Characters as tempered distributions: p-adic fields

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A character χ of the multiplicative group of a local field k defines a distribution $\varphi = \varphi_\chi$ on that group:

$$\langle \varphi, f \rangle = \int_{k^\times} \chi(x) f(x)\, d^\times x\,.$$

It satisfies the functional equation

$$\mu_a \varphi = \chi(a)\varphi\,,$$

which means that φ is χ**-equivariant**. Up to scalar multiplication, it is unique with respect to that property.

The multiplicative group $k^\times$ is an open set in k, and the Schwartz space of $k^\times$ is embedded into that of k. Under what circumstances does φ on $k^\times$ extend to a χ-equivariant tempered distribution on k? What does the space of all χ-equivariant distributions on k look like? What is the Fourier transform of the distribution φ_χ?

This material originated in [Tate:1951/1967], but the approach here amounts to working out details suggested in [Weil:1967]. What is slightly new is that the computation of the Fourier transform of χ is not quite the usual one.

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Let

$$\begin{aligned} k &= \text{a non-Archimedean local field} \\ \mathfrak{o} &= \text{integers in } k \\ \mathfrak{p} &= \text{prime ideal of } \mathfrak{o} \\ \mathfrak{d} &= \text{different of the extension } k/\mathbb{Q}_p \\ &= \text{inverse of } \{x \in k \,|\, \mathrm{trace}_{k/\mathbb{Q}_p}(x\mathfrak{o}) \subseteq \mathbb{Z}_p\} \\ &= \text{(say) } \mathfrak{p}^\delta \\ \varpi &= \text{generator of } \mathfrak{p} \\ \nu &= \text{the multiplicative character } x \mapsto |x| \\ \mathbb{F}_q &= \mathfrak{o}/\mathfrak{p}\,. \end{aligned}$$

For an ideal $\mathfrak{a} \subseteq \mathfrak{o}$ let $\mathrm{N}\mathfrak{a} = |\mathfrak{o}/\mathfrak{a}|$. For example, $\mathrm{N}\mathfrak{p} = q$.

There is a canonical embedding of $\mathbb{Q}_p/\mathbb{Z}_p$ into the quotient $\mathbb{Q}/\mathbb{Z}$, identifying it with the p-torsion. For every x in $\mathbb{Q}_p$ there exists a unique fraction m/p^k such that $x - m/p^k$ is in $\mathbb{Z}_p$. The integer m is uniquely determined modulo p^k, and $x \mapsto \psi_p(x) = e^{2\pi i m/p^k}$ is well defined, and determines a character of $\mathbb{Q}_p$ whose kernel is $\mathbb{Z}_p$. The map

$$x \longmapsto \psi(x) = \psi_p(\mathrm{trace}_{k/\mathbb{Q}_p}(x))$$

is a character of k such that

$$\mathfrak{d}^{-1} = \{x \in k \,|\, \psi(x\mathfrak{o}) = 1\}\,.$$

The character $\psi(x/\varpi^{\delta+m})$ is a primitive character of $\mathfrak{o}/\mathfrak{p}^m$.

Choose the measure on k such that

$$\mathrm{meas}(\mathfrak{o}) = |\mathfrak{o}/\mathfrak{d}|^{-1/2} = q^{-\delta/2}\,.$$

The measure $d^\times x = dx/|x|$ is a multiplicatively invariant measure on $k^\times$.

1. Characters as distributions on the multiplicative group

The Schwartz space $\mathcal{S}(k^\times)$ is the vector space of all locally constant complex-valued functions of compact support on $k^\times$. The multiplicative group acts on it by right multiplication:

$$\rho_a f(x) = f(xa)\,.$$

A distribution on $k^\times$ is any linear functional on its Schwartz space. The group $k^\times$ acts by the usual duality formula on the linear dual of $\mathcal{S}(k^\times)$, the space of distributions:

$$\langle \rho_a \varphi, f\rangle = \langle \varphi, \rho_{a^{-1}} f\rangle\,.$$

The integral

$$\langle \varphi\chi, f\rangle = \int_{k^\times} f(x)\chi(x)\, d^\times x = \int_{k^\times} f(x)\chi(x)|x|^{-1}\, dx\,.$$

defines a χ-equivariant distribution on $k^\times$. It is essentially unique:

1.1. Theorem. *Every distribution φ on $k^\times$ satisfying the functional equation $\rho_a\varphi = \chi(a)\varphi$ is a multiple of φ_χ.*

Proof. Suppose φ to be such a distribution. If $\chi = 1$ on $1 + \mathfrak{p}^f$, then

$$\langle \varphi, f\rangle = \langle \varphi, f_*\rangle$$

where

$$f_*(x) = \frac{1}{\mathrm{meas}(1+\mathfrak{p}^f)} \cdot \int_{1+\mathfrak{p}^f} f(xu)\, du\,.$$

Therefore φ amounts to integration against

$$F(x) = \frac{\langle \varphi, \mathfrak{char}_{x(1+\mathfrak{p}^f)}\rangle}{\mathrm{meas}(1+\mathfrak{p}^f)}\,.$$

We have a short exact sequence

$$1 \longrightarrow \mathfrak{o}^\times \longrightarrow k^\times \longrightarrow k^\times/\mathfrak{o}^\times \longrightarrow 1$$

The map from $\langle\varpi\rangle$ to the quotient is an isomorphism, so the quotient is isomorphic to the group of powers of ϖ, isomorphic to $\mathbb{Z}$. This isomorphism does not depend on the choice of ϖ, and I'll call the image of ϖ in the quotient a **canonical generator** of it. I'll write it as $\mathfrak{p}^\times$.

A character of $k^\times$ trivial on $\mathfrak{o}^\times$, or equivalently a character of $k^\times/\mathfrak{o}^\times$, is said to be **unramified**. It is determined by the image of ϖ, which can be any non-zero complex number z. It is often convenient to write it as $|x|^s$ with s in $\mathbb{C}$, but since $|x| = q^{-n}$ if $x = \varpi^n$ we have

$$|x|^s = q^{-ns} = e^{-ns\log q}$$

so s is only determined up to a term $2\pi in/\log q$. Nonetheless, because of global considerations it is convenient to use s as a parameter.

A splitting of the exact sequence above is determined by a single element of $k^\times$ whose image in $k^\times/\mathfrak{o}^\times$ is $\mathfrak{p}^\times$ or, equivalently, a generator of $\mathfrak{p}$. There is no best choice, in spite of personal prejudices. Given a generator ϖ of $\mathfrak{p}$, one can factor any x in $k^\times$ as $u\cdot\varpi^n$, thus factoring $k^\times = \mathfrak{o}^\times \times \langle\varpi\rangle$. In these circumstances one can write any character of $k^\times$ uniquely as $\sigma(x)\cdot z^{\mathrm{ord}(x)}$, where $\sigma(\varpi) = 1$ and z lies in $\mathbb{C}^\times$.

Remark. Suppose f to lie in $\mathcal{S}(k^\times)$, $\chi(x) = \omega(x)\cdot z^{\mathrm{ord}(x)}$. Then $\langle\chi, f\rangle$ is a polynomial in $z^{\pm1}$ (i.e. a Laurent polynomial in z.

2. As distributions on the additive group

The Schwartz space of k is that of all locally constant, complex-valued functions of compact support on k. We have an exact sequence of vector spaces

(2.1)
$$0 \longrightarrow \mathcal{S}(k^\times) \longrightarrow \mathcal{S}(k) \overset{f\mapsto f(0)}{\longrightarrow} \mathbb{C} \longrightarrow 0\,.$$

The group $k^\times$ acts on all of these compatibly—on the first two by ρ and on the last trivially. The triviality means that the image of each $\rho_a f - f$ in $\mathbb{C}$ is 0. Given χ, integration gives us a χ-equivariant distribution φ_χ on $k^\times$. Does it extend to a distribution on k? Is the extension unique?

If f lies in $\mathcal{S}(k)$, then $f - f(0)\mathfrak{char}_\mathfrak{o}$ lies in $\mathcal{S}(k^\times)$. Evaluating $\langle\varphi_\chi, f\rangle$ therefor reduces to evaluating $\langle\varphi_\chi, \mathfrak{char}_\mathfrak{o}\rangle$. But if $z = \chi(\varpi)$ and $|z| < 1$ we can write

$$\begin{aligned}\langle\varphi_\chi, \mathfrak{char}_\mathfrak{o}\rangle &= \int_\mathfrak{o} \chi(x)|x|^{-1}\,dx \\ &= \sum_{k=0}^\infty \int_{\mathfrak{p}^k-\mathfrak{p}^{k+1}} \chi(x)|x|^{-1}\,dx \\ &= \left(\int_{\mathfrak{o}^\times} \chi(x)\,dx\right)\left(\sum_{k=0}^\infty z^k\right)\,,\end{aligned}$$

which certainly converges, and defines an equivariant extension.

2.2. Theorem. *If $\chi \neq 1$ there is a unique extension. If $\chi = 1$ there is none, and the Dirac distribution*

$$\delta_0\colon\ f \longmapsto f(0)$$

spans the space of distributions φ such that $\rho_a\varphi = \varphi$ for all a.

Proof. Since $k^\times$ acts trivially on $\mathcal{S}(k)/\mathcal{S}(k^\times)$, any extension is certainly unique.

Suppose at first that φ did satisfy $\rho_a\varphi = \chi(a)\varphi$. Then

$$\langle \rho_a\varphi, f\rangle = \langle \varphi, \rho_{a^{-1}} f\rangle = \chi(a)\langle \varphi, f\rangle$$

so $\langle \varphi, \rho_{a^{-1}} f - f\rangle = (\chi(a) - 1)\langle \varphi, f\rangle$ and

(2.3) $$\langle \varphi, f\rangle = \frac{\langle \varphi, \rho_{a^{-1}} f - f\rangle}{\chi(a) - 1}\,.$$

as long as $\chi(a) \neq 1$. But this can be used to **specify** φ, as long as $\chi! = 1$. For any a $\rho_{a^{-1}} f - f$ lies in the Schwartz space of $k^\times$, so the numerator is always defined, and if we choose a with $\chi(a)! = 1$ this formula will define a suitable distribution.

If $\chi = 1$, the argument fails, and in fact there is no extension to k. For suppose φ were one. Let f be the characteristic function of some small neighbourhood of 0. Then on the one hand

$$\langle \varphi, \rho_{\varpi^{-1}} f\rangle = \langle \varphi, f\rangle, \quad \langle \varphi, \rho_{\varpi^{-1}} f - f\rangle = 0\,,$$

but on the other

$$\langle \varphi, \rho_{\varpi^{-1}} f - f\rangle = \int_{k^\times} \big(f(x) - f(\varpi^{-1}x)\big)\, d^\times x \neq 0\,.$$

Remark. There is another way to look at the same problem. Choose a fixed φ_* in $\mathcal{S}(k)$ with $\varphi_*(0) = 1$. Then for every φ in $\mathcal{S}(k)$ the function $\varphi - \varphi(0)\cdot\varphi_*$ will lie in $\mathcal{S}(k^\times)$. The integral

$$\int_{k^\times} \chi(x)\big(\varphi(x) - \varphi(0)\cdot\varphi_*(x)\big)\, d^\times x$$

defines a distribution that extends χ on $\mathcal{S}(k^\times)$. It is not the only such extension, since we can always add a multiple of δ_0 to it without modifying its effect on $\mathcal{S}(k^\times)$. So in looking for a χ-equivariant extension of χ we are looking for a distribution

$$\langle \varphi, \varphi\rangle = \int_{k^\times} \chi(x)\big(\varphi(x) - \varphi(0)\cdot\varphi_\#\big)\, d^\times x + c_\chi\varphi(0)$$

such that $\rho_a\varphi = \chi(a)\varphi$ for all a.

I leave as exercise to find the constant c_χ making φ a χ-equivariant distribution.

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Example. Suppose $\chi(x) = |x|^s = z^{\mathrm{ord}(x)}$ and f is the characteristic function of $\mathfrak{o}$. What is $\langle \chi, f\rangle$? For $\mathrm{RE}(s) > 0$

$$\begin{aligned}\langle \chi, f\rangle &= \int_{\mathfrak{o}} |x|^{-s-1}\, dx \\ &= \sum\nolimits_{k\ge 0} \int_{\mathfrak{p}^k - \mathfrak{p}^{-(k+1)}} |x|^s\, dx/|x| \\ &= \sum\nolimits_{k\ge 0} q^{-ks} = \frac{1}{1 - q^{-s}}\,.\end{aligned}$$

The residue of the distribution $|x|^s$ at $s = 0$ is a multiple of the Dirac δ_0.

Remark. It is potentially useful to consider these results in light of the long exact sequence of cohomology derived from (2.1) :

$$0 \longrightarrow \mathrm{Hom}_{k^\times}(\mathbb{C}, \mathbb{C}) \longrightarrow \mathrm{Hom}_{k^\times}(\mathcal{S}(k), \mathbb{C}) \longrightarrow \mathrm{Hom}_{k^\times}(\mathcal{S}(k^\times), \mathbb{C}) \longrightarrow \mathrm{Ext}_{k^\times}(\mathbb{C}, \mathbb{C}) \longrightarrow \ldots$$

3. Analysis on finite rings

Let $\mathfrak{r} = \mathfrak{o}/\mathfrak{p}^n$ for some $n > 0$. For the moment, suppose ω to be any primitive additive character of $\mathfrak{r}$, for example

$$x \longmapsto \psi(x/\varpi^{\delta+n}) .$$

The **Fourier transform** on $\mathbb{C}[\mathfrak{r}]$ is

$$\widehat{f}(y) = \frac{1}{\sqrt{\mathrm{N}\mathfrak{r}}} \cdot \sum \omega(-xy) f(x) .$$

It is an isometry of $\mathrm{L}^2(\mathfrak{r})$ with itself.

If χ is a multiplicative character of $\mathfrak{o}^\times$, it is said to have **conductor** $\mathfrak{p}^r$ is χ is trivial on $1 + \mathfrak{p}^r$ but not on $1 + \mathfrak{p}^{r-1}$. If χ has conductor $\mathfrak{p}^r$, extend it to be a function on all of $\mathfrak{r}$ by setting $\chi(x) = 0$ for x not a unit. This extension is, up to scalar factor, the unique χ-equivariant function on $\mathfrak{r}$.

In this situation, define

$$\mathfrak{g}(\chi) = \frac{1}{\sqrt{\mathrm{N}\mathfrak{r}}} \cdot \sum\nolimits_{\mathfrak{r}} \omega(-x)\chi(x) .$$

The following is easy to verify:

3.1. Proposition. *The Fourier transform of* χ *is* $\mathfrak{g}(\chi)\chi^{-1}$.

3.2. Corollary. *We have* $|\mathfrak{g}(\chi)| = 1$.

Proof. Because the L^2 norm of the Fourier transform of χ is equal to that of χ. ▯

4. The Fourier transform

The formula

$$\widehat{f}(y) = \int_k \psi(-xy) f(x)\, dx$$

defines a Fourier transform on $\mathcal{S}(k)$, which is an isomorphism of $\mathcal{S}(k)$ with itself. With the given choice of measure, the Fourier transform of $\mathfrak{char}_{\mathfrak{o}}$ is $\mathrm{N}\mathfrak{d}^{-1/2}\mathfrak{char}_{\mathfrak{d}^{-1}}$, and *vice-versa*.

For two functions f, φ in $\mathcal{S}(k)$

$$\langle \widehat{\varphi}, f \rangle = \langle \varphi, \widehat{f} \rangle .$$

When φ is a distribution, this **defines** the Fourier transform of φ.

How does the Fourier transform interact with the action of $k^\times$?

4.1. Lemma. *For any distribution* φ

$$\langle \rho_c \widehat{\varphi}, f \rangle = |c| \langle \widehat{\rho_{1/c}\varphi}, f \rangle .$$

Proof. For any f in $\mathcal{S}(k)$

$$\begin{aligned} \widehat{\rho_{1/c}f}(y) &= \int_k \psi(-xy) f(x/c)\, dx \\ &= \int_k \psi(-zcy) f(z)\, dcz \\ &= |c| \widehat{f}(cy)\,, \end{aligned}$$

and $\widehat{\rho_{1/c}f} = |c|\rho_c \widehat{f}$. Hence for a distribution φ

$$\begin{aligned} \langle \rho_c \widehat{\varphi}, f \rangle &= \langle \widehat{\varphi}, \rho(1/c) f \rangle \\ &= \langle \varphi, \widehat{\rho(1/c) f} \rangle \\ &= \langle \varphi, |c| \rho_c \widehat{f} \rangle \\ &= |c| \langle \varphi, \rho_c \widehat{f} \rangle \\ &= |c| \langle \rho_{1/c} \varphi, \widehat{f} \rangle \\ &= |c| \langle \widehat{\rho_{1/c} \varphi}, f \rangle\,. \end{aligned}$$

If φ is χ-equivariant, this gives us

$$\rho_c \widehat{\varphi} = |c| \chi^{-1}(c) \widehat{\varphi}\,,$$

so that $\widehat{\varphi}$ is equivariant for $\nu\chi^{-1}$. Since the space of χ-equivariant distributions has dimension one, this implies that the Fourier transform of χ is a scalar multiple of $\nu\chi^{-1}$. What is that scalar? The usual calculation uses suitably chosen test functions to answer this, but with the prospect of similar if more difficult calculations in mind, I'll do something a bit different.

Formally, we have

$$\begin{aligned} \int_k \chi(x)|x|^{-1} \widehat{f}(x)\, dx &= \int_k \chi(x)|x|^{-1} \left(\int_k \psi(-yx) f(y)\, dy \right) dx \\ &= \int_k f(y) \left(\int_k \psi(-xy)\chi(x)|x|^{-1}\, dx \right) dy\,. \end{aligned}$$

Making sense of this poses two problems. First of all, to calculate the factor $\gamma_\psi(\chi)$ such that the integral

$$\int_k \psi(-xy)\chi(x)|x|^{-1}\, dx$$

make sense and is equal to

$$\gamma_\psi(\chi)\chi^{-1}(y)\,.$$

Second, to justify the manipulation of integrals. The crucial step is this:

4.2. Lemma. *If* $|y| = q^{-m}$ *and* $\mathfrak{f} = \mathfrak{p}^f$ *is the conductor of* χ*, then*

$$\int_{\mathfrak{p}^n} \psi(-xy)\chi(x)\, d^\times x = \int_{\mathfrak{p}^{-\delta-m-f}} \psi(-xy)\chi(x)\, d^\times x$$

for $n \le -\delta - m - f$.

I'll prove this at the same time I calculate the integral explicitly.

4.3. Lemma. *If $y \sim \varpi^m$ then*

$$\int_{\mathfrak{p}^k} \psi(-xy)\,dx = \begin{cases} q^{-k-\delta/2} & \text{if } m \ge -\delta - k \\ 0 & \text{if } m \le -\delta - k - 1. \end{cases}$$

Now I begin the proof of Lemma 4.2. Say $y = \varpi^m u$ with u in $\mathfrak{o}^\times$.

Unramified. Assume $n \gg 0$, $\chi = |x|^s$.

$$\begin{aligned}
&\int_{\mathfrak{p}^n} \psi(-xy)\chi(x)|x|^{-1}\,dx \\
&\qquad = \sum\nolimits_{k\ge n} \left(\int_{\mathfrak{p}^k - \mathfrak{p}^{k+1}} \psi(-xy)\chi(x)|x|^{-1}\,dx \right) \\
&\qquad = \sum\nolimits_{k\ge n} q^{-ks} q^k \left(\int_{\mathfrak{p}^k - \mathfrak{p}^{k+1}} \psi(-xy)\,dx \right) \\
&\qquad = \sum\nolimits_{k\ge n} q^{-ks} q^k \left(\int_{\mathfrak{p}^k} \psi(-xy)\,dx \right) - \sum\nolimits_{k\ge n} q^{-ks} q^k \left(\int_{\mathfrak{p}^{k+1}} \psi(-xy)\,dx \right) \\
&\qquad = \sum\nolimits_{k\ge n} q^{-ks} q^k \left(\int_{\mathfrak{p}^k} \psi(-xy)\,dx \right) - \sum\nolimits_{\ell\ge n+1} q^{-(\ell-1)s} q^{\ell-1} \left(\int_{\mathfrak{p}^\ell} \psi(-xy)\,dx \right) \\
&\qquad = \sum\nolimits_{k\ge n} q^{-ks} q^k \left(\int_{\mathfrak{p}^k} \psi(-xy)\,dx \right) - \sum\nolimits_{\ell\ge n+1} q^{-\ell s} q^{s-1} q^{\ell} \left(\int_{\mathfrak{p}^\ell} \psi(-xy)\,dx \right) \\
&\qquad = \sum\nolimits_{k\ge -\delta-m} q^{-ks-\delta/2} - \sum\nolimits_{\ell\ge -\delta-m} q^{-\ell s-\delta/2} q^{s-1} \\
&\qquad = \left(1 - q^{-(1-s)}\right) \cdot \frac{q^{(\delta+m)s-\delta/2}}{1-q^{-s}} \\
&\qquad = \chi^{-1}(y) \cdot q^{\delta(s-1/2)} \cdot \frac{1-q^{-(1-s)}}{1-q^{-s}} \\
&\qquad = \gamma_\psi(\chi) \cdot \frac{|y|\chi^{-1}(y)}{|y|}\,.
\end{aligned}$$

Ramified. Say χ has conductor $\mathfrak{p}^f$.

$$\begin{aligned}
&\int_{\mathfrak{p}^n} \psi(-xy)\chi(x)|x|^{-1}\,dx \\
&\qquad = \sum\nolimits_{k\ge n} \left(\int_{\mathfrak{p}^k - \mathfrak{p}^{k+1}} \psi(-xy)\chi(x)\,dx/|x| \right) \\
&\qquad = \sum\nolimits_{k\ge n} q^{-ks} \left(\int_{\mathfrak{o}^\times} \psi(-\varpi^k uy)\chi(u)\,du \right) \\
&\qquad = \sum\nolimits_{k\ge n} q^{-ks} \left(\int_{\mathfrak{o}^\times} \psi(-\varpi^{k+m} u\varepsilon)\chi(u)\,du \right) \\
&\qquad = \sum\nolimits_{k\ge n} q^{-ks} \chi^{-1}(\varepsilon) \left(\int_{\mathfrak{o}^\times} \psi(-\varpi^{k+m} u)\chi(u)\,du \right)
\end{aligned}$$

If $\ell = k + m$ the inner integral is

$$\int_{\mathfrak{o}^\times} \psi(\varpi^\ell u)\chi(u)\,du = 0$$

It is a kind of Gauss sum.

There are now four cases to consider.

• We have $\ell \ge -\delta$. Then $\psi(-\varpi^\ell u) = 1$ identically, and the integral vanishes since χ is a nontrivial character.

• We have $-\delta - f < \ell < \delta$. The integral again vanishes since χ is non-trivial on each subgroup $(1+\mathfrak{p}^i)$.

• We have $\ell = -\delta - f$. The integral is the finite Gauss sum $\mathfrak{g}_\psi(\chi)$, and the corresponding term in the sum is

$$\chi^{-1}(y)(\mathrm{N}\mathfrak{d}\mathrm{N}\mathfrak{f})^{s-1/2}\mathfrak{g}_\psi(\chi)\,.$$

• We have $\ell < -\delta - f$. The character $\psi(\varpi^\ell u$ is non-trivial on each coset $u(1+\mathfrak{p}^f)$, and χ is constant on one of these, so the integral vanishes.

This concludes the proof of Lemma 4.2. ▯

4.4. Theorem. *We have*

$$\widehat{\chi} = \gamma_\psi(\chi)\nu\chi^{-1}$$

for some scalar $\gamma_\psi(\chi)$*. If* $\chi(x) = |x|^s$ *then*

$$\gamma_\psi(\chi) = \mathrm{N}\mathfrak{d}^{s-1/2}\cdot\frac{1-q^{-(1-s)}}{1-q^{-s}}\,.$$

If χ *has conductor* $\mathfrak{f} = \mathfrak{p}^f$ *then*

$$\gamma_\psi(\chi) = (\mathrm{N}\mathfrak{d}\mathrm{N}\mathfrak{f})^{s-1/2}\mathfrak{g}_\psi(\chi)\,.$$

Proof. Assume $\mathrm{RE}(s) > 0$. Choose f in $\mathcal{S}(k^\times)$. Then

$$\begin{aligned}&\int_k \widehat{f}(x)\chi(x)|x|^{-1}\,dx\\ &\qquad= \int_{\mathfrak{p}^n}\widehat{f}(x)\chi(x)|x|^{-1}\,dx\\ &\qquad= \int_{\mathfrak{p}^n}\left(\int_{\mathfrak{p}^r} f(y)\psi(-xy)\,dy\right)\chi(x)|x|^{-1}\,dx\end{aligned}$$

for $n, r \ll 0$. All integrals are bounded, so there is no problem reversing the order of integration, and this is

$$\int_{\mathfrak{p}^r} f(y)\left(\int_{\mathfrak{p}^n}\psi(-xy)\chi(x)|x|^{-1}\,dx\right)dy\,.$$

Since $f(0) = 0$, we have an upper bound on m, and then Lemma 4.2 tells us that for $n \gg 0$ the inner integral is independent of n and equal to what it should be. ▯

Remarks. The usual proof uses special test functions, whereas this one gets by with a somewhat arbitrary choice. This is perhaps only a curiousity. For global applications, one cannot escape special choices, because evaluating adelic integrals requires it.

5. References

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