega)\big) > 0\,.$$

We can therefore write every w as xvy with x in W_Θ, v in $[W_\Theta\backslash W/W_\Omega]$, y in W_Ω. The elements x and y are not unique without further restriction. For example, if x lies in $W_{\Theta\cap v\Omega}$ then $xv = v{\cdot}v^{-1}xv$. But the previous result suggests what one can expect.

Suppose given an arbitrary w in W. Factor it as uy with u in $[W/W_\Omega]$, y in W_Ω. Factor $u = xv$ with x in W_Θ, v in $[W_\Theta\backslash W]$. Then by (a), x will be in $W_{\Theta\cap x\Omega}$ and

$$\begin{aligned} \ell(w) &= \ell(u) + \ell(y) \\ \ell(u) &= \ell(x) + \ell(v)\,. \end{aligned}$$

Hence:

1.7. Proposition. *Every w in W may be expressed uniquely as $w = xvy$ with x in $[W_\Theta/W_{\Theta\cap v\Omega}]$, v in $[W_\Theta\backslash W/W_\Omega]$, y in W_Ω. In this situation, $\ell(w) = \ell(x) + \ell(v) + \ell(y)$.*

Uniqueness is because every w is determined by R_w or L_w.

1.8. Corollary. *If x is in $[W_\Theta\backslash W/W_\Omega]$ then $W_\Theta \cap xW_\Omega x^{-1} = W_{\Theta\cap x\Omega}$.*

Proof. Suppose w to be in $W_\Theta \cap xW_\Omega x^{-1}$. Write it as w_1w_2 with w_1 in $[W_\Theta/W_{\Theta\cap x\Omega}]$, w_2 in $W_{\Theta\cap x\Omega}$, Then

$$wx = w_1x\cdot x^{-1}w_2x = x\cdot x^{-1}wx\,.$$

Because of uniqueness, we deduce that $w_1 = 1$.

2. Associates

From now on, I'll assume W to be finite, although it should not be difficult to make suitable modifications if this is not the case. I'll also use a W-invariant Euclidean norm on $V^\vee$ to identify it with its dual.

The space $V^\vee$ is partitioned by chambers and their relatively open faces. The chambers partition the complement of the root hyperplanes, and are a principal homogeneous set with respect to the action of W. In this section I'll look at the induced partitions of lower dimensional linear subspaces in the partition.

Fix a fundamental chamber $C = C_\emptyset$ with associated Δ. With our assumption on finiteness, there exists a longest element $w_\ell = w_{\ell,\Delta}$ in W. Let s_λ be the root reflection in the hyperplane $\lambda = 0$.

For each $\Theta \subseteq \Delta$ let

$$V_\Theta = \bigcap_{\alpha\in\Theta} \mathrm{Ker}(\alpha), \quad V^\Theta = \text{ linear span of } \Theta\,.$$

The roots that vanish identically on V_Θ are those in Σ_Θ. The space V_Θ is partitioned into chambers by the hyperplanes $\lambda = 0$ for λ in $\Sigma - \Sigma_\Theta$. The **relative roots** of V_Θ are the open half-spaces $\lambda > 0$ for λ in $\Sigma - \Sigma_\Theta$.

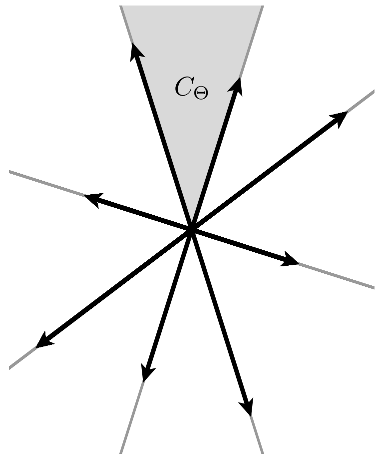
One of the chambers of V_Θ is the face C_Θ of the chamber C. I leave the following as an exercise:

2.1. Lemma. *If λ lies in $\Sigma - \Sigma_\Theta$, then $\lambda > 0$ if and only if its restriction to V_Θ is positive on C_Θ.*

Such restrictions are the **positive** relative roots.

In general, the chambers of V_Θ are the faces of full chambers, and in particular we know that each is the Weyl transform of a unique face of C. I want to make this more precise.

The basic problem to be dealt with is that the chambers in the partition of V_Θ do not generally form a homogeneous set for any group. They might well even have different shapes, as in the following figure, taken from the root system C_3. (The vectors shown are transforms of the fundamental weights.)



Any chamber in V_Θ will be wC_Ω for some w in $[W_\Theta \backslash W / W_\Omega]$. Because a chamber of V_Θ is open in it, we also have $w\Sigma_\Omega = \Sigma_\Theta$. In particular. $w\Sigma^+_\Omega = \Sigma^+_\Theta$.

2.2. Lemma. *Suppose Θ and Ω to be subsets of Δ, w in W. Then $w\Sigma^+_\Omega = \Sigma^+_\Theta$ if and only if $w\Omega = \Theta$.*

In these circumstances, Θ and Ω are said to be **associates**. Let $W(\Theta, \Omega)$ be the set of all w taking Ω to Θ. It is a subset of $[[W_\Theta \backslash W]]$.

Proof. This is a consequence of Lemma 1.5.

If $\Omega = w^{-1}\Theta \subseteq \Delta$—i.e. if w lies in $[[W_\Theta \backslash W]]$—then wC_Ω is a chamber in V_Θ. The following is now straightforward:

2.3. Proposition. *In these circumstances, the map taking w to wC_Ω is a bijection between $[[W_\Theta \backslash W]]$ and the chambers in V_Θ.*

There is a special case of importance. Suppose Θ to be a subset of Δ whose complement is a singleton $\{\alpha\}$. Then V_Θ is a line, the union of two rays, hence two chambers. One of these is C_Θ. According to the Proposition, the other corresponds to a unique $w \neq 1$ in $[[W_\Theta \backslash W]]$. It can be specified explicitly:

2.4. Lemma. *In these circumstances, the unique $w \neq 1$ in $[[W_\Theta \backslash W]]$ is the inverse of $w_{\ell,\Delta} w_{\ell,\Theta}$.*

Proof. The involution $w_{\ell,\Theta}$ takes Θ into its negative, which corresponds to a face of the negative chamber $-C$. The element $w_{\ell,\Delta}$ takes $-\Theta$ back to a subset $\overline{\Theta}$ of Δ. The involution $w_{\ell,\Theta}$ acts trivially on V_Θ, and $w_{\ell,\Delta}$ takes C_Θ to $-C_{\overline{\Theta}}$.

The set $\overline{\Theta}$ is the **conjugate** of Θ in Δ.

I call each $w_{\ell,\Delta} w_{\ell,\Theta}$ an **elementary conjugation**. Its inverse $w_{\ell,\Delta} w_{\ell,\overline{\Theta}}$ is also one, and lies in $[[W_\Theta \backslash W]]$.

2.5. Lemma. *If $s = w_{\ell,\Delta} w_{\ell,\overline{\Theta}}$ then $R_s = \Sigma^+_\Delta - \Sigma^+_{\overline{\Theta}}$.*

It can happen that Θ is its own conjugate. This will happen all the time if $\Theta = \emptyset$, but also happens in more interesting situations.

Examples. If the root system is A_2, with simple roots α and β, the conjugate of $\{\alpha\}$ is $\{\beta\}$.

If the root system is C_2 with simple roots α and β, each singleton in Δ is its own conjugate.

A **gallery** in V_Θ is a sequence of chambers $C_0, C_1, \ldots, C_n$ where $C_0 = C_\Theta$ and each C_{i-1} and C_i share a panel. If $\Theta = \emptyset$, galleries correspond bijectively to arbitrary sequences of simple reflections. What happens in other cases?

Each C_i in a gallery is of the form $w_i C_{\Theta_i}$ for a unique w_i in $[[W_\Theta \backslash W]]$ and $\Theta_i \subset \Delta$. Also, the common face of $w_{i-1}^{-1} C_{i-1}$ and $w_{i-1}^{-1} C_i$ must be equal to a face of C, say C_{Δ_i}. Here $\Delta_i = \Theta_{i-1} \cup \{\alpha_i\}$ for some α_i in $\Delta - \Theta_{i-1}$. Apply Lemma 2.4, with Θ_i, Δ_i replacing Θ, Δ. This tells us that

$$w_{i-1}^{-1} C_i = (w_{\ell,\Delta_i} w_{\ell,\Theta_{i-1}})^{-1} C_{\overline{\Theta}_{i-1}} .$$

Therefore

$$\begin{aligned}\Theta_i &= \overline{\Theta}_{i-1}\\ &= w_{\ell,\Delta_i} w_{\ell,\Theta_{i-1}} \Theta_{i-1}\\ w_i &= w_{i-1}(w_{\ell,\Delta_i} w_{\ell,\Theta_{i-1}})^{-1}\,.\end{aligned}$$

Conversely, suppose (Θ_i) (for $0 \le i \le m$) to be a sequence of subsets of Δ, and (α_i) (for $1 \le i \le m$) to be a sequence of simple roots. Suppose these satisfy the inductive conditions:

(a) $\Theta_0 = \Theta$;
(b) α_i is in $\Delta - \Theta_{i-1}$;
(c) $\Theta_i = w_{\ell,\Delta_i} w_{\ell,\Theta_{i-1}} \Theta_{i-1} \quad \big(\Delta_i = \Theta_{i-1} \cup \{\alpha_i\}\big)$.

I call such a pair of sequences a **gallery pair**.

Given a gallery pair (Θ_i), (αi), define by induction the sequences (w_i) in W, chambers C_i in V_Θ:

$$\begin{aligned}\Delta_i &= \Theta_{i-1} \cup \{\alpha_i\}\\ w_i &= w_{i-1}(w_{\ell,\Delta_i} w_{\ell,\Theta_{i-1}})^{-1}\\ C_i &= w_i C_{\Theta_i}\,.\end{aligned}$$

In this way, every gallery gives rise to a gallery pair, and every gallery pair gives rise to a gallery. These are clearly inverse to each other. Hence we have proved:

2.6. Theorem. *The map taking a gallery to the corresponding gallery pair is a bijection.*

If $\Theta = \emptyset$, then $\ell(ws_\alpha) > \ell(w)$ if and only if $w\alpha > 0$. What is the analogue here?

2.7. Proposition. *We have $\ell(w_i) > \ell(w_{i-1})$ if and only if $w_{i-1}\alpha_i > 0$.*

2.8. Proposition. *Every element of $W(\Theta, \Omega)$ can be expressed as a reduced product of elementary conjugations.*

Proof. Each such expression corresponds to a gallery from C_Θ to wC_Ω.

If $C_0, C_1, \ldots, C_n$ is a gallery in V_Θ, it is called **primitive** if Θ_{i-1} is never the same as Θ_i, and I'll call two chambers **primitively associated** if they can be connected by a primitive gallery.

2.9. Proposition. *(a) The group $W(\Theta, \Theta)$ acts transitively on the chambers of type Θ in V_Θ. If Θ and Ω are associates, and C a chamber of type Θ, there exists a primitive gallery from C to a chamber of type Ω.*

Proof. The first is immediate. For the second, because of the first claim it suffices to find a primitive gallery linking a chamber of type Θ to one of type Ω. But there certainly exists at least gallery linking the two types, and the tail of the gallery starting out from the last occurrence of type Θ is primitive.

The figure above suggests that the geometry of such equivalence classes might be of interest. For example, do they correspond to convex regions? I am not aware of anything in the literature taking up this suggestion.

3. Parabolic pairs

Fix a reductive group G, a minimal parabolic subgroup $P_\emptyset$, and a maximal split torus $A_\emptyset$ in $P_\emptyset$. Let Σ be the set of relative roots, Δ the subset of simple roots. For $\Theta \subseteq \Delta$ the Levi factor M_Θ is the centralizer of A_Θ.

CLASSIFICATION. Every parabolic subgroup of G is conjugate to a unique P_Θ, and every parabolic subgroup of G is its own normalizer. Thus $P_\Theta\backslash G$ parametrizes subgroups of G conjugate to P_Θ, and subsets of Δ parametrize conjugacy classes of parabolic subgroups. What about pairs of parabolic subgroups?

I start with these three facts: (i) Any two parabolic subgroups contain a maximal split torus in common; (ii) the minimal parabolic subgroups containing $A_\emptyset$ are those of the form $wP_\emptyset w^{-1}$ for some unique w in $[W/W_\Theta]$; (iii) all maximal split tori in an algebraic group are conjugate.

3.1. Theorem. *If P, Q are two parabolic subgroups, there exist unique subsets Θ, Ω of Δ, unique w in $[W_\Theta\backslash W/W_\Omega]$, and g such that*

$$gPg^{-1} = P_\Theta, \quad gQg^{-1} = wP_\Omega w^{-1}\,.$$

The conjugacy classes of such pairs is thus in bijection with such triples (Θ, Ω, w). The conjugacy class itself is in bijection with $P_\Theta \cap wP_\Omega w^{-1}\backslash G$.

Proof. Conjugating P if necessary, we may assume $P = P_\Theta$. If $Q_\emptyset$ is a minimal parabolic subgroup of Q, it contains a maximal split torus in common with $P_\emptyset$. We may conjugate this by an element of P to $A_\emptyset$, and we may therefore assume Q contains $A_\emptyset$. There will exist a minimal parabolic subgroup $Q_\emptyset$ in Q that contains $A_\emptyset$. If $Q_\emptyset = w^{-1}P_\emptyset w$ then $Q = wP_\Omega w^{-1}$ for some unique $\Omega \subseteq \Delta$.

It remains to show that if $(P_\Theta, xP_\Omega x^{-1})$ is conjugate to $(P_\Theta, yP_\Omega y^{-1})$ with x, y in W then x is in $W_\Theta y W_\Omega$. So suppose that

$$g(P_\Theta, xP_\Omega x^{-1})g^{-1} = (P_\Theta, yP_\Omega y^{-1}) \,.$$

Since a parabolic subgroup is its own normalizer, g must lie in P_Θ. The torus $gA_\emptyset g^{-1}$ lies in both P_Θ and $yP_\Omega y^{-1}$, and must be conjugate in the intersection to $A_\emptyset$. So we may assume g to be in the normalizer of $A_\emptyset$. If w is its image in the Weyl group, then on the one hand w lies in W_Θ, while on the other $wxW_\Omega x^{-1}w^{-1} = yW_\Omega$. This implies that $y^{-1}wx$ is in W_Ω, and hence x is in $W_\Theta x W_\Omega$. ∎

3.2. Corollary. *The map taking w to $P_\Theta w P_\Omega$ induces a bijection of $[W_\Theta\backslash W/W_\Omega]$ with $P_\Theta\backslash G/P_\Omega$.*

STRUCTURE. Suppose P, Q, to be parabolic subgroups. The group Q acts transitively on the double coset $P\backslash PwQ$. The orbit is isomorphic to the quotient $w^{-1}Pw \cap Q\backslash Q$. This leads naturally to the question, what can we say about the intersection $P \cap Q$ of two parabolic subgroups, or the quotient $P \cap Q\backslash Q$?

The Lie algebra $\mathfrak{p}_\Theta$ is the sum of $\mathfrak{p}_\emptyset$ and the direct sum of root spaces $\mathfrak{g}_\lambda$ for λ in Σ_Θ^-. The roots occurring are those in the union Ω of Σ^+ and Σ_Θ^-. The set Ω satisfies two conditions:

(a) it contains all positive roots;
(b) it is **closed** in the sense that if λ and μ are in it so is $\lambda + \mu$.

Since $-\Theta = -\Delta \cap \Omega$, the set Θ is uniquely determined by Ω. Conversely:

3.3. Lemma. *Suppose X to be a subset of Σ, $\Theta = -X \cap \Delta$. Then X satisfies the conditions (a) and (b) if and only if $X = \Sigma^+ \cup \Sigma_\Theta^-$.*

Proof. Closure implies immediately that $\Sigma^+ \cup \Sigma_\Theta^- \subseteq X$. For the other half, it suffices to show that $X \cap \Sigma^- \subseteq \Sigma_\Theta^-$. suppose λ to lie in $X \cap \Sigma^-$, say

$$\lambda = -\sum\nolimits_\Delta \lambda_\alpha \alpha$$

with λ_α in $\mathbb{N}$. Let $h(\lambda) = \sum \lambda_\alpha$. If $h(\lambda) = 1$ then λ lies in $-\Delta \cap X = -\Theta$, and we are done. Otherwise, suppose $\mu = \lambda - \alpha$ for some α in Δ. Because X is closed and contains all positive roots, $\mu = \lambda + \alpha$ also lies in X. By an induction argument, we may assume that μ lies in Σ_Θ^-. But since $-\alpha = \lambda - \mu$, $-\alpha$ also lies in X, hence in $-\Theta$. So λ must lie in Σ_Θ^- as well. ∎

3.4. Proposition. *Assume w to be in $[W_\Theta\backslash W/W_\Theta]$. The image of $P_\Theta \cap wP_\Omega w^{-1}$ modulo N_Θ is the parabolic subgroup $P^\Theta_{\Theta\cap w\Omega}$ of M_Θ. Its unipotent radical is the image in M_Θ of $wN_\Omega w^{-1} \cap P_\Theta$.*

Proof. Because $w^{-1}\Theta > 0$, the image of the intersection contains $P^\Theta_\emptyset$, and is hence some parabolic subgroup P_X. The root spaces of $P_\Theta \cap wP_\Omega w^{-1}$ are those with roots in

$$\big(\Sigma^+ \cup \Sigma_\Theta^-\big) \cap w\big(\Sigma^+ \cup \Sigma_\Omega^-\big) \,. \qquad \textbf{(3.5)}$$

The intersection of this with $-\Delta$ certainly contains $-(\Theta \cap w\Omega)$, and according to Lemma 3.3 it must be shown that it is equal to it. Suppose that $\alpha \in -\Delta$ lies in the intersection(3.5) . It lies in $-\Theta$, and since $w^{-1}\Theta > 0$, it also lies in $-w\Sigma_\Omega$. But then by Lemma 1.5 it lies in $-w\Omega$. Therefore the image of $P_\Theta \cap wP_\Omega w^{-1}$ modulo N_Θ is the parabolic indexed by $\Theta \cap s\Omega$. This implies also that

$$-(\Sigma_\Theta \cap w\Sigma_\Omega) = -\Sigma_{\Theta\cap w\Omega} \,.$$

The roots of the radical in M_Θ are then $\Sigma_\Theta^+ - \Sigma_{\Theta\cap w\Omega}^+$. But by the equation , this is the same as $\Sigma_\Theta \cap w\Sigma_\Omega$, which are the roots corresponding to the image of $wN_\Omega w^{-1} \cap P_\Theta$ in M_Θ.

As shown in the proof:

3.6. Corollary. *If w lies in $[W_\Theta\backslash W/W_\Omega]$ then $\Sigma_\Theta^+ \cap w\Sigma_\Omega^+ = \Sigma_{\Theta\cap w\Omega}^+$.*

Two parabolic subgroups P, Q are said to be **associates** if Levi subgroups M_P and M_Q are conjugate in G. I call them **immediate associates** if the image modulo N_P of $P \cap Q$ is all of M_P and that modulo N_Q is all of M_Q. The groups P_Θ and $wP_\Omega w^{-1}$ are immediate associates if and only if $w\Omega = \Theta$.

If $\Theta = w\Omega$ then P_Θ and $wP_\Omega w^{-1}$ share the Levi subgroup M_Θ, and $P_\Theta w P_\Omega = P_\Theta w N_\Omega$. The orbit is the homogeneous space $w^{-1}N_\Theta w \cap N_\Omega$.

3.7. Corollary. *If $\Theta = w\Omega$ then*

$$P_\Theta w P_\Omega = P_\Theta w N_w$$

in which N_w is the product of the N_λ with $\lambda > 0$, $w^{-1}\lambda < 0$.

One important special case is when $w = w_{\ell,\Delta} w_{\ell,\Omega}$, and $wP_\Omega w^{-1}$ is the opposite of P_Θ.

4. References

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