## On Chevalley's formula for structure constants

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Suppose

$$\begin{aligned} \mathfrak{g} &= \text{a semi-simple Lie algebra over } \mathbb{C} \\ G &= \text{the corresponding adjoint group} \\ B &= \text{a Borel subgroup of } G \\ T &= \text{a Cartan subgroup of } B \\ \mathfrak{b}, \mathfrak{t} &= \text{corresponding Lie algebras} \\ \Sigma &= \text{roots corresponding to the choice of } \mathfrak{t} \\ \Delta &= \text{simple roots corresponding to the choice of } \mathfrak{b} \\ W &= \text{the Weyl group } N_G(T)/T. \end{aligned}$$

For each $\alpha$ in $\Delta$, let $e_\alpha \neq 0$ be an element of the root space $\mathfrak{g}_\alpha$. The triple $(\mathfrak{b}, \mathfrak{t}, \{e_\alpha\})$ is called a **frame** for $\mathfrak{g}$. The set of frames is a principal homogeneous space for $G$.

For each $\alpha$ in $\Delta$ there exists a unique $e_{-\alpha}$ in $\mathfrak{g}_{-\alpha}$ such that

$$t_\alpha = -[e_\alpha, e_{-\alpha}]$$

lies in $\mathfrak{t}$ and satisfies the equation $\langle \alpha, t_\alpha \rangle = 2$. Let $\theta$ be the opposition involution of $\mathfrak{g}$ (and of $G$) taking each $e_\alpha$ to $e_{-\alpha}$, acting as $-1$ on $\mathfrak{t}$ (and $t \mapsto t^{-1}$ on the torus $T$). For example, if $\mathfrak{g} = \mathfrak{sl}_2$ then these could be

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\theta(X) = -{}^tX$. There is no canonical way to choose $e_\lambda \neq 0$ in $\mathfrak{g}_\lambda$, but it is possible to choose for every root $\lambda$ not in $\pm\Delta$ an $e_\lambda$ such that

$$\theta(e_\lambda) = e_{-\lambda} \, .$$

Given the choice of frame, it is unique up to sign. If $t_\lambda = -[e_{-\lambda}, e_\lambda]$, then $t_\lambda$ and $e_{\pm\lambda}$ make up a basis of a copy of $\mathfrak{sl}_2$.

The $t_\alpha$ and the $e_\lambda$ form a **Chevalley basis** of $\mathfrak{g}$. The associated structure constants are defined by the equation

$$[e_\lambda, e_\mu] = N_{\lambda,\mu} e_{\lambda+\mu}$$

whenever $\lambda + \mu$ is also a root. Because the $e_\lambda$ are determined up to sign, the absolute value of these structure constants is completely and uniquely determined by $\mathfrak{g}$, and does not depend on any choices. It is therefore natural to expect a simple formula for $|N_{\lambda,\mu}|$.

For every $\lambda$, $\mu$ there exist non-negative integers $p$, $q$ such that the root string

$$-p\lambda + \mu, \; \ldots \; , q\lambda + \mu$$

is $(\mu + \mathbb{Z}\lambda) \cap \Sigma$. Let $p_{\lambda,\mu}$ be that value of $p$. The following is found in [Chevalley:1955].

**1. Theorem.** *If $\lambda$, $\mu$, $\lambda + \mu$ are all roots then*

$$|N_{\lambda,\mu}| = p_{\lambda,\mu} + 1 \, .$$

Because of the ambiguity in signs, one cannot expect a statement that is both simple and more specific.

This formula is crucial in defining integral structures on $\mathfrak{g}$ and $G$ as well as on other reductive groups, and is ultimately responsible for the classification of reductive groups over finite fields. Most proofs of Chevalley's formula in the literature (for example, that in Chapter 7 of [Carter:2005]) follow roughly his own argument. It is relatively short, but somewhat unmotivated. [Tits:1966] offered a rather more elegant if longer argument, and I followed this in [Casselman:2013], which generalized the result to Kac-Moody algebras. However, it is not easy to disentangle either Tits' argument or mine from context. In particular, both of those papers are mainly concerned with the relationship between the extended Weyl group and structure constants, and it is difficult to see that Tits' proof of Chevalley's formula can be extracted without going into details of this relationship. But it can be, and in this note I'll present his argument as succinctly as I can.

One notable feature of Tits' argument is that he is able to exploit symmetries that arise because he makes a few changes in sign from conventional notation. For example, he writes the condition on $\lambda + \mu$ as a symmetric condition on three roots $\lambda$, $\mu$, $\nu$ such that $\lambda + \mu + \nu = 0$. I call such a triple a **Tits triple**. Also, his definition of $e_{-\alpha}$ differs in sign from the usual one.

Chevalley's formula remains valid for Kac-Moody algebras. The proof in that case is almost the same as that given below for Lie algebras of finite dimension, but in addition one has to be careful about Weyl-invariant norms on real roots. This is explained in [Casselman:2013], but I thought that it might be useful at the end of this note to comment on the how the argument for finite-dimensional Lie algebras has to be amended to cover those cases too.

**Contents**



## 1. Outline

The first step in the proof is a direct translation of Jacobi's identity:

**1.1. Lemma.** *If $\lambda$, $\mu$, $\nu$ are a Tits triple then*

$$N_{\lambda,\mu}\, t_\nu + N_{\mu,\nu}\, t_\lambda + N_{\nu,\lambda}\, t_\mu = 0\,.$$

*Proof.* Jacobi's identity asserts that

$$[[e_\lambda, e_\mu], e_\nu] + [[e_\mu, e_\nu], e_\lambda] + [[e_\nu, e_\lambda], e_\mu] = 0\,,$$

which is rotationally symmetric. This translates immediately to

$$N_{\lambda,\mu}[e_{-\nu}, e_\nu] + N_{\mu,\nu}[e_{-\lambda}, e_\lambda] + N_{\nu,\lambda}[e_{-\mu}, e_\mu] = -(N_{\lambda,\mu}\, t_\nu + N_{\mu,\nu}\, t_\lambda + N_{\nu,\lambda}\, t_\mu) = 0\,.$$ $\square$

There is another, somewhat similar, equation that can be derived easily. Let $\|\lambda\|$ be a Euclidean norm on roots and coroots invariant under the Weyl group. This is characterized by the condition that

$$\langle \lambda, v\rangle = 2\cdot\frac{t_\lambda \bullet v}{\|t_\lambda\|^2}\,, \quad \langle v, \lambda^\vee\rangle = 2\cdot\frac{v\bullet\lambda}{\|\lambda\|^2}\,. \tag{1.2}$$

In effect, $t_\lambda$ may be identified with the coroot $\lambda^\vee$. One consequence is that the sign of $\langle v, \lambda^\vee \rangle$ is the same as that of $v \bullet \lambda$.

**1.3. Lemma.** *If $\lambda$, $\mu$, $\nu$ are a Tits triple then*

$$\|\nu\|^2 \, t_\nu + \|\lambda\|^2 \, t_\lambda + \|\mu\|^2 \, t_\mu = 0 \, .$$

*Proof.* This is a direct translation of the equation $\lambda + \mu + \nu = 0$ because, as follows from (1.2), the map

$$\lambda \longmapsto \frac{t_\lambda}{\|t_\lambda\|^2}$$

is linear, and because the product $\|\lambda\| \cdot \|t_\lambda\|$ is constant. ∎

The next step in proving Theorem 1 is this result about root geometry, which is perhaps the basic reason why Chevalley's formula is valid:

**1.4. Lemma.** (Geometric Lemma) *If $\lambda$, $\mu$, $\nu$ are a Tits triple then*

$$\frac{p_{\lambda,\mu} + 1}{\|\nu\|^2} = \frac{p_{\mu,\nu} + 1}{\|\lambda\|^2} = \frac{p_{\nu,\lambda} + 1}{\|\mu\|^2} \, .$$

This will be proven in a later section. It and the previous result imply that

$$(p_{\lambda,\mu} + 1) \, t_\nu + (p_{\mu,\nu} + 1) \, t_\lambda + (p_{\nu,\lambda} + 1) \, t_\mu = 0 \, .$$

This and Jacobi's identity imply, by an elementary argument in linear algebra, that

$$\frac{N_{\lambda,\mu}}{p_{\lambda,\mu} + 1} = \frac{N_{\mu,\nu}}{p_{\mu,\nu} + 1} = \frac{N_{\nu,\lambda}}{p_{\nu,\lambda} + 1} \, .$$

Given this (i.e. given the Geometric Lemma), in order to prove Chevalley's formula it remains only to show that *one of these ratios is* $\pm 1$. This will follow from the same geometric analysis of Tits triples that goes into the proof of the Geometric Lemma, together with some computations involving the adjoint representation of copies of $\mathrm{SL}_2$ on $\mathfrak{g}$.

## 2. Root geometry

Throughout this section, suppose $\lambda$, $\mu$, and $\nu$ to be a Tits triple.

**2.1. Lemma.** *Let $n = \langle \mu, \lambda^\vee \rangle$.*

*(a) $n < -1$ if and only if $\|\mu\| > \|\nu\|$;*
*(b) $n = -1$ if and only if $\|\mu\| = \|\nu\|$;*
*(c) $n > -1$ if and only if $\|\mu\| < \|\nu\|$.*

*Proof.* I recall the reflection

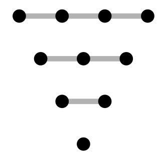
$$s_\lambda \colon \; v \longmapsto v - \langle v, t_\lambda \rangle \lambda \, .$$

If $n = -1$ then $s_\lambda \mu = -\nu$. If $n \geq 0$ then evaluate $(\mu + \lambda) \bullet (\mu + \lambda)$ to see that $\|\nu\|^2 > \|\mu\|^2$. In the remaining case, with $n \leq -2$, consider instead $s_\lambda \lambda = -\lambda$, $s_\lambda \nu$, $s_\lambda \mu$. ∎

Thid implies that there are a limited number of configurations of the $\lambda$-string through $\mu$. There are at most two distinct root lengths in any irreducible finite root system, hence:

**2.2. Lemma.** *There do not exist three real roots $\kappa$ in a $\lambda$-string with $\langle \kappa, \lambda^\vee \rangle \leq 0$.*

In other words, the following figures show all possibilities for root strings. The lengths in the string decrease until at most the half-way point, then increase.



*Possible root strings*

All these occur, as you can verify by perusing classical root diagrams.

**2.3. Lemma.** *The following are equivalent:*

*(a)* $s_\lambda \mu = -\nu$*;*
*(b)* $\langle \mu, \lambda^\vee \rangle = -1$*;*
*(c)* $\|\lambda\| \ge \|\mu\|, \|\nu\|$.

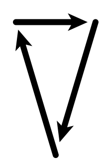
Of course the inequality in (c) can be strengthened to $\|\lambda\| \ge \|\mu\| = \|\nu\|$.

The point of this result is that whenever $\lambda + \mu + \nu = 0$ we can cycle to obtain $s_\lambda \mu = -\nu$, since we can certainly cycle to get the third condition by taking $\lambda$ of maximum length.

*Proof.* The equivalence of (a) and (b) is immediate.

Assume (b). Then $s_\lambda \mu = -\nu$, so $\|\mu\| = \|\nu\|$. But also by (1.2) $\langle \lambda, \mu^\vee \rangle \le -1$, so by Lemma 2.1 we have $\|\lambda\| \ge \|\mu\|$. Thus (b) implies (c).

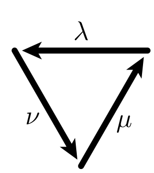
Assume (c). Only at most two lengths are possible. If $\|\lambda\| = \|\mu\| > \|\nu\|$, then by Lemma 2.1 we must have $s_\nu \lambda = -\mu$ and $\langle \lambda, \nu^\vee \rangle = -1$. Hence $\langle \nu, \lambda^\vee \rangle < 0$ as well. But then by Lemma 2.1 $\|\nu\| > \|\nu + \lambda\| = \|\mu\|$, a contradiction. In other words, the following figure depicts an impossible root configuration



Hence at most one has length greater than that of the others. Therefore $\|\mu\| = \|\nu\|$, and hence by Lemma 2.1 $s_\lambda \mu = -\nu$. ∎

I am now going to prove the Geometric Lemma. We may now assume that $\|\lambda\| \ge \|\mu\| = \|\nu\|$. There are two different cases.

**Case 1.** Suppose first that $\|\lambda\| = \|\mu\| = \|\nu\|$.



Then by Lemma 2.1

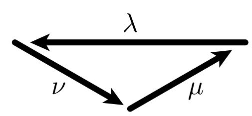
$$\begin{aligned} s_\lambda\colon \lambda &\longmapsto -\lambda \\ \mu &\longmapsto -\nu \\ \nu &\longmapsto -\mu \\ s_\mu\colon \lambda &\longmapsto -\nu \\ \mu &\longmapsto -\mu \\ \nu &\longmapsto -\lambda \end{aligned}$$

so $s_\lambda s_\mu$ rotates $(\lambda, \mu, \nu)$ to $(\mu, \nu, \lambda)$. Thus

$$p_{\lambda,\mu} = p_{\mu,\nu} = p_{\nu,\lambda}\,,$$

and the Geometric Lemma is verified in this case.

**Case 2.** Suppose now that $\|\lambda\| > \|\mu\| = \|\nu\|$.




We may set these last to 1 and set $\|\lambda\|^2 = n > 1$. By Lemma 2.3 we must have $\langle \nu, \lambda^\vee\rangle = -1$. By (1.2)

$$\langle \lambda, \nu^\vee\rangle \cdot \|\nu\|^2 = \langle \nu, \lambda^\vee\rangle \cdot \|\lambda\|^2$$

we must also have $\langle \lambda, \nu^\vee\rangle = -n < -1$. By symmetry $\langle \lambda, \mu^\vee\rangle = -n$, and hence $\lambda \bullet \mu < 0$.

But then $\lambda$ and $-\mu = \lambda + \nu$ must also lie in the initial half of the $\nu$-string through $\lambda$, and therefore by Lemma 2.2 its beginning must be $\lambda$. Therefore

$$p_{\nu,\lambda} = 0\,.$$

and

$$\frac{p_{\nu,\lambda} + 1}{\|\mu\|^2} = 1\,,$$

What must now be shown is that

$$\frac{p_{\mu,\nu} + 1}{\|\lambda\|^2} = 1, \quad \frac{p_{\lambda,\mu} + 1}{\|\nu\|^2} = 1$$

or, equivalently,

$$p_{\mu,\nu} = n - 1, \quad p_{\lambda,\mu} = 0\,.$$

If $\mu$ were not at the beginning of its $\lambda$-string, there would exist a root $\mu - \lambda$ of squared-length

$$\|\mu - \lambda\|^2 = \|\mu\|^2 - 2(\mu \bullet \lambda) + \|\lambda\|^2$$

greater than that of $\lambda$. Again by Lemma 2.2, this cannot happen. Hence $p_{\lambda,\mu} = 0$.

It remains to show that $p_{\mu,\nu} = n - 1$. But $\nu + \mu = -\lambda$, which by length considerations must be at the end of its $\mu$ string. But the start of this string is $-s_\mu\lambda = -\lambda - n\mu$, which therefore has length $n$. Since $\nu$ is one back from the terminus, $p_{\mu,\nu} = n - 1$.

This concludes the proof of the Geometric Lemma. ∎

### 3. Representations of SL(2)

Let $\pi_n$ be the representation of $\mathfrak{sl}_2$ on the space $S^nV$ of symbolic powers of the standard representation on $V = \mathbb{C}^2$. It has dimension $n$. If

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then a basis of $S^nV$ is made up of the products $w_{[k]} = u^{[k]}v^{[n-k]}$ in which

$$u^{[k]} = \frac{u^k}{k!}, \quad v^{[\ell]} = \frac{v^\ell}{\ell!} .$$

If

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .$$

then under $\pi_n$

$$e\colon\ w_{[k]} \longmapsto (k+1)w_{[k+1]} .$$

If

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

then

$$\sigma\colon\ w_{[k]} \longmapsto (-1)^k w_{[n-k]} .$$

Let $d = n+1$, assumed to be even. The weights of this with respect to $t$ are

$$-d, \ldots, -1, 1, \ldots, d \,,$$

The formulas above imply immediately:

**3.1. Proposition.** *Suppose $v$ to be a vector of weight $-1$ in an irreducible representation $\pi$ of $\mathrm{SL}_2$ of even dimension $d$. Then $\pi(e_+)v$ and $\pi(\sigma)v$ are both of weight $1$, and*

$$\pi(e)v = -(-1)^{d/2} \left(\frac{d}{2}\right) \pi(\sigma)v \,.$$

What does this have to do with structure constants? Let $\pi$ be the representation on $\mathfrak{g}$ corresponmding to the $\lambda$-string through $\mu$. Since $\theta(\sigma) = \sigma$ in $\mathrm{SL}_2$ and the root embeddings of $\mathrm{SL}_2$ commute with $\theta$, we have $\sigma_\lambda e_\mu \sigma_\lambda^{-1} = \pm e_{-\nu}$ since $s_\lambda \mu = -\nu$. In this case we have $d/2 = p_{\lambda,\mu} + 1$, so

$$[e_\lambda, e_\mu] = \pm(p_{\lambda,\mu} + 1)e_{-\nu}$$

in this case. The proof of Chevalley's theorem is concluded.

#### 4. Remarks on integral structures

If $\pi$ is any irreducible representation of $\mathrm{SL}_2$ with lowest weight vector $w_{[0]}$ and we choose basis as above, then

$$\frac{e_+^k}{k!} \cdot w_{[0]} = w_{[k]} \,.$$

If we choose this basis on a $\lambda$-string, then it is an immediate consequence of formulas given above that

$$[e_\lambda, e_\mu] = \pm(p_{\lambda,\mu} + 1)e_{\lambda+\mu}$$

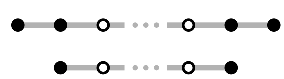
for all $\mu$ in the string (except the terminal weight). Furthermore, the $\mathbb{Z}$-module with the $w_{[k]}$ as basis is stable under $\mathrm{SL}_2(\mathbb{Z})$. (This follows from the formula for the exponential of $te_+$.) It is very natural to ask, can one find a basis of the adjoint action on all of $\mathfrak{g}$ which restricts to $\pm$ this basis on every root string? The answer is yes, as is shown in [Steinberg:1967]. But alas! this line of reasoning is circular, since Steinberg's proof relies on Chevalley's formula. However, it does explain why one might expect Chevalley's formula to hold, since it suggests that it is equivalent to the existence of a good integral structure on $\mathfrak{g}$ and its irreducible representations.

Chevalley's Tôhoku paper explains the connection between his formula and integral structure, but the clearest account I know of is in [Steinberg:1967]. [Kostant:1966] has a more direct, if apparently incomplete, construction of the affine ring of certain semi-simple groups. The recent paper [Lusztig:2009] elaborates on Kostant's construction. [Demazure-Grothendieck:1962–4] constructs split reductive group schemes over general bases, but basically just reduces the general case to that of groups over $\mathbb{C}$.

#### 5. Kac-Moody algebras

In this section I want to sketch briefly what modifications of the argument above are necessary to prove Chevalley's formula for Kac-Moody algebras. That it remains valid for these algebras was presumably first shown by Tits (see [Tits:1987]), and a proof along the lines of Chevalley's proof for classical algebras is sketched in [Morita:1987].

I have in fact tried to word my arguments so that very little change is needed to make it work for generally. The first modification is that in the statement of the theorem one must assume that $\lambda$, $\mu$, and $\lambda + \mu$ are all real roots, which is to say Weyl transforms of simple roots. So a Tits triple is now defined to be a set of real roots $\lambda$, $\mu$, $\nu$ such that $\lambda + \mu + \nu = 0$. Any such triple can be conjugated into a system of rank two, and it is easy to see that in this case there exists a Weyl-invariant inner product that is positive on real roots. It remains true that root strings involving real roots are finite, hence associated to representations of $\mathrm{SL}_2$, and also that real roots in a root string can have at most two lengths. In addition to the root string configurations we have seen earlier, these are also hence possible:



*Additional root strings for KM algebras—real roots dark*

No serious change has to be made in any of the proofs in the section on root geometry. Indeed, it was with this in mind that I chose the notation I used in the proof of the Geometric Lemma. Details can be found in [Casselman:2013].

## 6. References

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